# The Hopf algebra of Fliess operators and its dual pre-Lie algebra

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ABSTRACT. We study the Hopf algebra $H$ of Fliess operators coming from Control Theory in the one-dimensional case. We prove that it admits a graded, finte-dimensional, connected gradation. Dually, the vector space $\mathbb{R}\langle x_0, x_1\rangle$ is both a pre-Lie algebra for the pre-Lie product dual of the coproduct of $H$, and an associative, commutative algebra for the shuffle product. These two structures admit a compatibility which makes $\mathbb{R}\langle x_0, x_1\rangle$ a Com-pre-Lie algebra. We give a presentation of this object as a Com-pre-Lie algebra and as a pre-Lie algebra.

KEYWORDS. Fliess operators; pre-Lie algebras; Hopf algebras.

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## Contents

# Introduction

Right pre-Lie algebras, or shortly pre-Lie algebras [4, 1], are vector spaces with a bilinear product $\bullet$ satisfying the following axiom:

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y).$$

Consequently, the antisymmetrization of $\bullet$ is a Lie bracket. These objects are also called right-symmetric algebras or Vinberg algebra [10, 15]. If $A$ is a pre-Lie algebra, then the symmetric algebra $S(A)$ inherits a product $\star$ making it a Hopf algebra, isomorphic to the enveloping algebra of the Lie algebra $A$ [11, 12]. Whenever it is possible, we can consider the dual Hopf algebra $S(A)^*$ and its group of characters $G$, which is the exponentiation, in a certain sense, of the Lie algebra $A$.

We here consider the inverse construction, departing from a group used in Control Theory, namely the group fof Fliess operators [3, 5, 6]; this group is used to define the feedback product. We limit ourselves here to the one-dimensional case. This is the set $\mathbb{R}\langle\langle x_0, x_1\rangle\rangle$ of noncommutative formal series in two indeterminates, with a certain product generalizing the composition of formal series (definition 1). The Hopf algebra $H$ of coordinates of this group is described in [5], where it is also proved it is graded by the length of words; note that this gradation is not connected and not finite-dimensional. We first give a way to describe the composition in the group $\mathbb{R}\langle\langle x_0, x_1\rangle\rangle$ and the coproduct of $H$ by induction on the length of words(lemma 2 and proposition 3). We prove that $H$ admits a second gradation, which is connected; the dimensions of this gradation are given by the Fibonacci sequence (proposition 8). As the product of $\mathbb{R}\langle\langle x_0, x_1\rangle\rangle$ is left-linear, $H$ is a commutative, right-sided combinatorial Hopf algebra [9], so, dually, $\mathbb{R}\langle x_0, x_1\rangle$ inherits a pre-Lie product $\bullet$, which is inductively defined in proposition 11. We prove that the words $x_1^n$, $n \geq 0$, form a minimal subset of generators of this pre-Lie algebra (theorem 12).

The pre-Lie algebra $\mathbb{R}\langle x_0, x_1\rangle$ has also an associative, commutative product, namely the shuffle product $\sqcup\!\sqcup$ [13]. We prove that the following axiom is satisfied (proposition 14):

$$(x \sqcup\!\sqcup y) \bullet z = (x \bullet z) \sqcup\!\sqcup y + x \sqcup\!\sqcup (y \bullet z).$$

So $\mathbb{R}\langle x_0, x_1\rangle$ is a Com-pre-Lie algebra (definition 15). We give a presentation of this Com-pre-Lie algebra in theorem 27. We use for this a description of free Com-pre-Lie algebras in terms of partitioned trees (definition 17), which generalizes the construction of pre-Lie algebras in terms of rooted trees of [1]. We then deduce a presentation of $\mathbb{R}\langle x_0, x_1\rangle$ as a pre-Lie algebra in theorem 30. This presentation induces a new basis of $\mathbb{R}\langle x_0, x_1\rangle$ in terms of words with letters in $\mathbb{N}^*$, see corollary 31. The pre-Lie product of two elements of this basis uses a dendriform structure [2, 8] on the algebra of words with letters in $\mathbb{N}^*$ (theorem 34). The study of this dendriform structure is postponed to the appendix, as well as the enumeration of partitioned trees; we also prove that free Com-pre-Lie algebras are free as pre-Lie algebras, using the rigidity theorem of [7].

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**Notation.** We denote by $\mathbb{K}$ a commutative field of characteristic zero. All the objects (algebra, coalgebras, pre-Lie algebras...) in this text will be taken over $\mathbb{K}$.

# 1 Construction of the Hopf algebra

## 1.1 Definition of the composition

Let us consider an alphabet of two letters, denoted by $x_0$ and $x_1$. We denote by $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ the completion of the free algebra generated by this alphabet. Note that $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ is an algebra for the concatenation product and for the shuffle product, which we denote by $\sqcup\!\sqcup$.

**Exemples.** If $a, b, c, d \in \{x_0, x_1\}$:

$$\begin{aligned} abc \sqcup\!\sqcup d &= abcd + abdc + adbc + dabc, \\ ab \sqcup\!\sqcup cd &= abcd + acbd + cabd + acdb + cadb + cdab, \\ a \sqcup\!\sqcup bcd &= abcd + bacd + bcad + bcda. \end{aligned}$$

The unit for both these products is the empty word, which we denote by $\emptyset$. The algebra $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ is given its usual ultrametric topology.

**Definition 1** *[5].*

1. *For any $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, we define a continuous algebra map $\varphi_d$ from $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ to $End(\mathbb{K}\langle\langle x_0, x_1\rangle\rangle)$ in the following way: for all $X \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$,*

$$\varphi_d(x_0)(X) = x_0 X, \ \varphi_d(x_1)(X) = x_1 X + x_0(d \sqcup\!\sqcup X).$$

2. *We define a composition $\circ$ on $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ in the following way: for all $c, d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, $c \circ d = \varphi_d(c)(\emptyset) + d$.*

It is proved in [5] that this composition is associative.

**Notation**. For all $c, d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, we put $c \tilde{\circ} d = c \circ d - d = \varphi_d(c)(\emptyset)$.

**Remark.** If $c_1, c_2, d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, $\lambda \in \mathbb{K}$:

$$(c_1 + \lambda c_2)\tilde{\circ} d = \varphi_d(c_1 + \lambda c_2)(\emptyset) = (\varphi_d(c_1) + \lambda \varphi_d(c_2))(\emptyset) = \varphi_d(c_1)(\emptyset) + \lambda \varphi_d(c_2)(\emptyset) = c_1 \tilde{\circ} d + \lambda c_2 \tilde{\circ} d.$$

So the composition $\tilde{\circ}$ is linear on the left. As $\varphi_d$ is continuous, the map $c \longrightarrow c \tilde{\circ} d$ is continuous for any $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$. Hence, it is enough to know how to compute $c \tilde{\circ} d$ for any word $c$, which is made possible by the next lemma, using an induction on the length:

**Lemma 2** *For any words $c$, for any $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$:*

1. $\emptyset \tilde{\circ} d = \emptyset$.
2. $(x_0 c)\tilde{\circ} d = x_0(c \tilde{\circ} d)$.
3. $(x_1 c)\tilde{\circ} d = x_1(c \tilde{\circ} d) + x_0(d \sqcup\!\sqcup (c \tilde{\circ} d))$.

**Proof.** 1. $\emptyset \tilde{\circ} d = \varphi_d(\emptyset)(\emptyset) = Id(\emptyset) = \emptyset$.

2. $(x_0 c)\tilde{\circ} d = \varphi_d(x_0 c)(\emptyset) = \varphi_d(x_0) \circ \varphi_d(c)(\emptyset) = \varphi_d(x_0)(c \tilde{\circ} d) = x_0(c \tilde{\circ} d)$.

3. $(x_1 c)\tilde{\circ} d = \varphi_d(x_1 c)(\emptyset) = \varphi_d(x_1) \circ \varphi_d(c)(\emptyset) = \varphi_d(x_1)(c \tilde{\circ} d) = x_1(c \tilde{\circ} d) + x_0(d \sqcup\!\sqcup (c \tilde{\circ} d))$. $\square$

## 1.2 Dual Hopf algebra

We here give an inductive description of the Hopf algebra of the coordinates of the group $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ of [5].

For any word $c$, let us consider the map $X_c \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle^*$, such that for any $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, $X_c(d)$ is the coefficient of $c$ in $d$. We denote by $V$ the subspace of $A^*$ generated by these maps. Let $H = S(V)$, or equivalently the free commutative algebra generated by the $X_c$'s. The elements of

$H$ are seen as polynomial functions on $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$; the elements of $H \otimes H$ are seen as polynomial functions on $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle \times \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$. Then $H$ is given a multiplicative coproduct defined in the following way: for any word $c$, for any $f, g \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$,

$$\Delta(X_c)(f, g) = X_c(f \circ g).$$

As $\circ$ is associative, $\Delta$ is coassociative, so $H$ is a bialgebra.

**Notations.**

1. The space of words is a commutative algebra for the shuffle product $ш$. Dually, the space $V$ inherits a coassociative, cocommutative coproduct, denoted by $\Delta_{ш}$. For example, if $a, b, c \in \{x_0, x_1\}$:

$$\begin{array}{rcl} \Delta_{ш}(X_\emptyset) & = & X_\emptyset \otimes X_\emptyset, \\ \Delta_{ш}(X_a) & = & X_a \otimes X_\emptyset + X_\emptyset \otimes X_a, \\ \Delta_{ш}(X_{ab}) & = & X_{ab} \otimes X_\emptyset + X_a \otimes X_b + X_b \otimes X_a + X_\emptyset \otimes X_{ab}, \\ \Delta_{ш}(X_{abc}) & = & X_{abc} \otimes X_\emptyset + X_a \otimes X_{bc} + X_b \otimes X_{ac} + X_c \otimes X_{ab} \\ & & + X_{ab} \otimes X_c + X_{ac} \otimes X_b + X_{bc} \otimes X_a + X_\emptyset \otimes X_{abc}. \end{array}$$

2. We define two linear endomorphisms $\theta_0, \theta_1$ of $V$ by $\theta_i(X_c) = X_{x_i c}$ for any word $c$.

The following proposition allows to compute $\Delta(X_c)$ for any word $c$ by induction on the length of $c$.

**Proposition 3** *For all $x \in V$, we put $\tilde{\Delta}(x) = \Delta(x) - 1 \otimes x$.*

1. $\tilde{\Delta}(X_\emptyset) = X_\emptyset \otimes 1$.
2. $\tilde{\Delta} \circ \theta_0 = (\theta_0 \otimes Id) \circ \tilde{\Delta} + (\theta_1 \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{ш}$.
3. $\tilde{\Delta} \circ \theta_1 = (\theta_1 \otimes Id) \circ \tilde{\Delta}$.

**Proof.** For any word $c$, for any $f, g \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$:

$$\tilde{\Delta}(X_c)(f, g) = \Delta(X_c)(f, g) - (1 \otimes X_c)(f, g) = X_c(f \circ g) - X_c(g) = X_c(f \otimes g - g) = X_c(f \tilde{\circ} g).$$

As $\tilde{\circ}$ is linear on the left, $\tilde{\Delta}(X_c) \in V \otimes H$, so formulas in 2. and 3. make sense.

Let $f \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$. It can be uniquely written as $f = x_0 f_0 + x_1 f_1 + \lambda\emptyset$, with $f_0, f_1 \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, $\lambda \in K$. For all $g \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$:

$$\begin{array}{rcl} f \tilde{\circ} g & = & (x_0 f_0) \tilde{\circ} g + (x_1 f_1) \tilde{\circ} g + \lambda\emptyset \tilde{\circ} g \\ & = & x_0(f_0 \tilde{\circ} g + g ш (f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda\emptyset. \end{array}$$

1. We obtain:

$$\tilde{\Delta}(X_\emptyset)(f, g) = X_\emptyset(x_0(f_0 \tilde{\circ} g + g ш (f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda\emptyset) = 0 + 0 + \lambda = (X_\emptyset \otimes 1)(f, g).$$

So $\Delta(X_\emptyset) = X_\emptyset \otimes 1$.

2. Let $c$ be a word.

$$\begin{array}{rcl}
\tilde{\Delta}\circ\theta_0(X_c)(f,g) &=& \tilde{\Delta}(X_{x_0c})(f,g)\\
&=& X_{x_0c}(x_0(f_0\tilde{\circ}g+g\sqcup\!\sqcup(f_1\tilde{\circ}g))+x_1(f_1\tilde{\circ}g)+\lambda\emptyset)\\
&=& X_c(f_0\tilde{\circ}g+g\sqcup\!\sqcup(f_1\tilde{\circ}g))+0+0\\
&=& X_c(f_0\tilde{\circ}g+(f_1\tilde{\circ}g)\sqcup\!\sqcup g)+0+0\\
&=& \tilde{\Delta}(X_c)(f_0,g)+(\tilde{\Delta}\otimes Id)\circ\Delta_{\sqcup\!\sqcup}(X_c)(f_1,g,g)\\
&=& \tilde{\Delta}(X_c)(f_0,g)+(Id\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{\sqcup\!\sqcup}(X_c)(f_1,g)\\
&=& (\theta_0\otimes Id)\circ\tilde{\Delta}(X_c)(f,g)+(\theta_1\otimes Id)\circ(Id\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{\sqcup\!\sqcup}(X_c)(f,g),
\end{array}$$

so $\tilde{\Delta}\circ\theta_0(X_c)=(\theta_0\otimes Id)\circ\tilde{\Delta}(X_c)+(\theta_1\otimes Id)\circ(Id\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{\sqcup\!\sqcup}(X_c)$.

3. Let $c$ be a word.

$$\begin{array}{rcl}
\tilde{\Delta}\circ\theta_1(X_c)(f,g) &=& \tilde{\Delta}(X_{x_0c})(f,g)\\
&=& X_{x_1c}(x_0(f_0\tilde{\circ}g+g\sqcup\!\sqcup(f_1\tilde{\circ}g))+x_1(f_1\tilde{\circ}g)+\lambda\emptyset)\\
&=& 0+X_c(f_1\tilde{\circ}g)+0\\
&=& \tilde{\Delta}(X_c)(f_1,g)\\
&=& (\theta_1\otimes Id)\circ\tilde{\Delta}(X_c)(f,g),
\end{array}$$

so $\tilde{\Delta}\circ\theta_1(X_c)=(\theta_1\otimes Id)\circ\tilde{\Delta}(X_c)$. □

**Examples.**

$$\begin{array}{rcl}
\Delta(X_{x_0}) &=& X_{x_0}\otimes 1+1\otimes X_{x_0}+X_{x_1}\otimes X_\emptyset,\\
\Delta(X_{x_0^2}) &=& X_{x_0^2}\otimes 1+1\otimes X_{x_0^2}+X_{x_0x_1}\otimes X_\emptyset+X_{x_1x_0}\otimes X_\emptyset+X_{x_1x_1}\otimes X_\emptyset^2+X_{x_1}\otimes X_{x_0},\\
\Delta(X_{x_0x_1}) &=& X_{x_0x_1}\otimes 1+1\otimes X_{x_0x_1}+X_{x_1x_1}\otimes X_\emptyset+X_{x_1}\otimes X_{x_1},\\
\Delta(X_{x_1x_0}) &=& X_{x_1x_0}\otimes 1+1\otimes X_{x_1x_0}+X_{x_1x_1}\otimes X_\emptyset.
\end{array}$$

**Corollary 4** *For all $n\geq 1$, $\tilde{\Delta}(X_{x_1^n})=X_{x_1^n}\otimes 1$ and $\Delta(X_{x_1^n})=X_{x_1^n}\otimes 1+1\otimes X_{x_1^n}$.*

**Proof.** By induction on $n$. □

## 1.3 gradation

It is proved in [5] that the Hopf algebra $H$ is graded by the length of words, but this gradation is not connected, that is to say that the homogeneous component of degree 0 is not (0), as it contains $X_\emptyset$. We here define another gradation, which is connected.

**Definition 5** *Let $c=c_1\dots c_k$ be a word. We put:*

$$deg(c)=lg(c)+1+\sharp\{i\mid c_i=x_0\}.$$

*For all $k\geq 1$, we put:*

$$V_k=Vect(X_c\mid deg(x)=k).$$

*This define a connected gradation of $V$, that is to say:*

$$V=\bigoplus_{k\geq 1}V_k.$$

*This gradation induces a connected gradation of the algebra $H$:*

$$H=\bigoplus_{k\geq 0}H_k,\ \text{and}\ H_0=\mathbb{K};$$

**Proposition 6** *If $c$ is a word of degree $n$, then:*

$$\tilde{\Delta}(X_c) \in \bigoplus_{i+j=n} V_i \otimes H_j.$$

*So the gradation $(V_k)_{k\geq 1}$ is a gradation of the Hopf algebra $H$.*

**Proof.** Let us start by the following observations:

1. Let $c$ be a word of degree $k$. Then $x_0c$ is a word of degree $k+2$. Hence, $\theta_0$ is homogeneous of degree 2 on $V$.

2. Let $c$ be a word of degree $k$. Then $x_1c$ is a word of degree $k+1$. Hence, $\theta_1$ is homogeneous of degree 1 on $V$.

3. Let $c$ and $d$ be two words of respective degrees $k$ and $l$. Then any word obtained by shuffling $c$ and $d$ is of degree $k+l-1$: its length is the sum of the length of $c$ and $d$, and the number of $x_0$ in it is the sum of the numbers of $x_0$ in $c$ and $d$. Hence, the coproduct $\Delta_{ш}$ is homogeneous of degree 1 from $V$ to $V \otimes V$.

Let us prove the result by induction on the length $k$ of $c$. If $k=0$, then $c=\emptyset$ so $n=1$, and $\tilde{\Delta}(X_c) = X_c \otimes 1 \in V_1 \otimes H_0$. Let us assume the result for all words of length $< k-1$. Two cases can occur.

1. If $c = x_0d$, then $deg(d) = n-2$. we put $\Delta_{ш}(X_d) = \sum x_i' \otimes x_i''$. By the preceding third observation, we can assume that for all $i$, $x_i', x_i''$ are homogeneous elements of $V$, with $deg(x_i') + deg(x_i') = n-2+1 = n-1$. Then:

$$\tilde{\Delta}(X_c) = (\theta_0 \otimes Id) \circ \tilde{\Delta}(X_d) + \sum_i (\theta_1 \otimes m) \circ (\tilde{\Delta}(x_i') \otimes x_i'').$$

   By the induction hypothesis, $\tilde{\Delta}(X_d) \in (V \otimes H)_{n-1}$. By the second observation, $(\theta_0 \otimes Id) \circ \tilde{\Delta}(X_d) \in (V \otimes H)_n$. By the induction hypothesis applied to $x_i'$, for all $i$, $(\tilde{\Delta}(x_i') \otimes x_i'') \in (V \otimes H \otimes V)_{n-1}$, so by the first observation, $(\theta_1 \otimes m) \circ (\tilde{\Delta}(x_i') \otimes x_i'') \in (V \otimes H)_{n-1+1} \subseteq (V \otimes H)_n$. So $\Delta(X_c) \in (V \otimes H)_n$.

2. $c = x_1d$, then $deg(d) = n-1$. Moreover, $\tilde{\Delta}(X_c) = (\theta_1 \otimes Id) \circ \tilde{\Delta}(X_d)$. By the induction hypothesis, $\tilde{\Delta}(X_d) \in (V \otimes H)_{n-1}$. By the second observation, $\tilde{\Delta}(X_c) \in (V \otimes H)_n$.

So the result holds for any word $c$. □

**Corollary 7** *For all $n \geq 0$:*

$$\Delta(H_n) \subseteq \bigoplus_{i+j=n} H_i \otimes H_j.$$

**Proof.** The firs assertion comes from the multiplicativity of $\Delta$. As $H$ is a graded, connected bialgebra, it is a Hopf algebra. □

Let us now study the formal series of $V$ and $H$.

**Proposition 8** *1. For all $k$, let us put $p_k = dim(V_k)$ and $F_V = \sum_{k=1}^{\infty} p_k X^k$. Then:*

$$F_V = \frac{X}{1-X-X^2},$$

*and for all $k \geq 1$:*

$$p_k = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^k - \left( \frac{1-\sqrt{5}}{2} \right)^k \right).$$

*This is the Fibonacci sequence (A000045 in [14]).*

2. *We put* $F_H=\sum_{k=0}^\infty dim(H_k)X^k$. *Then:*

$$F_H=\prod_{k=1}^\infty \frac{1}{(1-X^k)^{p_k}}.$$

**Proof.** Let us consider the formal series:

$$F(X_0,X_1)=\sum_{i,j\geq 0}\sharp\{\text{words in } x_0,x_1 \text{ with } i\ x_0 \text{ and } j\ x_1\}X_0^iX_0^j.$$

Then $F(X_0,X_1)=\dfrac{1}{1-X_0-X_1}$. Moreover, by definition of the degree of a word:

$$F_V=XF(X^2,X)=\frac{X}{1-X-X^2}.$$

As $H$ is the symmetric algebra generated by $V$, its formal series is given by the second point. $\square$

**Examples.** We obtain:

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $dim(V_k)$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |
| $dim(H_k)$ | 1 | 1 | 2 | 4 | 8 | 15 | 30 | 56 | 108 | 203 | 384 |

The third row is sequence A166861 of [14].

**Remark.** Consequently, the space $V$ inherits a bigradation:

$$V_{k,n}=Vect(X_c\mid deg(c)=k \text{ and } lg(c)=n).$$

If $c$ is a word of length $n$ and of degree $k$, denoting by $a$ the number of its letters equal to $x_0$ and by $b$ the number of its letters equal to $x_1$, then:

$$\begin{cases} a+b &= n,\\ 2a+b+1 &= k,\end{cases}$$

so $a=k-n-1$. Hence:

$$dim(V_{k,n})=\binom{n}{k-n-1},$$

and the formal series of this bigradation is:

$$\sum_{k,n\geq 0} dim(V_{k,n})X^kY^n=\frac{X}{1-XY-X^2Y}.$$

# 2 Pre-Lie structure on $\mathbb{K}\langle x_0,x_1\rangle$

## 2.1 pre-Lie coproduct on $V$

As the composition $\circ$ is linear on the left, the dual coproduct satisfies $\tilde{\Delta}(V)\subseteq V\otimes H$, so $H$ is a commutative right-sided Hopf algebra in the sense of [9], and $V$ inherits a right pre-Lie coproduct: if $\pi$ is the canonical projection from $H=S(V)$ onto $V$,

$$\delta=(\pi\otimes\pi)\circ\Delta=(Id\otimes\pi)\circ\tilde{\Delta}.$$

It satisfies the right pre-Lie coalgebra axiom:

$$(23).((\delta\otimes Id)\circ\delta-(Id\otimes\delta)\circ\delta)=0.$$

The following proposition allows to compute $\delta(X_c)$ by induction on the length of $c$.

**Proposition 9** *1.* $\delta(X_\emptyset)=0$.

*2.* $\delta\circ\theta_0=(\theta_0\otimes Id)\circ\delta+(\theta_1\otimes Id)\circ\Delta_{ш}$.

*3.* $\delta\circ\theta_1=(\theta_1\otimes Id)\circ\delta$.

**Proof.** The first point comes from $\Delta(X_\emptyset)=X_\emptyset\otimes 1+1\otimes X_\emptyset$. Let $x\in V$. We put $\Delta_{ш}(x)=x'\otimes x''\in V\otimes V$. For any $y\in V$, we put $\tilde{\Delta}(y)-y\otimes 1=y^{(1)}\otimes y^{(2)}\in V\otimes H_+$. Then:

$$\begin{aligned}(\theta_1\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{ш}(x) &= (\theta_1\otimes m)(x'\otimes 1\otimes x''+x'^{(1)}\otimes x'^{(2)}\otimes x''\\ &= \theta_1(x')\otimes\underbrace{x''}_{\in V}+x'^{(1)}\otimes\underbrace{x'^{(2)}x''}_{\in Ker(\pi)}.\end{aligned}$$

Applying $Id\otimes\pi$, it remains:

$$(Id\otimes\pi)\circ(\theta_1\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{ш}(x)=(\theta_1\otimes Id)\circ\Delta_{ш}(x).$$

Let $i=0$ or $1$. Then:

$$(Id\otimes\pi)\circ(\theta_i\otimes Id)\circ\tilde{\Delta}=(\theta_i\otimes Id)\circ(Id\otimes\pi)\circ\tilde{\Delta}=(\theta_i\otimes Id)\circ\delta.$$

The result is induced by these remarks, combined with proposition 3. □

**Examples.**

$$\begin{aligned}\delta(X_{x_0}) &= X_{x_1}\otimes X_\emptyset,\\ \delta(X_{x_0^2}) &= X_{x_0x_1}\otimes X_\emptyset+X_{x_1x_0}\otimes X_\emptyset+X_{x_1}\otimes X_{x_0},\\ \delta(X_{x_0x_1}) &= X_{x_1x_1}\otimes X_\emptyset+X_{x_1}\otimes X_{x_1},\\ \delta(X_{x_1x_0}) &= X_{x_1x_1}\otimes X_\emptyset.\end{aligned}$$

**Proposition 10** $Ker(\delta)=Vect(X_{x_1^n},n\geq 0)$.

**Proof.**The inclusion $\supseteq$ is trivial by corollary 4. Let us prove the other inclusion.

*First step.* Let us prove the following property: if $x\in V_k$ is such that

$$\delta(x)=\lambda\sum_{i+j=k-2}\frac{(k-2)!}{i!j!}X_{x_1^i}\otimes X_{x_1^j},$$

then there exists $\mu\in\mathbb{K}$ such that $x=\mu x_1^{k-1}$. It is obvious if $k=1$, as then $x=\mu\emptyset$. Let us assume the result at all ranks $<k$. We put $x=x_1^\alpha(x_0f_0+x_1f_1)$, where $\alpha\geq 0$, $f_0$ is homogeneous of degree $k-2-\alpha$ and $f_1$ is homogeneous of degree $k-1-\alpha$.

$$\delta(x)=(\theta_1^\alpha\otimes Id)\left((\theta_0\otimes Id)\circ\delta(f_0)+(\theta_1\otimes Id)\circ\delta(f_1)+(\theta_1\otimes Id)\circ\Delta_{ш}(f_0)\right).$$

Let us consider the terms in this expression of the form $X_\emptyset\otimes X_c$, with $c$ a word. This gives:

$$\lambda X_\emptyset\otimes X_{x_1^{k-2}}=0,$$

so $\lambda=0$ and $\delta(x)=0$. Let us now consider the terms of the form $X_{x_1^\alpha x_0c}\otimes X_d$, with $c,d$ words. We obtain:

$$0=(\theta_1^\alpha\circ\theta_0\otimes Id)\circ\delta(f_0).$$

As both $\theta_0$ and $\theta_1$ are injective, we obtain $\delta(f_0)=0$. By the induction hypothesis, $f_0=\nu X_1x_1^l$, with $l=k-2-\alpha<k$. Hence:

$$\Delta_{ш}(f_0)=\nu\sum_{i+j=l}\frac{l!}{i!j!}X_{x_1^i}\otimes X_{x_1^j},$$

and:

$$(\theta_1^{\alpha+1} \otimes Id) \left( \delta(f_1) + \nu \sum_{i+j=l} \frac{l!}{i!j!} X_{x_1^i} \otimes X_{x_1^j} \right) = 0.$$

As $\theta_1$ is injective, we obtain with the induction hypothesis that $f_1 = \mu X_{x_1^{k-2-\alpha}}$, so:

$$x = \mu X_{x_1^{k-1}} + \nu X_{x_1^\alpha x_0 x_1^{k-\alpha-2}}.$$

This gives:

$$\begin{array}{rcl} \delta(x) & = & \nu(\theta_1^{\alpha+1} \otimes Id) \left( \displaystyle\sum_{i+j=k-\alpha-2} \frac{(k-\alpha-2)!}{i!j!} X_{x_1^i} \otimes X_{x_1^j} \right) \\ & = & \nu \displaystyle\sum_{i+j=k-\alpha-2} \frac{(k-\alpha-2)!}{i!j!} X_{x_1^{i+\alpha}} \otimes X_{x_1^j} \\ & = & 0, \end{array}$$

so necessarily $\nu = 0$ and $x = \mu X_{x_1^{k-1}}$.

*Second step.* Let $x \in Ker(\delta)$. As $\delta$ is homogeneous of degree 0, the homogeneous components of $x$ are in $Ker(\delta)$. By the first step, with $\lambda = 0$, these homogeneous components, hence $x$, belong to $Vect(X_{x_1^k}, k \geq 0)$. $\square$

## 2.2 Dual pre-Lie algebra

As $V$ is a graded pre-Lie coalgebra, its graded dual is a pre-Lie algebra. We identify this graded dual with $\mathbb{K}\langle x_0, x_1\rangle \subseteq \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$; for any words $c, d$, $X_c(d) = \delta_{c,d}$. The pre-Lie product of $\mathbb{K}\langle x_0, x_1\rangle$ is denoted by $\bullet$. Dualizing proposition 9, we obtain:

**Proposition 11** *1. For all word $c$, $\emptyset \bullet c = 0$.*

*2. For all words $c, d$, $(x_0 c) \bullet d = x_0(c \bullet d)$.*

*3. For all words $c, d$, $(x_1 c) \bullet d = x_1(c \bullet d) + x_0(c \sqcup\!\sqcup d)$.*

**Proof.** Let $u, v, w$ be words. Then $X_w(u \bullet v) = \delta(X_w)(u \otimes v)$. Hence, if $d$ is a word:

$$\begin{array}{rcl} X_\emptyset(u \bullet v) & = & 0, \\ X_{x_0 d}(u \bullet v) & = & (\theta_0 \otimes Id) \circ \delta(X_d)(u \otimes v) + (\theta_1 \otimes Id) \circ \Delta_{\sqcup\!\sqcup}(X_d)(u \otimes v) \\ & = & X_d(\theta_0^*(u) \bullet v + \theta_1^*(u) \sqcup\!\sqcup v), \\ X_{x_1 d}(u \bullet v) & = & (\theta_1 \otimes Id) \otimes \delta(X_d)(u \otimes v) \\ & = & X_d(\theta_1^*(u) \bullet v). \end{array}$$

Moreover, for all word $c$:

$$\begin{array}{rclcrclcrcl} \theta_0^*(\emptyset) & = & 0, & & \theta_0^*(x_0 c) & = & c, & & \theta_0^*(x_1 c) & = & 0 \\ \theta_1^*(\emptyset) & = & 0, & & \theta_1^*(x_0 c) & = & 0, & & \theta_1^*(x_1 c) & = & c. \end{array}$$

Hence, for any words $c, d$:

$$\begin{array}{rcl} X_{x_0d}(x_0c \bullet v) &=& X_d(c \bullet v) \\ &=& X_{x_0d}(x_0(x \bullet v)), \\ X_{x_1d}(x_0c \bullet v) &=& 0 \\ &=& X_{x_1d}(x_0(x \bullet v)); \end{array}$$

$$\begin{array}{rcl} X_{x_0d}(x_1c \bullet v) &=& X_d(c \sqcup\!\sqcup v) \\ &=& X_{x_0d}(x_1(c \bullet v) + x_0(c \sqcup\!\sqcup v)), \\ X_{x_1d}(x_1c \bullet v) &=& X_d(c \bullet v) \\ &=& X_{x_1d}(x_1(c \bullet v) + x_0(c \sqcup\!\sqcup v)). \end{array}$$

Hence, for any $w$, $X_w(x_0c \bullet v) = X_w(x_0(x \bullet v))$ and $X_w(x_1c \bullet v) = X_w((x_1(c \bullet v) + x_0(c \sqcup\!\sqcup v))$. $\Box$

**Examples.**

| | | |
|---|---|---|
| $x_0 \bullet x_0 = 0$ | $x_0 \bullet x_0x_0 = 0$ | $x_1 \bullet x_0x_0 = x_0x_0x_0$ |
| $x_0 \bullet x_1 = 0$ | $x_0 \bullet x_0x_1 = 0$ | $x_1 \bullet x_0x_1 = x_0x_0x_1$ |
| $x_1 \bullet x_0 = x_0x_0$ | $x_0 \bullet x_1x_0 = 0$ | $x_1 \bullet x_1x_0 = x_0x_1x_0$ |
| $x_1 \bullet x_1 = x_0x_1$ | $x_0 \bullet x_1x_1 = 0$ | $x_1 \bullet x_1x_1 = x_0x_1x_1$ |

| | |
|---|---|
| $x_0x_0 \bullet x_0 = 0$ | $x_0x_0 \bullet x_1 = 0$ |
| $x_0x_1 \bullet x_0 = x_0x_0x_0$ | $x_0x_1 \bullet x_1 = x_0x_0x_1$ |
| $x_1x_0 \bullet x_0 = 2x_0x_0x_0$ | $x_1x_0 \bullet x_1 = x_0x_0x_1 + x_0x_1x_0$ |
| $x_1x_1 \bullet x_0 = x_1x_0x_0 + x_0x_1x_0 + x_0x_0x_1$ | $x_1x_1 \bullet x_1 = x_1x_0x_1 + 2x_0x_1x_1$ |

Dualizing proposition 10:

**Theorem 12** $\mathbb{K}\langle x_0, x_1\rangle = Vect(x_1^n, n \geq 0) \oplus (\mathbb{K}\langle x_0, x_1\rangle \bullet \mathbb{K}\langle x_0, x_1\rangle)$. *Hence, $(x_1^n)_{n\geq 0}$ is a minimal system of generators of the pre-Lie algebra $\mathbb{K}\langle x_0, x_1\rangle$.*

**Proof.** As $\bullet = \delta^*$, $Im(\bullet) = Ker(\delta)^\perp = Vect(X_{x_1^n}, n \geq 0)^\perp$. The first assertion is then immediate. As $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ is a graded, connected pre-Lie coalgebra, $\mathbb{K}\langle x_0, x_1\rangle$ is a graded, connected pre-Lie algebra. The result then comes from the next lemma. $\Box$

**Lemma 13** *Let $A$ be a graded, connected pre-Lie algebra, and $V$ be a graded subspace of $A$.*

1. *$V$ generates $A$ if, and only if, $A = V + A \bullet A$.*
2. *$V$ is a minimal subspace of generators of $A$ if, and only if, $A = V \oplus A \bullet A$.*

**Proof.** 1. $\Longrightarrow$. Let $x \in A$. Then it can be written as an element of the pre-Lie subalgebra generated by $v$, so as the sum of an element of $V$ and of iterated pre-Lie products of elements of $V$. Hence, $x \in V + A \bullet A$. Note that we did not use the gradation of $A$ to prove this.

1. $\Longleftarrow$. Let $B$ be the pre-Lie subalgebra generated by $V$. Let $x \in A_n$, let us prove that $x \in B$ by induction on $n$. As $A_0 = (0)$, it is obvious if $n = 0$. Let us assume the result at all ranks $< n$. We obtain, by the gradation:

$$A_n = V_n \oplus \sum_{i=1}^{n-1} A_i \bullet A_{n-i}.$$

So we can write:

$$x = \lambda x_1^{n-1} + \sum x_i \bullet y_i,$$

where $x_i$, $y_i$ are homogeneous of degree $< n$. By the induction hypothesis, these elements belong to $B$, so $x \in B$.

2. $\Longrightarrow$. By 1. $\Longrightarrow$, $A = V + A \bullet A$. If $V \cap A \bullet A \neq (0)$, we can choose a graded subspace $W \subsetneq V$, such that $A = W \oplus A \bullet A$. By 1. $\Longleftarrow$, $W$ generates $A$, so $V$ is not a minimal system of generators of $A$: contradiction. So $A = V \oplus A \bullet A$.

2. $\Longleftarrow$. By 1. $\Longleftarrow$, $V$ is a space of generators of $A$. If $W \subsetneq V$, then $W \oplus A \bullet A \subsetneq A$. By 1. $\Longrightarrow$, $W$ does not generate $V$. So $V$ is a minimal subspace of generators. $\square$

**Proposition 14** *For all $x, y, z \in \mathbb{K}\langle x_0, x_1 \rangle$,*

$$(x ⧢ y) \bullet z = (x \bullet z) ⧢ y + x ⧢ (y \bullet z).$$

**Proof.** We prove it if $x, y, z$ are words. If $x = \emptyset$, then:

$$(\emptyset ⧢ y) \bullet z = y \bullet z = (\emptyset \bullet z) ⧢ y + \emptyset ⧢ (u \bullet z).$$

If $y = \emptyset$, the result is also true, using the commutativity of ⧢. We can now consider that $x, y$ are nonempty words.

Let us proceed by induction on $k = lg(x) + lg(y)$. If $k = 0$ or $1$, there is nothing to prove. Let us assume the result at all rank $< k$. Four cases can occur.

*First case.* $x = x_0 a$ and $y = x_0 b$. Then:

$$\begin{aligned}
(x ⧢ y) \bullet z &= (x_0(a ⧢ x_0 b) \bullet z + (x_0(x_0 a ⧢ b)) \bullet z \\
&= x_0((a ⧢ x_0 b) \bullet z) + x_0((x_0 a ⧢ b) \bullet z) \\
&= x_0((a \bullet z) ⧢ x_0 b) + x_0(a ⧢ ((x_0 b) \bullet z)) + x_0(((x_0 a) \bullet z) ⧢ b) + x_0(x_0 a ⧢ (b \bullet z)) \\
&= x_0((a \bullet z) ⧢ x_0 b) + x_0(a ⧢ (x_0(b \bullet z)) + x_0((x_0(a \bullet z)) ⧢ b) + x_0(x_0 a ⧢ (b \bullet z)) \\
&= x_0(a \bullet z) ⧢ x_0 b + x_0 a ⧢ x_0(b \bullet z) \\
&= (x \bullet z) ⧢ y + x ⧢ (y \bullet z).
\end{aligned}$$

*Second case.* $x = x_1 a$ and $y = x_0 b$. This gives:

$$\begin{aligned}
(x ⧢ y) \bullet z &= (x_1(a ⧢ x_0 b)) \bullet z + (x_0(x_1 a ⧢ b)) \bullet z \\
&= x_1((a \bullet z) ⧢ x_0 b) + x_1(a ⧢ x_0(b \bullet z)) \\
&\quad + x_0(a ⧢ x_0 b ⧢ z) + x_0(((x_1 a) \bullet z) ⧢ b) + x_0(x_1 a ⧢ (b \bullet z)) \\
&= x_1((a \bullet z) ⧢ x_0 b) + x_1(a ⧢ x_0(b \bullet z)) \\
&\quad + x_0(a ⧢ x_0 b ⧢ z) + x_0((x_1(a \bullet z)) ⧢ b) + x_0((x_0(a ⧢ z)) ⧢ b) + x_0(x_1 a ⧢ (b \bullet z)),
\end{aligned}$$

$$\begin{aligned}
(x \bullet z) ⧢ y &= (x_1(a \bullet z)) ⧢ x_0 b + (x_0(a ⧢ z)) ⧢ (x_0 b) \\
&= x_1((a \bullet z) ⧢ (x_0 b)) + x_0(x_1(a \bullet z) ⧢ b) \\
&\quad + x_0(a ⧢ z ⧢ x_0 b) + x_0((x_0(a ⧢ z)) ⧢ b),
\end{aligned}$$

$$\begin{aligned}
x ⧢ (y \bullet z) &= x_1 a ⧢ x_0(b \bullet z) \\
&= x_1(a ⧢ x_0(b \bullet z)) + x_0(x_1 a ⧢ (b \bullet z)).
\end{aligned}$$

These computations imply the required equality.

*Third case.* $x = x_0 a$ and $y = x_1 b$. This is a consequence of the second case, using the commutativity of ⧢.

*Last case.* $x = x_1a$ and $y = x_1b$. Similar computations give:

$$
\begin{aligned}
(x \sqcup\!\sqcup y) \bullet z &= x_1((a \bullet z) \sqcup\!\sqcup x_1b) + x_1(a \sqcup\!\sqcup x_1(b \bullet w)) + x_1(a \sqcup\!\sqcup x_0(b \sqcup\!\sqcup z)) + x_0(a \sqcup\!\sqcup x_1b \sqcup\!\sqcup z) \\
&\quad + x_1(x_1a \sqcup\!\sqcup (b \bullet z)) + x_1((x_1(a \bullet z)) \sqcup\!\sqcup b) + x_1((x_0(a \sqcup\!\sqcup z)) \sqcup\!\sqcup b) + x_0(a \sqcup\!\sqcup x_1b \sqcup\!\sqcup z), \\
(x \bullet z) \sqcup\!\sqcup y &= x_1((a \bullet z) \sqcup\!\sqcup x_1b) + x_1((x_1(a \bullet z)) \sqcup\!\sqcup b) + x_0(a \sqcup\!\sqcup x_1b \sqcup\!\sqcup z) + x_1((x_0(a \sqcup\!\sqcup z)) \sqcup\!\sqcup b), \\
x \sqcup\!\sqcup (y \bullet z) &= x_1(a \sqcup\!\sqcup x_1(b \bullet w)) + x_1(a \sqcup\!\sqcup x_0(b \sqcup\!\sqcup z)) + x_1(x_1a \sqcup\!\sqcup (b \bullet z)) + x_0(a \sqcup\!\sqcup x_1b \sqcup\!\sqcup z).
\end{aligned}
$$

So the result holds in all cases. $\square$

# 3 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a Com-pre-Lie algebra

Proposition 14 motivates the following definition:

**Definition 15** *An* Com-pre-Lie algebra *is a triple* $(V, \bullet, \sqcup\!\sqcup)$*, such that:*

1. $(V, \bullet)$ *is a pre-Lie algebra.*
2. $(V, \sqcup\!\sqcup)$ *is a commutative, associative algebra (non necessarily unitary).*
3. *For all* $a, b, c \in V$*,* $(a \sqcup\!\sqcup b) \bullet c = (a \bullet c) \sqcup\!\sqcup b + a \sqcup\!\sqcup (b \bullet c)$*.*

For example, $\mathbb{K}\langle x_0, x_1 \rangle$ is a Com-pre-Lie algebra.

## 3.1 Free Com-pre-Lie algebras

**Definition 16** 1. *A* partitioned forest *is a pair* $(F, I)$ *such that:*

   (a) *$F$ is a rooted forest (the edges of $F$ being oriented from the leaves to the roots).*

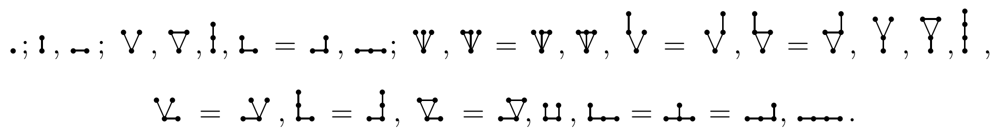
   (b) *$I$ is a partition of the vertices of $F$ with the following condition: if $x, y$ are two vertices of $F$ which are in the same part of $I$, then either they are both roots, or they have the same direct descendant.*

2. *We shall say that a partitioned forest is a* partitioned tree *if all the roots are in the same part of the partition.*

3. *Let $\mathcal{D}$ be a set. A* partitioned tree decorated by $\mathcal{D}$ *is a pair $(t, d)$, where $t$ is a partitioned tree and $d$ is a map from the set of vertices of $t$ into $\mathcal{D}$. For any vertex $x$ of $t$, $d(x)$ is called the* decoration *of $x$.*

4. *The set of isoclasses of partitioned trees will be denoted by $\mathcal{PT}$. For any set $\mathcal{D}$, the set of isoclasses of partitioned trees decorated by $\mathcal{D}$ will be denoted by $\mathcal{PT}(\mathcal{D})$.*

**Examples.** We represent partitioned trees by the Hasse graph of the underlying rooted forest, the partition being represented by horizontal edges. Here are partitioned trees with $\leq 4$ vertices:



**Definition 17** *Let $t = (t, I)$ and $t' = (t', J) \in \mathcal{PT}$.*

1. *Let $s$ be a vertex of $t'$. The partitioned tree $t \bullet_s t'$ is defined as follows:*

(a) *As a rooted forest, $t \bullet_s t'$ is obtained by grafting all the roots of $t'$ on the vertex $s$ of $t$.*

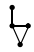
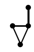
(b) *We put $I = \{I_1, \ldots, I_k\}$ and $J = \{J_1, \ldots, J_l\}$. The partition of the vertices of this rooted forest is $I \sqcup J = \{I_1, \ldots, I_k, J_1, \ldots, J_l\}$.*

2. *The partitioned tree $t \amalg t'$ is defined as follows:*

   (a) *As a rooted forest, $t \bullet_s t'$ is $tt'$.*

   (b) *We put $I = \{I_1, \ldots, I_k\}$ and $J = \{J_1, \ldots, J_l\}$ and we assume that the set of roots of $t$ is $I_1$ and the set of roots of $t'$ is $J_1$. The partition of the vertices of $t \bullet t'$ $\{I_1 \sqcup J_1, I_2, \ldots, I_k, J_1, \ldots, J_l\}$.*

**Examples.**

1. Here are the three possible graftings  $\bullet_s$ : ,  and .

2. Here are the two possible graftings  $\bullet_s$ :  and .

These operations can also be defined for decorated partitioned trees.

**Proposition 18** *Let $\mathcal{D}$ be a set. We denote by $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ the vector space generated by $\mathcal{PT}(\mathcal{D})$. We extend $\amalg$ by bilinearity on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ and we define a second product $\bullet$ on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ in the following way: if $t, t' \in \mathcal{PT}(\mathcal{D})$,*

$$t \bullet t' = \sum_{s \in V(t)} t \bullet_s t'.$$

*Then $(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}, \bullet, \amalg)$ is a Com-pre-Lie algebra.*

**Proof.** Let $t, t', t''$ be three partitioned trees.

If $s', s''$ are two vertices of $t$, we define by $t \bullet_{s,s'} (t', t'')$ the partitioned trees obtained by grafting the roots of $t'$ on $s'$, the roots of $t''$ on $s''$, the partition of the vertices of the obtained rootes forest being the union of the partitions of $t$, $t'$ and $t''$. Then:

$$\begin{aligned}
(t \bullet t') \bullet t'' &= \sum_{s' \in V(t)} (t \bullet_{s'} t') \bullet t'' \\
&= \sum_{s', s'' \in V(t)} (t \bullet_{s'} t') \bullet_{s''} t'' + \sum_{s' \in V(t), s'' \in V(t')} (t \bullet_{s'} t') \bullet_{s''} t'' \\
&= \sum_{s', s'' \in V(t)} t \bullet_{s's''} (t', t'') + \sum_{s' \in V(t), s'' \in V(t')} t \bullet_{s'} (t' \bullet_{s''} t'') \\
&= \sum_{s', s'' \in V(t)} t \bullet_{s's''} (t', t'') + t \bullet (t' \bullet t'').
\end{aligned}$$

So $(t \bullet t') \bullet t'' - t \bullet (t' \bullet t'')$ is clearly symmetric in $t$ and $t'$, and $\bullet$ is pre-Lie.

Moreover, $(t \amalg t') \amalg t'' = t \amalg (t' \amalg t'')$ is the rooted forest $tt't''$, the partition being $\{I_1 \cup J_1 \cup K_1, I_2, \ldots, I_k, J_2, \ldots, J_l, K_2, \ldots, K_m\}$, with immediate notations; $t \amalg t' = t' \amalg t$ is the rooted forest $tt'$, the partition being $\{I_1 \cup J_1, I_2, \ldots, I_k, J_2, \ldots, J_l\}$. So $\amalg$ is an associative, commutative product.

Finally:

$$\begin{aligned}
(t \amalg t') \bullet t'' &= \sum_{s \in V(t)} (t \amalg t') \bullet_s t'' + \sum_{s' \in V(t')} (t \amalg t') \bullet_{s'} t'' \\
&= \sum_{s \in V(t)} (t \bullet_s t'') \amalg t' + \sum_{s' \in V(t')} t \amalg (t' \bullet_{s'} t'') \\
&= (t \bullet t') \amalg t'' + t \amalg (t' \bullet t'').
\end{aligned}$$

So $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is Com-pre-Lie. $\square$

In particular, $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is pre-Lie. Let us use the extension of the pre-Lie product $\bullet$ to $S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$ defined by Oudom and Guin [11, 12]:

1. If $t_1, \ldots, t_k \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $t_1 \ldots t_k \bullet 1 = t_1 \ldots t_k$.
2. If $t, t_1, \ldots, t_k \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $t \bullet t_1 \ldots t_k = (t \bullet t_1 \ldots t_{k-1}) \bullet t_k - t \bullet (t_1 \ldots t_{k-1} \bullet t_k)$.
3. If $a, b, c \in S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$, $ab \bullet c = (a \bullet c^{(1)})(b \bullet c^{(2)})$, where $\Delta(c) = c^{(1)} \otimes c^{(2)}$ is the usual coproduct of $S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$. In particular, if $t_1, \ldots, t_k, t \in \mathcal{PT}(\mathcal{D})$:

$$t_1 \ldots t_k \bullet t = \sum_{i=1}^{k} t_1 \ldots (t_i \bullet t) \ldots t_k.$$

**Lemma 19** *Let $t = (t, I), t_1 = (t_1, I^{(1)}), \ldots, t_k = (t_k, I^{(k)})$ be partitioned trees ($k \geq 1$). Let $s_1, \ldots, s_k \in V(t)$. The partitioned tree $t \bullet_{s_1, \ldots, s_k} (t_1, \ldots, t_k)$ is obtained by grafting the roots of $t_i$ on $s_i$ for all $i$, the partition being $I \sqcup I^{(1)} \sqcup \ldots \sqcup I^{(k)}$. Then:*

$$t \bullet t_1 \ldots t_k = \sum_{s_1, \ldots, s_k \in V(t)} t \bullet_{s_1, \ldots, s_k} (t_1, \ldots, t_k).$$

**Proof.** By induction on $k$. This is obvious if $k = 1$. Let us assume the result at rank $k$.

$$\begin{aligned}
t \bullet t_1 \ldots t_{k+1} &= (t \bullet t_1 \ldots t_k) \bullet t_{k+1} - \sum_{i=1}^{k} t \bullet (t_1 \ldots (t_i \bullet t_{k+1}) \ldots t_k) \\
&= \sum_{s_1, \ldots, s_k \in V(t)} (t \bullet_{s_1, \ldots, s_k} (t_1, \ldots, t_k)) \bullet t_{k+1} - \sum_{i=1}^{k} \sum_{s \in V(t_i)} t \bullet (t_1 \ldots (t_i \bullet_s t_{k+1}) \ldots t_i) \\
&= \sum_{s_1, \ldots, s_{k+1} \in V(t)} (t \bullet_{s_1, \ldots, s_k} (t_1, \ldots, t_k)) \bullet_{s_{k+1}} t_{k+1} \\
&\quad + \sum_{i=1}^{k} \sum_{s \in V(t_i)} (t \bullet_{s_1, \ldots, s_k} (t_1, \ldots, t_k)) \bullet_s t_{k+1} \\
&\quad - \sum_{i=1}^{k} \sum_{s_1, \ldots, s_k \in V(t)} \sum_{s \in V(t_i)} t \bullet_{s_1, \ldots, s_k} (t_1, \ldots, t_i \bullet_s t_{k+1}, \ldots, t_i) \\
&= \sum_{s_1, \ldots, s_{k+1} \in V(t)} t \bullet_{s_1, \ldots, s_{k+1}} (t_1, \ldots, t_{k+1}).
\end{aligned}$$

Hence, the result holds for all $k$. $\square$

**Theorem 20** *Let $\mathcal{D}$ be a set, let $A$ be a Com-pre-Lie algebra, and let $a_d \in A$ for all $d \in \mathcal{D}$. There exists a unique morphism of Com-pre-Lie algebra $\phi : \mathfrak{g}_{\mathcal{PT}(\mathcal{D})} \longrightarrow A$, such that $\phi(\bullet_d) = a_d$ for all $d \in \mathcal{D}$. In other words, $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is the free Com-pre-Lie algebra generated by $\mathcal{D}$.*

**Proof.** *Unicity.* Let $t \in \mathcal{T}^d$. We denote by $r_1, \ldots, r_n$ its roots. For all $1 \leq i \leq n$, let $t_{i,1}, \ldots, t_{i,k_i}$ be the partitioned trees born from $r_i$ and let $d_i$ be the decoration of $r_i$. Then:

$$t = (\bullet_{d_1} \bullet t_{1,1} \ldots t_{1,k_1}) ⧢ \ldots ⧢ (\bullet_{d_n} \bullet t_{n,1} \ldots t_{n,k_n}).$$

So $\phi$ is inductively defined by:

$$\phi(t) = (a_{d_1} \bullet \phi(t_{1,1}) \ldots \phi(t_{1,k_1})) ⧢ \ldots ⧢ (a_{d_n} \bullet \phi(t_{n,1}) \ldots \phi(t_{n,k_n})). \tag{1}$$

*Existence.* As the product $\shuffle$ of $A$ is commutative and associative, (1) defines inductively a morphism $\phi$ from $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ to $A$. By definition, it is compatible with the product $\shuffle$. Let us prove the compatibility with the product $\bullet$. Let $t, t'$ be two partitioned trees, let us prove that $\phi(t \bullet t') = \phi(t) \bullet \phi(t')$ by induction on the number $N$ of vertices of $t$. If $N = 1$, then $t = \bullet_d$ and:

$$\phi(t \bullet t') = a_d \bullet \phi(t') = \phi(t) \bullet \phi(t'),$$

by definition of $t'$. If $N > 1$, two cases are possible.

*First case.* If $t$ has only one roots, then $t = \bullet_d \bullet t_1 \dots t_k$, and:

$$t \bullet t' = \bullet_d \bullet t_1 \dots t_k t' + \sum_{i=1}^{k} \bullet_d \bullet t_1 \dots t_i \circ t' \bullet t_k.$$

Using the induction hypothesis on $t_1, \dots, t_k$:

$$\begin{aligned}
\phi(t \bullet t') &= a_d \bullet \phi(t_1) \dots \phi(t_k)\phi(t') + \sum_{i=1}^{k} a_d \bullet \phi(t_1) \dots \phi(t_1 \circ t') \dots \phi(t_k) \\
&= a_d \bullet \phi(t_1) \dots \phi(t_k)\phi(t') + \sum_{i=1}^{k} a_d \bullet (\phi(t_1) \dots \phi(t_1) \circ \phi(t') \dots \phi(t_k)) \\
&= (a_d \bullet \phi(t_1) \dots \phi(t_k)) \bullet \phi(t') \\
&= \phi(t) \bullet \phi(t').
\end{aligned}$$

*Second case.* If $t$ has $k > 1$ roots, we put $t = t_1 \shuffle \dots \shuffle t_k$. The induction hypothesis holds for $t_1, \dots, t_k$, so:

$$\begin{aligned}
\phi(t \circ t') &= \sum_{i=1}^{k} \phi(t_1 \shuffle t_i \bullet t' \shuffle \dots \shuffle t_k) \\
&= \sum_{i=1}^{k} \phi(t_1) \shuffle \phi(t_i \bullet t') \shuffle \dots \shuffle \phi(t_k) \\
&= \sum_{i=1}^{k} \phi(t_1) \shuffle \phi(t_i) \bullet \phi(t') \shuffle \dots \shuffle \phi(t_k) \\
&= (\phi(t_1) \shuffle \dots \shuffle \phi(t_k)) \bullet \phi(t') \\
&= \phi(t) \bullet \phi(t').
\end{aligned}$$

Hence, $\phi$ is a morphism of Com-pre-Lie algebras. $\Box$

## 3.2 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a Com-pre-Lie algebra

**Proposition 21** *As a Com-pre-Lie algebra, $\mathbb{K}\langle x_0, x_1 \rangle$ is generated by $\emptyset$ and $x_1$.*

**Proof.** Let $A$ be the Com-pre-Lie subalgebra of $\mathbb{K}\langle x_0, x_1 \rangle$ generated by $\emptyset$ and $x_1$. For all $n \geq 1$, it contains $x_1^{\shuffle n} = n! x_1^n$, so it contains $x_1^n$ for all $n \geq 0$. As $\mathbb{K}\langle x_0, x_1 \rangle$ is generated by these elements as a pre-Lie algebra, $A = \mathbb{K}\langle x_0, x_1 \rangle$. $\Box$

We denote by $\phi_{APL} : \mathfrak{g}_{\mathcal{PT}(\{1,2\})} \longrightarrow \mathbb{K}\langle x_0, x_1 \rangle$ the unique morphism of Com-pre-Lie algebras which sends $\bullet_1$ to $\emptyset$ and $\bullet_2$ to $\bullet_2$. By proposition 21, it is surjective.

**Lemma 22** *Let $t_1, \dots, t_k \in \mathcal{PT}(\{1,2\})$.*

1. $\phi_{APL}(\bullet_1 \bullet t_1 \dots t_k) = 0$ *if* $k \geq 1$.

2. $\phi_{APL}(\bullet_2 \bullet t_1 \dots t_k) = 0$ *if* $k \geq 2$.

3. *If* $t \in \mathcal{PT}(\{1,2\})$, $\phi_{APL}(\bullet_2 \bullet t) = x_0 \phi_{APL}(t)$.

**Proof.** We proceed by induction on $k$. If $k = 1$:

$$\begin{array}{rcl} \phi_{APL}(\bullet_1 \bullet t) & = & \emptyset \bullet \phi_{APL}(t) \\ & = & 0, \\ \phi_{APL}(\bullet_2 \bullet t) & = & x_1 \bullet \phi_{APL}(t) \\ & = & x_0 \phi_{APL}(t). \end{array}$$

Let us assume the result at rank $k-1 \geq 1$. Then:

$$\begin{array}{rcl} \phi_{APL}(\bullet_1 \bullet t_1 \dots t_k) & = & \emptyset \bullet \phi_{APL}(t_1) \dots \phi_{APL}(t_k) \\ & = & (\emptyset \bullet \phi_{APL}(t_1) \dots \phi_{APL}(t_{k-1})) \bullet \phi_{APL}(t_k) \\ & & -\displaystyle\sum_{i=1}^{k} \emptyset \bullet \phi_{APL}(t_1) \dots \phi_{APL}(t_i \bullet t_k) \dots \phi_{APL}(t_{k-1}) \\ & = & 0 - 0, \\ \phi_{APL}(\bullet_2 \bullet t_1 \dots t_k) & = & x_1 \bullet \phi_{APL}(t_1) \dots \phi_{APL}(t_k) \\ & = & (x_1 \bullet \phi_{APL}(t_1) \dots \phi_{APL}(t_{k-1})) \bullet \phi_{APL}(t_k) \\ & & -\displaystyle\sum_{i=1}^{k} x_1 \bullet \phi_{APL}(t_1) \dots \phi_{APL}(t_i \bullet t_k) \dots \phi_{APL}(t_{k-1}). \end{array}$$

If $k \geq 3$, the induction hypothesis immediately allows to conclude that $\phi_{APL}(\bullet_2 \bullet t_1 \dots t_k) = 0 - 0 = 0$. If $k = 2$, this gives:

$$\begin{array}{rcl} \phi_{APL}(\bullet_2 \bullet t_1 t_2) & = & (x_1 \bullet \phi_{APL}(t_1)) \bullet \phi_{APL}(t_2) - x_1 \bullet \phi_{APL}(t_1 \bullet t_2) \\ & = & (x_0 \phi_{APL}(t_1)) \bullet \phi_{APL}(t_2) - x_0 \phi_{APL}(t_1 \bullet t_2) \\ & = & x_0(\phi_{APL}(t_1) \bullet \phi_{APL}(t_2)) - x_0 \phi_{APL}(t_1 \bullet t_2) \\ & = & 0. \end{array}$$

Hence, the result holds for all $k \geq 1$. $\square$

**Lemma 23** *For all* $t \in \mathcal{PT}(\{1,2\})$, $\phi_{APL}(t)$ *is a linear span of words of length the number of vertices of* $t$ *decorated by* 2.

**Proof.** By induction on the number of vertices $N$ of $t$. If $N = 1$, then $t = \bullet_1$ or $\bullet_2$ and the result is obvious. Let us assume the result at all rank $< N$.

*First case.* If $t$ has only one root, we put $t = \bullet_i \bullet t_1 \dots t_k$. By the preceding lemma, we can assume that $i = 2$ and $k = 1$. Then $\phi_{APL}(t) = x_0 \phi_{APL}(t_1)$ and the result is obvious.

*Second case.* If $t$ has $k > 1$ roots, we put $t = t_1 ⧢ \dots ⧢ t_k$. Then $\phi_{APL}(t_1)$ is equal to $\phi_{APL}(t_1) ⧢ \dots ⧢ \phi_{APL}(t_k)$ and the result is immediate. $\square$

**Lemma 24** *We define inductively a family* $F$ *of elements of* $\mathcal{PT}(\{1,2\})$ *by:*

1. $F(1) = \{\bullet_1, \bullet_2\}$.

2. $F(n+1) = (\bullet_2 \bullet F(n)) \cup \displaystyle\bigcup_{i=1}^{n} (F(i) ⧢ F(n+1-i))$.

3. $F = \displaystyle\bigcup_{n \geq 1} F(n)$.

*Let $t \in \mathcal{PT}(\{1,2\})$. Then $\phi_{APL}(t) \neq 0$ if, and only if, $t \in F$.*

**Proof.** $\Longrightarrow$. We proceed by induction on the number $N$ of vertices of $t$. This is obvious if $N = 1$. Let us assume the result at all rank $< N$.

*First case.* If $N$ has only one root, we put $N = \bullet_i \bullet t_1 \ldots t_k$. By lemma 22, $i = 2$ and $k = 1$. Then $\phi_{APL}(t) = x_0 \phi_{APL}(t_1)$. By the induction hypothesis, $t_1 \in F$, so $t \in F$.

*Second case.* If $N$ has $k > N$ roots, we put $t = t_1 \shuffle \ldots \shuffle t_k$. Then:

$$\phi_{APL}(t) = \phi_{APL}(t_1) \shuffle \phi_{APL}(t_2 \shuffle \ldots \shuffle t_k) \neq 0,$$

so by the induction hypothesis, $t_1$ and $t_2 \shuffle \ldots \shuffle t_k \in F$, and $t \in F$.

$\Longleftarrow$. Let $t \in T(n)$. We proceed by induction on $n$. It $n = 1$, this is obvious. If $n > 1$ then $t = \bullet_2 \bullet t'$, with $t' \in F(n-1)$, or $t = t' \shuffle t''$, with $t' \in F(i)$, $t'' \in F(n-i)$. In the first case, by the induction hypothesis, $\phi_{APL}(t') \neq 0$ and $\phi_{APL}(t) = x_0 \phi_{APL}(t') \neq 0$. In the second case, $\phi_{APL}(t'), \phi_{APL}(t'') \neq 0$ by the induction hypothesis, so $\phi_{APL}(t) = \phi_{APL}(t') \shuffle \phi_{APL}(t'') \neq 0$. $\Box$

We define a second family of elements of $\mathcal{PT}(\{1,2\})$ in the following way:

1. $F'(1) = \{\bullet_1, \bullet_2\}$.
2. $F'(2) = \{\bullet_2 \shuffle \bullet_2, \mathbf{I}^2_2, \mathbf{I}^1_2\}$.
3. $F'(n+1) = (\bullet_2 \bullet F'(n)) \cup \displaystyle\bigcup_{i=2}^{n-1} \big(F'(i) \shuffle F'(n+1-i)\big) \cup \big(\bullet_2 \shuffle F'(n)\big)$ if $n \geq 2$.
4. $F' = \displaystyle\bigcup_{n \geq 1} F'(n)$.

We define a map $\pi$ from $F$ to $\mathcal{PT}(\{1,2\})$ in the following way:

1. $\pi(\bullet_i) = \bullet_i$ if $i = 1, 2$.
2. $\pi(\bullet_1 \shuffle \ldots \shuffle \bullet_1) = \bullet_1$.
3. If $t = \bullet_1 \shuffle \ldots \shuffle \bullet_1 \shuffle t_1 \shuffle \ldots \shuffle t_k$, $k \geq 1$, with $t_1, \ldots, t_k \neq \bullet_1$, then $\pi(t) = \pi(t_1) \shuffle \ldots \shuffle \pi(t_k)$.
4. If $t = \bullet_2 \bullet t_1 \ldots t_k$, then $\pi(t) = \bullet_2 \bullet \pi(t_1) \ldots \pi(t_k)$.

**Lemma 25** *$\pi$ is a projection on $F'$ and $\phi_{APL} \circ \pi = \phi_{APL|F}$.*

**Proof.** Let $t \in F$. Let us prove by induction on the number $N$ of vertices of $t$ that:

1. $\pi(t) \in F'$.
2. If $t \in F'$, $\pi(t) = t$.
3. $\phi_{APL} \circ \pi(t) = \phi_{APL}(t)$.
4. If $\pi(t) = \bullet_1$, then $t = \bullet_1^{\shuffle n}$ for a particular $n$.

All these points are immediate if $N = 1$. Let us assume the result at all rank $< N$, $N \geq 2$. We put $t = \bullet_1 \shuffle \ldots \shuffle \bullet_1 \shuffle t_1 \shuffle \ldots \shuffle t_k$, $k \geq 0$, with $t_1, \ldots, t_k \neq \bullet_1$.

*First case.* If $k \geq 2$, then $\pi(t) = \pi(t_1) ⧢ \ldots ⧢ \pi(t_k)$. By the induction hypothesis, $\pi(t_1), \ldots, \pi(t_k) \in F'$ and are not equal to $\cdot_1$, so $\pi(t) \in F'$. By the induction hypothesis, $\pi(t_1) \neq \cdot_1$, so $\pi(t) \neq \cdot_1$. Moreover:

$$\begin{array}{rcl} \phi_{APL}(t) & = & \phi_{APL}(\cdot_1) ⧢ \ldots ⧢ \phi_{APL}(\cdot_1) ⧢ \phi_{APL}(t_1) ⧢ \ldots ⧢ \phi_{APL}(t_k) \\ & = & \emptyset ⧢ \ldots ⧢ \emptyset ⧢ \phi_{APL} \circ \pi(t_1) ⧢ \ldots ⧢ \phi_{APL} \circ \pi(t_k) \\ & = & \phi_{APL}(\pi(t_1) ⧢ \ldots ⧢ \pi(t_k)) \\ & = & \phi_{APL} \circ \pi(t). \end{array}$$

If $t \in F'$, necessarily $t = t_1 ⧢ \ldots ⧢ t_k$, and $t_1, \ldots, t_k \in F'$. By the induction hypothesis, $\pi(t_1) = t_1, \ldots, \pi(t_k) = t_k$, so $\pi(t) = t$.

*Second case.* If $k = 1$, as $t_1 \in F$, we put $t_1 = \cdot_2 \bullet s$. Then $\pi(t) = \cdot_2 \bullet \pi(s)$. By the induction hypothesis, $\pi(s) \in F'$, so $\pi(t) = F'$. Moreover:

$$\begin{array}{rcl} \phi_{APL}(t) & = & \phi_{APL}(\cdot_1) ⧢ \ldots ⧢ \phi_{APL}(\cdot_1) ⧢ (\phi_{APL}(\cdot_2) \bullet \phi_{APL}(s)) \\ & = & \emptyset ⧢ \ldots ⧢ \emptyset ⧢ (\phi_{APL}(\cdot_2) \bullet \phi_{APL}(s)) \\ & = & \phi_{APL} \circ \pi(\cdot_2) \bullet \phi_{APL} \circ \pi(s) \\ & = & \phi_{APL} \circ \pi(t). \end{array}$$

If $t' \in F'$, then $s \in F'$, and $t = \cdot_2 \bullet s$. Then $\pi(t) = \cdot_2 \bullet \pi(s) = \cdot_2 \bullet s = t$.

*Last case.* If $k = 0$, all the results are obvious. $\square$

**Lemma 26** *Let $t, t' \in \mathcal{PT}(\{1,2\})$. Then:*

$$\phi_{APL}\left((\cdot_2 \bullet t) ⧢ (\cdot_2 \bullet t')\right) = \phi_{APL}\left(\cdot_2 \bullet ((\cdot_2 \bullet t) ⧢ t' + t ⧢ (\cdot_2 \bullet t'))\right).$$

**Proof.** Indeed, putting $w = \phi_{APL}(t)$ and $w' = \phi_{APL}(t')$:

$$\begin{array}{rcl} \phi_{APL}\left((\cdot_2 \bullet t) ⧢ (\cdot_2 \bullet t')\right) & = & x_0 w ⧢ x_0 w' \\ & = & x_0(w ⧢ x_0 w') + x_0(x_0 w ⧢ w') \\ & = & \phi_{APL}\left(\cdot_2 \bullet ((\cdot_2 \bullet t) ⧢ t' + t ⧢ (\cdot_2 \bullet t'))\right). \end{array}$$

We used lemma 22 for the first and third equalities. $\square$

**Theorem 27** *The kernel of $\phi_{APL}$ is the Com-pre-Lie ideal generated by the elements:*

1. $\cdot_1 \bullet t_1 \ldots t_k$, *where* $k \geq 1$, $t_1, \ldots, t_k \in \mathcal{PT}(\{1,2\})$.
2. $\cdot_2 \bullet t_1 \ldots t_k$, *where* $k \geq 2$, $t_1, \ldots, t_k \in \mathcal{PT}(\{1,2\})$.
3. $\cdot_1 ⧢ t - t$, *where* $t \in \mathcal{PT}(\{1,2\})$.
4. $(\cdot_2 \bullet t) ⧢ (\cdot_2 \bullet t') - \cdot_2 \bullet ((\cdot_2 \bullet t) ⧢ t' - t ⧢ (\cdot_2 \bullet t'))$, *where* $t, t' \in \mathcal{PT}(\{1,2\})$.

**Proof.** Let $I$ be the ideal generated by these elements. Lemmas 22 and 26 prove that the elements 1., 2. and 4. belong to $Ker(\phi_{APL})$. Moreover, for all $t \in \mathcal{PT}(\{1,2\})$, $\pi(\cdot_1 ⧢ t) = \pi(t)$. For all $t \in \mathcal{PT}(\{1,2\})$:

$$\phi_{APL}(\cdot_1 ⧢ t) = \emptyset ⧢ \phi_{APL}(t) = \phi_{APL}(t),$$

so elements 3. also belong to $Ker(\phi_{APL})$. Hence, $I \subseteq Ker(\phi_{APL})$.

Let $h = \mathfrak{g}_{\mathcal{PT}(\{1,2\})}/I$. As the elements 1. and 2. belong to $I$, $h$ is linearly spanned by the elements $\overline{t}$, $t \in F$. As the elements 3. belong to $I$, for all $t \in F$, $\overline{\pi(t)} = \overline{t}$. As $\pi$ is a projection on $F'$, $h$ is linearly spanned by the elements $\overline{t}$, $t \in F'$.

We now define inductively two families of partitionned trees in the following way:

1. $T''(1) = \{\bullet_2\}$ and $F''(1) = \{\bullet_1, \bullet_2\}$.
2. $\mathcal{T}''(n+1) = \bullet_2 \bullet F''(n)$.
3. $F''(n+1) = \displaystyle\bigcup_{i=1}^{n+1} T''(i) \shuffle \bullet_2^{\shuffle (n+1-i)}$.
4. $F'' = \displaystyle\bigcup_{n\geq 1} F''(n)$.

Let us prove that for all $t \in F'$, there exists $t' \in Vect(F'')$ such that $\overline{t} = \overline{t'}$. We proceed by induction on the number $N$ of vertices of $t$. If $N = 1$, then $t = \bullet_1$ or $\bullet_2$ and we take $t' = t$. Let us assume the result at all rank $< N$. We put $t = t_1 \shuffle \ldots \shuffle t_k \shuffle \bullet_2 \shuffle \ldots \shuffle \bullet_2$, with $t_i = \bullet_2 \bullet s_i$, $s_i \neq 1$, for all $1 \leq i \leq k$. We proceed by induction on $k$. If $k = 0$, we take $t' = t = \bullet_2 \shuffle \ldots \shuffle \bullet_2$. If $k = 1$, then, by the induction hypothesis on $N$ applied to $s_1$:

$$\overline{t} = (\overline{\bullet_2} \bullet \overline{s_1}) \shuffle \overline{\bullet_2} \shuffle \ldots \shuffle \overline{\bullet_2} = (\overline{\bullet_2} \bullet \overline{s'_1}) \shuffle \overline{\bullet_2} \shuffle \ldots \shuffle \overline{\bullet_2} = \overline{(\bullet_2 \bullet s'_1) \shuffle \bullet_2 \shuffle \ldots \shuffle \bullet_2}.$$

We take $t' = (\bullet_2 \bullet s'_1) \shuffle \bullet_2 \shuffle \ldots \shuffle \bullet_2$, which clearly belongs to $Vect(F'')$, as $s'_1 \in Vect(F'')$. Let us assume the result at all rank $< k$. Then, as the elements 4. belong to $I$:

$$\overline{t_1 \shuffle t_2} = \underbrace{\overline{\bullet_2 \bullet (t_1 \shuffle s_2)}}_{t'_1} + \underbrace{\overline{\bullet_2 \bullet (s_1 \bullet t_2)}}_{t''_1},$$

so:

$$\overline{t} = \overline{t'_1 \shuffle t_3 \shuffle \ldots \shuffle t_k \shuffle \bullet_2 \shuffle \ldots \shuffle \bullet_2} + \overline{t''_1 \shuffle t_3 \shuffle \ldots \shuffle t_k \shuffle \bullet_2 \shuffle \ldots \shuffle \bullet_2}.$$

By the induction hypothesis on $k$ applied to these two partitionned trees, there exists $x'_1$ and $x''_1 \in Vect(F'')$, such that $\overline{t} = \overline{x'_1} + \overline{x''_1}$. We take $t' = x'_1 + x''_1$. Consequently, the elements $\overline{t}$, $t \in F''$, linearly span $h$.

Let $t \in F''(n)$. Then it has $n$ vertices, and at most one of them is decorated by 1. We denote by $F''_1(n)$ the set of elements of $F''(n)$ with one vertex decorated by 1, and we put $F''_2(n) = F''(n) \setminus F''_1(n)$. Let us prove that for all $n \geq 1$, $|F''_1(n+1)| \leq 2^{n-1}$ and $|F''_2(n)| \leq 2^{n-1}$. For $n = 0$, as $FF'_1(2) = \{\mathbf{I}^{1}_{2}\}$ and $F''_2(1) = \{\bullet_2\}$, this is immediate. Let us assume the result at all rank $\leq n$. Then:

$$\begin{aligned}
F''_2(n+1) &= \bigcup_{i=1}^{n+1} \bullet_2^{\shuffle (n+1-i)} \shuffle T''(i) \cap F''_2(i) \\
&= \{\bullet_2^{\shuffle (n+1)}\} \cup \bigcup_{i=1}^{n} \bullet_2^{\shuffle (n+1-i)} \shuffle \bullet_2 \bullet F''_2(i).
\end{aligned}$$

Hence, $|F''_2(n+1)| \leq 1 + 1 + 2 + \ldots + 2^{n-1} = 2^n$.

$$\begin{aligned}
F''_1(n+2) &= \bigcup_{i=1}^{n+2} \bullet_2^{\shuffle (n+2-i)} \shuffle T''(i) \cap F''_1(i) \\
&= \bigcup_{i=2}^{n+2} \bullet_2^{\shuffle (n+2-i)} \shuffle \bullet_2 \bullet F''_1(i-1)
\end{aligned}$$

Hence, $|F''_1(n+2)| \leq +1 + 1 + \ldots + 2^{n-1} = 2^n$.

Let $\overline{\phi}_{APL}$ be the linear map induced by $\phi_{APL}$ on $h$. If $t \in F_1''(n)$, by lemma 23, $\overline{\phi}_{APL}(\overline{t})$ is a linear span of word of length $n-1$. If $t \in F_2''(n)$, by lemma 23, $\overline{\phi}_{APL}(\overline{t})$ is a linear span of word of length $n$. Hence, for all $n \geq 0$:

$$\overline{\phi}_{APL}(Vect(F_2''(n)) + Vect(F_1''(n+1))) \subseteq Vect(\text{words of length } n).$$

As $\phi_{APL}$ is surjective, we obtain:

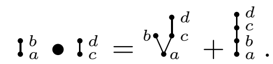
$$\overline{\phi}_{APL}(Vect(F_2''(n)) + Vect(F_1''(n+1))) = Vect(\text{words of length } n).$$

Moreover, as $dim(Vect(\text{words of length } n)) = 2^n$ and $dim(Vect(F_2''(n)) + Vect(F_1''(n+1))) \leq |F_2''(n)| + |F_1''(n)| \leq 2^{n-1} + 2^{n-1} = 2^n$, the restriction of $\overline{\phi}_{APL}$ to $Vect(F_2''(n)) + Vect(F_1''(n+1))$ is injective. Finally, $\overline{\phi}_{APL}$ is injective, so $Ker(\phi_{APL}) = I$. $\square$

# 4 Presentation of $\mathbb{K}\langle x_0, x_1\rangle$ as a pre-Lie algebra

## 4.1 A surjective morphism

Let $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ be the free pre-Lie algebra generated by $\mathbb{N}^*$, as described in [1]. It can be seen as the subspace of $\mathfrak{g}_{\mathcal{PT}(\mathbb{N}^*)}$ generated by rooted trees (which are seen as partitioned trees such that any part of the partition is a singleton), with the restriction of the pre-Lie product $\bullet$ defined by graftings. For example, in $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, if $a, b, c, d > 0$:



This pre-Lie algebra is graded, the degree of a tree being the sum of its decorations.

By theorem 12, there exists a unique surjective map of pre-Lie algebras $\Phi_{PL} : \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} \longrightarrow \mathbb{K}\langle x_0, x_1\rangle$, sending $\bullet_n$ to $x_1^{n-1}$ for all $n \geq 1$. As $x_1^{i-1}$ is homogeneous of degree $i$ for all $i$, this morphism is homogeneous of degree 0.

**Notation.** If $t_1 \dots t_k \in \mathcal{T}(\mathbb{N}^*)$ and $n \in \mathbb{N}^*$, we put:

$$B_n(t_1 \dots t_k) = \bullet_n \bullet t_1 \dots t_k.$$

This is the tree obtained by grafting $t_1, \dots, t_k$ on a common root decorated by $n$.

**Proposition 28** *Let $t = B_n(t_1 \dots t_k) \in \mathcal{T}(\mathbb{N}^*)$. We put $\phi_{PL}(t_i) = w_i$ for all $1 \leq i \leq k$. Then:*

$$\phi_{PL}(t) = \begin{cases} x_0 w_1 ⧢ \dots ⧢ x_0 w_k ⧢ x_1^{n-1-k} \text{ if } k < n, \\ 0 \text{ otherwise.} \end{cases}$$

**Proof.** As $\mathfrak{g}_{\mathcal{PT}(\{1,2\})}$ is pre-Lie, there exists a unique morphism of pre-Lie algebras:

$$\psi : \begin{cases} \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} & \longrightarrow & \mathfrak{g}_{\mathcal{PT}(\{1,2\})} \\ \bullet_n & \longrightarrow & \frac{1}{(n-1)!} \bullet_2^{⧢(n-1)}. \end{cases}$$

Then $\phi_{APL} \circ \psi$ is a pre-Lie algebra morphism sending $\bullet_n$ to $\frac{1}{(n-1)!} x_1^{⧢(n-1)} = x_1^{n-1}$ for all $n \geq 1$, so $\phi_{APL} \circ \psi = \phi_{PL}$. We obtain, by lemma 19:

$$\begin{array}{rcl} \psi(\bullet_n \bullet t_1 \dots t_k) & = & \frac{1}{(n-1)!} \bullet_2^{⧢(n-1)} \bullet (\psi(t_1) \dots \psi(t_k)) \\ & = & \frac{1}{(n-1)!} \sum_{I_1 \sqcup \dots \sqcup I_n = \{1,\dots,k\}} \bullet_2 \bullet \left(\prod_{i \in I_1} t_i\right) ⧢ \dots ⧢ \bullet_2 \bullet \left(\prod_{i \in I_k} t_i\right) \end{array}$$

Let us apply $\phi_{APL}$ to this expression. If $|I_j| \geq 2$, by theorem 27:

$$\phi_{APL}(\bullet_2 \bullet \left(\prod_{i\in I_j} t_i\right) = 0.$$

Consequently, if $k \geq n$, at least one of the $I_j$ contains two elements, so $\phi_{APL} \circ \psi(t) = \phi_{PL}(t) = 0$. Let us assume that $k < n$. Hence, using the commutativity of $ш$:

$$\begin{array}{rcl}
\phi_{PL}(\bullet_n \bullet t_1 \dots t_k) &=& \displaystyle\frac{1}{(n-1)!} \sum_{I_1 \sqcup \dots \sqcup I_n = \{1,\dots,k\},\, |I_j| \leq 1} x_1 \bullet \left(\prod_{i\in I_1} w_i\right) ш \dots ш x_1 \bullet \left(\prod_{i\in I_k} w_i\right)\\
&=& \displaystyle\frac{1}{(n-1)!} \sum_{\iota:\{1,\dots,k\}\longrightarrow\{1,\dots,n-1\},\text{injective}} x_1 \bullet w_1 ш \dots x_1 \bullet w_k ш x_1^{ш(n-1-k)}\\
&=& \displaystyle\frac{1}{(n-1)!} \sum_{\iota:\{1,\dots,k\}\longrightarrow\{1,\dots,n-1\},\text{injective}} x_0 w_1 ш \dots x_0 w_k ш x_1^{ш(n-1-k)}\\
&=& \displaystyle\frac{(n-1)\dots(n-k)}{(n-1)!} x_0 w_1 ш \dots x_0 w_k ш x_1^{ш(n-1-k)}\\
&=& \displaystyle\frac{(n-1)\dots(n-k)(n-1-k)!}{(n-1)!} x_0 w_1 ш \dots x_0 w_k ш x_1^{n-1-k}\\
&=& x_0 w_1 ш \dots x_0 w_k ш x_1^{n-1-k}.
\end{array}$$

□

**Corollary 29** *Let $s_1, \dots, s_k, t_1, \dots, t_l \in \mathcal{T}(\{N^*\})$, $k, l \geq 0$. For all $i, j, n \geq 1$:*

$$\begin{array}{rl}
& \phi_{PL}\left(B_{n+1}((B_i(s_1 \dots s_k) B_j(t_1 \dots t_l)))\right.\\
= & \phi_{PL}\left(B_n(B_{i+1}(s_1 \dots s_k B_j(t_1 \dots t_l)))\right) + \phi_{PL}\left(B_n(B_{j+1}(B_i(s_1 \dots s_k) t_1 \dots t_l))\right).
\end{array}$$

**Proof.** We note:

$$\begin{array}{rcl}
T_1 &=& B_{n+1}((B_i(s_1 \dots s_k) B_j(t_1 \dots t_l))\\
&=& \bullet_{n+1} \bullet ((\bullet_i \bullet s_1 \dots s_k)(\bullet_j \bullet t_1 \dots t_l))\\
T_2 &=& B_n(B_{i+1}(s_1 \dots s_k B_j(t_1 \dots t_l))\\
&=& \bullet_n \bullet (\bullet_{i+1} \bullet (s_1 \dots s_k(\bullet_j \bullet t_1 \dots t_l)))\\
T_3 &=& B_n(B_{j+1}(B_i(s_1 \dots s_k) t_1 \dots t_l)\\
&=& \bullet_n \bullet (\bullet_{j+1} \bullet ((\bullet_i \bullet s_1 \dots s_k) t_1 \dots t_l)).
\end{array}$$

If $k \geq i$, or $l \geq j$, or $n = 1$, all these elements are sent to zero by $\phi_{PL}$ by proposition 28. Let us assume now that $k < i$, $l < j$, $n < 1$. We put $v_i = \phi_{PL}(s_i)$ and $w_i = \phi_{PL}(t_i)$. Then:

$$\begin{array}{rcl}
\phi_{PL}(T_1) &=& x_0(\underbrace{x_0 v_1 ш \dots ш x_0 v_k ш x_1^{i-1-k}}_{X}) ш x_0(\underbrace{x_0 w_1 ш \dots ш x_0 w_l ш x_1^{j-1_l}}_{Y}) ш x_1^{n-2}\\
&=& x_0 X ш x_0 Y ш x_1^{n-2},\\
\phi_{PL}(T_2) &=& x_0(x_0 v_1 ш \dots ш x_0(x_0 w_1 ш \dots ш x_0 w_l ш x_1^{j-1-l}) ш x_1^{i-1-k}) ш x_1^{n-2}\\
&=& x_0(X ш x_0 Y) ш x_1^{n-2},\\
\phi_{PL}(T_3) &=& x_0(x_0(x_0 v_1 ш \dots ш x_0 v_k ш x_1^{i-1-k}) ш x_0 w_1 ш x_0 w_l ш x_1^{j-1-l}) ш x_1^{n-2}\\
&=& x_0(x_0 X ш Y) ш x_1^{n-2}.
\end{array}$$

As $x_0 X ш x_0 Y = x_0(X ш x_0 Y) + x_0(x_0 X ш Y)$, we obtain the result. □

**Theorem 30** *The kernel of $\phi_{PL}$ is the pre-Lie ideal generated by:*

1. $B_1(t_1 \ldots t_k)$, *where* $k \geq 1$, $t_1, \ldots, t_k \in \mathcal{T}(\mathbb{N}^*)$.
2. $B_{n+1}(B_i(s_1 \ldots s_k)B_j(t_1 \ldots t_l)) - B_n(B_{i+1}(s_1 \ldots s_k B_j(t_1 \ldots t_l)) - B_{j+1}(B_i(s_1 \ldots s_k)t_1 \ldots t_l))$, *where* $k, l \geq 0$, $s_1, \ldots, s_k, t_1, \ldots, t_l \in \mathcal{T}(\mathbb{N}^*)$.

**Proof.** Let $I$ be the ideal generated by these elements. By proposition 28 and corollary 29, $I \subseteq Ker(\phi_{PL})$. We put $h = \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}/I$. Applying repeatedly the relation given by elements 2., it is not difficult to prove that for any $t \in \mathcal{T}(\mathbb{N}^*)$, there exists a linear span of ladders $t'$ such that $\overline{t} = \overline{t'}$ in $h$. Moreover, by the relation given by elements 1., if one of the vertices of a ladder $t$ which is not the leaf is decorated by 1, then $\overline{t} = 0$. Let us denote by $L(n)$ the set of ladders decorated by $\mathbb{N}^*$, of weight $n$, such that all the vertices which are not the leaf are decorated by integer $> 1$. It turns out that $h$ is generated by the elements $\overline{t}$, $t \in L = \bigcup L(n)$.

Let $\overline{\phi_{PL}}$ be the morphism form $h$ to $\mathbb{K}\langle x_0, x_1 \rangle$ induced by $\phi_{PL}$. By homogeneity, as $\phi_{PL}$ is surjective, for all $n \geq 1$:

$$\overline{\phi}_{PL}(Vect(L(n))) = Vect(\text{words of degree } n).$$

In order to prove that $I = Ker(\phi_{PL})$, it is enough to prove that $\overline{\phi}_{PL}$ is injective. By homogeneity, it is enough to prove that $\overline{\phi}_{|Vect(L(n))}$ is injective for all $n \geq 1$. Hence, it is enough to prove that for all $n \geq 1$,

$$|L(n)| = dim(Vect(\text{words of degree } n)) = p_n,$$

where the $p_n$ are the integers defined in proposition 8. Let $l_n = |L(n)|$ and $q_n$ be the number of $t \in L(n)$ with no vertex decorated by 1. Then for all $n \geq 2$, $l_n = q_n + q_{n-1}$, and $l_1 = 1$. We put:

$$L = \sum_{n=1}^{\infty} l_n X^n, \; Q = \sum_{n=1}^{\infty} q_n X^n.$$

We obtain $P = X + Q + XQ$. Moreover:

$$Q = \frac{1}{1 - \sum_{i \geq 2} X^i} - 1 = \frac{1}{1 - \frac{X^2}{1-X}} - 1 = \frac{X^2}{1 - X - X^2},$$

Finally:

$$L = \frac{X}{1 - X - X^2} = F.$$

So, for all $n \geq 1$, $|L(n)| = p_n$. $\square$

As an immediate corollary, a basis of $h$ is given by the classes of the elements of $L$. Turning to $\mathbb{K}\langle x_0, x_1 \rangle$, we obtain:

**Corollary 31** *Let $w = a_1 \ldots a_k$ be a word with letters in $\mathbb{N}^*$.*

1. *We put:*
$$m_w = x_1^{a_1 - 1} \bullet (x_1^{a_1 - 1} \bullet (\ldots (x_1^{a_{k-1} - 1} \bullet x_1^{a_k}) \ldots)).$$
2. *We shall say that $w$ is* admissible *if $a_1, \ldots, a_{k-1} > 1$. The set of admissible words is denoted by $\mathcal{A}dm$.*

*Then $(m_w)_{w \in \mathcal{A}dm}$ is a basis of $\mathbb{K}\langle x_0, x_1 \rangle$.*

**Remark.** If $w$ is not admissible, that is to say if there exists $1 \leq i < k$, such that $a_i = 1$, then $m_w = 0$ by proposition 28.

We extend the map $w \longrightarrow m_w$ by linearity.

## 4.2 Pre-Lie product in the basis of admissible words

**Notations.**

1. For all $k, l$, we denote by $Sh(k,l)$ the set of $(k,l)$- shuffles, that is to say $k+l$-permutations $\zeta$ such that $\zeta(1) < \ldots < \zeta(k)$, $\zeta(k+1) < \ldots < \zeta(k+l)$.
2. For all $k, l$ we denote by $Sh_{\prec}(k,l)$ the set of $(k,l)$-shuffles $\zeta$ such that $\zeta^{-1}(k+l) = k$.
3. For all $k, l$ we denote by $Sh_{\succ}(k,l)$ the set of $(k,l)$-shuffles $\zeta$ such that $\zeta^{-1}(k+l) = k+l$.
4. The symmetric group $\mathfrak{S}_n$ acts on the set of words with letters in $\mathbb{N}^*$ of length $n$ by permutation of the letters:
$$\sigma.(a_1 \ldots a_n) = a_{\sigma^{-1}(1)} \ldots a_{\sigma^{-1}(n)}.$$

**Proposition 32** *Let $\mathbb{K}\langle \mathbb{N}^* \rangle$ be the space generated by words with letters in $\mathbb{N}^*$. We define a dendriform structure on this space by:*

$$\begin{aligned}
(a_1 \ldots a_k) \prec (b_1 \ldots b_l) &= \sum_{\zeta \in Sh_{\prec}(k,l)} \zeta.a_1 \ldots a_k b_1 \ldots b_{k-1}(b_k+1)\\
(a_1 \ldots a_k) \succ (b_1 \ldots b_l) &= \sum_{\zeta \in Sh_{\succ}(k,l)} \zeta.a_1 \ldots a_{k-1}(a_k+1) b_1 \ldots b_k.
\end{aligned}$$

*The associative product $\prec + \succ$ is denoted by $\star$.*

**Proof.** We denote by $Sh(k,l,m)$ the set of $k+l+m$-permutations such that $\zeta(1) < \ldots < \zeta(k)$, $\zeta(k+1) < \ldots < \zeta(k+l)$, $\zeta(k+l+1) < \ldots \zeta(k+l+m)$. Then:

$$\begin{aligned}
& (a_1 \ldots a_k \prec b_1 \ldots b_l) \prec c_1 \ldots c_m = a_1 \ldots a_k \prec (b_1 \ldots b_l \star c_1 \ldots c_m)\\
= & \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k} \zeta.a_1 \ldots a_k b_1 \ldots (b_l+1) c_1 \ldots (c_m+1);\\
& (a_1 \ldots a_k \succ b_1 \ldots b_l) \prec c_1 \ldots c_m = a_1 \ldots a_k \succ (b_1 \ldots b_l \prec c_1 \ldots c_m)\\
= & \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k+l} \zeta.a_1 \ldots (a_k+1) b_1 \ldots b_l c_1 \ldots (c_m+1);\\
& (a_1 \ldots a_k \star b_1 \ldots b_l) \succ c_1 \ldots c_m = a_1 \ldots a_k \succ (b_1 \ldots b_l \succ c_1 \ldots c_m)\\
= & \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k+l+m} \zeta.a_1 \ldots (a_k+1) b_1 \ldots (b_l+1) c_1 \ldots c_m.
\end{aligned}$$

So $\mathbb{K}\langle\langle \mathbb{N}^* \rangle\rangle$ is a dendriform algebra. $\square$

We postpone the study of this dendriform algebra to section 5.2.

**Notations.** For all $a_1, \ldots, a_k \in \mathbb{N}^*$, we denote by $l(a_1 \ldots a_k) = B_{a_1} \circ \ldots \circ B_{a_k}(1)$ the ladder decorated from the root to the leaf by $a_1, \ldots, a_k$. Note that $m_{a_1 \ldots a_k} = \phi_{PL}(l(a_1 \ldots a_k))$.

**Lemma 33** *Let $k, l \geq 1$ and let $a_1, \ldots, a_l, b_1, \ldots, b_l \in \mathbb{N}^*$. Then:*

$$\phi_{PL}(B_{a_1+1}(l(a_2 \ldots a_k) l(b_1 \ldots b_l)) + B_{b_1+1}(l(a_1 \ldots a_k) l(b_2 \ldots b_l)) = m_{a_1 \ldots a_k \star b_1 \ldots b_l}.$$

**Proof.** By induction on $k+l$. If $k = l = 1$, then:

$$\phi_{PL}(\mathfrak{l}^{b_1}_{a_1+1} + \mathfrak{l}^{a_1}_{b_1+1}) = m_{(a_1+1)b_1+(b_1+1)a_1} = m_{a_1 \star b_1}.$$

Let us assume the result at all ranks $< k+l$. If $k=1$, then:

$$\begin{array}{rcl} & = & \phi_{PL}(B_{a_1+1}(l(b_2\ldots b_l))+B_{b_1+1}(l(a_1)l(b_2\ldots b_l))\\ & = & \phi_{PL}(\bullet_{a_1+1}\bullet l(b_2\ldots b_l)+\bullet_{b_1+1}\bullet(l(a_1)l(b_2\ldots b_l)))\\ & = & \phi_{PL}(l((a_1+1)b_2\ldots b_l))+\phi_{PL}(\bullet_{b_1}\bullet(l((a_1+1)b_2\ldots b_l)+\bullet_{b_2+1}\bullet(l(a_1)l(b_3\ldots b_l)))\\ & = & m_{(a_1+1)b_2\ldots b_l}+m_{b_1(a_1\star b_2\ldots b_l)}\\ & = & m_{(a_1+1)b_2\ldots b_l}+\displaystyle\sum_{i=1}^{l-1} m_{b_1\ldots b_i(a_1+1)\ldots b_l}+m_{b_1\ldots(b_l+1)a_1}\\ & = & m_{a_1\star b_1\ldots b_l}. \end{array}$$

If $l=1$, a similar computation, permuting the $a_i$'s and the $b_j$'s, proves the result. If $k,l>1$, then:

$$\begin{array}{rl} & \phi_{PL}(B_{a_1+1}(l(a_2\ldots a_k)l(b_1\ldots b_l))+B_{b_1+1}(l(a_1\ldots a_k)l(b_2\ldots b_l))\\ = & \phi_{PL}(\bullet_{a_1}\bullet(\bullet_{a_2+1}\bullet l(a_3\ldots a_k)l(b_1\ldots b_l))+\bullet_{b_1+1}\bullet l(a_1\ldots a_k)l(b_2\ldots b_l)))\\ & +\phi_{PL}(\bullet_{b_1}\bullet(\bullet_{a_1+1}\bullet l(a_2\ldots a_k)l(b_2\ldots b_l))+\bullet_{b_2+1}\bullet l(a_1\ldots a_k)l(b_3\ldots b_l)))\\ = & m_{a_1(a_2\ldots a_k\star b_1\ldots b_l)+b_1(a_1\ldots a_k\star b_2\ldots b_l)}\\ = & m_{a_1\ldots a_k\star b_1\ldots b_l}. \end{array}$$

Hence, the result holds for all $k,l\geq 1$. $\Box$

**Theorem 34** *For all $a_1,\ldots,a_k,b_1,\ldots,b_l\in\mathbb{N}^*$:*

$$m_{a_1\ldots a_k}\bullet m_{b_1\ldots b_l}=\sum_{i=1}^{k-1} m_{a_1\ldots a_{i-1}(a_i-1)(a_{i+1}\ldots a_k\star b_1\ldots b_l)}+m_{a_1\ldots a_k b_1\ldots b_l}.$$

**Proof.** By definition of $m_{a_1b_1\ldots b_l}$, if $k=1$, $m_{a_1}\bullet m_{b_1\ldots b_l}=m_{a_1b_1\ldots b_l}$. So the result holds if $k=1$. Let us assume that $k\geq 2$. In $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, we have:

$$l(a_1\ldots a_k)\bullet l(b_1\ldots b_l)=\bullet_{a_1}\bullet(l(a_2\ldots a_k)\bullet l(b_1\ldots b_l))+\bullet_{a_1}\bullet l(a_2\ldots a_k)l(b_1\ldots b_l).$$

Applying $\phi_{PL}$:

$$\begin{array}{rcl} m_{a_1\ldots a_k}\bullet m_{b_1\ldots b_l} & = & m_{a_1(a_2\ldots a_k)\bullet(b_1\ldots b_l)}\\ & & +\phi_{PL}(\bullet_{a_1-1}\bullet(\bullet_{a_2+1}l(a_3\ldots a_k)l(b_1\ldots b_l))+\bullet_{b_1+1}\bullet l(a_1\ldots a_k)l(b_2\ldots b_l)))\\ & = & m_{a_1(a_2\ldots a_k)\bullet(b_1\ldots b_l)}+m_{(a_1-1)(a_2\ldots a_k\star b_1\ldots b_l)}, \end{array}$$

by the preceding lemma. The result follows from an easy induction. $\Box$

**Remark.** In particular, $m_1\circ m_{b_1\ldots b_l}=0$.

**Corollary 35** *Let $a_1\ldots a_k, b_1\ldots b_l$ be two words with letters in $\mathbb{N}^*$. Then $m_{a_1\ldots a_k}\bullet m_{b_1\ldots b_l}$ is a span of $m_w$, where $w$ is a word with $k+l$ letters and of weight $a_1+\ldots+a_k+b_1+\ldots+b_l$.*

Hence, $\mathbb{K}\langle x_0,x_1\rangle$ is a bigraded pre-Lie algebra, with:

$$\mathbb{K}\langle x_0,x_1\rangle_{n,k}=Vect(m_{a_1\ldots a_k}\mid a_1+\ldots+a_k=n).$$

We put:

$$G=\sum_{k,n\geq 0} dim(\mathbb{K}\langle x_0,x_1\rangle_{n,k})X^nY^k.$$

**Proposition 36** $G = \frac{XY}{1-X-X^2Y} = \sum_{k=1}^{\infty}\sum_{l=2k-1}^{\infty}\binom{l-k}{k-1}X^lY^k.$

**Proof.** Note that $dim(\mathbb{K}\langle x_0,x_1\rangle_{n,k})$ is the number of words $a_1\dots a_k$ of length $k$, such that $a_1,\dots,a_{k-1}\geq 2$, and $a_1+\dots+a_k=n$. Hence:

$$\begin{aligned}
G &= \sum_{k=1}^{\infty}\left(\frac{X^2Y}{1-X}\right)^{k-1}\frac{XY}{1-X}\\
&= \frac{XY}{1-X}\frac{1}{1-\frac{X^2Y}{1-X}}\\
&= \frac{XY}{1-X-X^2Y},\\
&= \sum_{k=1}^{\infty}\frac{X^{2k-1}Y^k}{(1-X)^n}\\
&= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\binom{k+n-1}{k-1}X^{2k+n-1}Y^k\\
&= \sum_{k=1}\sum_{l=2k-1}^{\infty}\binom{l-k}{k-1}X^lY^k.
\end{aligned}$$

$\square$

### 4.3 An associative product on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$

We now define an associative product on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, in such a way that $\phi_{PL}$ becomes a morphism of Com-pre-Lie algebras.

**Proposition 37** *We define a product $\sqcup\!\sqcup$ on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ by:*

$$B_p(s_1\dots s_k)\sqcup\!\sqcup B_q(t_1\dots t_l)=\binom{p+q-k-l-2}{p-k-1}B_{p+q-1}(s_1\dots s_kt_1\dots t_l).$$

*Then $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ is a Com-pre-Lie algebra and $\phi_{PL}$ is a morphism of Com-pre-Lie algebras.*

**Proof.** As $\binom{p+q-k-l-2}{p-k-1}=\binom{p+q-k-l-2}{q-l-1}$, $\sqcup\!\sqcup$ is commutative. Let $t=B_p(s_1\dots s_k)$, $t'=B_q(\bullet t_1\dots t_l)$ and $t''=B_r(u_1\dots u_m)$. Then:

$$t\sqcup\!\sqcup(t'\sqcup\!\sqcup t'')=\underbrace{\binom{q+r-l-m-2}{q-l-1}\binom{p+q+r-k-l-m-3}{q+r-l-m-2}}_{A}B_{p+q+r-2}(s_1\dots s_kt_1\dots t_lu_1\dots u_m),$$

$$(t\sqcup\!\sqcup t')\sqcup\!\sqcup t''=\underbrace{\binom{p+q-k-l-2}{p-k-1}\binom{p+q+r-k-l-m-3}{p+q-k-l-2}}_{B}B_{p+q+r-2}(s_1\dots s_kt_1\dots t_lu_1\dots u_m).$$

If $p\leq k$ or $q\leq l$ or $r\leq m$, then $A=B=0$. If $p>k$ and $q>l$ and $r>m$, then:

$$A=B=\frac{(p+q+r-k-l-m-3)!}{(p-k-1)!(q-l-1)!(r-m-1)!}.$$

So $\sqcup\!\sqcup$ is associative.

Let $t_1 = B_p(s_1 \dots s_k)$, $t_2 = B_q(t_1 \dots t_l)$ and $t \in \mathcal{T}(\mathbb{N}^*)$. Then:

$$\begin{array}{rcl}
(t_1 \sqcup\!\sqcup t_2) \circ T & = & \binom{p+q-k-l-2}{m-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t) \\
& & + \displaystyle\sum_{i=1}^{k} \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots (s_i \bullet t) \dots s_k t_1 \dots t_l) \\
& & + \displaystyle\sum_{j=1}^{l} \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots (t_j \bullet t) \dots t_l), \\
(t_1 \bullet t) \sqcup\!\sqcup t_2 & = & \left( \displaystyle\sum_{i=1}^{k} B_p(s_1 \dots (s_i \bullet t) \dots s_k) + B_p(s_1 \dots s_k t) \right) \sqcup\!\sqcup t_2 \\
& = & \displaystyle\sum_{i=1}^{k} \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots (s_i \bullet t) \dots s_k t_1 \dots t_l) \\
& & + \binom{p+q-k-l-3}{p-k-2} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t), \\
t_1 \sqcup\!\sqcup (t_2 \bullet t) & = & t_1 \sqcup\!\sqcup \left( \displaystyle\sum_{j=1}^{l} B_q(t_1 \dots (t_j \bullet t) \dots t_l) + B_q(t_1 \dots t_j t) \right) \\
& = & \displaystyle\sum_{j=1}^{l} \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots (t_j \bullet t) \dots t_l) \\
& & + \binom{p+q-k-l-3}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t).
\end{array}$$

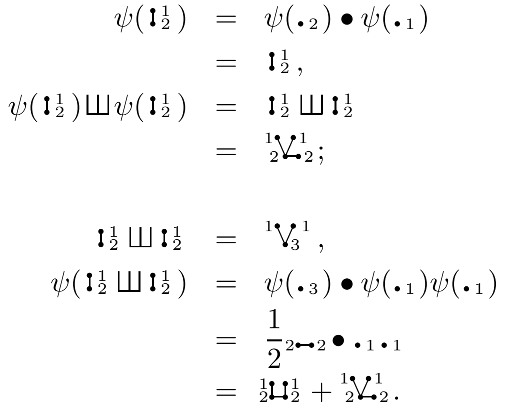
As $\binom{p+q-k-l-3}{p-k-2} + \binom{p+q-k-l-3}{p-k-1} = \binom{p+q-k-l-2}{p-k-1}$, $(t_1 \sqcup\!\sqcup t_2) \bullet t = (t_1 \bullet t) \sqcup\!\sqcup t_2 + t_1 \sqcup\!\sqcup (t_2 \bullet t)$. So $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ is Com-pre-Lie.

Let $t_1 = B_p(s_1 \dots s_k)$ and $t_2 = B_q(t_1 \dots t_l)$. If $k \geq p$, then $\binom{p+q-k-l-2}{p-k-1} = 0$, so $t_1 \sqcup\!\sqcup t_2 = 0$. By proposition 28, $\phi_{PL}(t_1) = 0$, so $\phi_{PL}(t_1 \sqcup\!\sqcup t_2) = \phi_{PL}(t_1) \sqcup\!\sqcup \phi_{PL}(t_2) = 0$. Similarly, if $l \geq q$, $\phi_{PL}(t_1 \sqcup\!\sqcup t_2) = \phi_{PL}(t_1) \sqcup\!\sqcup \phi_{PL}(t_2) = 0$. If $k < p$ and $l < q$, we put $w_i = \phi_{PL}(s_i)$ and $w'_j = \phi_{PL}(t_j)$. Then:

$$\begin{array}{rcl}
\phi_{PL}(t_1) \sqcup\!\sqcup \phi_{PL}(t_2) & = & x_0 w_1 \sqcup\!\sqcup \dots \sqcup\!\sqcup x_0 w_k \sqcup\!\sqcup x_1^{p-1-k} \sqcup\!\sqcup x_0 w'_1 \sqcup\!\sqcup \dots \sqcup\!\sqcup x_0 w'_l \sqcup\!\sqcup x_1^{q-1-l} \\
& = & \binom{p+q-k-l-2}{p-k-1} x_0 w_1 \sqcup\!\sqcup \dots x_0 w'_l \sqcup\!\sqcup x_1^{p+q-k-l-2} \\
& = & \binom{p+q-k-l-2}{p-k-1} \phi_{PL}(B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l)) \\
& = & \phi_{PL}(t_1 \sqcup\!\sqcup t_2).
\end{array}$$

So $\phi_{PL}$ is a Com-pre-Lie algebra morphism. $\square$

**Remark.** $\psi$ is not compatible with $\sqcup\!\sqcup$. Indeed:



# 5 Appendix

## 5.1 Enumeration of partitioned trees

Let $d \geq 1$. For all $n \geq 1$, let $f_n$ be the number of partitioned trees decorated by $\{1, \ldots, d\}$ with $n$ vertices and let $t_n$ be the number of partitioned trees decorated by $\{1, \ldots, d\}$ with $n$ vertices and one root. By convention, $f_0 = 1$. We put:

$$T = \sum_{n=1}^{\infty} t_n X^n,\ F = \sum_{n=0}^{\infty} f_n X^n.$$

Let $V_T$ be the vector space generated by the set of partitioned trees decorated by $\{1, , \ldots, d\}$ and $V_F$ be the vector space generated by the set of partitioned trees decorated by $\{1, , \ldots, d\}$ with only one root. There is a bijection:

$$\left\{ \begin{array}{rcl} S(V_T) & \longrightarrow & V_F \\ t_1 \ldots t_k & \longrightarrow & t_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup t_k. \end{array} \right.$$

Hence:

$$F = \prod_{i=1}^{\infty} \frac{1}{(1 - X^k)^{t_k}}. \tag{2}$$

There is a bijection:

$$\left\{ \begin{array}{rcl} \displaystyle\bigoplus_{i=1}^{d} S(V_F) & \longrightarrow & V_T \\ (F_{1,1} \ldots, F_{1,k_1}, \ldots, F_{d,1} \ldots F_{d,k_d}) & \longrightarrow & \displaystyle\sum_{i=1}^{d} \bullet_i \bullet (F_{i,1} \ldots F_{i,k_i}). \end{array} \right.$$

This gives:

$$T = dX \prod_{i=1}^{\infty} \frac{1}{(1 - X^k)^{f_{k-1}}}. \tag{3}$$

Formulas (2) and (3) allow to compute inductively $f_k$ and $t_k$ for all $k \geq 1$. This gives for example:

$$\left\{ \begin{array}{rcl} f_1 & = & d \\ f_2 & = & \dfrac{d(3d+1)}{2} \\ f_3 & = & \dfrac{d(19d^2+9d+2)}{6} \\ f_4 & = & \dfrac{d(63d^2+34d^2+13d+2)}{8} \\ f_5 & = & \dfrac{d(644d^4+400d^3+175d^2+35d+6)}{30} \end{array} \right.$$

Here are examples of $f_n$ for $d = 1$ or 2:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $d = 1$ | 1 | 2 | 5 | 14 | 42 | 134 | 444 | 1518 | 5318 | 18989 |
| $d = 2$ | 2 | 7 | 32 | 167 | 952 | 5759 | 36340 | 236498 | 1576156 | 10702333 |

The row $d = 1$ is sequence $A035052$ of [14].

## 5.2 Study of the dendriform structure on admissible words

We here study the dendriform algebra $\mathbb{K}\langle\mathbb{N}^*\rangle$ of proposition 32. It is clearly commutative, via the bijection from $Sh_\prec(k,l)$ to $Sh_\succ(l,k)$ given by the composition (on the left) by the permutation $(l+1\ldots l+k\,1\ldots l)$.

Let $V$ be a vector space. The shuffle dendriform algebra $Sh(V)$ is $T_+(V)$, with the produts given by:

$$\begin{aligned}
(a_1\ldots a_k) \prec (b_1 \ldots b_l) &= \sum_{\zeta \in Sh_\prec(k,l)} \zeta.a_1\ldots a_k b_1 \ldots b_{k-1} b_k\\
(a_1\ldots a_k) \succ (b_1 \ldots b_l) &= \sum_{\zeta \in Sh_\succ(k,l)} \zeta.a_1\ldots a_{k-1} a_k b_1 \ldots b_k.
\end{aligned}$$

Moreover, this is the free commutative dendriform algebra generated by $V$, that is to say if $A$ is a commutative dendriform algebra and $f : V \longrightarrow A$ is any linear map, there exists a morphism of dendriform algebras $\phi : Sh(V) \longrightarrow A$ such that $\phi_{|}V = f$. As $a_1 \ldots a_k \succ b = a_1 \ldots a_k b$ in $Sh(V)$ for all $a_1,\ldots,a_k, b \in V$, this morphism $\phi$ is defined by:

$$\phi(a_1\ldots a_k) = (\ldots(a_1 \succ a_2) \succ a_3)\ldots) \succ a_k.$$

**Proposition 38** *1. Let $V$ be the space generated by the words $1^k i$, $k \in \mathbb{N}$, $i \geq 1$. Then $K\langle\mathbb{N}^*\rangle$ is isomorphic, as a dendriform algebra, to $Sh(V)$.*

*2. Let $A$ be the subspace of $K\langle\mathbb{N}^*\rangle$ generated by admissible words. Then it is a dendriform subalgebra of $K\langle\mathbb{N}^*\rangle$. Moreover, if $W$ is the space generated by the letters $i$, $i \geq 1$, then $A$ is isomorphic, as a dendriform algebra, to $Sh(W)$.*

**Proof.** Let $w = a_1 \ldots a_k$ be a word with letters in $\mathbb{N}^*$. We denote by $o(w)$ the sequence of indices $j \in \{1,\ldots,k-1\}$ such that $a_j \neq 1$. This sequences are totally ordered in this way: $(j_1,\ldots,j_k) < (j'_1,\ldots,j'_l)$ if there exists a $p$ such that $j_k = j'_l$, $j_{k-1} = j'_{l-1}$, …, $j_{k-p+1} = j'_{l-p+1}$, $j_{k-p} < j'_{l-p}$, with the convention $j_0 = j_{-1} = \ldots = j'_0 = j'_{-1} = \ldots = 0$.

Let $\phi : Sh(V) \longrightarrow K\langle\mathbb{N}^*\rangle$ be the unique morphism of dendriform algebras which extends the identity of $V$. Then:

$$\begin{aligned}
\phi((1^{k_1-1}a_1)\ldots(1^{k_n-1}a_n)) &= = 1^{k_1-1}(a_1+1)\ldots 1^{k_{n-1}-1}(a_{n-1}+1)1^{k_n-1}a_n\\
&\quad +\text{words } w' \text{ such that } o(w') > (k_1,\ldots,k_{n-1}).
\end{aligned}$$

By thriangularity, $\phi$ is an isomorphism. Moreover, for all $a_1,\ldots,a_n \geq 1$:

$$\phi(a_1\ldots a_n) = (a_1+1)\ldots(a_{n-1}+1)a_n.$$

Consequently, $\phi(Sh(W)) = A$, so $A$ is a dendriform subalgebra of $K\langle\mathbb{N}^*\rangle$ and is isomorphic to $Sh(W)$. $\square$

### 5.3 Freeness of the pre-Lie algebra $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$

**Notations.** Let $k \geq 1$, $d_1, \ldots, d_k \in \mathcal{D}$ and let $F_1, \ldots, F_k$ be decorated partitioned forests. We put:

$$B_{d_1,\ldots,d_k}(F_1, \ldots F_k) = (\bullet_{d_1} \bullet F_1) ⧢ \ldots ⧢ (\bullet_{d_k} \bullet F_k).$$

Note that any partitioned tree can be written under the form $B_{d_1,\ldots,d_k}(F_1, \ldots F_k)$. This writing is unique up to a permutation of the $d_i$'s and the $F_i$'s.

**Proposition 39** *We define a coproduct $\delta$ on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ in the following way: for any decorated partitioned tree $t = B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{k,1} \ldots t_{k,n_k})$,*

$$\delta(t) = \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{n_i} B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,j-1} t_{i,j+1} \ldots t_{i,n_i}, \ldots, t_{k,1} \ldots t_{k,n_k}) \otimes t_{i,j}.$$

1. *For all $x \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $(\delta \otimes Id) \circ \delta(x) = (23)(\delta \otimes Id) \circ \delta(x)$.*
2. *For all $x, y \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $\delta(x \bullet y) = x \otimes y + \delta(x) \bullet y$.*

**Proof.** 1. Let $t = B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{k,1} \ldots t_{k,n_k})$. For all $i, j$, we put:

$$t/t_{i,j} = B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,j-1} t_{i,j+1} \ldots t_{i,n_i}, \ldots, t_{k,1} \ldots t_{k,n_k}).$$

Then:

$$\delta(t) = \frac{1}{k} \sum_{i,j} t/t_{i,j} \otimes t_{i,j}.$$

Hence:

$$(\delta \otimes Id) \circ \delta(t) = \sum_{(i,j) \neq (i',j')} (t/t_{i,j})/t_{i',j'} \otimes t_{i',j'} \otimes t_{i,j}$$

As $(t/t_{i,j})/t_{i',j'}$ and $(t/t_{i',j'})/t_{i,j}$ are both the partitioned tree obtained by cutting $t_{i,j}$ and $t_{i',j'}$ in $t$, they are equal, so $(\delta \otimes Id) \circ \delta(t)$ is invariant under the action of $(23)$.

2. Let $t'$ be a decorated partitioned tree.

$$\begin{aligned}
\delta(t \bullet t') &= \sum_{i=1}^{k} \delta(B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,n_i} t', \ldots, t_{k,1} \ldots t_{k,n_k})) \\
&\quad + \sum_{i,j} \delta(B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,j} \bullet t' \ldots t_{i,n_i}, \ldots, t_{k,1} \ldots t_{k,n_k})) \\
&= \frac{1}{k} kt \otimes t' + \frac{1}{k} \sum_{i} \sum_{i',j'} B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,n_i} t', \ldots, t_{k,1} \ldots t_{k,n_k})/t_{i',j'} \otimes t_{i',j'} \\
&\quad + \frac{1}{k} \sum_{(i,j) \neq (i',j')} B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,j} \bullet t' \ldots t_{i,n_i}, \ldots, t_{k,1} \ldots t_{k,n_k})/t_{i',j'} \otimes t_{i',j'} \\
&\quad + \frac{1}{k} \sum_{i,j} t/t_{i,k} \otimes t_{i,j} \bullet t' \\
&= t \otimes t' + \sum t^{(1)} \otimes t^{(2)} \bullet t' + \sum t^{(1)} \otimes t^{(2)} \bullet t'.
\end{aligned}$$

So $\delta(t \bullet t') = t \otimes t' + \delta(t) \bullet t'$. $\square$

By Muriel Livernet's pre-Lie rigidity theorem [7]:

**Corollary 40** *The pre-Lie algebra $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is freely generated by $Ker(\delta)$.*

**Remarks.**

1. It is not difficult to prove that for any $x, y \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$:

$$\delta(x ⧢ y) = \sum x^{(1)} \otimes x^{(2)} ⧢ y + \sum y^{(1)} \otimes x ⧢ y^{(2)}.$$

Hence, $Ker(\delta)$ is an algebra for the product ⧢.

2. Here are elements of $Ker(\delta)$ in the non decorated case. Let $t_1, t_2, t_3, t_4$ be partitioned trees.

$$\begin{array}{rcl} X &=& B(t_1t_2, 1) - B(t_1, t_2), \\ Y &=& B(t_1t_2t_3, 1, 1) - B(t_1t_2, t_3, 1) - B(t_1t_3, t_2, 1) - B(t_2t_3, t_1, 1) + 2B(t_1, t_2, t_3), \\ Z &=& B(t_1t_2t_3t_4, 1) - B(t_1t_2t_3, t_4) - B(t_1t_2t_4, t_3) - B(t_1t_3t_4, t_2) - B(t_2t_3t_4, t_1) \\ && +B(t_1t_2, t_3t_4) + B(t_1t_3, t_2t_4) + b(t_1t_4, t_2t_3), \\ T &=& B(t_1t_2, t_3t_4, 1, 1) + B(t_1t_3, t_2t_4, 1, 1) + B(t_1t_4, t_2t_3, 1, 1) - B(t_1t_2, t_3, t_4, 1) \\ && -B(t_1t_3, t_2, t_4, 1) - B(t_1t_4, t_2, t_3, 1) - B(t_2t_3, t_1, t_4, 1) - B(t_2t_4, t_1, t_3, 1) \\ && -B(t_3t_4, t_1, t_2, 1) + 3B(t_1, t_2, t_3, t_4). \end{array}$$

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