# THE MOD 2 COHOMOLOGY RINGS OF $\mathrm{SL}_2$ OF THE IMAGINARY QUADRATIC INTEGERS

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(WITH AN APPENDIX BY AUREL PAGE)

Abstract. When a group acts cellularly on a space, the torsion subcomplex contains all of the homological information at a given prime number. The recently introduced technique of torsion subcomplex reduction has been successfully applied under certain hypotheses to determine the Farrell cohomology of discrete groups. In this paper, we extend the torsion subcomplex reduction technique to the case where the kernel of the group action is nontrivial. This is accomplished by algebraically excising the torsion subcomplex from the equivariant spectral sequence. We demonstrate the power of this "découpage in equivariant cohomology" by establishing general dimension formulae for the second page of the equivariant spectral sequence of the action of the $\mathrm{SL}_2$ groups over imaginary quadratic integers on their associated symmetric space. Then, we show how to obtain the mod 2 cohomology rings of our groups from this information.

## 1. Introduction

The objects of study in this paper are the groups $\mathrm{SL}_2$ over the ring $\mathcal{O}_{-m}$ of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$, with $m$ a square-free positive integer. These groups, as well as their central quotients $\mathrm{PSL}_2(\mathcal{O}_{-m})$, are known as Bianchi groups. The determination of the (co)homology of Bianchi groups, motivated in [17], has a long history of case-by-case computations (see [15] for a list of references). This changed with the recently introduced technique of torsion subcomplex reduction, which provided general formulae for the Farrell cohomology of the $\mathrm{PSL}_2(\mathcal{O}_{-m})$ groups for any $m$ [13]. In this article, we extend the subcomplex reduction technique to the case where the kernel of a group action is nontrivial in order to obtain the mod 2 cohomology rings of the $\mathrm{SL}_2(\mathcal{O}_{-m})$ groups. For this purpose, we excise the non-central torsion subcomplex from the equivariant spectral sequence. This new method of découpage in equivariant cohomology is developed in a fairly general setting in Section 4 . The reason why we consider only $\mathbb{F}_2$–coeffients is that for any prime $\ell > 2$, the Lyndon–Hochschild–Serre spectral sequence with $\mathbb{F}_\ell$–coefficients associated to the central extension

$$1 \to \{\pm 1\} \longrightarrow \mathrm{SL}_2(\mathcal{O}_{-m}) \longrightarrow \mathrm{PSL}_2(\mathcal{O}_{-m}) \to 1$$

is concentrated in the horizontal axis, yielding $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}); \mathbb{F}_\ell) = \mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{F}_\ell)$. Furthermore, Bianchi groups contain only 2– and 3–torsion; results for $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{F}_3)$ can be found in [13].

Philosophically, the cohomology of the Bianchi groups can be thought of as coming from two different places. Since the Bianchi groups act on a 2–dimensional retract of hyperbolic 3–space $\mathcal{H}^3$, the Bianchi groups have virtual (co)homological dimension 2. Consequently, in dimensions 2 and below, many of the cohomology classes of the Bianchi groups come from the topology of the quotient space and are detected with rational coefficients [16]; calculations exclusively for rational coefficients have been carried out in [19]. Above the virtual cohomological dimension, all cohomology classes are torsion and originate from finite subgroups. In particular, one can use the equivariant spectral sequence to determine the (co)homology of Bianchi groups. Results of the second author have made this precise. The article [12] introduced the $\ell$–torsion subcomplex, which was used to prove that the homology above dimension 2 of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ is determined by this subcomplex; in particular, that this homology is a direct sum of generic modules associated to the homeomorphism types of connected components of the subcomplex.

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In this work, we will show how to extend the torsion subcomplex results from the projective special linear group to the linear group. The difficulty that we need to overcome is that the kernel of the action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on $\mathcal{H}^3$ is nontrivial. We solve this problem with our new "découpage" technique in Section 4 . We then apply it in Section 6 to establish general dimension formulae for polynomial classes on the second page of the equivariant spectral sequence with $\mathbb{F}_2$–coefficients for the action of the $\mathrm{SL}_2(\mathcal{O}_{-m})$ groups on their associated symmetric space. The last piece of information that we need in order to determine the mod 2 cohomology of the $\mathrm{SL}_2$ Bianchi groups is the rank of the second page differential. We can say a lot about this rank by individually analyzing component types of the reduced 2–torsion subcomplex using the cohomology ring structure. This is the subject of Section 9 .

We conclude with some sample calculations. For instance, Example ($\iota$) in Section 10 states that for $m \equiv 3 \mod 8$ such that $\mathbb{Q}(\sqrt{-m})$ has precisely one finite ramification place over $\mathbb{Q}$, and such that the ideal class number of the totally real number field $\mathbb{Q}(\sqrt{m})$ is 1, our cohomology ring has the following dimensions;

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-m})\,;\,\mathbb{F}_2) = \begin{cases} \beta^1+\beta^2, & q = 4k+5, \\ \beta^1+\beta^2+2, & q = 4k+4, \\ \beta^1+\beta^2+3, & q = 4k+3, \\ \beta^1+\beta^2+1, & q = 4k+2, \\ \beta^1, & q = 1, \end{cases}$$

where $\beta^q := \dim_{\mathbb{F}_2} \mathrm{H}^q({}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3;\,\mathbb{F}_2)$.
For absolute values of the discriminant less than 296,

$$\beta^2+1 = \beta^1 = \dim_{\mathbb{Q}} \mathrm{H}_1({}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3;\,\mathbb{Q}) =: \beta_1;$$

$m$ and $\beta_1$ in this range are given as follows (the Betti numbers are taken from [15]).

| $m$ | 11 | 19 | 43 | 59 | 67 | 83 | 107 | 131 | 139 | 163 | 179 | 211 | 227 | 251 | 283 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ | 1 | 1 | 2 | 4 | 3 | 5 | 6 | 8 | 7 | 7 | 10 | 10 | 12 | 14 | 13 |

We have used here that our above conditions on $\mathbb{Q}(\sqrt{-m})$ and $\mathbb{Q}(\sqrt{m})$ are equivalent to the quotient of the 2–torsion subcomplex having the shape $\bullet\!-\!\bullet$, as worked out in [13].

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## 2. The non-central torsion subcomplex

In this section we recall the $\ell$–torsion subcomplexes theory of [13] and compare it with the *non-central* $\ell$–torsion subcomplexes we are going to study. We require any discrete group $\Gamma$ under our study to be provided with a *$\Gamma$–cell complex*, that is a finite-dimensional cell complex $X$ with cellular $\Gamma$–action such that each cell stabilizer fixes its cell point-wise. Let $\ell$ be a prime number.

***Definition* 1.** The *$\ell$–torsion subcomplex* of a $\Gamma$–cell complex $X$ consists of all the cells of $X$ whose stabilizers in $\Gamma$ contain elements of order $\ell$.

We further require that the fixed point set $X^G$ be acyclic for every nontrivial finite $\ell$–subgroup $G$ of $\Gamma$. Then Brown's proposition X.(7.2) [4] specializes as follows.

**Proposition 2.** *There is an isomorphism between the $\ell$–primary parts of the Farrell cohomology of $\Gamma$ and the $\Gamma$–equivariant Farrell cohomology of the $\ell$–torsion subcomplex.*

For a given Bianchi group, the non-central torsion subcomplex can be quite large. It turns out to be useful to reduce this subcomplex, and we identify two conditions under which we can do this in a way that Proposition 2 still holds.

***Condition A.*** In the $\ell$–torsion subcomplex, let $\sigma$ be a cell of dimension $n-1$ which lies in the boundary of precisely two $n$–cells representing different orbits, $\tau_1$ and $\tau_2$. Assume further that no higher-dimensional cells of the $\ell$–torsion subcomplex touch $\sigma$; and that the $n$–cell stabilizers admit an isomorphism $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$.

***Condition B.*** The inclusion of the cell stabilizer $\Gamma_\sigma$ into $\Gamma_{\tau_1}$ and $\Gamma_{\tau_2}$ induces isomorphisms on mod $\ell$ cohomology.

When both conditions are satisfied in the $\ell$–torsion subcomplex, we merge the cells $\tau_1$ and $\tau_2$ along $\sigma$ and do so for their entire orbits. The effect of this merging is to decease the size of the $\ell$–torsion subcomplex without changing its $\Gamma$–equivariant Farrell cohomology. This process can often be repeated: by a "terminal vertex," we will denote a vertex with no adjacent higher-dimensional cells and precisely one adjacent edge in the quotient space, and by "cutting off" the latter edge, we will mean that we remove the edge together with the terminal vertex from our cell complex.

***Definition 3.*** The *reduced $\ell$–torsion subcomplex* associated to a $\Gamma$–cell complex $X$ is the cell complex obtained by recursively merging orbit-wise all the pairs of cells satisfying conditions $A$ and $B$, and cutting off edges that admit a terminal vertex when condition $B$ is satisfied.

The following theorem, stating that Proposition 2 still holds after reducing, is proved in [13].

**Theorem 4.** *There is an isomorphism between the $\ell$–primary parts of the Farrell cohomology of $\Gamma$ and the $\Gamma$–equivariant Farrell cohomology of the reduced $\ell$–torsion subcomplex.*

In the case of a trivial kernel of the action on the $\Gamma$–cell complex, this allows one to establish general formulae for the Farrell cohomology of $\Gamma$. In contrast, our action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on $\mathcal{H}^3$ has the 2-torsion group $\{\pm 1\}$ in the kernel; since every cell stabilizer contains 2-torsion, the 2–torsion subcomplex does not ease our calculation in any way. We can remedy this situation by considering the following object, on whose cells we impose a supplementary property.

***Definition 5.*** The *non-central $\ell$–torsion subcomplex* of a $\Gamma$–cell complex $X$ consists of all the cells of $X$ whose stabilizers in $\Gamma$ contain elements of order $\ell$ that are not in the center of $\Gamma$.

We note that this definition yields a correspondence between the *non-central* $\ell$–torsion subcomplex for a group acting by conjugation and the $\ell$–torsion subcomplex for its central quotient group. We use this correspondence in order to identify the *non-central* $\ell$–torsion subcomplex for the action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on hyperbolic 3–space as the $\ell$–torsion subcomplex of $\mathrm{PSL}_2(\mathcal{O}_{-m})$. On the latter subcomplex, we have the information from [11, 13]. However, incorporating the non-central condition for $\mathrm{SL}_2(\mathcal{O}_{-m})$ introduces significant technical obstacles, and we develop the "découpage" technique in Section 4 in order to overcome them.

## 3. Spectral sequences and Central Extensions

The calculations in this paper involve two spectral sequences. We use the Lyndon–Hochschild–Serre spectral sequence to determine cohomology of groups via group extensions. We also use the equivariant spectral sequence since the Bianchi groups act cellularly on low-dimensional contractible complexes. We introduce both spectral sequences briefly in this section; more details can be found in [1], [4], and [9].

For the development of the Lyndon–Hochschild–Serre spectral sequence we follow the approach in [1]. Given a short exact sequence of groups

$$1 \to H \to \Gamma \to \Gamma/H \to 1 \tag{1}$$

there is an induced surjection of classifying spaces $B_\rho : B_\Gamma \longrightarrow B_{\Gamma/H}$. The mapping cylinder of $B_\rho$ is homotopy equivalent to $B_H$. One can then apply the Leray-Serre spectral sequence ([9], Chapters 5 and 6) to the resulting fibration. The spectral sequence that emerges has $E_2^{i,j} \cong \mathrm{H}^i(\Gamma/H; \mathrm{H}^j(H; M))$ for untwisted coefficients $M$ and converges to $\mathrm{H}^{i+j}(\Gamma; M)$. We note that this spectral sequence can also be developed more algebraically, like in [4], VII.5.

In the short exact sequence of groups in Equation 1, when the normal subgroup $H$ is central in $\Gamma$, it is possible to say more. This is a central extension, and in such cases $\Gamma/H$ acts trivially on the homotopy fiber $B_H$. Then with field coefficients $M$, the $E_2$–term of the resulting spectral sequence has the form

$$E_2^{i,j} \cong \mathrm{H}^i(\Gamma/H; M) \otimes \mathrm{H}^j(H; M).$$

Furthermore, Lemma IV.1.12 in [1] identifies a unique cohomology class $k$, called the *$k$–invariant*, which generates the kernel of $B_\pi^* : \mathrm{H}^2(\Gamma/H) \to \mathrm{H}^2(\Gamma)$. The $k$–invariant is also the cohomology class associated to the extension.

The Leray-Serre spectral sequence, of which the Lyndon–Hochschild–Serre spectral sequence is just one example, has many useful properties. It is compatible with cup products, for example. We will also use the fact that the action of the Steenrod algebra commutes with the differentials in the spectral sequence. That is, for a cohomology class $u$ on the $E_r$ page, $d_r(Sq^I u) = Sq^I d_r(u)$.

For the equivariant spectral sequence, we follow the development in [4, Chapter VII]. Let the group $\Gamma$ act cellularly on a space $X$ in such a way that the stabilizer of a cell $\sigma$ fixes the cell pointwise. Consider the *equivariant cohomology groups* $\mathrm{H}^*(\Gamma, C^\bullet(X))$ with coefficients in the cochain complex $C^\bullet(X)$. We define these groups by taking a projective resolution, $F$, of $\mathbb{F}_2\Gamma$ over $\mathbb{F}_2$, then setting $\mathrm{H}^*(\Gamma, C^\bullet(X)) = \mathrm{H}^*(\mathrm{Hom}_{\mathbb{F}_2}(F, C^\bullet(X)))$. When $X$ is a contractible space, the equivariant cohomology groups can be identified with the cohomology of $\Gamma$, as

$$\mathrm{H}^*(\Gamma, C^\bullet(X; \mathbb{F}_2)) \cong \mathrm{H}^*(\Gamma, C^\bullet(\text{point}; \mathbb{F}_2)) \cong \mathrm{H}^*(\Gamma; \mathbb{F}_2).$$

Using the horizontal and vertical filtrations of the double complex $\mathrm{Hom}_{\mathbb{F}_2}(F, C^\bullet(X))$ we get a spectral sequence with

$$E_1^{i,j} \cong \mathrm{H}^j(\Gamma, C^i(X; \mathbb{F}_2))$$

converging to $\mathrm{H}^{i+j}(\Gamma; \mathbb{F}_2)$. Let $X_i$ be a set of representatives of $i$–cells in $X$ and let $\Gamma_\sigma$ be the stabilizer in $\Gamma$ of a cell $\sigma$. Then

$$C^i(X) = \bigoplus_{\sigma \in X_i} \mathbb{F}_2 \cong \bigoplus_{\sigma \,\in\, {}_\Gamma\backslash X_i} \mathrm{Coind}_{\Gamma_\sigma}^{\Gamma} \mathbb{F}_2$$

Once we apply Shapiro's Lemma and sum over the representatives of $i$–cells in $\Gamma\backslash X_i$, the spectral sequence takes the form

$$E_1^{i,j} \cong \bigoplus_{\sigma \,\in\, {}_\Gamma\backslash X_i} \mathrm{H}^j(\Gamma_\sigma; \mathbb{F}_2).$$

The equivariant spectral sequence has a number of additional desirable properties that we will use in the sequel.

(1) There is a product on the spectral sequence, $E_r^{pq} \otimes E_r^{st} \to E_r^{p+s,q+t}$, which is compatible with the standard cup product on $\mathrm{H}^*(\Gamma)$ ([4, VII.5])
(2) On the $E_2$–page, the products in $E_2^{0,*}$, the vertical edge, are compatible with the products in $\prod_{\sigma \in \Sigma_0} \mathrm{H}^q(\Gamma_\sigma)$. ([4, X.4.5.vi]).
(3) The differential $d_1$ is a difference of restriction maps (cohomology analog of [4, VII.8])

$$\bigoplus_{\sigma \,\in\, {}_\Gamma\backslash X_i} \mathrm{H}^j(\Gamma_\sigma; \mathbb{F}_2) \xrightarrow{d_1^{i,j}} \bigoplus_{\tau \,\in\, {}_\Gamma\backslash X_{i+1}} \mathrm{H}^j(\Gamma_\tau; \mathbb{F}_2).$$

When ${}_\Gamma\backslash X$ is a graph, say a line segment or a single edge and vertex loop, then the spectral sequence consists of only two columns and collapses at the $E_2$ term to give a Wang sequence where the appropriate map is the $d_1$ differential. This observation allows for the straightforward calculations for cohomology of amalgamated products and HNN extensions.

**Remark 6.** In both the equivariant spectral sequence and the Lyndon–Hochschild–Serre spectral sequence the differentials are also derivations. Given cohomology classes $u$ and $v$ and working mod 2, this means that

$$d_r(uv) = d_r(u)v + ud_r(v).$$

This helpful property implies two useful results: 1) when $d_r(u) = 0$, then $d_r(uv) = ud_r(v)$; and 2) when $u$ is a class on the $E_r$ page, then $d_r(u^2) = 2d_r(u) = 0$.

## 4. Découpage in equivariant cohomology

Let $\Gamma$ be a discrete group, $n$ and $q$ be integers, and $\ell$ be a prime number. Let $X$ be a cellular $\Gamma$–complex on which the cell stabilizers fix their cells pointwise and the center $\mathcal{Z}$ of $\Gamma$ acts trivially. Denote by $\Gamma_\sigma$ the stabilizer in $\Gamma$ of a cell $\sigma$, and by $T$ the non-central $\ell$–torsion subcomplex of $(X, \Gamma)$ specified in Definition 5. Throughout this section, we exclusively consider cohomology with trivial $\mathbb{F}_\ell$–coefficients. We distinguish the classes in the polynomial subrings of the cohomology rings of the finite subgroups in $\Gamma$, and call them "polynomial cohomology classes." They enter into the $E_1$ page of the equivariant spectral sequence of $(X, \Gamma)$ via the cell stabilizers. By abuse of notation, we will also call their quotients on the latter pages of this sequence "polynomial cohomology classes", and do the same with classes in $\mathrm{H}^*(\Gamma)$ to which they converge. We will denote the collection of all such classes by $\mathrm{H}^*_{\mathrm{poly}(X)}(\Gamma)$, where the symbol "$X$" in the subscript refers to the $\Gamma$–action on $X$. We will call any cohomology class in $\Gamma$ not satifying this condition a "non-polynomial cohomology class." The collection of all such classes will be denoted by $\mathrm{H}^*_{\mathrm{non}(X)}(\Gamma)$. $\mathrm{H}^*_{\mathrm{poly}(X)}(\Gamma)$ will contain all polynomial classes of $H^*(\Gamma)$, but may contain nilpotent classes as well, for example, any class that comes from a polynomial generator in the first column rightward of the spectral sequence. Our aim is now to calculate $\mathrm{H}^*_{\mathrm{poly}(X)}(\Gamma)$.

***Hypothesis* 7.** We require that the cohomology rings of cell stabilizers can be expressed as a polynomial subring tensored with the ring generated by nilpotent cohomology classes. We also require that for any cell $\tau \in T$, the inclusion of the center $\mathcal{Z} \hookrightarrow \Gamma_\tau$ into the cell stabilizer of $\tau$ induces a monomorphism on the polynomial parts of the cohomology rings, $\mathrm{H}^*_{\mathrm{poly}(X)}(\Gamma_\tau) \hookrightarrow \mathrm{H}^*_{\mathrm{poly}(X)}(\mathcal{Z})$. Furthermore, for any cell $\sigma \in X$ not in $T$, we require that the inclusion $\mathcal{Z} \hookrightarrow \Gamma_\sigma$ induces an isomorphism $\mathrm{H}^*_{\mathrm{poly}(X)}(\Gamma_\sigma) \to \mathrm{H}^*_{\mathrm{poly}(X)}(\mathcal{Z})$.

Denote by ${}_\Gamma\backslash X^{(n)}$ a system of representatives of the $n$–cells of $X$ under the action of $\Gamma$, and analogously ${}_\Gamma\backslash T^{(n)}$. For fixed $q$, consider the cokernel of the map of $\Gamma$–equivariant co-chain complexes

$$0 \to \bigoplus_{\tau \in {}_\Gamma\backslash T^{(n)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\Gamma_\tau) \hookrightarrow \bigoplus_{\tau \in {}_\Gamma\backslash T^{(n)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\mathcal{Z}) \to \mathrm{Cokern}^q_n \to 0. \tag{2}$$

Because $\mathrm{H}^*_{\mathrm{poly}(X)}(\Gamma_\sigma) \to \mathrm{H}^*_{\mathrm{poly}(X)}(\mathcal{Z})$ is an isomorphism for $\sigma \in X$ not in $T$, there is another induced short exact sequence of co-chain complexes:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & \bigoplus\limits_{\sigma \in {}_\Gamma\backslash X^{(2)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\Gamma_\sigma) \arrow[r] \arrow[u, "d_1^{2,q}"] & \bigoplus\limits_{\sigma \in {}_\Gamma\backslash X^{(2)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\mathcal{Z}) \arrow[r] \arrow[u, "\delta^{2,q}"] & \mathrm{Cokern}^q_2 \arrow[r] \arrow[u, "\partial^{2,q}"] & 0 \\
0 \arrow[r] & \bigoplus\limits_{\sigma \in {}_\Gamma\backslash X^{(1)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\Gamma_\sigma) \arrow[r] \arrow[u, "d_1^{1,q}"] & \bigoplus\limits_{\sigma \in {}_\Gamma\backslash X^{(1)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\mathcal{Z}) \arrow[r] \arrow[u, "\delta^{1,q}"] & \mathrm{Cokern}^q_1 \arrow[r] \arrow[u, "\partial^{1,q}"] & 0 \\
0 \arrow[r] & \bigoplus\limits_{\sigma \in {}_\Gamma\backslash X^{(0)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\Gamma_\sigma) \arrow[r] \arrow[u, "d_1^{0,q}"] & \bigoplus\limits_{\sigma \in {}_\Gamma\backslash X^{(0)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\mathcal{Z}) \arrow[r] \arrow[u, "\delta^{0,q}"] & \mathrm{Cokern}^q_0 \arrow[r] \arrow[u, "\partial^{0,q}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

(3)

The differentials $\delta^{n,q}$ are given by the topological co-boundary maps of the cell complex $X$, and the maps $\partial^{n,q}$ are induced by the short exact sequence (2).

The column in the above diagram consisting of the terms $\bigoplus\limits_{\sigma \in {}_\Gamma\backslash X^{(n)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\Gamma_\sigma)$ is the $q$-th row of the polynomial part of the $E_1$ page of the equivariant spectral sequence associated to the action of $\Gamma$ on $X$. Hence, passing to the homology of its differentials produces the polynomial part of the $E_2$ page of this sequence. Similarly, we obtain the polynomial part of the $q$-th row of the $E_2$ page of the equivariant spectral sequence associated to the trivial action of $\mathcal{Z}$ on ${}_\Gamma\backslash X$ as the homology of the column containing the $\bigoplus\limits_{\tau \in {}_\Gamma\backslash X^{(n)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\mathcal{Z})$ terms. Applying the snake lemma, this yields the long exact sequence

$$\begin{tikzcd}
 & \cdots & & \\
 & \mathrm{poly}(E_2^{2,q}(X,\Gamma)) \arrow[r] & \mathrm{poly}(E_2^{2,q}({}_\Gamma\backslash X, \mathcal{Z})) \arrow[r] & \ker \partial^{2,q}/\mathrm{image}\, \partial^{1,q} \arrow[ull] \\
 & \mathrm{poly}(E_2^{1,q}(X,\Gamma)) \arrow[r] & \mathrm{poly}(E_2^{1,q}({}_\Gamma\backslash X, \mathcal{Z})) \arrow[r] & \ker \partial^{1,q}/\mathrm{image}\, \partial^{0,q} \arrow[ull] \\
0 \arrow[r] & \mathrm{poly}(E_2^{0,q}(X,\Gamma)) \arrow[r] & \mathrm{poly}(E_2^{0,q}({}_\Gamma\backslash X, \mathcal{Z})) \arrow[r] & \ker \partial^{0,q} \arrow[ull]
\end{tikzcd}$$

Here, $\mathrm{poly}(E_2^{n,q}(X,\Gamma))$ is the polynomial part of the $(n, q)$–th term of the second page of the spectral sequence converging to the $\Gamma$–equivariant cohomology of $X$ and $\mathrm{poly}(E_2^{n,q}({}_\Gamma\backslash X, \mathcal{Z}))$ the one for the trivial action of $\mathcal{Z}$ on ${}_\Gamma\backslash X$.

## 5. Maps induced by finite subgroups in the Bianchi groups

The polynomial and non-polynomial cohomology of Bianchi groups are determined by their originating classes in the equivariant spectral sequence associated to the action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on hyperbolic 3–space, $\mathcal{H}^3$. Therefore, we need to classify the possible cell stabilizers of the $\mathrm{SL}_2(\mathcal{O}_{-m})$ action and determine the restriction maps between subgroups. Those tasks are the goals of this section.

Since the action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on $\mathcal{H}^3$ is properly discontinuous, cell stabilizers are finite subgroups of $\mathrm{SL}_2(\mathcal{O}_{-m})$. The enumeration of the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ is a classical result, and the list of finite subgroups which appear in the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ is also well known [7]: They are the cyclic groups 1, $\mathbb{Z}/2$ and $\mathbb{Z}/3$, the Klein four group $\mathbf{D}_2$, the symmetric group on three letters $\mathbf{S}_3$, and the alternating group on four letters $\mathbf{A}_4$. The preimages of these groups in $\mathrm{SL}_2(\mathcal{O}_{-m})$ are the cyclic groups $\mathbb{Z}/2$, $\mathbb{Z}/4$, $\mathbb{Z}/6$, the quaternion group $\mathbf{Q}_8$ of order 8, the dicyclic group **Di** of order 12, and the binary tetrahedral group **Te** of order 24.

We start by recalling some mod 2 cohomology rings. In what follows, the symbol $\mathbb{F}_2$ represents the field of two elements and all the cohomology rings and groups in which we omit the coefficients, are meant to be taken with (obviously trivial) $\mathbb{F}_2$–coefficients. We write polynomial cohomology classes [in square brackets] and nilpotent cohomology classes (in parentheses). The index of a class specifies its degree.

**Theorem 8.** [1] *The cohomology rings for finite subgroups of Bianchi groups are given below, where a subscript denotes the degree of the generator.*

$$\begin{aligned}
\mathrm{H}^*(\mathbb{Z}/2) &\cong \mathrm{H}^*(\mathbf{S}_3) \cong \mathbb{F}_2[x_1] \\
\mathrm{H}^*(\mathbb{Z}/3) &\cong \mathrm{H}^*(1) \cong \mathbb{F}_2 \\
\mathrm{H}^*(\mathbf{D}_2) &\cong \mathbb{F}_2[x_1, y_1] \\
\mathrm{H}^*(\mathbf{A}_4) &\cong \mathbb{F}_2[u_2, v_3, w_3]/(u_2^3 + v_3^2 + w_3^2 + v_3 w_3)
\end{aligned}$$

□

We use the Lyndon–Hochschild–Serre spectral sequence to determine the cohomology rings of the finite subgroups of $\mathrm{SL}_2(\mathcal{O}_{-m})$. In particular, for each finite subgroup $G$ of $\mathrm{SL}_2(\mathcal{O}_{-m})$, there is a corresponding finite subgroup $Q$ of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ which fits into the central extension

$$1 \to \mathbb{Z}/2 \longrightarrow G \longrightarrow Q \to 1$$

where $\mathbb{Z}/2$ is the subgroup of $\mathrm{SL}_2(\mathcal{O}_{-m})$ containing $\{\pm I\}$. We note that all finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ also sit inside $\mathrm{SU}(2)$ and act freely on it. Since $\mathrm{SU}(2)$ can be identified with the 3–sphere, the cohomology rings for the finite subgroups of $\mathrm{SL}_2(\mathcal{O}_{-m})$ are periodic of period dividing 4 [4]. In particular, the cohomology rings can all be expressed as a tensor product of a polynomial ring on one generator with an exterior algebra.

**Theorem 9.** *The following are the mod* 2 *cohomology rings of the finite subgroups of* $\mathrm{SL}_2(\mathcal{O}_{-m})$.

$$\begin{aligned}
\mathrm{H}^*(\mathbb{Z}/4) &\cong \mathrm{H}^*(\mathbf{Di}) \cong \mathbb{F}_2[e_2](b_1)\\
\mathrm{H}^*(\mathbb{Z}/2) &\cong \mathrm{H}^*(\mathbb{Z}/6) \cong \mathbb{F}_2[e_1]\\
\mathrm{H}^*(\mathbf{Q}_8) &\cong \mathbb{F}_2[e_4](x_1, y_1, x_2, y_2, x_3)\\
\mathrm{H}^*(\mathbf{Te}) &\cong \mathbb{F}_2[e_4](b_3)
\end{aligned}$$

*Proof.* The cohomology results for cyclic group results are straightforward, and the calculation of the other cohomology rings (and a classification for all periodic groups) are contained in [1]. However, we briefly review the derivations here as the descriptions will be useful when we determine restriction maps to subgroups.

The dicyclic group $\mathbf{Di}$ is a semidirect product and fits into the short exact sequence

$$1 \to \mathbb{Z}/3 \longrightarrow \mathbb{Z}/3 \rtimes \mathbb{Z}/4 \longrightarrow \mathbb{Z}/4 \to 1.$$

Since $\mathrm{H}^*(\mathbb{Z}/3)$ has trivial mod 2 homology, in the Lyndon–Hochschild–Serre spectral sequence the only nontrivial cohomology occurs on the horizontal axis and is isomorphic to $\mathrm{H}^*(\mathbb{Z}/4)$.

For $\mathbb{Z}/4$, $\mathbf{Q}_8$ and $\mathbf{Te}$, we again use the Lyndon–Hochschild–Serre spectral sequence for central extensions. In the spectral sequence of the extension, let $\mathbb{F}_2[e_1]$ be the cohomology ring corresponding to the central subgroup $\mathbb{Z}/2$. On the $E_2$ page of the spectral sequence, the image of $d_2(e_1)$ can be identified with the $k$–invariant. In addition, the Lyndon–Hochschild–Serre spectral sequence is compatible with the Steenrod squaring operation, and so the Kudo transgression theorem describes the images of $d_{2n+1}((e_1)^{2n})$.

We start with the calculation of $\mathrm{H}^*(\mathbb{Z}/4)$. Let $\mathrm{H}^*(\mathbb{Z}/2) \cong \mathbb{F}_2[x_1]$, and consider the spectral sequence associated to the central extension. In this case, the $k$–invariant $d_2(e_1) = x_1^2$, and in particular $d_2(e_1 x_i^k) = x_i^{k+2}$ and $d_2(e_1^2) = 0$. As the result, the spectral sequence collapses at the $E_2$ page with classes in even rows and in columns 0 and 1 only. The class $e_1^2$ represents the polynomial class in $\mathrm{H}^2(\mathbb{Z}/4)$, and the class in $E_2^{0,1}$ is the one-dimensional exterior class.

We next calculate $\mathrm{H}^*(\mathbf{Q}_8)$. The extension is central, so $E_1^{p,q} \cong \mathrm{H}^p(\mathbf{D}_2) \otimes \mathrm{H}^q(\mathbb{Z}/2)$, and by [1, Proposition IV.2.10], the $k$–invariant $d_2(e_1) = x_1^2 + x_1 y_1 + y_1^2 \in \mathrm{H}^*(\mathbf{D}_2)$. Also, $d_2(e_1^2) = 0$, which completely determines the differentials on the $E_2$ page. Therefore, the only classes which survive to the $E_3$ page lie in even rows. Next, by compatibility with the Steenrod operations,

$$d_3(e_1^2) = d_3\left(Sq^1(e_1)\right) = Sq^1(x_1^2 + x_1 y_1 + y_1^2) = x_1^2 y_1 + x_1 y_1^2.$$

Through careful accounting, one can show that the classes that remain in $E_3$ are in columns 0 through 3, and that the spectral sequence has non-zero terms in rows congruent to 0 mod 4. Consequently, all higher differentials vanish and $E_3 \cong E_\infty$. The only classes left correspond to 1, $x_1$, $y_1$, $x_1^2$, $y_1^2$, $x_1^2 y_1 = x_1 y_1^2$ and the product of these classes with powers of $e_1^4$, which represents a four-dimensional polynomial class. The result follows.

Finally, we consider the calculation of $\mathrm{H}^*(\mathbf{Te})$, which proceeds in a similar manner as the case for $\mathrm{H}^*(\mathbf{Q}_8)$, except now the quotient group is $\mathbf{A}_4$ and we have to consider

$$\mathrm{H}^*(\mathbf{A}_4) \cong \mathbb{F}_2[u_2, v_3, w_3] \ /(u_2^3 + v_3^2 + w_3^2 + v_3 w_3).$$

Briefly, $d_2(e_1) = u_2$, and $d_3(e_1^2) = v_3 = Sq^1(u_2)$ [3]. Once again, after a careful check, the spectral sequence collapses at the $E_3$ page and the only classes left correspond to 1, the four-dimensional class represented by $e_1^4$, $w_3 \in E_3^{3,0}$, and the product of $w_3$ with powers of $e_1^4$. $\square$

**Remark 10.** Results in [1] also provide the structure of $\mathrm{H}^*(\mathbf{Q}_8)$ as a ring over three generators: $\mathrm{H}^*(\mathbf{Q}_8) \cong \mathbb{F}_2[e_4](x_1, y_1)/ < R >$, where $R$ is the ideal generated by $x_1^2+x_1y_1+y_1^2$ and $x_1^2y_1+x_1y_1^2$.

Given the following commuting diagram of groups,

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d, "i_2"] & Q_1 \arrow[d, "i_1"] \\ \mathbb{Z}/2 \arrow[r] & G_2 \arrow[r] & Q_2 \end{tikzcd}$$

if one knows the induced map $i_1^* : \mathrm{H}^*(Q_2) \longrightarrow \mathrm{H}^*(Q_1)$, it is possible to calculate the effect of $i_2^* : \mathrm{H}^*(G_2) \longrightarrow \mathrm{H}^*(G_1)$. The commutative diagram gives rise to two Lyndon–Hochschild–Serre spectral sequences, one for each extension, and the two are compatible via the isomorphism on the fiber. Specifically, the maps between the spectral sequences are given by $i_1^*$ along the $x$–axis and the identity along the $y$–axis. Once the spectral sequences converge, the result gives the effect of the restriction map $res^{G_2}_{G_1} = i_2^*$.

**Theorem 11.** *The following are the nontrivial restriction maps on cohomology generators for finite subgroups of* $\mathrm{SL}_2(\mathcal{O}_{-m})$*:*

$$\begin{aligned}
res^{\mathbb{Z}/4}_{\mathbb{Z}/2}(e_2) &= e_1^2 \\
res^{\mathbf{Di}}_{\mathbb{Z}/6}(e_2) &= res^{\mathbf{Di}}_{\mathbb{Z}/2}(e_2) = e_1^2 \\
res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(e_4) &= e_2^2, res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1) = (b_1) \\
res^{\mathbf{Q}_8}_{\mathbb{Z}/2}(e_4) &= e_1^4 \\
res^{\mathbf{Te}}_{\mathbb{Z}/6}(e_4) &= res^{\mathbf{Te}}_{\mathbb{Z}/2}(e_4) = e_1^4 \\
res^{\mathbf{Te}}_{\mathbb{Z}/4}(e_4) &= e_2^2
\end{aligned}$$

*In addition,* $res^{\mathbb{Z}/6}_{\mathbb{Z}/2}$ *and* $res^{\mathbf{Di}}_{\mathbb{Z}/4}$ *are isomorphisms.*

*Proof.* The isomorphism cases follow as $\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$ and $\mathrm{H}^*(S_3) \cong \mathrm{H}^*(\mathbb{Z}/2)$. For the rest of the restriction maps, we can prove the results concerning the polynomial classes using a Lyndon–Hochschild–Serre spectral sequence calculation in cohomology associated to the central extension

$$1 \to \mathbb{Z}/2 \longrightarrow G \longrightarrow Q \to 1.$$

In every calculation, the polynomial class of $\mathrm{H}^*(G)$ arises from the 0 column of the spectral sequence, so it corresponds to some power of $e_1^k$. The choice of $k$ invariant of the extension determines when $d_{2n}$ vanishes on the vertical axis; at this point, $e_1^{2n}$ survives to $E_\infty$ and represents the polynomial class in the abutment. This observation determines the restriction maps on polynomial classes since the map between spectral sequences on the fiber (the $y$–axis) is the identity.

To calculate $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1)$, recall that $\mathrm{H}^*(\mathbf{D}_2) = \mathbb{F}_2[s_1, t_1]$, and that if $s_1$ and $x_1 \in \mathrm{H}^*(\mathbb{Z}/2)$ represent the same group element in $\mathbf{D}_2$, then $res^{\mathbf{D}_2}_{\mathbb{Z}/2}(s_1^j) = x_1^j$ and $res^{\mathbf{D}_2}_{\mathbb{Z}/2}(s_1^j t_1^k) = 0$ when $k > 0$. After the Lyndon–Hochschild–Serre spectral sequence is applied, $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(s_1)$ surjects onto the stated class in $\mathrm{H}^1(\mathbb{Z}/4)$. We note that the other nilpotent classes of $\mathrm{H}^*(\mathbf{Q}_8)$ in dimensions 2 and 3 must restrict to 0, since from Theorem 9, the $E_2$ page for the calculation of $\mathrm{H}^*(\mathbb{Z}/4)$ is zero in the second column and to the right.

Finally, we need to show that $res^{\mathbf{Te}}_{\mathbb{Z}/4}(b_3) = 0$. We recall from the calculation of $\mathrm{H}^*(\mathbf{Te})$ in Theorem 9 that the exterior class $b_3$ is in $E_\infty^{3,0}$. However, in the calculation of $\mathrm{H}^*(\mathbb{Z}/4)$, $E_\infty^{i,j} = 0$ and $i > 1$. Therefore, the restriction map must send $b_3$ to 0. □

**Remark 12.** We note that the restriction map is natural with respect to cup products, so the results of Theorem 11 can be used to determine the effect of the restriction map on any cohomology class. In particular, the only nontrivial restriction map on classes with nilpotent components is $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(e_4^i x_1) = e_2^{2i} b_1$. In addition, in the cohomology of the groups we consider, only pure polynomial classes restrict nontrivially to pure polynomial classes.

## 6. Application of the technique to the Bianchi groups at the prime 2

We make use of a tessellation $X$ of hyperbolic 3–space $\mathcal{H}^3$ by a Bianchi group $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$ in which each cell stabilizer fixes its cell pointwise. We will apply the method of Section 4 in order to obtain a dimension formula for cohomology at the prime $\ell = 2$; this is our main result and it is stated in Theorem 18, for which we introduce the following notation.

**Notation 13.** Denote by $\chi(_\Gamma\backslash T) = \beta_0(_\Gamma\backslash T) - \beta_1(_\Gamma\backslash T)$ the Euler characteristic of the orbit space $_\Gamma\backslash T$ of the non-central 2–torsion subcomplex.

Recall that for $\mathcal{O}$ not the Gaussian or Eisensteinian integers, the Euler characteristic of the hyperbolic orbit space $_\Gamma\backslash\mathcal{H}^3$ vanishes because its boundary consists of disjoint 2–tori [17, p. 513]. Consequently, the Betti numbers $\beta_1$ and $\beta_2$ of the hyperbolic orbit space $_\Gamma\backslash\mathcal{H}^3$ satisfy $1 - \beta_1 + \beta_2 = 0$, so we can replace $\beta_2$ by $\beta_1 - 1$ when it is convenient.

**Notation 14.** For $q \in \{1, 2\}$, denote the dimension $\dim_{\mathbb{F}_2} \mathrm{H}^q(_\Gamma\backslash\mathcal{H}^3; \mathbb{F}_2)$ by $\beta^q$.

The Universal Coefficient Theorem yields

$$\mathrm{H}^q(_\Gamma\backslash\mathcal{H}^3; \mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{H}_q(_\Gamma\backslash\mathcal{H}^3; \mathbb{Z}), \mathbb{F}_2) \oplus \mathrm{Ext}(\mathrm{H}_{q-1}(_\Gamma\backslash\mathcal{H}^3; \mathbb{Z}), \mathbb{F}_2).$$

As $\mathrm{H}_2(_\Gamma\backslash\mathcal{H}^3; \mathbb{Z})$ contains no torsion, we obtain that for $q \in \{1, 2\}$, the dimension $\beta^q$ equals the Betti number $\beta_q$ plus the number of 2–torsion summands of $\mathrm{H}_1(_\Gamma\backslash\mathcal{H}^3; \mathbb{Z})$. The latter number vanishes for all absolute values of the discriminant less than 296 and has been determined in [15] on a database which includes all the Bianchi groups of ideal class numbers 1, 2, 3 and 5, most of the cases of ideal class number 4, as well as all of the cases of discriminant absolute value bounded by 500.

**Notation 15.** Denote by $c$ the co-rank of the map $\mathrm{H}^1(_\Gamma\backslash\mathcal{H}^3; \mathbb{F}_2) \to \mathrm{H}^1(_\Gamma\backslash T; \mathbb{F}_2)$ induced by the inclusion $T \subset \mathcal{H}^3$.

**Remark 16.** By definition of the co-rank $c$, we have the lower bound $0 \leq c$. While obviously the co-rank $c$ vanishes for $\beta_1(_\Gamma\backslash T) = 0$, we can for $\beta_1(_\Gamma\backslash T) > 0$ expect the upper bound

$$c \leq \beta_1(_\Gamma\backslash T) - 1.$$

We arrive at this conclusion because a rotation axis can only be quotiented into a loop by a translation of the axis itself, and this has to be carried out by an element of infinite order in the Bianchi groups. This implies that there can be no facets of the Bianchi fundamental polyhedron filling in such a loop. Thus, every loop in $_\Gamma\backslash T$ comes from a loop in $_\Gamma\backslash\mathcal{H}^3$. So for the case $\beta_1(_\Gamma\backslash T) > 0$, the map

$$\mathrm{H}^1(_\Gamma\backslash\mathcal{H}^3; \mathbb{F}_2) \to \mathrm{H}^1(_\Gamma\backslash T; \mathbb{F}_2)$$

has at least rank one, making its co-rank $c$ fulfill $1 \leq \beta_1(_\Gamma\backslash T) - c$. Hence $c$ can be non-zero only for non-central 2–torsion subcomplex quotients containing at least two distinct loops.

**Notation 17.** Let $v$ denote the number of conjugacy classes of subgroups of quaternionic type $\mathbf{Q}_8$ in $\mathrm{SL}_2(\mathcal{O}_{-m})$, whether or not they are contained in a binary tetrahedral group, and define

$$\mathrm{sign}(v) := \begin{cases} 0, & v = 0, \\ 1, & v > 0. \end{cases}$$

Note that there is a formula for $v$ in terms of the prime divisors of the discriminant of the ring $\mathcal{O}_{-m}$ of integers [8]. Also, by the results of Section 5, Hypothesis 7 is satisfied. This allows us to prove

**Theorem 18.** *The polynomial part of the $E_2^{n,q}$–page of the equivariant spectral sequence with $\mathbb{F}_2$–coefficients associated to the action of $\Gamma$ on $\mathcal{H}^3$ is concentrated in the columns $n \in \{0, 1, 2\}$ with the following dimensions over $\mathbb{F}_2$.*

| | | | |
|---|---|---|---|
| $q = 4k + 3$ | $0$ | $\chi(_\Gamma\backslash T) - 1 + \beta^1(_\Gamma\backslash\mathcal{H}^3) + c$ | $\beta^2(_\Gamma\backslash\mathcal{H}^3) + c$ |
| $q = 4k + 2$ | $1 - \mathrm{sign}(v)$ | $\beta^1(_\Gamma\backslash\mathcal{H}^3) + v - \mathrm{sign}(v)$ | $\beta^2(_\Gamma\backslash\mathcal{H}^3)$ |
| $q = 4k + 1$ | $0$ | $\chi(_\Gamma\backslash T) - 1 + \beta^1(_\Gamma\backslash\mathcal{H}^3) + c$ | $\beta^2(_\Gamma\backslash\mathcal{H}^3) + c$ |
| $q = 4k$ | $1$ | $\beta^1(_\Gamma\backslash\mathcal{H}^3)$ | $\beta^2(_\Gamma\backslash\mathcal{H}^3)$ |
| $k \in \mathbb{N} \cup \{0\}$ | $n = 0$ | $n = 1$ | $n = 2$ |

*Proof in odd degrees $q$.* We note that since $T$ is non-central, it admits no cell stabilizers of type $\mathbb{Z}/2$, so by Theorem 9, no cell stabilizer of $T$ admits polynomial classes in odd degrees $q$ of its cohomology ring. Then for $q$ odd, the short exact sequence (2) becomes an isomorphism of cochain complexes

$$0 \to 0 \to \bigoplus_{\tau \in \Gamma\backslash T^{(n)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\mathbb{Z}/2) \to \mathrm{Cokern}^q_n \to 0,$$

which makes the short exact sequence (3) become

$$0 \to \bigoplus_{\sigma \in \Gamma\backslash \mathcal{H}^{(n)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\Gamma_\sigma) \to \bigoplus_{\sigma \in \Gamma\backslash \mathcal{H}^{(n)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\mathbb{Z}/2) \to \bigoplus_{\sigma \in \Gamma\backslash T^{(n)}} \mathrm{H}^q_{\mathrm{poly}(X)}(\mathbb{Z}/2) \to 0.$$

From this, the snake lemma provides us the following long exact sequence, which is concentrated as specified because of the cohomological dimensions of the orbit spaces.

$$\begin{tikzcd}
 & \mathrm{poly}(E_2^{2,q}(\mathcal{H}^3,\Gamma)) \arrow[r] & \mathrm{H}^2(\Gamma\backslash\mathcal{H}^3;\, \mathbb{F}_2) \arrow[r] & 0 \arrow[r] & \dots \\
 & \mathrm{poly}(E_2^{1,q}(\mathcal{H}^3,\Gamma)) \arrow[r] & \mathrm{H}^1(\Gamma\backslash\mathcal{H}^3;\, \mathbb{F}_2) \arrow[r] & \mathrm{H}^1(\Gamma\backslash T;\, \mathbb{F}_2) \arrow[ull] & \\
0 \arrow[r] & \mathrm{poly}(E_2^{0,q}(\mathcal{H}^3,\Gamma)) \arrow[r] & \mathrm{H}^0(\Gamma\backslash\mathcal{H}^3;\, \mathbb{F}_2) \arrow[r] & \mathrm{H}^0(\Gamma\backslash T;\, \mathbb{F}_2) \arrow[ull] &
\end{tikzcd}$$

Since $T \subset \mathcal{H}^3$, we immediately see that there is a natural inclusion

$$\mathrm{H}^0(\Gamma\backslash\mathcal{H}^3;\, \mathbb{F}_2) \hookrightarrow \mathrm{H}^0(\Gamma\backslash T;\, \mathbb{F}_2)$$

yielding $\mathrm{poly}(E_2^{0,q}(\mathcal{H}^3,\Gamma)) = 0$. As $\Gamma\backslash\mathcal{H}^3$ is connected, we remain with an inclusion of an $\mathbb{F}_2$–subspace of dimension $\beta_0(\Gamma\backslash T) - 1$ into $\mathrm{poly}(E_2^{1,q}(\mathcal{H}^3,\Gamma))$. On the other side, with the co-rank $c$ of Notation 15, we have to complement $\mathrm{poly}(E_2^{1,q}(\mathcal{H}^3,\Gamma))$ by an $\mathbb{F}_2$–subspace of dimension $\beta_1(\Gamma\backslash\mathcal{H}^3) - (\beta_1(\Gamma\backslash T) - c)$. This yields the claimed formula $\chi(\Gamma\backslash T) - 1 + \beta^1(\Gamma\backslash\mathcal{H}^3) + c$ for the dimension of $\mathrm{poly}(E_2^{1,q}(\mathcal{H}^3,\Gamma))$. The remaining terms of the long exact sequence produce

$$\mathrm{poly}(E_2^{2,q}(\mathcal{H}^3,\Gamma)) \cong \mathrm{H}^2(\Gamma\backslash\mathcal{H}^3;\, \mathbb{F}_2) \oplus (\mathbb{F}_2)^c.$$

□

*Proof in degrees $q = 4k$.* By Theorem 11, there is an isomorphism $\mathrm{H}^{4k}_{\mathrm{poly}(X)}(\Gamma_\tau) \cong \mathrm{H}^{4k}_{\mathrm{poly}(X)}(\mathbb{Z}/2)$ for all $\tau \in T$, so $\mathrm{Cokern}^{4k}_n = 0$ for all $n$; and the short exact sequence (3) reduces to an isomorphism of cochain complexes

$$0 \to \bigoplus_{\sigma \in \Gamma\backslash \mathcal{H}^{(n)}} \mathrm{H}^{4k}_{\mathrm{poly}(X)}(\Gamma_\sigma) \to \bigoplus_{\sigma \in \Gamma\backslash \mathcal{H}^{(n)}} \mathrm{H}^{4k}_{\mathrm{poly}(X)}(\mathbb{Z}/2) \to 0 \to 0.$$

Taking the homology of the cochain complexes yields an isomorphism

$$\mathrm{poly}(E_2^{n,4k}(\mathcal{H}^3,\Gamma)) \cong \mathrm{H}^n(\Gamma\backslash\mathcal{H}^3).$$

□

*Proof in degrees $q = 4k + 2$.* Consider the reduced non-central 2–torsion subcomplex $T$ of a Bianchi group $\mathrm{SL}_2(\mathcal{O}_{-m})$. We split $T$ into the union $U$ of connected components of quotient type and the remaining graph $W$ without connected components of quotient type . Denote by $e$, respectively $v$, the number of edges, respectively vertices, of $W$. Then using Theorem 9, we observe

$$\bigoplus_{\tau \in \Gamma\backslash W^{(0)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\Gamma_\tau) = 0 \text{ and } \bigoplus_{\tau \in \Gamma\backslash W^{(1)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\Gamma_\tau) \cong (\mathbb{F}_2)^e;$$

which implies we have an isomorphism $\mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\Gamma_\tau) \cong \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\mathbb{Z}/4) \cong \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\mathbb{Z}/2)$ for all $\tau$ in $U$. Then in degrees $q = 4k + 2$, the short exact sequence (2) of $\Gamma$–equivariant co-chain complexes becomes concentrated in the two rows $n = 0$ and $n = 1$; reading as follows.

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{\tau\in\Gamma\backslash U^{(1)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\mathbb{Z}/2)\oplus(\mathbb{F}_2)^e \arrow[r] & \bigoplus_{\tau\in\Gamma\backslash U^{(1)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\mathbb{Z}/2)\oplus(\mathbb{F}_2)^e \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{\tau\in\Gamma\backslash U^{(0)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\mathbb{Z}/2)\oplus 0 \arrow[r, "\mathrm{id}\oplus 0"] \arrow[u] & \bigoplus_{\tau\in\Gamma\backslash U^{(0)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\mathbb{Z}/2)\oplus(\mathbb{F}_2)^v \arrow[r, "0\oplus\mathrm{id}"] \arrow[u, "\delta^{0,4k+2}|_{\Gamma\backslash T}"] & \mathrm{Cokern}_0^{4k+2} \arrow[r] \arrow[u, "\partial^{0,4k+2}"] & 0,
\end{tikzcd}$$

so we have $\mathrm{Cokern}_0^{4k+2} \cong (\mathbb{F}_2)^v$ and $\mathrm{Cokern}_1^{4k+2} = 0$.

Therefore, the short exact sequence of chain complexes (3) becomes

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^{(2)}} \mathbb{F}_2 \arrow[r] \arrow[u, "d_1^{2,4k+2}"] & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^{(2)}} \mathbb{F}_2 \arrow[r] \arrow[u, "\delta^{2,4k+2}"] & 0 \arrow[r] \arrow[u, "\partial^{2,4k+2}"] & 0 \\
0 \arrow[r] & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^{(1)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\Gamma_\sigma) \arrow[r] \arrow[u, "d_1^{1,4k+2}"] & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^{(1)}} \mathbb{F}_2 \arrow[r] \arrow[u, "\delta^{1,4k+2}"] & 0 \arrow[r] \arrow[u, "\partial^{1,4k+2}"] & 0 \\
0 \arrow[r] & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^{(0)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\Gamma_\sigma) \arrow[r] \arrow[u, "d_1^{0,4k+2}"] & \bigoplus_{\sigma\in\Gamma\backslash\mathcal{H}^{(0)}} \mathbb{F}_2 \arrow[r] \arrow[u, "\delta^{0,4k+2}"] & (\mathbb{F}_2)^v \arrow[r] \arrow[u, "\partial^{0,4k+2}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Applying the snake lemma yields the long exact sequence with nontrivial first connecting map

$$\begin{tikzcd}
 & \mathrm{poly}(E_2^{2,4k+2}(\mathcal{H}^3,\Gamma)) \arrow[r] & \mathrm{H}^2(\Gamma\backslash\mathcal{H}^3) \arrow[r] & 0 \arrow[r] & \dots \\
 & \mathrm{poly}(E_2^{1,4k+2}(\mathcal{H}^3,\Gamma)) \arrow[r] & \mathrm{H}^1(\Gamma\backslash\mathcal{H}^3) \arrow[r] & 0 \arrow[llu, out=-10, in=170] & \\
0 \arrow[r] & \mathrm{poly}(E_2^{0,4k+2}(\mathcal{H}^3,\Gamma)) \arrow[r] & \mathbb{F}_2 \arrow[r] & (\mathbb{F}_2)^v \arrow[llu, out=-10, in=170] &
\end{tikzcd}$$

So, $\mathrm{poly}(E_2^{2,4k+2}(\mathcal{H}^3,\Gamma)) \cong \mathrm{H}^2(\Gamma\backslash\mathcal{H}^3)$. As $\partial^{0,4k+2}$ is the zero map, the isomorphism

$$\bigoplus_{\tau\in\Gamma\backslash W^{(0)}} \mathrm{H}^{4k+2}_{\mathrm{poly}(X)}(\mathbb{Z}/2) \to \mathrm{Cokern}_0^{4k+2}$$

induces for $W \neq \emptyset$ an inclusion of the kernel of $\delta^{0,4k+2}$ into $\ker\partial^{0,4k+2} \cong (\mathbb{F}_2)^v$ by composition with the restriction map given by the connected component inclusion $W \subset \mathcal{H}^3$. Hence, $\mathrm{poly}(E_2^{0,4k+2}(\mathcal{H}^3,\Gamma)) = (\mathbb{F}_2)^{1-\mathrm{sign}(v)}$, where we use the fact that $\mathrm{sign}(v)$ takes the same value as the indicator function of $\{W \neq \emptyset\}$. Therefore, the first connecting map is of rank $(v - \mathrm{sign}(v))$, which implies

$$\dim_{\mathbb{F}_2} \mathrm{poly}(E_2^{1,4k+2}(\mathcal{H}^3,\Gamma)) = \beta^1(\Gamma\backslash\mathcal{H}^3) + v - \mathrm{sign}(v).$$

□

A matrix for the map $\delta^{0,4k+2}|_{\Gamma\backslash T}$ is given by the matrix block $\left(\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}\right)$ on the connected components of quotient type ; by the matrix block $\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)$ on the connected components of quotient type  and by the zero block on ${}_{\Gamma}\backslash U^{(0)}$. However, note that we do not need this explicit form of the map.

## 7. Calculation of the non-polynomial part of the cohomology

From Section 6, we have a complete understanding of the polynomial part of the $E_2$ page of the equivariant spectral sequence converging to $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}))$. In this section, we will obtain the non-polynomial part of this $E_2$ page. Recall that the only 2-torsion elements which stabilize cells not in the non-central 2–torsion subcomplex are in the $\mathbb{Z}/2$ center. Therefore, the non-polynomial part of the mod 2 cohomology of the Bianchi groups has its support on the non-central 2–torsion subcomplex. Furthermore, this cohomology splits as a direct sum indexed by the connected components of this subcomplex, as we shall see in the following lemma. The main work in this section will be to use the cohomology and restriction information gathered in Section 5 in order to determine the equivariant cohomology supported on the individual connected components of the quotient of the reduced non-central 2–torsion subcomplex. By the reduction of the subcomplexes, we only need to do this for each homeomorphism type of connected components.

**Lemma 19.** *The non-polynomial part of* $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}))$ *splits into direct summands each with support on one connected component of the quotient of the reduced non-central* 2*–torsion subcomplex.*

*Proof.* An argument in Section 6 of [13] explains why the mod 2 cohomology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ splits into a direct sum above the virtual cohomological dimension. The argument references the calculations in [3], and notes that classes that arise in one component of the non-central 2–torsion subcomplex do not restrict to subgroups in others. This implies that products between classes that come from distinct components multiply trivially in cohomology.

We need to extend this result from the projective special linear group to the special linear group. In other words, we need to make sure that the result is compatible with the central extension of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ by $\{\pm 1\} \cong \mathbb{Z}/2$. Summarizing results in [11] where the 2–torsion subcomplex is developed, we note that all edge fusions which happen during the reduction of the subcomplex for $\mathrm{PSL}_2(\mathcal{O}_{-m})$ remove a vertex with stabilizer $\mathbb{Z}/2$ or $S_3$. The adjacent edges which are fused both have stabilizer $\mathbb{Z}/2$. Since $\mathrm{H}^*(S_3) \cong \mathrm{H}^*(\mathbb{Z}/2)$ and this is an isomorphism of rings, the mod 2 cohomology of the component corresponding to the 2–torsion subcomplex is ring-isomorphic to the mod 2 cohomology coming from the original component.

Now the central $\{\pm 1\}$ group acts trivially on the retracted cell complex, so as noted in the discussion after Definition 5, the *non-central* 2–torsion subcomplex for $\mathrm{SL}_2(\mathcal{O}_{-m})$ is identical to the 2–torsion subcomplex for $\mathrm{PSL}_2(\mathcal{O}_{-m})$. The stabilizers for $\mathrm{SL}_2(\mathcal{O}_{-m})$ are extensions of the stabilizers in $\mathrm{PSL}_2(\mathcal{O}_{-m})$ by $\{\pm 1\}$. In particular, $\mathbb{Z}/2$ is extended to $\mathbb{Z}/4$ and $S_3$ is extended to $\mathbf{Di}$. Since there is a ring isomorphism $\mathrm{H}^*(\mathbf{Di}) \cong \mathrm{H}^*(\mathbb{Z}/4)$, we conclude that at vertices where there was an edge fusion in the 2–torsion subcomplex in the $\mathrm{PSL}_2(\mathcal{O}_{-m})$ case, there will also be an edge fusion in the non-central 2–torsion subcomplex in the $\mathrm{SL}_2(\mathcal{O}_{-m})$ case. Therefore, the non-central 2–torsion subcomplex can be reduced as in the $\mathrm{PSL}_2(\mathcal{O}_{-m})$ case presented in [11]. As all the cohomology classes that are supported by the quotient space ${}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3$ are in the polynomial part, the above yields a splitting for all of the non-polynomial part of $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}))$. □

The key observation for our task in this section is that the individual connected components of the quotient of the reduced non-central 2–torsion subcomplex correspond to groups which can be described as amalgamated products and HNN extensions. We recall the definition of an HNN extension.

**Definition 20.** Let $\varphi : G_2 \to G_1$ be an injection of $G_2$ into $G_1$. An *HNN extension of* $G_1$ is a group with presentation

$$G_1 *_{G_2} = < t, G_1 \mid t^{-1} g t = \varphi(g), g \in G_2 > .$$

TABLE 1. Connected components of non-central reduced 2–torsion subcomplex quotients

| | Type of group $G$ | Quotient of tree acted on by $G$ |
|---|---|---|
| $(o)$ | HNN extension $\mathbb{Z}/4 *_{\mathbb{Z}/4}$ | $\mathbb{Z}/4$ |
| $(\iota)$ | Amalgamated product $\mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te}$ | $\mathbf{Te} \bullet\!\!-\!\!\bullet \mathbf{Te}$ |
| $(\theta)$ | Double HNN extension $\big((\mathbf{Q}_8 *_{\mathbb{Z}/4} \mathbf{Q}_8) *_{\mathbb{Z}/4}\big) *_{\mathbb{Z}/4}$ | $\mathbf{Q}_8$ $\mathbf{Q}_8$ |
| $(\rho)$ | Iterated construction $(\mathbf{Q}_8 *_{\mathbb{Z}/4}) *_{\mathbb{Z}/4} \mathbf{Te}$ | $\mathbf{Q}_8$ $\mathbf{Te}$ |

The element $t$ is often referred to as the *free letter*.

We note that the notation $G_1 *_{G_2}$ is not completely descriptive, since there may be many choices in $G_1$ for $(G_2)^t$.

Via Bass–Serre theory, it is known that amalgamated products and HNN extensions both act on trees. In the amalgamated product $G = G_1 *_H G_2$, there is an action where the fundamental domain is given by two vertices with stabilizers $G_1$ and $G_2$, connected by an edge with stabilizer $H$. In the HNN extension $G = G_1 *_{G_2}$, the fundamental domain is a single vertex with a loop where the vertex stabilizer is $G_1$ and the edge stabilizer is $G_2$. We can calculate the cohomology of HNN extensions and amalgamated products using the next result.

**Theorem 21.** *Let $\sigma_0$ and $\sigma_1$ be 0–cells and 1–cells in a fundamental domain, and let $G_v$ and $G_e$ be the vertex and edge stabilizers respectively. Then there is a long exact sequence in cohomology*

$$\cdots \xrightarrow{\alpha} \bigoplus_{e \in \sigma_1} \mathrm{H}^{i-1}(G_e) \xrightarrow{\delta} \mathrm{H}^i(G) \xrightarrow{\beta} \bigoplus_{v \in \sigma_0} \mathrm{H}^i(G_v) \xrightarrow{\alpha} \bigoplus_{e \in \sigma_1} \mathrm{H}^i(G_e) \xrightarrow{\delta} \cdots \tag{4}$$

*The direct sum is over one edge and two vertices if $G$ is an amalgamated product (resulting in a Mayer–Vietoris sequence), and over one edge and one vertex if $G$ is an HNN extension.*

The following is another result from Bass–Serre theory.

**Proposition 22.** *In the long exact sequence of Theorem 21, the map $\beta$ is the restriction map. In an amalgamated product, $\alpha = res^{G_1}_H - res^{G_2}_H$. In an HNN extension, $\alpha = res^{G_1}_{G_2} - \varphi^*$, where $\varphi^*$ is the map induced by conjugating $G_2$ by the free letter.*

In fact, when the equivariant spectral sequence only has two non-zero columns, the $E_1$ page degenerates into a Wang sequence which is precisely the long exact sequence (4) in Theorem 21 . Furthermore, the map $\alpha$ is the $d_1$ differential. We note that although we are primarily interested in an additive calculation of the cohomology, the long exact sequence can also be used to determine ring information, as $\ker(\alpha) = \mathrm{im}(\beta)$ and this is a ring isomorphism.

We now have the background to determine the equivariant cohomology for the four observed non-central reduced 2–torsion subcomplex quotient types. A connected component of the 2–torsion subcomplex can be considered as a tree with action of the subgroup $G$ of the Bianchi group that sends the tree to itself. We obtain $G$ as the *groupe fondamental du graphe de groupes* ([18], not to be confused with the fundamental group of the underlying graph) of the quotient of the non-central reduced 2–torsion subcomplex with attached stabilizer groups and monomorphisms from the edge stablizers into the vertex stabilizers. In Table 1, we specify the stabilizer types of the vertices, while all of the edges have stabilizer type $\mathbb{Z}/4$.

We will use the equivariant spectral sequence to carry out our cohomology calculations. We recall that Remark 12 states that concerning the occurring restriction maps on cohomology, only pure polynomial classes restrict nontrivially to pure polynomial classes. Using the notation we established in Section 4, we denote the passage to these subspaces by $\mathrm{H}_{\mathrm{poly}(X)}$, and the passage to the complements of these subspaces by $\mathrm{H}_{\mathrm{non}(X)}$.

($o$) **Lemma 23.** Let $G \cong \mathbb{Z}/4*_{\mathbb{Z}/4}$. Then the polynomial and non-polynomial components of $\mathrm{H}^*(G)$ are:

$$\mathrm{H}^q_{\mathrm{non}(X)}(G) \;\cong\; \mathbb{F}_2, \quad q > 0; \qquad \mathrm{H}^q_{\mathrm{poly}(X)}(G) \;\cong\; \mathbb{F}_2, \quad q \geq 0.$$

*Proof.* The quotient of the tree acted on by $G$ has shape , with vertex and edge stabilizers both isomorphic to $\mathbb{Z}/4$. Using the restriction maps from Theorem 11 we set up the $E_1$ and $E_2$ pages of the equivariant spectral sequence for this HNN extension below. Classes on the $E_2$ page which arise from polynomial classes are boxed on the right side diagram.

$$\begin{array}{c|ccc} & & E_1 \text{ page} & \\ 4 & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 \\ 3 & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 \\ 2 & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 \\ 1 & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 \\ 0 & \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 \\ \hline & 0 & & 1 \end{array} \quad \overset{\mathrm{H}_*(d_1)}{\Longrightarrow} \quad \begin{array}{c|cc} & E_2 \text{ page} & \\ 4 & \boxed{\mathbb{F}_2} & \boxed{\mathbb{F}_2} \\ 3 & \mathbb{F}_2 & \mathbb{F}_2 \\ 2 & \boxed{\mathbb{F}_2} & \boxed{\mathbb{F}_2} \\ 1 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \boxed{\mathbb{F}_2} & \boxed{\mathbb{F}_2} \\ \hline & 0 & 1 \end{array}$$

The arrows in the spectral sequence are the $d_1$ differentials given by $d_1 = res^{\mathbb{Z}/4}_{\mathbb{Z}/4} - \varphi^*$. Since both maps $res^{\mathbb{Z}/4}_{\mathbb{Z}/4}$ and $\varphi^*$ have identical effects on $\mathrm{H}^*(\mathbb{Z}/4)$, we obtain $d_1 = 0$. The polynomial classes occur in the even rows, and the non-polynomial classes are in the odd rows. Passing to the abutment, there are polynomial classes originating in all dimensions starting in dimension 0; non-polynomial classes start in dimension 1. $\square$

($\iota$) **Lemma 24.** Let $G \cong \mathbf{Te}*_{\mathbb{Z}/4}\mathbf{Te}$. Then the dimensions of the polynomial and non-polynomial components of $\mathrm{H}^*(G)$ are:

$$\dim_{\mathbb{F}_2} \mathrm{H}^q_{\mathrm{non}(X)}(G) = \begin{cases} 0, & q = 0 \\ 1, & q = 4k, \quad k \geq 1; \\ 0, & q = 4k+1; \\ 1, & q = 4k+2; \\ 2, & q = 4k+3; \end{cases} \qquad \dim_{\mathbb{F}_2} \mathrm{H}^q_{\mathrm{poly}(X)}(G) = \begin{cases} 1, & q = 4k; \\ 0, & q = 4k+1; \\ 0, & q = 4k+2; \\ 1, & q = 4k+3. \end{cases}$$

*Proof.* The quotient of the tree acted on by $G$ has shape . Both vertex stabilizers are isomorphic to $\mathbf{Te}$, and the edge stabilizer is isomorphic to $\mathbb{Z}/4$. Theorems 9 and 11 allow us to build the $E_1$ and $E_2$ pages of the equivariant spectral sequence for this amalgamated product. Both pages are below.

$$\begin{array}{c|ccc} & & E_1 \text{ page} & \\ 6 & 0 & \longrightarrow & \mathbb{F}_2 \\ 5 & 0 & \longrightarrow & \mathbb{F}_2 \\ 4 & (\mathbb{F}_2)^2 & \longrightarrow & \mathbb{F}_2 \\ 3 & (\mathbb{F}_2)^2 & \longrightarrow & \mathbb{F}_2 \\ 2 & 0 & \longrightarrow & \mathbb{F}_2 \\ 1 & 0 & \longrightarrow & \mathbb{F}_2 \\ 0 & (\mathbb{F}_2)^2 & \longrightarrow & \mathbb{F}_2 \\ \hline & 0 & & 1 \end{array} \quad \overset{\mathrm{H}_*(d_1)}{\Longrightarrow} \quad \begin{array}{c|cc} & E_2 \text{ page} & \\ 6 & & \boxed{\mathbb{F}_2} \\ 5 & & \mathbb{F}_2 \\ 4 & \boxed{\mathbb{F}_2} & \\ 3 & (\mathbb{F}_2)^2 & \mathbb{F}_2 \\ 2 & & \boxed{\mathbb{F}_2} \\ 1 & & \mathbb{F}_2 \\ 0 & \boxed{\mathbb{F}_2} & \\ \hline & 0 & 1 \end{array}$$

On the $E_1$ page, the polynomial classes appear in rows $4k$ in the zeroth column and $4k+2$ in the first column, and the only non-zero restriction maps are the ones between these classes. $\square$

($\theta$) **Lemma 25.** Let $G \cong \left(\left(\mathbf{Q}_8 *_{\mathbb{Z}/4} \mathbf{Q}_8\right) *_{\mathbb{Z}/4}\right) *_{\mathbb{Z}/4}$. Then the dimensions of the polynomial and non-polynomial components of $\mathrm{H}^*(G)$ are:

$$\dim_{\mathbb{F}_2} \mathrm{H}^q_{\mathrm{non}(X)}(G) = \begin{cases} 0, & q = 0 \\ 3, & q = 4k, \quad k \geq 1; \\ 2, & q = 4k+1; \\ 5, & q = 4k+2; \\ 2, & q = 4k+3; \end{cases} \qquad \dim_{\mathbb{F}_2} \mathrm{H}^q_{\mathrm{poly}(X)}(G) = \begin{cases} 1, & q = 4k; \\ 2, & q = 4k+1; \\ 0, & q = 4k+2; \\ 3, & q = 4k+3. \end{cases}$$

*Proof.* The quotient of the tree acted on by $G$ has shape . The two vertex stabilizers are isomorphic to $\mathbf{Q}_8$, and the three edge stabilizers are all isomorphic to $\mathbb{Z}/4$. Therefore, the $E_1$ page of the spectral sequence has two copies of $\mathrm{H}^*(\mathbf{Q}_8)$ in column 0 and three copies of $\mathrm{H}^*(\mathbb{Z}/4)$ in column 1. Our main concern is to determine the action of the $d_1$ differential from $E_1^{0,1}$ to $E_1^{1,1}$.

By Theorem 11, we know that in dimension 1, a non-zero restriction $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}$ can be traced back to a non-zero restriction $res^{\mathbf{D}_2}_{\mathbb{Z}/2}$. In cohomology, let $b_{11}, b_{12}$, and $b_{13}$ denote the exterior classes in $\oplus_3 \mathrm{H}^1(\mathbb{Z}/4)$ corresponding to the three edge stabilizers, and let $x_{11}, y_{11}, x_{12}, y_{12}$ be the nilpotent classes in $\mathrm{H}^1(\mathbf{Q}_8) \oplus \mathrm{H}^1(\mathbf{Q}_8)$ corresponding to the two vertex stabilizers. Further, assume that $x_{11}$ and $x_{12}$ both restrict isomorphically to $b_{11}$ via the correspondence from Theorem 11, and that $y_{11}$ and $y_{12}$ both restrict isomorphically to $b_{12}$. In $\mathbf{D}_2$, the product of any two non-zero elements is the final non-zero element. Therefore, in cohomology, $x_{11}, y_{11}, x_{12}$ and $y_{12}$ all restrict to $b_{13}$. That is,

$$\begin{aligned}
d_1^{0,1}(x_{11}) &= b_{11} + b_{13}\\
d_1^{0,1}(y_{11}) &= b_{12} + b_{13}\\
d_1^{0,1}(x_{12}) &= b_{11} + b_{13}\\
d_1^{0,1}(y_{12}) &= b_{12} + b_{13}.
\end{aligned}$$

So $d_1^{0,1}$ has rank 2. The only other non-zero differential is between polynomial classes in rows $4k$. Let $e_{41}$ and $e_{42}$ be the polynomial generators of $\mathrm{H}^4(\mathbf{Q}_8) \oplus \mathrm{H}^4(\mathbf{Q}_8)$, and let $e_{21}^2$, $e_{22}^2$, $e_{23}^2$ be the squares of the two-dimensional polynomial generators in $\oplus_3 \mathrm{H}^4(\mathbb{Z}/4)$. Based on the geometry of the quotient of the tree acted on by $G$, we have

$$d_1(e_{41}) = d_1(e_{42}) = e_{21}^2 + e_{22}^2 + e_{23}^2.$$

This implies that both the kernel and image of $d_1$ are 1–dimensional. We can now completely determine the $E_1$ and $E_2$ pages of the equivariant spectral sequence.

$$\begin{array}{c|ccc}
\multicolumn{4}{c}{E_1 \text{ page}}\\
6 & (\mathbb{F}_2)^4 & \longrightarrow & (\mathbb{F}_2)^3\\
5 & (\mathbb{F}_2)^4 & \longrightarrow & (\mathbb{F}_2)^3\\
4 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^3\\
3 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^3\\
2 & (\mathbb{F}_2)^4 & \longrightarrow & (\mathbb{F}_2)^3\\
1 & (\mathbb{F}_2)^4 & \longrightarrow & (\mathbb{F}_2)^3\\
0 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^3\\
\hline
 & 0 & & 1
\end{array}
\quad \overset{\mathrm{H}_*(d_1)}{\Longrightarrow} \quad
\begin{array}{c|cc}
\multicolumn{3}{c}{E_2 \text{ page}}\\
6 & (\mathbb{F}_2)^4 & \boxed{(\mathbb{F}_2)^3}\\
5 & (\mathbb{F}_2)^2 & \mathbb{F}_2\\
4 & \boxed{\mathbb{F}_2} & \boxed{(\mathbb{F}_2)^2}\\
3 & (\mathbb{F}_2)^2 & (\mathbb{F}_2)^3\\
2 & (\mathbb{F}_2)^4 & \boxed{(\mathbb{F}_2)^3}\\
1 & (\mathbb{F}_2)^2 & \mathbb{F}_2\\
0 & \boxed{\mathbb{F}_2} & \boxed{(\mathbb{F}_2)^2}\\
\hline
 & 0 & 1
\end{array}$$

□

($\rho$) **Lemma 26.** Let $G \cong (\mathbf{Q}_8 *_{\mathbb{Z}/4}) * \mathbf{Te}$. Then the polynomial and non-polynomial components of $\mathrm{H}^*(G)$ are:

$$\dim_{\mathbb{F}_2} \mathrm{H}^q_{\mathrm{non}(X)}(G) = \begin{cases} 0 & q = 0\\ 2, & q = 4k, \quad k \geq 1;\\ 1, & q = 4k+1;\\ 3, & q = 4k+2;\\ 2, & q = 4k+3; \end{cases} \qquad \dim_{\mathbb{F}_2} \mathrm{H}^q_{\mathrm{poly}(X)}(G) = \begin{cases} 1, & q = 4k;\\ 1, & q = 4k+1;\\ 0, & q = 4k+2;\\ 2, & q = 4k+3. \end{cases}$$

*Proof.* The quotient of the tree acted on by $G$ has shape . The vertex stabilizer incident to the loop is isomorphic to $\mathbf{Q}_8$. The other vertex stabilizer is isomorphic to $\mathbf{Te}$, and the two edge stabilizers are isomorphic to $\mathbb{Z}/4$. This is the most complicated case, since the fundamental domain for $G$ consists of two vertices and two edges, and the restriction maps arising from the edges have to be taken into account.

We follow the approach used for a similar mod 2 calculation in Lemma 3.1 of [3]. We start with an analysis of the component of $d_1 : \mathrm{H}^*(\mathbf{Q}_8) \to \mathrm{H}^*(\mathbb{Z}/4)$, where the copy of $\mathbb{Z}/4$ is the stabilizer of the loop. Referring back to the long exact sequence given in Equation 4, this component of $d_1$ is the difference $res^{\mathbf{Q}_8}_{\mathbb{Z}/4} - \varphi^*$, where the restriction map is induced by subgroup injection on one side of the loop, and by the twisting from the HNN extension on the other one. We tweak the notation for classes from Theorem 9 by denoting by $b_{11} \in \mathrm{H}^1(\mathbb{Z}/4)$ the class in the loop stabilizer and $b_{12} \in \mathrm{H}^1(\mathbb{Z}/4)$ the class from the edge stabilizer. (We use the same convention for polynomial classes.) By the mod 2 calculation in [3], which carefully tracks the effect of the twisting map on cohomology, one can show that the generators $x_1$ and $y_1$ mentioned in Theorem 9 have images

$$res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1) = b_{11}, \text{ but } res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(y_1) = 0.$$

That is, the restriction only detects the subgroup on one side of the HNN extension. The map on cohomology induced by $\varphi$, on the other hand, detects the subgroup on both sides:

$$\varphi^*(x_1) = \varphi^*(y_1) = b_{11}.$$

Consequently, the component of $d_1$ mapping to the loop sends $x_1$ to 0 and $y_1$ to $b_{11}$. In addition, as $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(e_4) = \varphi^*(e_4) = e_{21}^2$, the same component of $d_1$ sends $e_4$ to 0. The map $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}$ to the unlooped edge with $\mathbb{Z}/4$ stabilizer is given by Theorem 11, although we need to determine whether the nilpotent class with a nontrivial restriction is $x_1$, $y_1$, or $x_1 + y_1$. In fact, a basis for $\mathrm{H}^*(\mathbf{Q}_8)$ can be chosen so that $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}$ sends $x_1$ to 0 and $y_1$ to $b_{12}$. Summarizing,

$$\begin{array}{rcl} res^{\mathbf{Q}_8}_{\mathbb{Z}/4\oplus\mathbb{Z}/4}(y_1) & = & b_{11} + b_{12} \\ res^{\mathbf{Q}_8}_{\mathbb{Z}/4\oplus\mathbb{Z}/4}(e_4) & = & e_{22}^2 \end{array}$$

and all other classes are sent to 0. The rest of $d_1$ is given by the restriction map $res^{\mathbf{Te}}_{\mathbb{Z}/4}$, and this map is nontrivial only on the polynomial class. We have now completely determined the $d_1$ differential, so we can write down the $E_1$ and $E_2$ pages.

$$\begin{array}{c|ccc} \multicolumn{4}{c}{E_1 \text{ page}} \\ 6 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^2 \\ 5 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^2 \\ 4 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^2 \\ 3 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^2 \\ 2 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^2 \\ 1 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^2 \\ 0 & (\mathbb{F}_2)^2 & \longrightarrow & (\mathbb{F}_2)^2 \\ \hline & 0 & & 1 \end{array} \quad \overset{\mathrm{H}_*(d_1)}{\Longrightarrow} \quad \begin{array}{c|cc} \multicolumn{3}{c}{E_2 \text{ page}} \\ 6 & (\mathbb{F}_2)^2 & \boxed{(\mathbb{F}_2)^2} \\ 5 & \mathbb{F}_2 & \mathbb{F}_2 \\ 4 & \boxed{\mathbb{F}_2} & \boxed{\mathbb{F}_2} \\ 3 & (\mathbb{F}_2)^2 & (\mathbb{F}_2)^2 \\ 2 & (\mathbb{F}_2)^2 & \boxed{(\mathbb{F}_2)^2} \\ 1 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \boxed{\mathbb{F}_2} & \boxed{\mathbb{F}_2} \\ \hline & 0 & 1 \end{array}$$

□

## 8. Useful Periodicity Properties

We noted in the discussion after Theorem 8 that the cohomology of the finite subgroups of $\mathrm{SL}_2(\mathcal{O}_{-m})$ are all periodic of period dividing 4. We would like to make a similar statement about the groups associated to the four known components of the reduced non-central 2–torsion subcomplex shown in Table 1 and, more generally, the Bianchi groups themselves. Adem and Smith give a "homotopy-theoretic" characterization of cohomological periodicity.

***Definition* 27.** A CW-complex $X$ has *periodic cohomology in the sense of Adem and Smith* if there is a cohomology class $\alpha \in \mathrm{H}^*(X, \mathbb{Z})$ of positive degree $|\alpha|$ and an integer $d \geq 0$ such that the map

$$\alpha \cup - : \mathrm{H}^n_{loc}(X, \mathcal{B}) \to \mathrm{H}^{n+|\alpha|}_{loc}(X, \mathcal{B})$$

induced by the cup product with $\alpha$ is an isomorphism for every local coefficient system $\mathcal{B}$ and every integer $n > d$.

In this notion of periodicity, there is a class $\alpha$ which *eventually* provides an isomorphism. We note that $\alpha$ class must be a polynomial class. Adem and Smith prove:

**Theorem 28.** *([2], Corollary 1.3) A discrete group $\Gamma$ acts freely and properly on $\mathbb{R}^n \times \mathbb{S}^k$ for some $k, n > 0$ if and only if $\Gamma$ is a countable group with periodic cohomology in the sense of Adem and Smith.*

This result applies to the groups we are studying and their subgroups. The Bianchi groups act properly on $\mathcal{H}^3$, which is homeomorphic to $\mathbb{R}^3$. In addition, from the discussion after Theorem 8, we know that the finite subgroups in the Bianchi groups act freely on $\mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbb{C})$. Therefore the groups $\mathrm{SL}_2(\mathcal{O}_{-m})$ and their subgroups act freely and properly on $\mathbb{R}^3 \times \mathbb{S}^3$.

**Corollary 29.** *Subgroups $\Gamma$ in $\mathrm{SL}_2(\mathcal{O}_{-m})$ have periodic cohomology in the sense of Adem and Smith. The periodicity can be realized by cup product with a 4–dimensional class $\alpha$.*

*Proof.* By Theorem 28, we know about the existence of $\alpha$ and only need to find its degree. We essentially follow the argument in [5] and set $X := \mathbb{R}^3 \times \mathbb{S}^3$ with the above described action of $\Gamma$. Then there is a fibration

$$X \to E\Gamma \times_\Gamma X \to B\Gamma,$$

where $B\Gamma$ is the classifying space for $\Gamma$ and $E\Gamma$ is its universal cover. Since $X$ has the homotopy type of $\mathbb{S}^3$, this is a spherical fibration which is orientable as we are working with $\mathbb{F}_2$–coefficients. The result follows by applying the Gysin exact sequence. $\square$

Looking back at the four groups associated to the non-central 2–torsion subcomplex components, we see that the groups associated to the quotient types $\bullet\!-\!\bullet$, $\ominus$, and $\circ\!\!-\!\bullet$ have periodic cohomology of period 4 starting above degree $d = 0$, where $\alpha$ is the unique polynomial class in degree 4 which comes from the polynomial classes in $\mathrm{H}^4(\mathbf{Q}_8)$ and/or $\mathrm{H}^4(\mathbf{Te})$. In contrast, for the quotient type $\circ\!\bullet$, the periodic cohomology has period 2, where $\alpha$ is again the unique polynomial class in dimension 2, this time with its origins in $\mathrm{H}^2(\mathbb{Z}/4)$. We conclude that the equivariant spectral sequence of $\mathrm{SL}_2\,(\mathcal{O}_{-m})$ acting on $\mathcal{H}^3$ is vertically periodic, so once we know what happens in the first four rows, we know what happens in the entire spectral sequence. The following lemma, describing the nature of classes whose product is a polynomial class in $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$, will be useful for the differential calculations for $\mathrm{SL}_2\,(\mathcal{O}_{-m})$ in the next section.

**Lemma 30.** *Given $b$ and $c \in \mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$, if $bc$ is a polynomial class in $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$, then $b$ and $c$ are also polynomial classes coming from the cohomology of vertex stabilizers in the action of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ on $\mathcal{H}^3$.*

*Proof.* The implication from $bc$ being polynomial towards $b$ and $c$ being polynomial is clear. We have to show that the latter two classes come from the cohomology of vertex stabilizers. The fixed point sets in $\mathcal{H}^3$ of nontrivial finite subgroups in $\mathrm{PSL}_2(\mathcal{O}_{-m})$ are at most 1–dimensional; and the action of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ on $\mathcal{H}^3$ can be restricted to a 2–dimensional deformation retract of $\mathcal{H}^3$. Consequently, the equivariant spectral sequence converging to $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$ is concentrated in columns 0 and 1 with potentially a non-zero $E_2^{2,0}$ term. The product compatibility properties (1) and (2) of spectral sequences stated in Section 3 yield that polynomial classes in the abutment must originate on the vertical edge of the spectral sequence, and therefore arise from polynomial classes in the cohomology of vertex stabilizers. $\square$

## 9. Decomposition of the second page differential

In the equivariant spectral sequence for the action of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on $\mathcal{H}^3$, which we will denote by E3SL, we have completely determined the $d_1$ differentials. The final goal is to determine as much as possible about the $d_2$ differentials. For this purpose, we are going to compare the E3SL with the Lyndon–Hochschild–Serre spectral sequence associated to the extension

$$1 \to \mathbb{Z}/2 \to \mathrm{SL}_2(\mathcal{O}_{-m}) \to \mathrm{PSL}_2(\mathcal{O}_{-m}) \to 1,$$

and which we will denote by LH3S. We start with proving that the $d_2$ differential is trivial on polynomial classes in the zeroth column, using the tools of the previous section.

**Lemma 31.** *The $d_2$ differential is trivial on polynomial classes in $E_2^{0,*}$ in the equivariant spectral sequence.*

*Proof.* The equivariant spectral sequence is non-zero in the first three columns only, so Property (1) of spectral sequences from Section 3 implies that all polynomial classes must live in the zero column. From the equivariant cohomology calculations of non-central 2–torsion subcomplex components in Section 7, there is a single polynomial generator in dimension 4 for all four cases, and these classes are the restriction to each subcomplex of the polynomial class $\alpha$ which generates the periodicity in $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}))$ from Corollary 29. This follows from a similar argument as in Theorem 11, as $\alpha$ and the 4–dimensional polynomial classes all originate from the $\mathbb{Z}/2$ fiber of the extension. Since these classes survive to $E_\infty$, they must have trivial image under the $d_2$ differential.

Referring back to Theorem 18, we still have to address the case of quotient components of type $\circ\!\!\bullet$ when $\mathrm{sign}(v) = 0$, where there is a polynomial class of degree 2. This happens when $v = 0$, that is, when the Bianchi group contains no subgroups isomorphic to $\mathbf{Q}_8$. This implies that all components are of type $\circ\!\!\bullet$. Referring back to Lemma 23,

$$\mathrm{H}^*(\mathbb{Z}/4 *_{\mathbb{Z}/4}) \cong \mathbb{F}_2[e_2](b_1) \cong \mathrm{H}^*(\mathbb{Z}/4).$$

That is, the mod 2 cohomology is actually 2–periodic, with the periodicity induced by a cup product with the polynomial class in degree 2. Therefore, as in the previous paragraph, this class survives to $E_\infty$ as well. □

As some supplementary remarks, when one knows the type and number of non-central 2–torsion subcomplex quotient components associated to a Bianchi group $\mathrm{SL}_2(\mathcal{O}_{-m})$, one can construct $\alpha$ explicitly by summing the polynomial classes in dimension 4 from each component, or by summing the polynomial classes in dimension 2 from each component when $\mathrm{sign}(v) = 0$. An alternate direct construction of this last case is implicit in the proof of Lemma 37.

**Corollary 32.** *The $d_2$ differential can only be nontrivial on non-polynomial classes on the vertical edge of the equivariant spectral sequence associated to* $\mathrm{SL}_2(\mathcal{O}_{-m})$.

We note that the $d_2^{0,q}$ differentials are block-diagonalizable with matrix blocks supported each by one connected component of the quotient of the non-central 2–torsion subcomplex.

Working one component type at a time, we are going to show that in the equivariant spectral sequence, the $d_2$ differential is trivial on components of type $\bullet\!\!-\!\!\bullet$ and $\ominus\!\!\bullet$, and that $d_2^{0,1}$ is trivial if and only if $d_2^{0,3}$ is trivial for components of type $\circ\!\!\bullet$. At the time of the writing of this paper we have been unable to determine $d_2$'s behavior on components of type $\circ\!\!-\!\!\bullet$. However, this component type only appears for two of the Bianchi groups in the database summarized in Appendix A.

($\iota$) **Lemma 33.** The $d_2^{\mathrm{E3SL}}$ differential is trivial on connected components of type $\bullet\!\!-\!\!\bullet$.

*Proof.* Since $\mathrm{H}^1(\mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te}) = 0$, we only need to consider classes in dimensions $4k + 3$. Consider the Mayer–Vietoris sequence associated to the amalgamated product $\mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te}$ (cf. Theorem 21).

$$\ldots \xrightarrow{\delta} \mathrm{H}^*(\mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te}) \longrightarrow \mathrm{H}^*(\mathbf{Te}) \oplus \mathrm{H}^*(\mathbf{Te}) \longrightarrow \mathrm{H}^*(\mathbb{Z}/4) \xrightarrow{\delta} \mathrm{H}^{*+1}(\mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te}) \longrightarrow \ldots$$

By Lemma 24, $\dim_{\mathbb{F}_2}(\mathrm{H}^3(\mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te})) = 3$. Since $\mathrm{H}^*(\mathbf{Te}) \cong \mathbb{F}_2[e_4](b_3)$ and $\mathrm{H}^*(\mathbb{Z}/4) \cong \mathbb{F}_2[e_2](b_1)$, the Mayer–Vietoris sequence shows that two of the generators of this 3-dimensional $\mathbb{F}_2$–vector space surject onto $\mathrm{H}^3(\mathbf{Te}) \oplus \mathrm{H}^3(\mathbf{Te})$. If $d_2^{0,3}$ is non-zero in the equivariant spectral sequence, it will send one of these generators to a class in $E_2^{2,1}$.

We will track how these two generators arise in the Lyndon–Hochschild–Serre spectral sequence associated to

$$1 \to \mathbb{Z}/2 \longrightarrow \mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te} \longrightarrow \mathcal{A}_4 *_{\mathbb{Z}/2} \mathcal{A}_4 \to 1.$$

The mod 2 cohomology ring of the quotient group is $\mathrm{H}^*(\mathcal{A}_4 *_{\mathbb{Z}/2} \mathcal{A}_4) \cong \mathbb{F}_2[u_2, v_3, v_3', w_3](\sigma_2)$. The calculation of this ring via the Mayer–Vietoris sequence associated to $\mathcal{A}_4 *_{\mathbb{Z}/2} \mathcal{A}_4$ is implicit in the calculations in [3, Lemma 3.3]; we give a quick summary here. We recall from Theorem 8 that $\mathrm{H}^*(\mathbf{A}_4)$ is generated by three polynomial classes, $u_2, v_3$, and $w_3$, subject to one relation. The classes $u_2$ and $w_3$ in the cohomology of the amalgamated product are the sum of their corresponding images in $\mathrm{H}^*(\mathcal{A}_4) \oplus \mathrm{H}^*(\mathcal{A}_4)$, whereas $v_3$ and $v_3'$ each hit a class in separate copies of $\mathrm{H}^3(\mathcal{A}_4)$. The class $\sigma_2$ is the generator of the image $\delta\left(\mathrm{H}^1(\mathbb{Z}/2)\right) \subseteq \mathrm{H}^2(\mathcal{A}_4 *_{\mathbb{Z}/2} \mathcal{A}_4)$. It's an exterior class, and $Sq^1(\sigma_2) = 0$.

Denote by $z_1$ the generator of $\mathrm{H}^*(\mathbb{Z}/2) \cong \mathbb{F}_2[z_1]$. From the arguments in Theorem 9, in the Lyndon–Hochschild–Serre spectral sequence we have $d_2(z_1) = u_2 + \varepsilon\sigma_2$, where $\varepsilon = 0$ or 1. Referring back to Lemma 24, $\dim_{\mathbb{F}_2}\left(\mathrm{H}^2(\mathbf{Te} *_{\mathbb{Z}/4} \mathbf{Te})\right) = 1$; since this class sits in $E_2^{1,1}$ it is not polynomial. We will show below that $d_3(z_1^2) \neq 0$. Then the 2–dimensional class in the abutment must come from $E_2^{2,0}/\operatorname{im}(d_2)$, and this class must be $\sigma_2$ since $u_2$ is polynomial. That is, $\varepsilon = 0$ and $d_2(z_1) = u_2$.

On the $E_3$ page, we again use the compatibility of the differential with Steenrod squares along with results in [3] and Theorem 9.

$$d_3(z_1^2) = d_3(Sq^1(z_1)) = Sq^1(d_2(z_1)) = Sq^1(u_2 + \varepsilon\sigma_2) = v_3 + v_3'.$$

Attention: From Remark 6, we know that $d_3(u^2) = 0$ for any class $u$ on the $E_3$ page of the spectral sequence. In the line above, $z_1^2$ represents a primitive element on the $E_3$ page which is a 2-dimensional polynomial class; but $z_1$ is *not* an element on the $E_3$ page.

In low dimensions, the $E_4$ page of the LH3S looks like:

$$\begin{array}{c|cccc}
4 & z_1^4 & & & \\
3 & & & & \\
2 & & & z_1^2\sigma_2 & \\
1 & & & z_1\sigma_2 & \\
0 & & & \sigma_2 & \overline{v}_3, w_3 \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

where $\overline{v}_3$ represents the class $v_3 = v_3'$. Comparing dimensions with Lemma 24, these classes are the ones which appear in the first two columns of the $E_2$ page of the E3SL. Therefore, if $d_2^{\mathrm{E3SL}} \neq 0$ on some class in $E_2^{0,3}$, then that class will not survive to $E_\infty$ in the LH3S. We go back to the LH3S and show this cannot happen.

Assume that $d_2^{0,3}$ is non-zero in the E3SL. By the Mayer–Vietoris sequence calculation, $d_2$ is nontrivial on one of the two polynomial classes in dimension 3, and that class does not survive to $E_\infty$. Since $d_3(z_1^2)$ already has a known 1–dimensional image in the LH3S, one of the two polynomial classes must be in the image of $d_2^{\mathrm{LH3S}}$ to not be in the abutment. That is, we need to assume the existence of a new class $b \in E_2^{1,0}$ with $d_2^{\mathrm{LH3S}}(z_1 b) = \overline{v}_3$ or $w_3$.

We consider the extension $1 \to \mathbb{Z}/2 \longrightarrow \mathrm{SL}_2(\mathcal{O}_{-m}) \longrightarrow \mathrm{PSL}_2(\mathcal{O}_{-m}) \to 1$. The differential is a derivation, so $d_2^{\mathrm{LH3S}}(z_1 b) = b\left(d_2^{\mathrm{LH3S}}(z_1)\right)$, where $b$ is a class from $\mathrm{H}^1(\mathrm{PSL}_2(\mathcal{O}_{-m}))$. Furthermore, $c := d_2^{\mathrm{LH3S}}(z_1)$ is a cohomology class in $\mathrm{H}^2(\mathrm{PSL}_2(\mathcal{O}_{-m}))$, whence $bc$ is a polynomial class in $\mathrm{H}^3(\mathrm{PSL}_2(\mathcal{O}_{-m}))$. By Lemma 30, $b$ and $c$ must originate from polynomial classes in the cohomology of vertex stabilizers of the action of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ on $\mathcal{H}^3$. However, in $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$ there are no classes whose cup product yields $\overline{v}_3$ or $w_3$ [3]. This contradicts the assumed nontriviality of $d_2^{0,3\,\mathrm{E3SL}}$, establishing the complete vanishing of $d_2^{\mathrm{E3SL}}$. $\square$

For the $\Theta$ case, we will directly show that $d_2^{\mathrm{E3SL}}$ vanishes on $E_2^{0,3}$. However, to show that $d_2^{\mathrm{E3SL}}$ vanishes on $E_2^{0,1}$, we need to compare computations from the E3SL and LH3S. We start by recalling the following result:

**($\theta$) Lemma 34** ([3, Theorems 4.3 and 4.6])**.** In $\mathrm{PSL}_2(\mathcal{O}_{-m})$, the group associated to $\Theta$ is

$$\Gamma_0 = \left(\left(\mathbf{D}_2 *_{\mathbb{Z}/2} \mathbf{D}_2\right) *_{\mathbb{Z}/2}\right) *_{\mathbb{Z}/2}.$$

Its mod 2 cohomology ring is $\mathrm{H}^*(\Gamma_0) \cong \mathbb{F}_2[l_1, m_1, m_3](\sigma_1, \sigma_2, \tau_1)/m_3(m_3 + l_1^2 m_1 + l_1 m_1^2)$, where all products with nilpotent classes are trivial.

**($\theta$) Lemma 35.** The $d_2$ differential of the E3SL is trivial on $E_2^{0,3}$ on components of type $\Theta$.

*Proof.* We start by carefully determining the ring structure of the cohomology of the subgroup associated to $\Theta$ in $\mathrm{SL}_2\left(\mathcal{O}_{-m}\right)$. Using the notation introduced in Lemma 25, let $\mathrm{H}^1(\mathbf{Q}_8) \oplus \mathrm{H}^1(\mathbf{Q}_8)$ be generated by $x = x_{11}, y = y_{11}$ and $s = x_{12}, t = y_{12}$. Then in the 0–column, the two

1–dimensional classes can be represented by $a = (y,t)$ and $b = (x,s)$. Continuing in the 0–column, the four 2–dimensional classes can be represented by $c = (0,t^2)$, $a^2 + c = (y^2, 0)$, $d = (0,s^2)$, $b^2 + d = (x^2, 0)$; and the two 3–dimensional classes by $b_{31} = b(a^2 + c) = a(b^2 + d)$ and $b_{32} = ad = bc$; these relations follow from the ring structure detailed in Remark 10. The ring relations also imply that $a^2b = ab^2$ and $ac = bd = 0$, because $x^3 = y^3 = s^3 = t^3 = 0$. We now move to the $d_2$ differential. Since the image of $d_2$ can be the sum over a number of classes, we will first assume that there is only one possible class in the second column that can be in the image. We have two cases to consider.

**Case 1**: Either $d_2(a)$ or $d_2(b)$ is zero. Without loss of generality we assume $d_2(a) = 0$. As $d_2$ is a derivation, $d_2(av) = ad_2(v)$ for $v$ arbitrary, and $d_2(w^2) = 2d_2(w) = 0$ for any 1–dimensional class $w$. Then,

$$\begin{aligned} d_2(b_{31}) &= d_2\left(a\left(b^2+d\right)\right) &&= ad_2(b^2+d) &&= 0;\\ d_2(b_{32}) &= d_2(ad) &&= ad_2(d) &&= 0. \end{aligned}$$

**Case 2**: $d_2(a) = d_2(b) = z$, where $z$ is a one dimensional generator in $E_2^{2,0}$. Then $d_2(a+b) = 0$. Now as $a^3 = ac = 0$,

$$b_{31} = b(a^2 + c) = a^3 + a^2b + ac + bc = (a + b)(a^2 + c).$$

and

$$b_{32} = bc = ac + bc = (a + b)c.$$

Since $(a^2 + c)$ is $(y^2, 0)$, $d_2(a^2 + c) = 2d_2(y, 0) = 0$. We conclude that

$$d_2\left((a + b)(a^2 + c)\right) = d_2(b_{31}) = 0.$$

Similarly, $d_2(c) = 0$, so $d_2\left((a + b)\, c\right) = d_2(b_{32}) = 0$.

That is, regardless of $d_2$'s behavior on $E_2^{0,1}$, $d_2$ vanishes on both 3–dimensional classes. For the general case, we note that the cohomology in the second column are copies of $H^*(\mathbb{Z}/2)$. Therefore, the two-case argument above holds word-for-word when we restrict the image of $d_2$ to each possible subgroup. □

($\theta$) **Lemma 36.** The differential $d_2^{\mathrm{E3SL}}$ vanishes on $E_2^{0,1}$ on connected components of type $\Theta$.

*Proof.* We start with the Lyndon–Hochschild–Serre spectral sequence associated to the extension

$$1 \to \mathbb{Z}/2 \longrightarrow \left(\left(\mathbf{Q}_8 *_{\mathbb{Z}/4} \mathbf{Q}_8\right) *_{\mathbb{Z}/4}\right) *_{\mathbb{Z}/4} \longrightarrow \left(\left(\mathbf{D}_2 *_{\mathbb{Z}/2} \mathbf{D}_2\right) *_{\mathbb{Z}/2}\right) *_{\mathbb{Z}/2} \to 1$$

with the following low-dimensional rows of its $E_2$ page.

| | | | | |
|---|---|---|---|---|
| 3 | $z_1^3$ | four classes | four classes | five classes |
| 2 | $z_1^2$ | $l_1z_1^2, m_1z_1^2, \sigma_1z_1^2, \tau_1z_1^2$ | four classes | five classes |
| 1 | $z_1$ | $l_1z_1, m_1z_1, \sigma_1z_1, \tau_1z_1$ | $l_1^2z_1, l_1m_1z_1, m_1^2z_1, \tau_2z_1$ | five classes |
| 0 | | $l_1, m_1, \sigma_1, \tau_1$ | $l_1^2, l_1m_1, m_1^2, \tau_2$ | $l_1^3, l_1^2m_1, l_1m_1^2, m_1^3, m_3$ |
| | 0 | 1 | 2 | 3 |

We follow the argument used for the calculation of $\mathrm{H}^*(\mathbf{Q}_8)$ from $\mathrm{H}^*(\mathbf{D}_2)$ in Theorem 9, in order to determine $d_2^{0,1}$:

$$d_2(z_1) = l_1^2 + l_1m_1 + m_1^2.$$

As $d_2$ is a derivation, $d_2(z_1^2) = 0$. In addition, for any class $c \in \mathrm{H}^*\left(\left(\left(\mathbf{D}_2 *_{\mathbb{Z}/2} \mathbf{D}_2\right) *_{\mathbb{Z}/2}\right) *_{\mathbb{Z}/2}\right)$ on the horizontal edge of the spectral sequence, $d_2(cz_1) = cd_2(z_1)$. These observations completely determine the $d_2$ differential. In dimensions ranging from 0 to 3, the resulting $E_3$ page is concentrated in the following terms.

| | | | | |
|---|---|---|---|---|
| 2 | $z_1^2$ | $\sigma_1z_1^2, \tau_1z_1^2$ | | |
| 1 | | $\sigma_1z_1, \tau_1z_1$ | $\tau_2z_1$ | |
| 0 | | $l_1, m_1, \sigma_1, \tau_1$ | three classes | three classes |
| | 0 | 1 | 2 | 3 |

On the $E_3$ page, the differential $d_3$ is given by

$$d_3(z_1^2) = d_3\left(Sq^1(z_1)\right) = Sq^1\left(d_2(z_1)\right) = l_1^2 m_1 + l_1 m_1^2.$$

This latter class is non-zero in $E_3$; passing to the $E_4$ page, the number of resulting cohomology generators exactly matches the results in Lemma 25 up through dimension 3.

Assume that there is a Bianchi group $\mathrm{SL}_2(\mathcal{O}_{-m})$ so that $d_2^{\mathrm{E3SL}}$ is nontrivial on $E_2^{0,1}$ on a $\Theta$ component. The classes supported on the $\Theta$ component in $E_2^{0,1}$ of the E3SL correspond to the classes $l_1$ and $m_1$ in the calculation of $\mathrm{H}^*\left(\left(\left(\mathbf{Q}_8 *_{\mathbb{Z}/4} \mathbf{Q}_8\right) *_{\mathbb{Z}/4}\right) *_{\mathbb{Z}/4}\right)$, so assume without loss of generality that $d_2^{\mathrm{E3SL}}(l_1) \neq 0$ and the class $l_1$ does not survive to the $E_3$ page. Then the class $l_1$ will be missing from the horizontal edge of the LH3S in the calculation of $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}); \mathbb{F}_2)$. We claim this cannot happen. By [13, Theorem 2], in dimensions 3 and higher, $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{F}_2)$ is isomorphic to the direct sum of the cohomology rings associated to the components of its 2–torsion subcomplex. Therefore, regardless of the $d_2$ differential in the equivariant spectral sequence for $\mathrm{PSL}_2(\mathcal{O}_{-m})$, there will be a 3–dimensional class $l_1(l_1^2+l_1m_1+m_1^2) \in \mathrm{H}^3(\mathrm{PSL}_2(\mathcal{O}_{-m}))$ in the horizontal edge of the LH3S. In addition, since we accounted for all classes up to dimension 3 from the cohomology of the $\Theta$ component above, the class $l_1(l_1^2+l_1m_1+m_1^2)$, which is $d_2(l_1z_1)$, cannot survive to $E_\infty$. The only possibility is that there is a class $b \in \mathrm{H}^1(\mathrm{PSL}_2(\mathcal{O}_{-m}))$ so that

$$d_2(bz_1) = b(l_1^2 + l_1m_1 + m_1^2) = l_1(l_1^2 + l_1m_1 + m_1^2).$$

The class $b$ must be a geometric class coming from ${}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3$. However, since $b$ is a class that arises from the geometry, it must be nilpotent, hence $d_2(bz_1) = bd_2(z_1)$ is also nilpotent. On the other hand, $d_2(l_1z_1)$ is a polynomial class. We conclude that there can be no class $b$ with the required properties; and hence $l_1$ survives to $E_\infty$. □

The previous two lemmata imply that in the E3SL, $d_2^{0,1}$ is the zero map on cohomology classes which come from components of type $\Theta$. We move next to components of type $\bigcirc$.

($o$) **Lemma 37.** The $d_2$ differential is nontrivial on cohomology on components of type $\bigcirc$ in degrees $q \equiv 1 \mod 4$ if and only if it is nontrivial on these components in degrees $q \equiv 3 \mod 4$.

*Proof.* We prove this result in two cases.
**Case 1**: The quotient of the reduced non-central 2–torsion subcomplex of ${}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3$ consists precisely of $k$ connected components of type $\bigcirc$.

We proceed by direct calculation. We will use the equivariant spectral sequence to determine $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$, then apply this data in the LH3S to determine an additive answer for $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}))$. We write $1, \beta_1$ and $\beta_2$ for the Betti numbers of ${}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3$. Since we can exclude the cases of the Gaussian and Eisensteinian integers, the Euler characteristic of ${}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3$ vanishes, whence $\beta_2 = \beta_1 - 1$. Some of the loops in the quotient of the non-central 2–torsion subcomplex will contribute to $\beta_1$, but it is possible that some of the loops may be homotopic in ${}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3$. We assume that $k-c$ loops contribute to $\beta_1$. Then, the $E_2$ page of the equivariant spectral sequence converging to $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$ has this form:

| | | | |
|---|---|---|---|
| 2 | $\oplus_k\mathbb{F}_2$ | $\oplus_k\mathbb{F}_2$ | |
| 1 | $\oplus_k\mathbb{F}_2$ | $\oplus_k\mathbb{F}_2$ | |
| 0 | $\mathbb{F}_2$ | $\oplus_{k-c}\mathbb{F}_2 \oplus_{\beta_1-k+c}\mathbb{F}_2$ | $\oplus_{\beta_1-1}\mathbb{F}_2$ |
| | 0 | 1 | 2 |

We identify the classes in low dimensions.

$$\begin{aligned} E_2^{0,1} &: \quad t_1, \ldots, t_k \\ E_2^{1,0} &: \quad s_1, \ldots, s_{k-c}, b_1 \ldots, b_{\beta_1-k+c} \\ E_2^{2,0} &: \quad c_1, \ldots c_{\beta_1-1} \end{aligned}$$

The $s_i$ and $t_i$ are each associated to a single loop in the $k$ copies of $\bigcirc$ with the $t_i$ polynomial classes and the $s_i$ exterior. The only possible exception to this count is in $E_2^{1,0}$, where $c$ of the

$s_i$ may be identified in the quotient space. The $b_i$ are the other classes which contribute to $\beta_1$, and the $c_i$ correspond to generating 2–cycles counted by $\beta_2$. The $d_2$ differential vanishes on $E_2$ with the possible exception of $d_2^{0,1}$. Let us define $r(m)$ to be the rank of $d_2^{0,1}$ for $\mathrm{PSL}_2(\mathcal{O}_{-m})$.

On the $E_3$ page, we have only the following changes.

$$\begin{array}{ll} E_2^{0,1}: & \bar{t}_1, \ldots, \bar{t}_k, \text{which generate an } \mathbb{F}_2\text{–vector space of dimension } k - r(m) \\ E_2^{2,0}: & c_1, \ldots c_{\beta_1 - 1 - r(m)} \end{array}$$

This is our initial data in the LH3S. Specifically, we identify the following classes on the horizontal edge.

$$\begin{array}{ll} E_2^{1,0}: & s_1, \ldots s_{k-c}, b_1 \ldots, b_{\beta_1 - k + c}, \bar{t}_1, \ldots, \bar{t}_k \\ E_2^{2,0}: & s_1 t_1, \ldots s_k t_k, t_1^2, \ldots t_k^2, c_1, \ldots c_{\beta_1 - 1 - r(m)} \\ E_2^{3,0}: & s_1 t_1^2, \ldots s_k t_k^2, t_1^3, \ldots t_k^3 \end{array}$$

There are classes $z_1^j \in E_2^{0,j}$. Following Theorem 9, in the LH3S, $d_2^{0,1}(z_1) = \sum_i t_i^2 + \sum_j \varepsilon_j c_j$, where $\varepsilon_j = 0$ or 1. The latter sum will have no impact on our calculations, since we are only concerned with the additive answer and products with $c_j$ are trivial. (This follows as the $c_j$ are the only classes in the second column in the equivariant spectral sequence calculation of $\mathrm{PSL}_2(\mathcal{O}_{-m})$.) In addition, $Sq^1(c_j)$ cannot be a polynomial class. For otherwise, although $tc_j = 0$,

$$Sq^1(tc_j) = c_j Sq^1(t) = c_j t^2 \neq 0.$$

Therefore, without loss of generality, we can assume that all $\varepsilon_j$ are zero, yielding $d_2^{0,1}(z_1) = \sum_i t_i^2$. We also note that

$$d_3(z_1^2) = d_3(Sq^1(z_1)) = Sq^1(d_2(z_1)) = 0.$$

That is, as in Theorem 9, the spectral sequence will collapse at $E_3$. Consequently, it is sufficient to determine ranks and coranks of the $d_2$ differential to determine the additive answer for $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}))$. We calculate:

$$\begin{array}{ll} d_2: E_2^{0,1} \to E_2^{2,0}: & \ker(d_2) = 0, \ \dim(\operatorname{im} d_2) = 1 \\ d_2: E_2^{1,1} \to E_2^{3,0}: & d_2(z_1 b_i) = \left(\sum_i t_i^2\right) b_i = 0, \ \dim(\operatorname{im} d_2) = (k - c) + (k - r(m)) \\ d_2: E_2^{2,1} \to E_2^{4,0}: & d_2(c_j) = 0, \text{ so } \dim(\ker(d_2)) = \beta_1 - 1 - r(m) \text{ and } d_2 \text{ is a surjection} \\ d_2: E_2^{p,1} \to E_2^{p+2,0}: & \text{The } d_2 \text{ differential is an isomorphism for } p > 2. \end{array}$$

We write down the $E_3$ page, showing the dimensions of entries only.

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 3 | 0 | $\beta_1 - (k - c)$ | $\beta_1 - r(m) - 1$ | 0 |
| 2 | 1 | $\beta_1 + k - r(m)$ | $2k + \beta_1 - r(m) - 2$ | $c + r(m)$ |
| 1 | 0 | $\beta_1 - (k - c)$ | $\beta_1 - r(m) - 1$ | 0 |
| 0 | 1 | $\beta_1 + k - r(m)$ | $2k + \beta_1 - r(m) - 2$ | $c + r(m)$ |

Summing, we find that

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(k \times \bigcirc\!\!\!\bullet; \mathbb{F}_2) = \begin{cases} 1, & q = 0, \\ k + \beta_1 - r(m), & q = 1, \\ k + 2\beta_1 - r(m) + c - 1, & q \geq 2. \end{cases}$$

Looking at this answer, we can account for all of the classes using the E3SL to calculate $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}))$ with the exception of the ones which vanish under the $d_2^{\mathrm{E3SL}}$ differential. The ones that vanish are taken into account by $r(m)$, which implies that $\operatorname{im}(d_2^{\mathrm{E3SL}})$ has the same rank in all odd degrees.

**Case 2**: The quotient of the reduced non-central 2–torsion subcomplex of ${}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3$ contains other components as well as $\bigcirc\!\!\!\bullet$ .

One direction is straightforward. In the E3SL, the cohomology on the vertical edge of a $\bigcirc\!\!\!\bullet$ component is $\mathrm{H}^*(\mathbb{Z}/4) \cong \mathbb{F}_2[e_2](b_1)$. Since $d_2(e_2 b_1) = e_2 d_2(b_1)$, if $d_2(b_1) = 0$, so does $d_2(e_2 b_1)$.

For the other direction, we use a similar argument as in Lemma 36. In the LH3S, the data for the horizontal edge is given in Case 1 above. If we assume that $d_2(e_2 b_1)$ is non-zero, then there is a class that does not survive to $E_\infty$ in the E3SL, and so it does not show up on the horizontal edge of the LH3S either. We recall [13, Theorem 2], that in dimensions 3 and greater, $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$ is determined by the components of its 2–torsion subcomplex. This implies that the missing class in the LH3S cannot be in

(1) $E_2^{3,0}$, since these classes are determined by the 2–torsion subcomplex structure.
(2) $E_2^{2,1}$, as the only classes in $\ker d_2$ are the $c_j$, and the other classes have nontrivial images in degree 4 which we have determined explicitly.

In addition, the class cannot be in $E_2^{0,3}$, since $d_2(z_1^3)$ is non-zero. We conclude that the missing class in the LH3S must live in $E_2^{1,2}$. However, the classes in $E_2^{1,2}$ are determined by the classes in $E_2^{1,0}$. That is, if there is a class missing in $E_2^{1,2}$, there must also be a class missing in $E_2^{1,0}$. This can only happen if $d_2^{0,1}$ is nontrivial in the equivariant spectral sequence for $\mathrm{H}^*(\mathrm{PSL}_2(\mathcal{O}_{-m}))$. From the calculations above, we see that when this happens, $d_2^{0,1}$ is nontrivial in the E3SL as well. $\square$

**Remark 38.** There is a more direct proof of the first case of Lemma 37. We note that $\circ\!\!\bullet$ corresponds to the group $\mathbb{Z}/4*_{\mathbb{Z}/4}$. The group $\mathbb{Z}/4$ acts on the circle $\mathbb{S}^1$, and following the proof of Corollary 29, we see that in isolation, $\circ\!\!\bullet$ is 2–periodic. By necessity, this periodicity is given by multiplication with $e_2 \in E_2^{0,2}$. This implies that $e_2 d_2(b_1) = 0$ if and only if $d_2(b_1) = 0$.

This argument does not apply to Case 2 for the following reason. Theorem 18 shows that $e_2$, the class which provides the 2-periodicity, survives to $E_\infty$ precisely when the quotient of the reduced non-central 2–torsion subcomplex of ${}_{\mathrm{SL}_2(\mathcal{O}_{-m})}\backslash\mathcal{H}^3$ consists exclusively of connected components of type $\circ\!\!\bullet$, or equivalently, if there are no conjugacy classes of type $\mathbf{Q}_8$ nor $\mathbf{Te}$ in $\mathrm{SL}_2(\mathcal{O}_{-m})$. When there are other connected components besides $\circ\!\!\bullet$, the resulting cohomology is 4-periodic, not 2-periodic.

## 10. Example calculations

In the examples computed in this section, we use Lemma 19 to decompose the non-polynomial part of $E_2$ page of the equivariant spectral sequence converging to the cohomology of $\mathrm{SL}_2(\mathcal{O}_{-m})$ into classes which are supported by connected components of the quotient of the non-central 2–torsion subcomplex. Inserting the results of Sections 7 for these components then yields a complete description of the $E_2$ page. Since the equivariant spectral sequence collapses at the $E_3$ page, once we understand the $d_2$ differential we can read off $\mathrm{H}^*(\mathrm{SL}_2\,(\mathcal{O}_{-m})\,;\,\mathbb{F}_2)$. On components of types $\bullet\!-\!\bullet$ and $\ominus\!\!\bullet$, results in Section 9 shows that this $d_2$ differential vanishes. On components of type $\circ\!\!\bullet$, Lemma 37 shows that we need only to know the rank of $d_2^{0,1}$. For the latter rank, Appendix A provides numerical results. When we cannot completely determine the rank of $d_2$, or when the value of the co-rank $c$ is unclear (see Notation 15), we still can use Bianchi.gp [14] and HAP [6] to compute $\mathrm{H}^*(\mathrm{SL}_2\,(\mathcal{O}_{-m})\,;\,\mathbb{F}_2)$ in order to resolve this indeterminacy.

**Example.** ($\iota$). Let ${}_\Gamma\backslash T = \bullet\!-\!\bullet$. Then $v = 2$, $\chi({}_\Gamma\backslash T) = 1$ and $c = 0$. By Lemma 33, the differential $d_2^{\mathrm{E3SL}}$ vanishes. Then applying Theorem 18 and the proof of Lemma 24, we obtain the following dimensions.

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2\,(\mathcal{O}_{-m})\,;\,\mathbb{F}_2) = \begin{cases} \beta^1({}_\Gamma\backslash\mathcal{H}^3) + \beta^2({}_\Gamma\backslash\mathcal{H}^3), & q = 4k+5, \\ \beta^1({}_\Gamma\backslash\mathcal{H}^3) + \beta^2({}_\Gamma\backslash\mathcal{H}^3) + 2, & q = 4k+4, \\ \beta^1({}_\Gamma\backslash\mathcal{H}^3) + \beta^2({}_\Gamma\backslash\mathcal{H}^3) + 3, & q = 4k+3, \\ \beta^1({}_\Gamma\backslash\mathcal{H}^3) + \beta^2({}_\Gamma\backslash\mathcal{H}^3) + 1, & q = 4k+2, \\ \beta^1({}_\Gamma\backslash\mathcal{H}^3), & q = 1. \end{cases}$$

The type ${}_\Gamma\backslash T = \bullet\!-\!\bullet$ occurs for instance for the cases $\Gamma = \mathrm{SL}_2\,(\mathcal{O}_{-m})$ with $m \in \{11, 19, 43, 67, 163\}$, where $\beta^2 = \beta_2 = \beta_1 - 1$ and $\beta^1 = \beta_1$ is given as follows.

| $m$ | 11 | 19 | 43 | 67 | 163 |
|---|---|---|---|---|---|
| $\beta_1$ | 1 | 1 | 2 | 3 | 7. |

In these cases, the results for $\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-m})\,;\,\mathbb{F}_2)$ have been checked numerically using Bianchi.gp and HAP.

**Example.** $(\theta)$**.** Let ${}_{\Gamma}\backslash T =$ . Then $v = 2$ and $\chi({}_{\Gamma}\backslash T) = -1$. By Lemma 35, the differential $d_2^{\text{E3SL}}$ vanishes. Applying Theorem 18 and the proof of Lemma 25, we obtain the following dimensions.

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-m})\,;\,\mathbb{F}_2) = \begin{cases} c + \beta^1({}_{\Gamma}\backslash\mathcal{H}^3) + \beta^2({}_{\Gamma}\backslash\mathcal{H}^3) + 2, & q = 4k+5, \\ c + \beta^1({}_{\Gamma}\backslash\mathcal{H}^3) + \beta^2({}_{\Gamma}\backslash\mathcal{H}^3) + 2, & q = 4k+4, \\ c + \beta^1({}_{\Gamma}\backslash\mathcal{H}^3) + \beta^2({}_{\Gamma}\backslash\mathcal{H}^3) + 3, & q = 4k+3, \\ c + \beta^1({}_{\Gamma}\backslash\mathcal{H}^3) + \beta^2({}_{\Gamma}\backslash\mathcal{H}^3) + 3, & q = 4k+2, \\ \beta^1({}_{\Gamma}\backslash\mathcal{H}^3) + 2, & q = 1. \end{cases}$$

Instances are $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = \begin{cases} 5. & \text{Then } \beta^1 = \beta_1 = 2 \text{ and } \beta^2 = \beta_2 = 1. \\ 10 \text{ or } 13. & \text{Then } \beta^1 = \beta_1 = 3 \text{ and } \beta^2 = \beta_2 = 2. \\ 58. & \text{Then } \beta^1 = \beta_1 = 12 \text{ and } \beta^2 = \beta_2 = 11. \end{cases}$

The authors' numerical calculation produces compatible results for $\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-m})\,;\,\mathbb{F}_2)$ at a vanishing co-rank $c$ in all these four cases.

**Example.** $(o)$**.** For the case ${}_{\Gamma}\backslash T =$ , we have obtained in Section 7 that the non-polynomial part of the $E_2^{n,q}$–page is concentrated in the columns $n = 0$ and $n = 1$, where it is subject to the module types

$$\text{non-polynomial}(E_2^{n,q}(T,\Gamma)) \cong \begin{cases} \mathbb{F}_2, & q \text{ odd}, \\ 0, & q \text{ even}. \end{cases}$$

As there is just one loop in ${}_{\Gamma}\backslash T$, which by Remark 16 does not become contractible in ${}_{\Gamma}\backslash\mathcal{H}^3$, the co-rank $c$ vanishes. Also $v$ and $\chi({}_{\Gamma}\backslash T)$ are zero, yielding the boxed dimensions of polynomial classes in the below $E_2^{n,q}(\mathcal{H}^3,\Gamma)$–page.

| | | | |
|---|---|---|---|
| $q = 4k+3$ | $1$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3) - 1} + 1$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $q = 4k+2$ | $\boxed{1}$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3)}$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $q = 4k+1$ | $1$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3) - 1} + 1$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $q = 4k$ | $\boxed{1}$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3)}$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $k \in \mathbb{N} \cup \{0\}$ | $n = 0$ | $n = 1$ | $n = 2$ |

Let $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = 7$. Making use of the fact that in this case the quotient space ${}_{\Gamma}\backslash\mathcal{H}^3$ is homotopy equivalent to a Möbius strip [16], $\beta^1 = \beta_1 = 1$, $\beta^2 = \beta_2 = 0$ and hence $d_2 = 0$. This allows us to conclude that $\mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-7})\,;\,\mathbb{F}_2) \cong (\mathbb{F}_2)^2$ for all $q \geq 1$.

Let $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = 15$. Then $\beta^1 = \beta_1 = 2$ and $\beta^2 = \beta_2 = 1$. The numerical computation yields

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-15})\,;\,\mathbb{F}_2) = \begin{cases} 4, & q \geq 2, \\ 3, & q = 1. \end{cases}$$

So we infer that the $d_2$–differential vanishes.

Let $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = \begin{cases} 35. \text{ Then } \beta^1 = \beta_1 = 3 \text{ and } \beta^2 = \beta_2 = 2. \\ 91. \text{ Then } \beta^1 = \beta_1 = 5 \text{ and } \beta^2 = \beta_2 = 4. \\ 115. \text{ Then } \beta^1 = \beta_1 = 7 \text{ and } \beta^2 = \beta_2 = 6. \end{cases}$ The numerical computation yields

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-35})\,;\,\mathbb{F}_2) = \begin{cases} 5, & q \geq 2, \\ 3, & q = 1, \end{cases} \qquad \dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-91})\,;\,\mathbb{F}_2) = \begin{cases} 9, & q \geq 2, \\ 5, & q = 1, \end{cases}$$

$$\text{and } \dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-115})\,;\,\mathbb{F}_2) = \begin{cases} 13, & q \geq 2, \\ 7, & q = 1. \end{cases}$$

So we infer that the $d_2$–differential has full rank on the non-polynomial classes in the column $n = 0$, both in degrees $q = 4k + 1$ and $q = 4k + 3$ (while it is zero on the polynomial classes).

**Example.** $(\rho)$**.** As there is just one loop in ${}_{\Gamma}\backslash T :=$ , which by Remark 16 does not become contractible in ${}_{\Gamma}\backslash\mathcal{H}^3$, the co-rank $c$ vanishes. We observe $v = 2$ and $\chi({}_{\Gamma}\backslash T) = 0$. We have obtained in Section 7 that the non-polynomial part of the $E_2^{n,q}$–page consists of the following non-boxed module types.

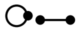
| | | |
|---|---|---|
| $q = 4k+4$ | $\boxed{\mathbb{F}_2}$ | $\boxed{\mathbb{F}_2}$ |
| $q = 4k+3$ | $(\mathbb{F}_2)^2$ | $(\mathbb{F}_2)^2$ |
| $q = 4k+2$ | $(\mathbb{F}_2)^2$ | $\boxed{(\mathbb{F}_2)^2}$ |
| $q = 4k+1$ | $\mathbb{F}_2$ | $\mathbb{F}_2$ |
| $q = 0$ | $\boxed{\mathbb{F}_2}$ | $\boxed{\mathbb{F}_2}$ |
| | $n = 0$ | $n = 1$ |

For $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = 2$, the orbit space ${}_{\Gamma}\backslash\mathcal{H}^3$ is homotopy equivalent to a cylinder [16], so the polynomial part of the $E_2$–page of the equivariant spectral sequence of $(\mathcal{H}^3, \Gamma)$ is concentrated in the columns $n = 0$ and $n = 1$ in the above boxed modules.

This allows us to conclude that

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-2}); \mathbb{F}_2) = \begin{cases} 2, & q = 4k+5, \\ 3, & q = 4k+4, \\ 4, & q = 4k+3, \\ 3, & q = 4k+2, \\ 2, & q = 1. \end{cases}$$

The examples of Euclidean rings $\mathcal{O}_{-2}$, $\mathcal{O}_{-7}$ and $\mathcal{O}_{-11}$ have been checked in HAP [6] with the cellular complex imported from Bianchi.gp [14]; and the example $\mathcal{O}_{-2}$ additionally by a paper-and-pencil calculation by the first author using classical methods. We also observe that these three Euclidean examples are compatible with the homology with Steinberg coefficients calculated in [16] up to a minor typo in the case $\mathcal{O}_{-2}$.

**Example.** $(o\ \iota)$**.** Let ${}_{\Gamma}\backslash T =$ . Then $c = 0$, $v = 2$ and $\chi({}_{\Gamma}\backslash T) = 1$. This yields

| | | | |
|---|---|---|---|
| $q = 4k+3$ | $1+2$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3)} + 1 + 1$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $q = 4k+2$ | $0$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3) + 1}$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $q = 4k+1$ | $1$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3)} + 1 + 1$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $q = 4k$ | $\boxed{1}$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3)}$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $k \in \mathbb{N} \cup \{0\}$ | $n = 0$ | $n = 1$ | $n = 2$ |

This case is realized for $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = \begin{cases} 6. & \text{Then } \beta^1 = \beta_1 = 2 \text{ and } \beta^2 = \beta_2 = 1. \\ 22. & \text{Then } \beta^1 = \beta_1 = 5 \text{ and } \beta^2 = \beta_2 = 4. \end{cases}$

With our numerical calculations, we obtain

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-6}); \mathbb{F}_2) = \begin{cases} 4, & q = 4k+5, \\ 6, & q = 4k+4, \\ 7, & q = 4k+3, \\ 5, & q = 4k+2, \\ 3, & q = 1, \end{cases} \qquad \dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-22}); \mathbb{F}_2) = \begin{cases} 10, & q = 4k+5, \\ 12, & q = 4k+4, \\ 13, & q = 4k+3, \\ 11, & q = 4k+2, \\ 6, & q = 1, \end{cases}$$

so we conclude that $d_2 = 0$.

**Example.** $(o\ o\ o)$. Let ${}_{\Gamma}\backslash T$ = . Then $v = 0$ and $\chi({}_{\Gamma}\backslash T) = 0$. This yields on $E_2$:

| | | | |
|---|---|---|---|
| $q = 4k+3$ | 3 | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3) - 1 + c} + 3$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3) + c}$ |
| $q = 4k+2$ | $\boxed{1}$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3)}$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $q = 4k+1$ | 3 | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3) - 1 + c} + 3$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3) + c}$ |
| $q = 4k$ | $\boxed{1}$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3)}$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $k \in \mathbb{N}\cup\{0\}$ | $n = 0$ | $n = 1$ | $n = 2$ |

Lemma 31 yields $\operatorname{rank} d_2^{0,2} = 0$.

This case is realized for $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with

- $m = 235$. Then $\beta^1 = \beta_1 = 13$ and $\beta^2 = \beta_2 = 12$. With our numerical calculations, we obtain $\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-235}); \mathbb{F}_2) = \begin{cases} 27, & q \geq 2, \\ 14, & q = 1. \end{cases}$
- $m = 427$. Then $\beta_1 = \beta^1 = 21$, $\beta_2 = \beta^2 = 20$ and the machine obtains $\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-427}); \mathbb{F}_2) = \begin{cases} 43, & q \geq 2, \\ 22, & q = 1. \end{cases}$

In both cases, we infer that $c = 1$ and we confirm $\operatorname{rank} d_2^{0,1} = \operatorname{rank} d_2^{0,3} = 2$ that we obtain from Appendix A and Lemma 37.

**Example.** $(\theta\ o\ o)$. Let ${}_{\Gamma}\backslash T$ = . Then $v = 2$ and $\chi({}_{\Gamma}\backslash T) = -1$. This yields

| | | | |
|---|---|---|---|
| $q = 4k+3$ | $2+2$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3) - 2 + c} + 3 + 2$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3) + c}$ |
| $q = 4k+2$ | 4 | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3) + 1}$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $q = 4k+1$ | $2+2$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3) - 2 + c} + 1 + 2$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3) + c}$ |
| $q = 4k$ | $\boxed{1}$ | $\boxed{\beta^1({}_{\Gamma}\backslash\mathcal{H}^3)}$ | $\boxed{\beta^2({}_{\Gamma}\backslash\mathcal{H}^3)}$ |
| $k \in \mathbb{N}\cup\{0\}$ | $n = 0$ | $n = 1$ | $n = 2$ |

This case is realized for $\Gamma = \mathrm{SL}_2(\mathcal{O}_{-m})$ with $m = 37$. Then $\beta^1 = \beta_1 = 8$ and $\beta^2 = \beta_2 = 7$. With our numerical calculations, we obtain

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2(\mathcal{O}_{-37}); \mathbb{F}_2) = \begin{cases} 19, & q = 4k+5, \\ 19, & q = 4k+4, \\ 20, & q = 4k+3, \\ 20, & q = 4k+2, \\ 11, & q = 1, \end{cases}$$

so we conclude that $\operatorname{rank} d_2^{0,1} = \operatorname{rank} d_2^{0,3} = 1$, $\operatorname{rank} d_2^{0,2} = 0$ and $c = 1$. To illustrate the difficulty in manually determining the co-rank $c$, we print, in Figure 1, a fundamental domain for the non-central 2–torsion subcomplex as a dashed graph contained in the boundary of the Bianchi fundamental polyhedron for the action of $\mathrm{SL}_2(\mathcal{O}_{-37})$ on $\mathcal{H}^3$. In this figure, vertices labeled by Bianchi.gp with the same number are identified. However, in order to preserve readability, only the vertices of the non-central 2–torsion subcomplex are labeled; and identifications of cells outside of it need to be taken into account for the Betti number $\beta_1 = 8$.

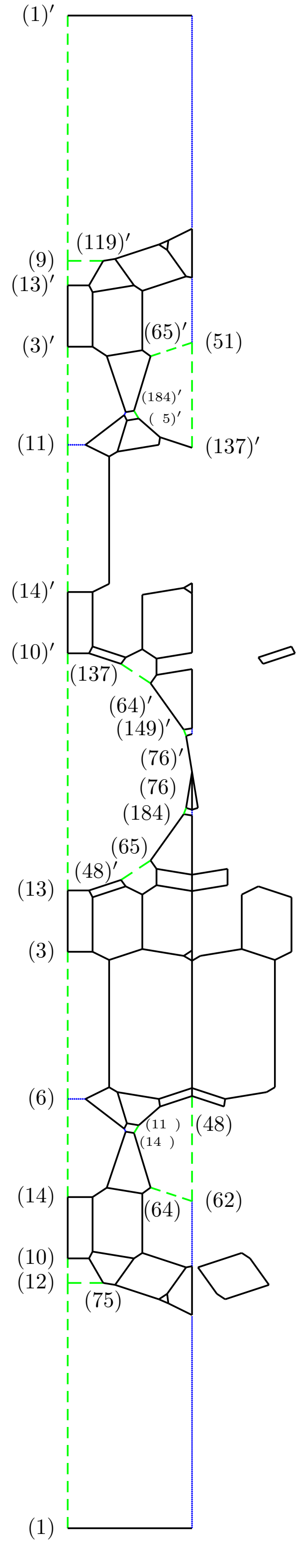



FIGURE 1. Fundamental domain in the case $m = 37$.

## 11. A CHECK USING THE SECOND PAGE DIFFERENTIAL OF PSL

In order to check our calculation of the second page differential of the spectral sequence, we compute numerically the $d^2$–differential of the equivariant homology spectral sequence associated to the action of $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$ on hyperbolic 3–space $\mathcal{H}^3$ with integral coefficients. We use the cell structure which is subdivided until each cell $\sigma$ is fixed pointwise by its stabilizer $\Gamma_\sigma$. The $E^2_{p,q}$–page of this spectral sequence is concentrated as follows in the three columns $p \in \{0, 1, 2\}$,

$$\begin{tikzcd}
q > 1 & \text{2-torsion} \oplus \text{3-torsion} & \text{2-torsion} \oplus \text{3-torsion} & 0 \\
q = 1 & \text{Farrell supplement} & \text{2-torsion} \oplus \text{3-torsion} & 0 \\
q = 0 & \mathbb{Z} & \mathrm{H}_1(\Gamma\backslash\mathcal{H}^3;\, \mathbb{Z}) & \mathrm{H}_2(\Gamma\backslash\mathcal{H}^3;\, \mathbb{Z}) \arrow[ull, "d^2_{2,0}"']
\end{tikzcd}$$

where the "Farrell supplement" is the cokernel of the map

$$\bigoplus_{\sigma \in \Gamma\backslash\mathcal{H}^{(0)}} \mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}) \xleftarrow{d^1_{1,1}} \bigoplus_{\sigma \in \Gamma\backslash\mathcal{H}^{(1)}} \mathrm{H}_1(\Gamma_\sigma; \mathbb{Z})$$

induced by inclusion of cell stabilizers. The Farrell supplement and $\mathrm{H}_1(\Gamma\backslash\mathcal{H}^3;\, \mathbb{Z})$ have been computed on a database of Bianchi groups [15].

For the cases in this database for which the origin of $d^2_{2,0}$ is nontrivial and its target contains 2–torsion, Aurel Page has computed the abelianization $\Gamma^{ab} \cong \mathrm{H}_1(\Gamma;\, \mathbb{Z})$, i.e. the commutator factor subgroup, of $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$. As the above spectral sequence converges to the group homology of $\Gamma$ with integer coefficients, we obtain a short exact sequence

$$0 \to \text{Farrell supplement}/_{\text{image}\, d^2_{2,0}} \to \mathrm{PSL}_2(\mathcal{O}_{-m})^{ab} \to \mathrm{H}_1(\Gamma\backslash\mathcal{H}^3;\, \mathbb{Z}) \to 1;$$

and we deduce the image of $d^2_{2,0}$ from it in Appendix A.

***Observation* 39.** Within the scope of the database in Appendix A, the dimension over $\mathbb{F}_2$ of the 2–primary part of the image of the $d^2_{2,0}$–differential of the equivariant spectral sequence associated to $\mathrm{PSL}_2(\mathcal{O}_{-m})$ is at most the number of connected components of type in the non-central 2–torsion subcomplex quotient. It is clear from $\mathrm{H}_1(\mathcal{A}_4;\, \mathbb{Z}) \cong \mathbb{Z}/3$ that the target of $d^2_{2,0}$ has no 2–torsion on the connected components of type , and the same property follows for connected components of type by a lemma in [11] specifying the matrix block induced in the $d^1_{1,1}$–differential by the inclusions into $\mathcal{D}_2$. The connected components on which our observation is backed only by the numerical results are the ones of type .

## Appendix A. Numerical results

The program [10] computes a presentation of the Bianchi groups. Aurel Page has carried this out and calculated the commutator factor groups (abelianizations) $\mathrm{SL}_2\left(\mathcal{O}_{-m}\right)^{\mathrm{Ab}}$ and $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)^{\mathrm{Ab}}$ in the cases that we need in order to deduce the image of $d^2_{2,0}$ in the way described in Section 11 . The outcome of this procedure is included in the below tables. We denote by $\Delta$ the discriminant of $\mathcal{O}_{-m}$, i.e.

$$\Delta = \begin{cases} -m, & m \equiv 3 \mod 4, \\ -4m, & \text{otherwise.} \end{cases}$$

and by $\beta_1$ the first Betti number of both $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$ and $\mathrm{SL}_2\left(\mathcal{O}_{-m}\right)$. This Betti number has been computed with the two independent programs [10] and [14] with identical results. We provide the remaining torsion parts $\mathrm{SL}_2\left(\mathcal{O}_{-m}\right)^{\mathrm{Ab}}_{\mathrm{tors}}$ and $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)^{\mathrm{Ab}}_{\mathrm{tors}}$ obtained with [10]. We insert the quotient of the reduced non-central 2–torsion subcomplex from [13] and [14]:
Let $o$ denote the number of connected components of homeomorphism type , let $\iota$ the one for , $\theta$ for and $\rho$ for . By Theorem 18 and Lemmata 23 through 26, we deduce the rank of $d_2^{0,1}$ for the equivariant spectral sequence with $\mathbb{F}_2$–coefficients associated to the action of $\mathrm{SL}_2\left(\mathcal{O}_{-m}\right)$ on $\mathcal{H}^3$ as the difference

$$\mathrm{rank}_{\mathbb{F}_2}\, d_2^{0,1} = o + 2\theta + \rho - \mathrm{rank}_{\mathbb{F}_2}\left(\mathrm{SL}_2\left(\mathcal{O}_{-m}\right)^{\mathrm{Ab}}_{\mathrm{tors}}\right).$$

| $-\Delta$ | $m$ | $o$ | $\iota$ | $\theta$ | $\rho$ | $\beta_1$ | $\mathrm{PSL}_2(\mathcal{O}_{-m})^{\mathrm{Ab}}_{\mathrm{tors}}$ | image $d^2_{2,0}(\mathrm{PSL})$ | $\mathrm{SL}_2(\mathcal{O}_{-m})^{\mathrm{Ab}}_{\mathrm{tors}}$ | $\mathrm{rank}_{\mathbb{F}_2}\, d_2^{0,1}(\mathrm{SL})$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 35 | 35 | 1 | 0 | 0 | 0 | 3 | $\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | 1 |
| 40 | 10 | 0 | 0 | 1 | 0 | 3 | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2$ | 0 |
| 47 | 47 | 1 | 0 | 0 | 0 | 5 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 52 | 13 | 0 | 0 | 1 | 0 | 3 | $(\mathbb{Z}/2)^2$ | 0 | $(\mathbb{Z}/2)^2$ | 0 |
| 55 | 55 | 1 | 0 | 0 | 0 | 5 | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/4$ | 0 |
| 56 | 14 | 1 | 1 | 0 | 0 | 5 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 |
| 68 | 17 | 0 | 1 | 1 | 0 | 5 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 |
| 79 | 79 | 3 | 0 | 0 | 0 | 6 | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2\oplus\mathbb{Z}/4$ | 1 |
| 84 | 21 | 3 | 0 | 0 | 0 | 7 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 1 |
| 87 | 87 | 1 | 0 | 0 | 0 | 8 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 88 | 22 | 1 | 1 | 0 | 0 | 5 | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/2$ | 0 |
| 91 | 91 | 1 | 0 | 0 | 0 | 5 | 0 | $\mathbb{Z}/2$ | 0 | 1 |
| 95 | 95 | 1 | 0 | 0 | 0 | 9 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 103 | 103 | 1 | 0 | 0 | 0 | 7 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/4$ | 0 |
| 104 | 26 | 0 | 0 | 1 | 0 | 8 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 |
| 111 | 111 | 1 | 0 | 0 | 0 | 10 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 115 | 115 | 1 | 0 | 0 | 0 | 7 | 0 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | 1 |
| 116 | 29 | 0 | 0 | 1 | 0 | 9 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 |
| 119 | 119 | 2 | 0 | 0 | 0 | 11 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 1 |
| 120 | 30 | 3 | 0 | 0 | 0 | 10 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus(\mathbb{Z}/3)^2$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 2 |
| 127 | 127 | 1 | 0 | 0 | 0 | 8 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/4$ | 0 |
| 132 | 33 | 3 | 2 | 0 | 0 | 10 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus(\mathbb{Z}/3)^2$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 1 |
| 136 | 34 | 2 | 0 | 0 | 2 | 8 | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2$ | 2 |
| 143 | 143 | 1 | 0 | 0 | 0 | 12 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/3$ | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 148 | 37 | 2 | 0 | 1 | 0 | 8 | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^3$ | 1 |
| 151 | 151 | 1 | 0 | 0 | 0 | 10 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/4$ | 0 |
| 152 | 38 | 1 | 1 | 0 | 0 | 10 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 |
| 155 | 155 | 1 | 0 | 0 | 0 | 10 | $\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | 1 |
| 159 | 159 | 1 | 0 | 0 | 0 | 14 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 164 | 41 | 0 | 1 | 1 | 0 | 12 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 |
| 167 | 167 | 1 | 0 | 0 | 0 | 13 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 168 | 42 | 3 | 0 | 0 | 0 | 13 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 2 |
| 183 | 183 | 1 | 0 | 0 | 0 | 14 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 184 | 46 | 2 | 0 | 0 | 0 | 11 | $\mathbb{Z}/2$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2$ | 1 |
| 191 | 191 | 1 | 0 | 0 | 0 | 15 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 195 | 195 | 2 | 0 | 0 | 0 | 15 | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/3$ | 2 |
| 199 | 199 | 1 | 0 | 0 | 0 | 13 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/4$ | 0 |
| 203 | 203 | 1 | 0 | 0 | 0 | 12 | $\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | 1 |
| 212 | 53 | 0 | 0 | 1 | 0 | 14 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 0 |
| 215 | 215 | 1 | 0 | 0 | 0 | 18 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 219 | 219 | 1 | 2 | 0 | 0 | 13 | $\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | 1 |
| 223 | 223 | 3 | 0 | 0 | 0 | 15 | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/4$ | 2 |
| 228 | 57 | 3 | 2 | 0 | 0 | 16 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 1 |
| 231 | 231 | 2 | 0 | 0 | 0 | 21 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 1 |
| 232 | 58 | 0 | 0 | 1 | 0 | 12 | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2$ | 0 |
| 235 | 235 | 3 | 0 | 0 | 0 | 13 | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2$ | 2 |
| 239 | 239 | 1 | 0 | 0 | 0 | 18 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 244 | 61 | 0 | 0 | 1 | 0 | 15 | $(\mathbb{Z}/2)^2$ | 0 | $(\mathbb{Z}/2)^2$ | 0 |
| 247 | 247 | 1 | 0 | 0 | 0 | 14 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/4$ | 0 |
| 248 | 62 | 2 | 0 | 0 | 0 | 16 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 1 |
| 255 | 255 | 2 | 0 | 0 | 0 | 23 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus(\mathbb{Z}/3)^2$ | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 1 |
| 259 | 259 | 1 | 0 | 0 | 0 | 14 | 0 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | 1 |
| 260 | 65 | 1 | 0 | 2 | 0 | 20 | $(\mathbb{Z}/2)^4\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^4\oplus\mathbb{Z}/3$ | 1 |
| 263 | 263 | 1 | 0 | 0 | 0 | 18 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 264 | 66 | 2 | 2 | 0 | 0 | 20 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus(\mathbb{Z}/3)^2$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 1 |
| 271 | 271 | 1 | 0 | 0 | 0 | 17 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/4$ | 0 |
| 276 | 69 | 3 | 0 | 0 | 0 | 23 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 1 |
| 280 | 70 | 3 | 0 | 0 | 0 | 19 | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2\oplus(\mathbb{Z}/3)^2$ | $\mathbb{Z}/2$ | 2 |
| 287 | 287 | 2 | 0 | 0 | 0 | 21 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 1 |
| 291 | 291 | 1 | 2 | 0 | 0 | 17 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 |
| 292 | 73 | 0 | 1 | 1 | 0 | 16 | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2$ | 0 |
| 295 | 295 | 1 | 0 | 0 | 0 | 19 | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/4$ | 0 |
| 296 | 74 | 0 | 0 | 1 | 0 | 19 | $(\mathbb{Z}/2)^4\oplus\mathbb{Z}/3$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^4\oplus\mathbb{Z}/3$ | 0 |
| 299 | 299 | 1 | 0 | 0 | 0 | 18 | $(\mathbb{Z}/3)^2$ | $\mathbb{Z}/2\oplus(\mathbb{Z}/3)^2$ | $(\mathbb{Z}/3)^2$ | 1 |
| 303 | 303 | 1 | 0 | 0 | 0 | 22 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |
| 308 | 77 | 3 | 0 | 0 | 0 | 23 | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2\oplus\mathbb{Z}/3$ | 1 |
| 311 | 311 | 1 | 0 | 0 | 0 | 23 | $\mathbb{Z}/2\oplus\mathbb{Z}/3$ | 0 | $\mathbb{Z}/4\oplus\mathbb{Z}/3$ | 0 |

| $-\Delta$ | $m$ | $o$ | $\iota$ | $\theta$ | $\rho$ | $\beta_1$ | $\mathrm{PSL}_2(\mathcal{O}_{-m})^{\mathrm{Ab}}_{\mathrm{tors}}$ | image $d^2_{2,0}(\mathrm{PSL})$ | $\mathrm{SL}_2(\mathcal{O}_{-m})^{\mathrm{Ab}}_{\mathrm{tors}}$ | $\mathrm{rk} d_2^{0,1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 312 | 78 | 3 | 0 | 0 | 0 | 22 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 1 |
| 319 | 319 | 1 | 0 | 0 | 0 | 20 | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/4$ | 0 |
| 323 | 323 | 1 | 2 | 0 | 0 | 16 | $\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | 1 |
| 327 | 327 | 1 | 0 | 0 | 0 | 24 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | $\mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 0 |
| 328 | 82 | 1 | 1 | 1 | 0 | 17 | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^3$ | 0 |
| 335 | 335 | 1 | 0 | 0 | 0 | 26 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | $\mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 0 |
| 340 | 85 | 0 | 0 | 2 | 0 | 23 | $(\mathbb{Z}/2)^4$ | $(\mathbb{Z}/3)^2$ | $(\mathbb{Z}/2)^4$ | 0 |
| 344 | 86 | 1 | 1 | 0 | 0 | 21 | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$ | 0 | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$ | 0 |
| 355 | 355 | 1 | 0 | 0 | 0 | 20 | 0 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | 1 |
| 356 | 89 | 0 | 1 | 1 | 0 | 24 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 0 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 0 |
| 359 | 359 | 3 | 0 | 0 | 0 | 25 | $(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 1 |
| 367 | 367 | 1 | 0 | 0 | 0 | 20 | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/4 \oplus (\mathbb{Z}/3)^2$ | 0 |
| 371 | 371 | 1 | 0 | 0 | 0 | 22 | $\mathbb{Z}/3$ | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | 1 |
| 372 | 93 | 3 | 0 | 0 | 0 | 27 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 1 |
| 376 | 94 | 2 | 0 | 0 | 0 | 22 | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2$ | 1 |
| 383 | 383 | 1 | 0 | 0 | 0 | 25 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | $\mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 0 |
| 388 | 97 | 0 | 1 | 1 | 0 | 21 | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2$ | 0 |
| 391 | 391 | 2 | 0 | 0 | 0 | 25 | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/4$ | 1 |
| 395 | 395 | 1 | 0 | 0 | 0 | 24 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 1 |
| 399 | 399 | 4 | 0 | 0 | 0 | 33 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 2 |
| 403 | 403 | 1 | 0 | 0 | 0 | 19 | 0 | $\mathbb{Z}/2$ | 0 | 1 |
| 404 | 101 | 2 | 0 | 1 | 0 | 28 | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$ | 1 |
| 407 | 407 | 1 | 0 | 0 | 0 | 29 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/3$ | $\mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 0 |
| 408 | 102 | 1 | 2 | 0 | 0 | 27 | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$ | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^4$ | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$ | 0 |
| 415 | 415 | 1 | 0 | 0 | 0 | 28 | $\mathbb{Z}/2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/4$ | 0 |
| 420 | 105 | 8 | 0 | 0 | 0 | 41 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/3)^3$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 6 |
| 424 | 106 | 0 | 0 | 1 | 0 | 23 | $(\mathbb{Z}/2)^4$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^4$ | 0 |
| 427 | 427 | 3 | 0 | 0 | 0 | 21 | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/2$ | 2 |
| 431 | 431 | 1 | 0 | 0 | 0 | 29 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 0 | $\mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 0 |
| 435 | 435 | 2 | 0 | 0 | 0 | 31 | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^5$ | $\mathbb{Z}/3$ | 2 |
| 436 | 109 | 0 | 0 | 1 | 0 | 25 | $(\mathbb{Z}/2)^4$ | 0 | $(\mathbb{Z}/2)^4$ | 0 |
| 439 | 439 | 5 | 0 | 0 | 0 | 26 | $(\mathbb{Z}/2)^3$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4$ | 2 |
| 440 | 110 | 3 | 0 | 0 | 0 | 32 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 1 |
| 443 | 443 | 1 | 1 | 0 | 0 | 21 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 |
| 447 | 447 | 1 | 0 | 0 | 0 | 32 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | $\mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 0 |
| 452 | 113 | 0 | 1 | 1 | 0 | 27 | $(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3$ | 0 | $(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3$ | 0 |
| 455 | 455 | 2 | 0 | 0 | 0 | 39 | $(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 0 |
| 456 | 114 | 2 | 2 | 0 | 0 | 32 | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^3$ | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$ | 1 |
| 463 | 463 | 1 | 0 | 0 | 0 | 23 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/4$ | 0 |
| 471 | 471 | 1 | 0 | 0 | 0 | 34 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | $\mathbb{Z}/4 \oplus \mathbb{Z}/3$ | 0 |
| 472 | 118 | 1 | 1 | 0 | 0 | 25 | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^3$ | 0 |
| 479 | 479 | 1 | 0 | 0 | 0 | 33 | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^3$ | 0 | $\mathbb{Z}/4 \oplus (\mathbb{Z}/3)^3$ | 0 |
| 483 | 483 | 2 | 0 | 0 | 0 | 33 | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/3$ | 2 |
| 487 | 487 | 1 | 0 | 0 | 0 | 24 | $\mathbb{Z}/2 \oplus (\mathbb{Z}/13)^2$ | 0 | $\mathbb{Z}/4 \oplus (\mathbb{Z}/13)^2$ | 0 |
| 488 | 122 | 0 | 0 | 1 | 0 | 28 | $(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3$ | 0 |
| 499 | 499 | 2 | 1 | 0 | 0 | 22 | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$ | 0 | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$ | 0 |
| 520 | 130 | 0 | 0 | 2 | 0 | 32 | $(\mathbb{Z}/2)^6$ | $(\mathbb{Z}/3)^2$ | $(\mathbb{Z}/2)^6$ | 0 |
| 532 | 133 | 3 | 0 | 0 | 0 | 33 | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$ | $(\mathbb{Z}/2)^2$ | 1 |
| 555 | 555 | 2 | 0 | 0 | 0 | 39 | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/3$ | 2 |
| 568 | 142 | 6 | 0 | 0 | 0 | 29 | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/4$ | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/4$ | 4 |
| 595 | 595 | 2 | 0 | 0 | 0 | 37 | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$ | $\mathbb{Z}/3$ | 2 |
| 667 | 667 | 1 | 0 | 0 | 0 | 32 | 0 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | 1 |
| 696 | 174 | 3 | 0 | 0 | 0 | 50 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 2 |
| 715 | 715 | 2 | 0 | 0 | 0 | 43 | 0 | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$ | 0 | 2 |
| 723 | 723 | 1 | 2 | 0 | 0 | 41 | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$ | 1 |
| 760 | 190 | 3 | 0 | 0 | 0 | 46 | $(\mathbb{Z}/2)^4$ | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$ | $(\mathbb{Z}/2)^4$ | 1 |
| 763 | 763 | 1 | 0 | 0 | 0 | 38 | 0 | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | 0 | 1 |
| 795 | 795 | 2 | 0 | 0 | 0 | 55 | $\mathbb{Z}/3$ | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$ | $\mathbb{Z}/3$ | 2 |
| 955 | 955 | 1 | 0 | 0 | 0 | 50 | $(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/3)^2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/3)^2$ | 1 |
| 1003 | 1003 | 1 | 2 | 0 | 0 | 48 | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$ | 0 |
| 1027 | 1027 | 1 | 0 | 0 | 0 | 48 | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$ | 1 |
| 1227 | 1227 | 1 | 2 | 0 | 0 | 69 | $\mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/3$ | 0 | $\mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/3$ | 1 |
| 1243 | 1243 | 1 | 2 | 0 | 0 | 58 | $(\mathbb{Z}/3)^4$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/3)^4$ | 1 |
| 1387 | 1387 | 1 | 2 | 0 | 0 | 62 | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/167)^2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/167)^2$ | 0 |
| 1411 | 1411 | 1 | 2 | 0 | 0 | 64 | $\mathbb{Z}/2 \oplus (\mathbb{Z}/16)^2 \oplus (\mathbb{Z}/43)^2$ | $\mathbb{Z}/3$ | $\mathbb{Z}/2 \oplus (\mathbb{Z}/16)^2 \oplus (\mathbb{Z}/43)^2$ | 0 |
| 1507 | 1507 | 1 | 2 | 0 | 0 | 70 | $(\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/5)^4$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/5)^4$ | 1 |
| 1555 | 1555 | 1 | 0 | 0 | 0 | 80 | $(\mathbb{Z}/4)^8 \oplus (\mathbb{Z}/11)^2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/3$ | $(\mathbb{Z}/4)^8 \oplus (\mathbb{Z}/11)^2$ | 1 |

## References

[1] Alejandro Adem and R. James Milgram, *Cohomology of finite groups*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 309, Springer-Verlag, Berlin, 2004. MR2035696 (2004k:20109)

[2] Alejandro Adem and Jeff H. Smith, *Periodic complexes and group actions*, Ann. of Math. (2) **154** (2001), no. 2, 407–435, DOI 10.2307/3062102. MR1865976 (2002i:57031)

[3] Ethan Berkove, *The mod 2 cohomology of the Bianchi groups*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4585–4602. MR1675241 (2001b:11043)

[4] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR1324339 (96a:20072)

[5] Francis X. Connolly and Stratos Prassidis, *Groups which act freely on* $\mathbf{R}^m \times S^{n-1}$, Topology **28** (1989), no. 2, 133–148, DOI 10.1016/0040-9383(89)90016-5. MR1003578 (90h:57052)

[6] Graham Ellis, *Homological algebra programming*, Computational group theory and the theory of groups, Contemp. Math., vol. 470, Amer. Math. Soc., Providence, RI, 2008, pp. 63–74, DOI 10.1090/conm/470/09186, (to appear in print). MR2478414 (2009k:20001)

[7] Felix Klein, *Ueber binäre Formen mit linearen Transformationen in sich selbst*, Math. Ann. **9** (1875), no. 2, 183–208. MR1509857

[8] Norbert Krämer, *Imaginärquadratische Einbettung von Maximalordnungen rationaler Quaternionenalgebren, und die nichtzyklischen endlichen Untergruppen der Bianchi-Gruppen*, preprint, 2012 `http://hal.archives-ouvertes.fr/hal-00720823/en/` (German).

[9] John McCleary, *User's Guide to Spectral Sequences*, Mathematics Lecture Series, vol. 12, Publish or Perish, Inc., Washington, Delaware, 1985.

[10] Aurel Page, *Computing arithmetic Kleinian groups*, preprint, `http://hal.archives-ouvertes.fr/hal-00703043`, 2012.

[11] Alexander D. Rahm, *The homological torsion of* $\mathrm{PSL}_2$ *of the imaginary quadratic integers*, Trans. Amer. Math. Soc. **365** (2013), no. 3, 1603–1635. MR3003276

[12] ———, *Homology and K-theory of the Bianchi groups*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 11-12, 615–619. MR2817377 (2012e:20116)

[13] ———, *Accessing the Farrell-Tate cohomology of discrete groups*, preprint, `http://hal.archives-ouvertes.fr/hal-00618167`, 2012.

[14] ———, *Bianchi.gp*, Open source program (GNU general public license), validated by the CNRS: `http://www.projet-plume.org/fiche/bianchigp` Part of the Pari/GP Development Center scripts library, 2010.

[15] ———, *Higher torsion in the Abelianization of the full Bianchi groups*, accepted for publication in LMS J. Comput. Math., `http://arxiv.org/abs/1207.7133v1s`, 2012.

[16] Joachim Schwermer and Karen Vogtmann, *The integral homology of* $\mathrm{SL}_2$ *and* $\mathrm{PSL}_2$ *of Euclidean imaginary quadratic integers*, Comment. Math. Helv. **58** (1983), no. 4, 573–598, DOI 10.1007/BF02564653. MR728453 (86d:11046)

[17] Jean-Pierre Serre, *Le problème des groupes de congruence pour SL2*, Ann. of Math. (2) **92** (1970), 489–527 (French). MR0272790 (42 #7671)

[18] ———, *Arbres, amalgames,* $\mathrm{SL}_2$, Société Mathématique de France, Paris, 1977 (French). Avec un sommaire anglais; Rédigé avec la collaboration de Hyman Bass; Astérisque, No. 46. Note that there exists an English translation (MR1954121), *Trees.* MR0476875 (57 #16426)

[19] Karen Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), no. 3, 399–419, DOI 10.1007/BF01455567. MR799670 (87a:22025)

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