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Totally reducible holonomies of torsion-free affine connections

Lionel BÉRARD BERGERY

En hommage à Marcel BERGER

Summary

That announcement gives the structure of **totally reducible linear Lie algebras** which are the **Lie algebra of the holonomy group** of (at least) one torsion-free connection. The result uses the (already known) classification of the irreducible ones and some previous (unpublished) works by the author giving the classification for the pseudo-riemannian totally reducible case.

One describes those Lie subalgebras through a general structure theorem involving two constructions and some lists. These constructions give new examples of non irreducible totally reducible holonomy algebras and also recover some irreducible ones which seem missing in the previous classification.

1. Introduction

A torsion-free linear connection (on a connected n -dimensional manifold M) defines a holonomy group $H \subset Gl(n, \mathbb{R})$ (up to conjugacy class). The Lie subalgebra $h \subset gl(n, \mathbb{R})$ of H is called the holonomy algebra. It is more precisely the Lie subalgebra of the restricted holonomy group H_0 and it characterizes that connected Lie subgroup of $Gl(n, \mathbb{R})$. For more informations on the holonomy groups, see for example the chapter 10 in the book [Bes].

Throughout the paper, for short, a linear Lie algebra which is the holonomy algebra of at least one torsion free connection will be called a "holonomy". Not any linear Lie algebra is a "holonomy", so it is a natural question to investigate which linear Lie algebras are indeed holonomies. The answer is "known" in the case where h is an irreducible subalgebra i.e., if the linear representation of h in \mathbb{R}^n (given by the inclusion $h \subset gl(n, \mathbb{R})$) is irreducible. The "final" classification was given by S. Merkulov and L. Schwachhöfer [M-S], after the works of many authors, starting with the seminal paper by M. Berger [Be1]. A detailed history of the intermediate contributions is explained in two survey papers by R. Bryant [Br1,2].

Here a "totally reducible" linear Lie algebra is a Lie subalgebra h of a linear algebra $gl(n, \mathbb{R})$ such that the linear representation of h on \mathbb{R}^n is totally reducible (or equivalently semi-simple) i.e., is the direct sum of irreducible representations. Notice that the Lie algebra h is reductive, but not necessarily semi-simple.

Now many holonomies are obtained only for linear connections of locally symmetric spaces. M. Berger already gave in [Be2] a complete classification of (1-connected) symmetric spaces with totally reducible holonomy. Some of them have a (totally reducible) non irreducible holonomy and are not products of symmetric spaces. Now one will focus on the non-symmetric case and a "non-symmetric" holonomy will be the holonomy of at least one non locally symmetric torsion free connection.

Remarks : 1) One will **not** follow the usual presentation of the lists which were given in the irreducible case. Indeed our constructions give a different insight on those lists.

2) The associated "structures" (for example pseudo-riemannian or conformal) will not be studied here in order to shorten that announcement.

3) Notice also that the proof of the results below uses the previous classifications of the irreducible case and that paper is not a new proof in that case. On the other hand, there are some irreducible holonomies which were missed in the previous classifications (see the remark

at the end of paragraph 8).

4) There are some holonomies which are the holonomy algebra of both symmetric spaces and non locally symmetric manifolds. There are quite few of them in the totally reducible case and they are all pseudo-riemannian (or "metric") holonomies.

Paragraph 2 contains some definitions and conventions in order to avoid any misleading, and paragraph 3 contains the structure theorem.

2. Some definitions for totally reducible linear algebras

Definition 1 : Indecomposable linear Lie algebras

1) Let $h_1 \subset gl(n_1, \mathbb{R})$ and $h_2 \subset gl(n_2, \mathbb{R})$ two linear Lie subalgebras. The direct product of h_1 and h_2 is the Lie algebra $h = h_1 \oplus h_2$, included as a subalgebra of $gl(n, \mathbb{R})$ with $n = n_1 + n_2$ through the following formula : for all $A \in h_1, B \in h_2, x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}$,

we have $(A, B)(x, y) = (Ax, By) \in \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

2) A linear Lie subalgebra is **indecomposable** if it is not the direct product of two linear Lie subalgebras (with $n_1 n_2 \neq 0$).

Definition 2 : Totally complex, real-complex and totally real linear Lie algebras

1) A linear algebra h is called **totally complex** if it is a **complex** Lie subalgebra of the complex Lie algebra $gl(m, \mathbb{C})$.

2) A linear algebra h is called **real-complex** if it is a **real non-complex** Lie subalgebra of the complex Lie algebra $gl(m, \mathbb{C})$.

3) A linear algebra h is called **totally real** if it is a Lie subalgebra of $gl(n, \mathbb{R})$ and, if $n = 2m$, it is not a subalgebra of the complex Lie algebra $gl(m, \mathbb{C})$ (even up to conjugacy).

Remarks : 1) The Lie algebra $gl(m, \mathbb{C})$ is a (real) Lie subalgebra of $gl(2m, \mathbb{R})$, hence a totally complex linear Lie algebra (in complex dimension m) is also a (real) linear Lie algebra (in real dimension $n = 2m$).

2) The above denominations are used in that paper because quite often, what is called a "complex" holonomy may be either a complex or a non complex Lie subalgebra of $gl(n, \mathbb{C})$. It is important to distinguish carefully these two cases since the proof of the main theorem use the "total complexification" for linear Lie algebras, and that may introduce some confusions with the usual complexification in representation theory.

In that direction, one recalls the following well known :

Proposition 1 : Complexification of indecomposable linear Lie algebras

Let $h \subset gl(n, \mathbb{R})$ be an indecomposable linear Lie algebra. Then its **total complexification** $h^{\mathbb{C}} = (h \otimes \mathbb{C}) \subset gl(n, \mathbb{C})$ is indecomposable if and only if $h \subset gl(n, \mathbb{R})$ is **not** a totally complex linear Lie algebra.

3. Structure of totally reducible holonomies

With all the above preliminaries, one may state the following structure theorem :

Theorem 1 : Structure of totally reducible holonomies

(1) Decomposition : Any totally reducible holonomy in dimension $n \geq 2$ is a direct product of linear Lie algebras which are :

- either **indecomposable totally reducible holonomies**,
- or $\mathbb{R} = gl(1, \mathbb{R})$ in dimension 1 (which is not a holonomy).

(2) Indecomposable case : Any totally reducible, indecomposable, non symmetric holonomy h is isomorphic (up to conjugacy) to a linear Lie algebra in one of the five "types" I, II, III, IV, V described below, where type I, III, IV, V holonomies are given by lists I, III, IV, V in paragraphs 4, 8, 9, 10 respectively, and type II holonomies are given by one of the two constructions in paragraphs 5 and 6.

Types of indecomposable totally reductive holonomies

For the proof of the theorem, it is easier to study together the holonomies with the same properties. Here are the "characteristics" of these 5 "types" :

Type I holonomies are irreducible linear Lie algebras with non-zero first prolongation and some further properties. They are the **list I** in paragraph 4. They form the "building blocks" for the constructions of type II holonomies in paragraph 5.

Type II holonomies are indecomposable (totally reducible) holonomies which are given by **two constructions** described in the propositions of paragraphs 5 and 6. As a result, the decomposition as a direct sum of irreducible factors for an indecomposable holonomy representation may have any number of irreducible factors. But there are some irreducible type II holonomies, which were already known. One will not give their list, since it may be deduced easily from the list I (see paragraph 7).

Type III holonomies are indecomposable holonomies with a special property for the representation of the semi-simple factor in h . They are the **list III** in paragraph 8.

Type IV holonomies are irreducible symplectic holonomies which are not in previous lists, and their classification is related with that of complex simple Lie algebras. They are the **list IV** in paragraph 9.

Type V holonomies are 10 exceptional irreducible holonomies in low dimensions which will complete the lists of indecomposable holonomies in the structure theorem. They are the **list V** in paragraph 10.

Remarks : 1) In part (1) of the theorem the condition $n \geq 2$ is necessary since $\mathbb{R} = gl(1, \mathbb{R})$ is not a holonomy. Indeed, in **real** dimension 1 the holonomy is always the "null" algebra $h = \{0\} \subset \mathbb{R} = gl(1, \mathbb{R})$. Here all connections are locally symmetric and $\{0\} \subset gl(1, \mathbb{R})$ is a irreducible (symmetric) holonomy.

2) If h is the totally reducible holonomy algebra of a **locally symmetric** space, one knows that it is a direct product of indecomposable totally reducible holonomy algebras of symmetric spaces. And $\mathbb{R} = gl(1, \mathbb{R})$ cannot be one of the factors, but $\{0\} \subset gl(1, \mathbb{R})$ may be a factor.

3) Conversely, if h is a **non symmetric** totally reducible holonomy algebra, necessarily the real dimension n is ≥ 2 . And then at least one of its indecomposable factors is non symmetric or is the exception $\mathbb{R} = gl(1, \mathbb{R})$.

4) Notice that the holonomies in the lists in paragraphs 4, 9, 10 are irreducible. On the other hand, the two constructions in paragraphs 5 and 6 and the lists III provide both irreducible

and non irreducible indecomposable holonomies.

5) Notice the following consequence of the structure theorem :

Corollary 1 : Complexification of totally reducible holonomies

A totally reducible linear Lie algebra h in dimension $n \geq 2$ is a holonomy if and only if its (total) complexification $h^{\mathbb{C}} = (h \otimes \mathbb{C}) \subset gl(n, \mathbb{C})$ is a totally complex totally reducible holonomy.

4. Type I holonomies : Irreducible holonomies with $h^{(1)} \neq 0$ and property \mathcal{C} .

Here $h^{(1)} = ((\mathbb{R}^n \circ \mathbb{R}^n)^* \otimes \mathbb{R}^n) \cap ((\mathbb{R}^n)^* \otimes h) \subset ((\mathbb{R}^n \otimes \mathbb{R}^n)^* \otimes \mathbb{R}^n)$ is the first prolongation of the linear Lie algebra h . The irreducible linear Lie algebras $h \subset gl(n, \mathbb{R})$ with $h^{(1)} \neq 0$ were classified by S. Kobayashi and T. Nagano [K-N], after a previous work by E. Cartan [Ca]. For future use in the constructions of paragraphs 5 and 6, one selects among them those which satisfy a further property \mathcal{C} , defined below :

Definition 3 : Property \mathcal{C}

Let $h \subset gl(n, \mathbb{R})$ be a linear Lie subalgebra and $h^{(1)}$ its first prolongation. Denote by $C(h)$ the subspace of h generated by all the maps $B(x, \cdot) \in h$ for all $x \in \mathbb{R}^n$ and all $B \in h^{(1)}$ (viewed as bilinear maps). Then $h \subset gl(n, \mathbb{R})$ satisfies property \mathcal{C} if $C(h) = h$ and h has a non trivial center.

It is an easy exercise to check which ones of the irreducible linear Lie algebras satisfy the above property \mathcal{C} . They are exactly the linear Lie algebras of the list I (A and B) below and the "exception" $\mathbb{R} = gl(1, \mathbb{R})$. With property \mathcal{C} , the (non trivial) center of h has to be \mathbb{C} in the totally complex case and \mathbb{R} in the real case. The following result is already well-known :

Proposition 2 : Type I holonomies (irreducible holonomies with property \mathcal{C})

(i) All the linear Lie algebras in list I-A and I-B below are irreducible, with non-zero first prolongation, satisfy property \mathcal{C} and they are **non symmetric holonomies**.

(ii) The linear algebra $\mathbb{R} = gl(1, \mathbb{R})$ in dimension 1 is irreducible, with non-zero first prolongation, satisfies property \mathcal{C} but it is **not a holonomy**.

Remark : All conventions and notations for the lists below are gathered in paragraph 11 at the end of the paper.

List I-A : Totally complex type I holonomies

	complex h	complex ρ	$dim^{\mathbb{C}}$	conditions
(1)	$gl(m, \mathbb{C})$	can	m	$m \geq 1$
(2)	$\mathbb{C} \oplus so(m, \mathbb{C})$	$\gamma \otimes_{\mathbb{C}} can$	m	$m \geq 3$
(3)	$\mathbb{C} \oplus sl(p, \mathbb{C}) \oplus sl(q, \mathbb{C})$	$\gamma \otimes_{\mathbb{C}} can_p \otimes_{\mathbb{C}} can_q$	pq	$p \geq q \geq 2$ $(p, q) \neq (2, 2)$
(4)	$gl(m, \mathbb{C})$	$Sym^2(can)$	$\frac{m(m+1)}{2}$	$n \geq 3$
(5)	$gl(m, \mathbb{C})$	$Ext^2(can)$	$\frac{m(m-1)}{2}$	$n \geq 5$
(6)	$\mathbb{C} \oplus so(10, \mathbb{C})$	$\gamma \otimes_{\mathbb{C}} (half\ spin)$	16	
(7)	$\mathbb{C} \oplus E_6^{\mathbb{C}}$	$\gamma \otimes_{\mathbb{C}} can$	27	

List I-B : Totally real type I holonomies

	real h	totally real ρ	$dim^{\mathbb{R}}$	conditions
(1a)	$gl(n, \mathbb{R})$	can	n	$n \geq 2$
(2a)	$\mathbb{R} \oplus so(p, q)$	$\gamma \otimes can$	$p + q$	$p \geq q \geq 0$ $p + q \geq 3$
(3a)	$\mathbb{R} \oplus sl(p, \mathbb{R}) \oplus sl(q, \mathbb{R})$	$\gamma \otimes can_p \otimes can_q$	pq	$p \geq q \geq 2$ $(p, q) \neq (2, 2)$
(3b)	$\mathbb{R} \oplus sl(p, \mathbb{C})$	$\gamma \otimes Herm(can_p)$	p^2	$p \geq 3$
(3c)	$\mathbb{R} \oplus sl(p, \mathbb{H}) \oplus sl(q, \mathbb{H})$	$\gamma \otimes (can_p \otimes_{\mathbb{H}} can_q)$	$4pq$	$p \geq q \geq 1$ $(p, q) \neq (1, 1)$
(4a)	$gl(m, \mathbb{R})$	$Sym^2(can)$	$\frac{m(m+1)}{2}$	$m \geq 3$
(4b)	$gl(m, \mathbb{H})$	$Antiherm(can)$	$m(2m + 1)$	$m \geq 2$
(5a)	$gl(m, \mathbb{R})$	$Ext^2(can)$	$\frac{m(m-1)}{2}$	$m \geq 5$
(5b)	$gl(m, \mathbb{H})$	$Herm(can)$	$m(2m - 1)$	$m \geq 3$
(6a)	$\mathbb{R} \oplus so(5, 5)$	$\gamma \otimes (half\ spin^{\mathbb{R}})$	16	
(6b)	$\mathbb{R} \oplus so(9, 1)$	$\gamma \otimes (half\ spin^{\mathbb{R}})$	16	
(7a)	$\mathbb{R} \oplus E_6^1$	$\gamma \otimes can$	27	
(7b)	$\mathbb{R} \oplus E_6^4$	$\gamma \otimes can$	27	

5. Type II holonomies : totally complex construction

Now two constructions will give a lot of other indecomposable totally reducible holonomies starting from the list I in paragraph 4 (and also \mathbb{R}). Most of them are not irreducible, but some are. Notice that in the non-irreducible case, the center of h gives the indecomposable property for the holonomy. First, here is the complex construction which gives totally complex holonomies.

Data for the complex construction

Let $h_j \subset gl(n_j, \mathbb{C})$, $j = 1, \dots, p$ be p complex linear Lie algebras from the list 1-A above ($p \geq 1$). Such a Lie algebra may be written $h_j = a_j \oplus s_j$, where s_j is semi-simple and a_j is the center. More precisely, using the linear representation, $a_j = \mathbb{C}Id_j$, where $Id_j \in gl(n_j, \mathbb{C})$ is the identity map. Denote by a the complex p -dimensional abelian Lie algebra which is the direct product of all those 1-dimensional centers i.e., $a = \bigoplus_{j=1}^p a_j$.

The basis of the complex m -dimensional vector space a given by the Id_j identifies a with \mathbb{C}^p . Inside a , define a "generic" complex hyperplane z by one homogeneous equation $z = \{(w_1, \dots, w_p) \in \mathbb{C}^p = a; \sum_{j=1}^p \lambda_j w_j = 0\}$, with the following property (\mathcal{P}_1) :

Property (\mathcal{P}_1) : all the complex numbers λ_j ($j = 1, \dots, p$) are non zero.

(Obviously, for any given complex hyperplane z the complex numbers λ_j are unique up to a non zero common factor.)

Let h be the complex Lie algebra which is the direct product of the complex abelian Lie algebra z and all the semi-simple s_j above i.e., $h = z \oplus s$ where $s = \bigoplus_{j=1}^p s_j$.

Let $m = n_1 + \dots + n_p$. The vector space \mathbb{C}^m may be viewed as the direct product $\bigoplus_{j=1}^p \mathbb{C}^{n_j}$ of all the representation spaces of the h_j . Then h becomes a complex Lie subalgebra of $gl(m, \mathbb{C})$ through the representation ρ given by the following formula :

for all $(w_1, \dots, w_p) \in z$, $(A_1, \dots, A_p) \in s$, $(X_1, \dots, X_p) \in \mathbb{C}^m$,

we have $\rho(w_1, \dots, w_p, A_1, \dots, A_p)(X_1, \dots, X_p) = (w_1 X_1 + A_1(X_1), \dots, w_p X_p + A_p(X_p))$.

Proposition 3 : "Totally complex Type II holonomies"

With the above "data for the complex construction", $h \subset gl(m, \mathbb{C})$ is a **totally complex type II indecomposable totally reducible holonomy**.

Remark : In that construction, the linear representation ρ of the Lie algebra h in the vector space \mathbb{C}^m is the direct **sum** of p irreducible representations. But it is not a direct **product** since z is "generic" in a .

6. Type II holonomies : real construction

A similar construction applies to the real case and gives the totally real forms of the totally complex linear algebras given by the complex construction. Once again the real 1-dimensional case plays a special role.

Data for the real construction

Let $h_j \subset gl(n_j, \mathbb{C})$, $j = 1, \dots, p$ be p **complex** linear Lie algebras from the list 1-A above. Let $h_k \subset gl(n_k, \mathbb{R})$, $k = p + 1, \dots, p + q$ be q (real) linear Lie algebras which are either from the list 1-B above or $\mathbb{R} = gl(1, \mathbb{R})$. Such a Lie algebra may be written $h_i = a_i \oplus s_i$, where s_i is semi-simple and a_i is the center. More precisely, $a_i = \mathbb{C}Id_i$ or $\mathbb{R}Id_i$, where Id_i is the identity in $gl(n_i, \mathbb{C})$ or $gl(n_i, \mathbb{R})$ respectively.

Let $p \geq 0$, $q \geq 0$ and $m = 2p + q \geq 1$. Denote by a the **real** m -dimensional abelian Lie algebra which is the direct product of all the centers i.e., $a = \bigoplus_{i=1}^{i=p+q} a_i$. There is a basis for a given by the Id_j and $i Id_j$ for $j = 1, \dots, p$ and the Id_k for $k = p + 1, \dots, p + q$. That basis identify the real m -dimensional vector space a with $\mathbb{R}^m = \mathbb{C}^p \oplus \mathbb{R}^q$.

Inside a , let z be a **real hyperplane** given by one homogeneous equation $z = \{(w_1, \dots, w_p, u_{p+1}, \dots, u_{p+q}) \in a = \mathbb{C}^p \times \mathbb{R}^q = \mathbb{R}^m ; \sum_{j=1}^{j=p} Im(\lambda_j w_j) + \sum_{k=p+1}^{k=p+q} \mu_k u_k = 0\}$, with the following property (\mathcal{P}_2) :

Property (\mathcal{P}_2) : all the complex numbers λ_j ($j = 1, \dots, p$) and all the real numbers μ_k ($k = p + 1, \dots, p + q$) are non zero.

(Here z is a "generic" real hyperplane in a . Obviously, for any given hyperplane z the complex numbers λ_j and the real numbers μ_k are unique up to a non zero **real** factor.)

Let h be the (non complex) Lie algebra which is the direct product of the abelian Lie algebra z and all the semi-simple s_i above i.e., $h = z \oplus_{i=1}^{i=p+q} s_i$.

Let $n = 2(n_1 + \dots + n_p) + (n_{p+1} + \dots + n_{p+q})$. Assume $n \geq 2$ in order to avoid the 1 dimensional case. The vector space \mathbb{R}^n may be viewed as the direct product $\bigoplus_{j=1}^{j=m} \mathbb{C}^{n_j} \oplus \bigoplus_{k=p+1}^{k=p+q} \mathbb{R}^{n_k}$ of all the representation spaces of the h_i 's.

Then h becomes a real Lie subalgebra of $gl(n, \mathbb{R})$ through the representation ρ given by the following formula :

for all $(w_1, \dots, w_p, u_{p+1}, \dots, u_{p+q}) \in z$, $(A_1, \dots, A_{p+q}) \in s$, $(X_1, \dots, X_p, Y_{p+1}, \dots, Y_{p+q}) \in \mathbb{R}^n$, we have $\rho(w_1, \dots, w_p, u_{p+1}, \dots, u_{p+q}, A_1, \dots, A_{p+q})(X_1, \dots, X_p, Y_{p+1}, \dots, Y_{p+q}) = (w_1 X_1 + A_1(X_1), \dots, w_p X_p + A_p(X_p), u_{p+1} Y_{p+1} + A_{p+1}(Y_{p+1}), \dots, u_{p+q} Y_{p+q} + A_{p+q}(Y_{p+q}))$.

Proposition 4 : "Real (non complex) Type III holonomies"

With the above "data for the real construction", $h \subset gl(n, \mathbb{R})$ is a **real (non complex) type II indecomposable totally reducible holonomy**.

Remarks : In that construction, the linear representation ρ of the Lie algebra h in the vector space \mathbb{R}^n is the direct **sum** of $p + q$ irreducible representations. But it is not a direct **product** since z is "generic" in a . When $q = 0$ then the Lie algebra h is real since z is a real hyperplane,

but the holonomy representation ρ is complex. Then we have a real-complex holonomy. On the other hand, when $p = 0$ all the irreducible factors are totally real.

7. Two remarks on type II holonomies

(1) Irreducible type II holonomies

Notice that applying the above constructions **with only one factor** give all the **irreducible type II holonomies**. First, here are some examples :

(A)	$sl(m, \mathbb{C})$	can	$dim^{\mathbb{C}} = m$	$m \geq 2$	totally complex
(B)	$sl(n, \mathbb{R})$	can	$dim^{\mathbb{R}} = n$	$n \geq 2$	totally real
(C)	$\mathbb{R} \oplus sl(m, \mathbb{C})$	$\sigma_{\theta} \otimes_{\mathbb{C}} can$	$dim^{\mathbb{C}} = m$	$m \geq 1$	real – complex

Now the general construction of type II **irreducible** holonomies is the following :

II-A Totally complex irreducible type II holonomies : they result from the complex construction with only one factor and there is one such example $h \subset gl(n, \mathbb{C})$ for each member $\mathbb{C} \oplus h \subset gl(n, \mathbb{C})$ of List I-A. (Do not forget $gl(, n\mathbb{C}) = \mathbb{C} \oplus sl(n, \mathbb{C})$).

II-B Totally real irreducible type II holonomies : they result from the real construction with only one real non complex factor and there is one such example $h \subset gl(n, \mathbb{R})$ for each member $\mathbb{R} \oplus h \subset gl(n, \mathbb{R})$ of List I-B.

II-C Real-complex irreducible type II holonomies : they result from the real construction with only one totally complex factor and there is one such example $\mathbb{R} \oplus h \subset gl(n, \mathbb{C})$ for each member $\mathbb{C} \oplus h \subset gl(n, \mathbb{C})$ of List I-A and representation σ_{θ} for the center \mathbb{R} with $0 \leq \theta \leq \frac{\pi}{2}$.

In those ways, one gets all the irreducible type II holonomies.

Remark : Notice that in the classical lists for the irreducible case in [Br1], [Br2] or [M-S], they are listed with the same semi-simple factor of h and various factors for the center.

(2) The following corollary is an obvious consequence of the structure theorem :

Corollary 2 : Totally reducible indecomposable holonomies with many factors

If a totally reducible indecomposable holonomy has at least 3 irreducible factors in the decomposition of the holonomy in a direct sum, then it is a type II holonomy.

Notice that there are other (non type II) indecomposable totally reducible holonomies whose decomposition in direct sum contains precisely two factors. They are type III holonomies below in paragraph 8 (list III A and B).

8. Type III holonomies : indecomposable holonomies with property \mathcal{S}

As in that title, type III holonomies are indecomposable totally reducible holonomies such that the restriction of the holonomy representation to the semi-simple factor in h satisfies the following (called property \mathcal{S} for short) :

Definition 4 : Property \mathcal{S}

Let $h \subset gl(n, \mathbb{R})$ be an indecomposable totally reducible holonomy and let s be the semi-simple factor in h . Denote by ρ^{ss} the restriction of the holonomy representation ρ to s .

Then h satisfies property \mathcal{S} if and only if it is not of type I or II and it satisfies one of the following :

(i) ρ^{ss} is not irreducible,

(ii) s is non complex and ρ^{ss} is complex,

(iii) $s = s_1 \oplus s_2$, with $s_2 = sl(2, \mathbb{C})$ or $su(2)$ or $sl(2, \mathbb{R})$ and ρ^{ss} is the tensor product of a representation of s_1 with the canonical representation of s_2 .

Remark : The condition (iii) is some sort of "global" quaternionic or paraquaternionic structure for the holonomy representation (cases (7) of List III).

In the totally complex case one get only (i) or (iii) in property \mathcal{S} . In the case (iii), the representation ρ is irreducible. In the case (i), the direct sum decomposition of ρ^{ss} has exactly 2 factors i.e., $\rho^{ss} = \rho_1^{ss} \oplus \rho_2^{ss}$. In the lists below, there are different properties for these two restricted representations

- in the cases (1), (2), (3) and (6) : $\rho_2^{ss} = \rho_1^{ss}$,
- in the cases (4), (5), (6) and also, if $m = 2$, in the cases (1), (2), (3) : $\rho_2^{ss} = (\rho_1^{ss})^*$
- in the cases (8), (9), (10), (11), ρ_1^{ss} and ρ_2^{ss} have different dimensions.

List III-A : Totally complex type III holonomies

	complex h	complex ρ	$dim^{\mathbb{C}}$	conditions
(1)	$sl(m, \mathbb{C})$	$can \oplus can$	$2m$	$m \geq 2$
(2)	$\mathbb{C} \oplus sl(m, \mathbb{C})$	$(\gamma^{\mathbb{C}} \otimes_{\mathbb{C}} can) \oplus (\gamma^{\mathbb{C}}(\lambda) \otimes_{\mathbb{C}} can)$	$2m$	$m \geq 2$
(3)	$\mathbb{C}^2 \oplus sl(m, \mathbb{C})$	$(\pi_1^{\mathbb{C}} \otimes_{\mathbb{C}} can) \oplus (\pi_2^{\mathbb{C}} \otimes_{\mathbb{C}} can)$	$2m$	$m \geq 2$
(4)	$sl(m, \mathbb{C})$	$can \oplus can^*$	$2m$	$m \geq 3$
(5)	$gl(m, \mathbb{C})$	$can \oplus can^*$	$2m$	$m \geq 3$
(6)	$sp(m, \mathbb{C})$	$can \oplus can$	$4m$	$m \geq 2$
(7)	$sp(m, \mathbb{C}) \oplus sp(1, \mathbb{C})$	$can \otimes_{\mathbb{C}} can$	$4m$	$m \geq 2$
(8)	$\mathbb{C} \oplus sl(p, \mathbb{C})$ $\oplus sl(q, \mathbb{C})$	$(\gamma^{\mathbb{C}} \otimes_{\mathbb{C}} \rho_0^{\mathbb{C}} \otimes_{\mathbb{C}} can_q) \oplus$ $(\gamma^{\mathbb{C}}(\frac{p}{p+q}) \otimes_{\mathbb{C}} can_p \otimes_{\mathbb{C}} can_q)$	$(p+1)q$	$p \geq 2$ $q \geq 2$
(9)	$\mathbb{C}^2 \oplus sl(m, \mathbb{C})$ $\oplus sl(2, \mathbb{C})$	$(\pi_1^{\mathbb{C}} \otimes_{\mathbb{C}} \rho_0^{\mathbb{C}} \otimes_{\mathbb{C}} can_2) \oplus$ $(\pi_2^{\mathbb{C}} \otimes_{\mathbb{C}} can_m \otimes_{\mathbb{C}} can_2)$	$2m+2$	$m \geq 2$
(10)	$\mathbb{C} \oplus sl(m, \mathbb{C})$	$(\gamma^{\mathbb{C}} \otimes_{\mathbb{C}} can) \oplus$ $(\gamma^{\mathbb{C}}(\frac{1}{2}) \otimes_{\mathbb{C}} Sym^2(can))$	$\frac{1}{2}m(m+3)$	$m \geq 2$
(11)	$\mathbb{C}^2 \oplus sl(2, \mathbb{C})$	$(\pi_1^{\mathbb{C}} \otimes_{\mathbb{C}} can) \oplus$ $(\pi_2^{\mathbb{C}} \otimes_{\mathbb{C}} Sym^2(can))$	5	

Liste III-B : Totally real type III holonomies

	real h	totally real ρ	$dim^{\mathbb{R}}$	conditions
(2a)	$\mathbb{R} \oplus sl(m, \mathbb{R})$	$(\gamma^{\mathbb{R}} \otimes can) \oplus (\gamma^{\mathbb{R}}(\mu) \otimes can)$	$2m$	$m \geq 2$ $\mu \neq 1$
(3a)	$\mathbb{R}^2 \oplus sl(m, \mathbb{R})$	$(\pi_1^{\mathbb{R}} \otimes can) \oplus (\pi_2^{\mathbb{R}} \otimes can)$	$2m$	$m \geq 2$
(4a)	$sl(m, \mathbb{R})$	$can \oplus can^*$	$2m$	$m \geq 3$
(5a)	$gl(m, \mathbb{R})$	$can \oplus can^*$	$2m$	$m \geq 3$
(7a)	$sp(m, \mathbb{R}) \oplus sp(1, \mathbb{R})$	$can \otimes can$	$4m$	$m \geq 2$
(7b)	$sp(p, q) \oplus sp(1)$	$can \otimes_{\mathbb{H}} can$	$4(p+q)$	$p+q \geq 2$ $p \geq q \geq 0$
(8a)	$\mathbb{R} \oplus sl(p, \mathbb{R})$ $\oplus sl(q, \mathbb{R})$	$(\gamma^{\mathbb{R}} \otimes \rho_0 \otimes can_q) \oplus$ $(\gamma^{\mathbb{R}}(\frac{p}{p+q}) \otimes can_p \otimes_{\mathbb{C}} can_q)$	$(p+1)q$	$p \geq 2$ $q \geq 2$
(9a)	$\mathbb{R}^2 \oplus sl(m, \mathbb{R})$ $\oplus sl(2, \mathbb{R})$	$(\pi_1^{\mathbb{R}} \otimes \rho_0 \otimes can_2) \oplus$ $(\pi_2^{\mathbb{R}} \otimes can_m \otimes can_2)$	$2m+2$	$m \geq 2$
(10a)	$\mathbb{R} \oplus sl(m, \mathbb{R})$	$(\gamma^{\mathbb{R}} \otimes can) \oplus$ $(\gamma^{\mathbb{R}}(\frac{1}{2}) \otimes Sym^2(can))$	$\frac{1}{2}m(m+3)$	$m \geq 2$
(11a)	$\mathbb{R}^2 \oplus sl(2, \mathbb{R})$	$(\pi_1^{\mathbb{R}} \otimes can) \oplus$ $(\pi_2^{\mathbb{R}} \otimes Sym^2(can))$	5	

Liste III-C : Real-complex type III holonomies

	real h	complex ρ	$dim^{\mathbb{C}}$	conditions
(1a)	$sl(m, \mathbb{R})$	$can \oplus can$ $= can \otimes \mathbb{C}$	m	$m \geq 2$
(1b)	$sl(m, \mathbb{H})$	can	$2m$	$m \geq 1$
(2b)	$\mathbb{R} \oplus sl(m, \mathbb{R})$	$\sigma_{\theta}^{\mathbb{C}} \otimes can$	m	$m \geq 2$ $0 \leq \theta \leq \frac{\pi}{2}$
(2c)	$\mathbb{R} \oplus sl(m, \mathbb{H})$	$\sigma_{\theta}^{\mathbb{C}} \otimes_{\mathbb{C}} can$	$2m$	$m \geq 1$ $0 \leq \theta \leq \frac{\pi}{2}$
(3b)	$\mathbb{C} \oplus sl(m, \mathbb{R})$	$\gamma^{\mathbb{C}} \otimes can$	m	$m \geq 2$
(3c)	$\mathbb{C} \oplus sl(m, \mathbb{H})$	$\gamma^{\mathbb{C}} \otimes_{\mathbb{C}} can$	$2m$	$m \geq 1$
(4b)	$su(p, q)$	can	$(p+q)$	$p \geq q \geq 0$ $p+q \geq 3$
(5b)	$u(p, q)$	can	$(p+q)$	$p \geq q \geq 0$ $p+q \geq 3$
(6a)	$sp(m, \mathbb{R})$	$can \oplus can$ $= can \otimes \mathbb{C}$	$2m$	$m \geq 2$
(6b)	$sp(p, q)$	can	$2(p+q)$	$p \geq q \geq 0$ $p+q \geq 2$

Remarks : 1) The cases (2b) (with $\theta \neq 0$), (2c), (3b), (3c) give irreducible holonomies. They appear in the previous classifications (for the irreducible case) only for $m = 2$ in the cases (2b) and (3b) and for $m = 1$ in the cases (2c) and (3c). That some of them may exist also for higher m was already noticed by R. Bryant [Br3] and D. Joyce. But (it seems to me that) all of them are indeed holonomies.

2) Notice that the cases (4), (5), (6), (4a), (5a), (6a) (with also the cases $m = 2$ of (1), (1a) and the cases $m = 2$ and $\lambda = -1$ of (2) and $m = 2$ and $\mu = -1$ of (2a)) are the non irreducible indecomposable totally reducible pseudo-riemannian holonomies already studied in [BB-I] and [BB].

9. Type IV holonomies : special irreducible symplectic holonomies

In 1996, Q.S. Chi, S.A. Merkulov and L.J. Schwachhöfer discovered new "exotic" irreducible holonomies [CMS]. They related them with the list of "Wolf spaces" (compact quaternion-Kähler symmetric spaces), and they studied their properties (with relations to supersymmetry). It is possible to describe these holonomies in the following way.

Let g be a complex **simple** Lie algebra. A construction due initially to J.A. Wolf [W] associates to each g a (non simple) complex subalgebra $k = h \oplus sp(1, \mathbb{C})$ such that the representation of $h \oplus sp(1, \mathbb{C})$ on the quotient space g/k is the tensorial product of a symplectic representation ρ of h and the canonical 2-dimensional representation of $sp(1, \mathbb{C})$ (and such a subalgebra k is unique up to conjugacy in g). Then the associated homogeneous space $G/(H Sp(1, \mathbb{C}))$ is symmetric and its "compact real form" is a compact quaternion-kähler symmetric spaces (they are called a Wolf's spaces).

Furthermore, with that representation ρ , the Lie algebra h becomes a holonomy. If $g = sl(n + 2, \mathbb{C})$, that construction gives $h = gl(n, \mathbb{C})$ with $\rho = can \oplus can^*$, and such a holonomy is already in List III-A (5). Then the other simple Lie algebras give the holonomies in the list IV-A below. Notice that $g = sp(n + 1, \mathbb{C})$ gives $h = sp(n, \mathbb{C})$ with its canonical representation, which was already known to Berger. The other cases are the new cases discovered by Q.S. Chi, S.A. Merkulov and L.J. Schwachhöfer.

On the other hand, the tensorial product representation for $h \oplus sp(1, \mathbb{C})$ is usually only symmetric. There is one family which is both symmetric and non symmetric, with $h = sp(n, \mathbb{C})$. This gives the holonomies (7), (7a) and (7b) in the list III.

List IV-A : : Totally complex type IV holonomies

	complex h	complex ρ	$dim^{\mathbb{C}}$	conditions
(1)	$sp(m, \mathbb{C})$	can	$2m$	$m \geq 2$
(2)	$so(m, \mathbb{C}) \oplus sp(1, \mathbb{C})$	$can \otimes_{\mathbb{C}} can$	$2m$	$m \geq 3$
(3)	$sp(1, \mathbb{C})$	$Sym^3(can)$	4	
(4)	$sp(3, \mathbb{C})$	$Ext_0^3(can)$	14	
(5)	$sl(6, \mathbb{C})$	$Ext^3(can)$	20	
(6)	$so(12, \mathbb{C})$	$half\ spin$	32	
(7)	$E_7^{\mathbb{C}}$	can	56	

List IV-B : Totally real type IV holonomies

	real h	totally real ρ	$dim^{\mathbb{R}}$	conditions
(1a)	$sp(m, \mathbb{R})$	can	$2m$	$m \geq 2$
(2a)	$so(p, q) \oplus sp(1, \mathbb{R})$	$can \otimes can$	$2(p + q)$	$p + q \geq 3$ $p \geq q \geq 0$
(2b)	$so(m, \mathbb{H}) \oplus sp(1)$	$can \otimes_{\mathbb{H}} can$	$4m$	$m \geq 2$
(3a)	$sp(1, \mathbb{R})$	$Sym^3(can)$	4	
(4a)	$sp(3, \mathbb{R})$	$Ext_0^3(can)$	14	
(5a)	$sl(6, \mathbb{R})$	$Ext^3(can)$	20	
(5b)	$su(3, 3)$	$Ext^3(can)^{\mathbb{R}}$	20	
(5c)	$su(5, 1)$	$Ext^3(can)^{\mathbb{R}}$	20	
(6a)	$so(6, 6)$	$half\ spin^{\mathbb{R}}$	32	
(6b)	$so(10, 2)$	$half\ spin^{\mathbb{R}}$	32	
(6c)	$so(6, \mathbb{H})$	$half\ spin^{\mathbb{R}}$	32	
(7a)	E_7^1	can	56	
(7b)	E_7^3	can	56	

10. Type V holonomies : exceptional irreducible holonomies

In the end, there are 10 irreducible holonomies which do not fit in the above lists or constructions. They are called exceptional holonomies (obviously a list of "exceptional" holonomies may depend on various choices for the lists).

Among them, six may be related to the theory of octonions : cases (1), (2) and their real forms.

Finally, there is no "conformally symplectic" holonomy in dimension $n \geq 6$, but such holonomies do exist in dimension 2 and 4. Dimension 2 is already contained in the previous lists. In dimension 4, we get the "exceptional conformally symplectic" holonomies of cases (3), (3a), (4) and (4a).

List V-A : Totally complex type V holonomies

	complex h	complex ρ	$dim^{\mathbb{C}}$
(1)	$G_2^{\mathbb{C}}$	can	7
(2)	$so(7, \mathbb{C})$	$spin$	8
(3)	$\mathbb{C} \oplus sl(2, \mathbb{C})$	$\gamma^{\mathbb{C}} \otimes_{\mathbb{C}} Sym^3(can)$	4
(4)	$\mathbb{C} \oplus sp(2, \mathbb{C})$	$\gamma^{\mathbb{C}} \otimes_{\mathbb{C}} can$	4

List V-B : Totally real type V holonomies

	real h	totally real ρ	$dim^{\mathbb{R}}$
(1a)	G_2^1	can	7
(1b)	G_2	can	7
(2a)	$so(4, 3)$	$spin^{\mathbb{R}}$	8
(2b)	$so(7)$	$spin^{\mathbb{R}}$	8
(3a)	$\mathbb{R} \oplus sl(2, \mathbb{R})$	$\gamma^{\mathbb{R}} \otimes Sym^3(can)$	4
(4a)	$\mathbb{R} \oplus sp(2, \mathbb{R})$	$\gamma^{\mathbb{R}} \otimes can$	4

11. Notations and conventions

Notations for the lists

In all the lists of that paper, holonomies are described through the following presentation in 4 columns : (a) h , (b) ρ , (c) \dim , (d) conditions, where

- (a) h is an "abstract" Lie algebra,
- (b) ρ is the representation which turns h into a subalgebra of some $gl(n, \mathbb{R})$ (or $gl(m, \mathbb{C})$), (and the notations and conventions for those ρ are below),
- (c) $\dim^{\mathbb{R}}$ [resp. $\dim^{\mathbb{C}}$] is the real [resp. complex] dimension of the representation space,
- (d) "conditions" are some conditions on the parameters in order to avoid repetitions (mainly due to classical isomorphisms in low dimensions).

Moreover, all the lists of that paper are divided in 2 or 3 parts (A), (B), (C) according to the following properties for the holonomies :

- (A) totally complex holonomies** i.e., h is a complex Lie subalgebra of $gl(n, \mathbb{C})$,
- (B) totally real holonomies** i.e., the representation space has no $\rho(h)$ -invariant complex structure,
- (C) real-complex holonomies** i.e., h is non complex, but the representation ρ is complex.

Notations for the representation ρ

Usually, the **representation** ρ is described through a tensorial product of representations of the center a (if there is one) and each of the simple factors in h . The only exceptions are for the (full) linear algebras $gl(m, \mathbb{R})$, $gl(m, \mathbb{C})$ or $gl(m, \mathbb{H})$ which are considered in the same way than simple factors. Also $so(4, \mathbb{C})$, $so(4)$ and $so(2, 2)$ are sometimes treated in the same way.

If ρ is not irreducible and is the direct sum of two irreducible factors, it is denoted by $\rho = \rho_1 \otimes \rho_2$ and now each factor is described as above. In the special case where ρ is complex-irreducible and not real irreducible, one describes both.

Conventions for the center of h

The canonical 1-dimensional representation of $\mathbb{R} = gl(1, \mathbb{R})$ in \mathbb{R} [resp. of $\mathbb{C} = gl(1, \mathbb{C})$ on \mathbb{C}] is always denoted by $\gamma^{\mathbb{R}}$ [resp. $\gamma^{\mathbb{C}}$].

In the case of a 1-dimensional factor \mathbb{C} in a 2-dimensional (complex) center, the representation $\gamma^{\mathbb{C}}(\lambda)$ of \mathbb{C} on \mathbb{C} is given by $\gamma^{\mathbb{C}}(\lambda) = \lambda Id$, where the parameter λ is a complex number. In order to avoid isomorphisms, one restricts furthermore to $|\lambda| \leq 1$ and if $|\lambda| = 1$ then $Im(\lambda) \geq 0$.

In the case of a 1-dimensional factor \mathbb{R} in a 2-dimensional (real) center, the representation $\gamma^{\mathbb{R}}(\mu)$ of \mathbb{R} on \mathbb{R} is given by $\gamma^{\mathbb{R}}(\mu) = \mu Id$, where the parameter μ is a real number. In order to avoid isomorphisms, one restricts furthermore to $|\mu| \leq 1$.

The complex representations $\sigma_{\theta}^{\mathbb{C}}$ of \mathbb{R} in the **complex** 1-dimensional space \mathbb{C} are given by $\sigma_{\theta}^{\mathbb{C}}(t) = t e^{i\theta} Id$, where θ is real (some angle). In order to avoid isomorphisms, one restricts furthermore to $\theta \in [0, \frac{\pi}{2}]$. Now $\sigma_{\theta}^{\mathbb{C}}$ is \mathbb{R} -irreducible if and only if $\theta \neq 0$.

If there is a 2-dimensional center (for a non-irreducible ρ), $\pi_1^{\mathbb{R}}$ and $\pi_2^{\mathbb{R}}$ [resp. $\pi_1^{\mathbb{C}}$ and $\pi_2^{\mathbb{C}}$] are the canonical representations of \mathbb{R}^2 on \mathbb{R} [resp. of \mathbb{C}^2 on \mathbb{C}] given by $\pi_1(t, u) = t Id$ and $\pi_1(t, u) = u Id$.

Conventions or the simple factors of h

Here the notation **can** is used either for any "canonical" representation of a classical Lie algebra or for the lowest dimensional representation of exotic ones.

And $\rho_0^{\mathbb{C}}$ [resp. $\rho_0^{\mathbb{R}}$] is the trivial 1-dimensional complex [resp. real] representation i.e., the corresponding simple factor is in the kernel of the representation.

Sometimes a complex-irreducible representation η of a real Lie algebra may have a "real structure" i.e., be the complexification of a real representation, here denoted by $\eta^{\mathbb{R}}$.

Notice a very special case : **only one** of the two (complex) half-spin representations of $so(6, \mathbb{H})$ has a real structure and gives a half-spin ^{\mathbb{R}} (and a holonomy).

And one recall that $\mathbb{H}^r \otimes_{\mathbb{H}} \mathbb{H}^s = \mathbb{R}^{4rs}$ is only a real vector space, with no invariant complex structure for the corresponding (tensorial product) representation.

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