The Hopf algebra of Fliess operators and its dual pre-Lie algebra

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ABSTRACT. We study the Hopf algebra H of Fliess operators coming from Control Theory in the one-dimensional case. We prove that it admits a graded, finte-dimensional, connected gradation. Dually, the vector space $\mathbb{R}\langle x_0, x_1\rangle$ is both a pre-Lie algebra for the pre-Lie product dual of the coproduct of H, and an associative, commutative algebra for the shuffle product. These two structures admit a compatibility which makes $\mathbb{R}\langle x_0, x_1\rangle$ a Com-pre-Lie algebra. We give a presentation of this object as a Com-pre-Lie algebra and as a pre-Lie algebra.

KEYWORDS. Fliess operators; pre-Lie algebras; Hopf algebras.

AMS CLASSIFICATION. 16W30, 17B60, 93B25, 05C05.

Contents

Introduction

Right pre-Lie algebras, or shortly pre-Lie algebras [4, 1], are vector spaces with a bilinear product $\bullet$ satisfying the following axiom:

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y).$$

Consequently, the antisymmetrization of $\bullet$ is a Lie bracket. These objects are also called right-symmetric algebras or Vinberg algebra [10, 15]. If A is a pre-Lie algebra, then the symmetric algebra $S(A)$ inherits a product $\star$ making it a Hopf algebra, isomorphic to the enveloping algebra of the Lie algebra A [11, 12]. Whenever it is possible, we can consider the dual Hopf algebra $S(A)^*$ and its group of characters G, which is the exponentiation, in a certain sense, of the Lie algebra A.

We here consider the inverse construction, departing from a group used in Control Theory, namely the group fof Fliess operators [3, 5, 6]; this group is used to define the feedback product. We limit ourselves here to the one-dimensional case. This is the set $\mathbb{R}\langle\langle x_0, x_1\rangle\rangle$ of noncommutative formal series in two indeterminates, with a certain product generalizing the composition of formal series (definition 1). The Hopf algebra H of coordinates of this group is described in [5], where it is also proved it is graded by the length of words; note that this gradation is not connected and not finite-dimensional. We first give a way to describe the composition in the group $\mathbb{R}\langle\langle x_0, x_1\rangle\rangle$ and the coproduct of H by induction on the length of words(lemma 2 and proposition 3). We prove that H admits a second gradation, which is connected; the dimensions of this gradation are given by the Fibonacci sequence (proposition 8). As the product of $\mathbb{R}\langle\langle x_0, x_1\rangle\rangle$ is left-linear, H is a commutative, right-sided combinatorial Hopf algebra [9], so, dually, $\mathbb{R}\langle x_0, x_1\rangle$ inherits a pre-Lie product $\bullet$, which is inductively defined in proposition 11. We prove that the words x_1^n, $n \geq 0$, form a minimal subset of generators of this pre-Lie algebra (theorem 12).

The pre-Lie algebra $\mathbb{R}\langle x_0, x_1\rangle$ has also an associative, commutative product, namely the shuffle product $\sqcup\!\sqcup$ [13]. We prove that the following axiom is satisfied (proposition 14):

$$(x \sqcup\!\sqcup y) \bullet z = (x \bullet z) \sqcup\!\sqcup y + x \sqcup\!\sqcup (y \bullet z).$$

So $\mathbb{R}\langle x_0, x_1\rangle$ is a Com-pre-Lie algebra (definition 15). We give a presentation of this Com-pre-Lie algebra in theorem 27. We use for this a description of free Com-pre-Lie algebras in terms of partitioned trees (definition 17), which generalizes the construction of pre-Lie algebras in terms of rooted trees of [1]. We then deduce a presentation of $\mathbb{R}\langle x_0, x_1\rangle$ as a pre-Lie algebra in theorem 30. This presentation induces a new basis of $\mathbb{R}\langle x_0, x_1\rangle$ in terms of words with letters in $\mathbb{N}^*$, see corollary 31. The pre-Lie product of two elements of this basis uses a dendriform structure [2, 8] on the algebra of words with letters in $\mathbb{N}^*$ (theorem 34). The study of this dendriform structure is postponed to the appendix, as well as the enumeration of partitioned trees; we also prove that free Com-pre-Lie algebras are free as pre-Lie algebras, using the rigidity theorem of [7].

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Notation. We denote by $\mathbb{K}$ a commutative field of characteristic zero. All the objects (algebra, coalgebras, pre-Lie algebras...) in this text will be taken over $\mathbb{K}$.

1 Construction of the Hopf algebra

1.1 Definition of the composition

Let us consider an alphabet of two letters, denoted by x_0 and x_1. We denote by $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ the completion of the free algebra generated by this alphabet. Note that $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ is an algebra for the concatenation product and for the shuffle product, which we denote by $\sqcup\!\sqcup$.

Exemples. If $a, b, c, d \in \{x_0, x_1\}$:

$$\begin{array}{rcl} abc \sqcup\!\sqcup d &=& abcd + abdc + adbc + dabc, \\ ab \sqcup\!\sqcup cd &=& abcd + acbd + cabd + acdb + cadb + cdab, \\ a \sqcup\!\sqcup bcd &=& abcd + bacd + bcad + bcda. \end{array}$$

The unit for both these products is the empty word, which we denote by $\emptyset$. The algebra $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ is given its usual ultrametric topology.

Definition 1 *[5].*

1. *For any $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, we define a continuous algebra map φ_d from $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ to $End(\mathbb{K}\langle\langle x_0, x_1\rangle\rangle)$ in the following way: for all $X \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$,*

$$\varphi_d(x_0)(X) = x_0 X,\ \varphi_d(x_1)(X) = x_1 X + x_0(d \sqcup\!\sqcup X).$$

2. *We define a composition $\circ$ on $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ in the following way: for all $c, d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, $c \circ d = \varphi_d(c)(\emptyset) + d$.*

It is proved in [5] that this composition is associative.

Notation. For all $c, d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, we put $c \tilde{\circ} d = c \circ d - d = \varphi_d(c)(\emptyset)$.

Remark. If $c_1, c_2, d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, $\lambda \in \mathbb{K}$:

$$(c_1 + \lambda c_2)\tilde{\circ} d = \varphi_d(c_1 + \lambda c_2)(\emptyset) = (\varphi_d(c_1) + \lambda\varphi_d(c_2))(\emptyset) = \varphi_d(c_1)(\emptyset) + \lambda\varphi_d(c_2)(\emptyset) = c_1 \tilde{\circ} d + \lambda c_2 \tilde{\circ} d.$$

So the composition $\tilde{\circ}$ is linear on the left. As φ_d is continuous, the map $c \longrightarrow c\tilde{\circ} d$ is continuous for any $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$. Hence, it is enough to know how to compute $c\tilde{\circ} d$ for any word c, which is made possible by the next lemma, using an induction on the length:

Lemma 2 *For any words c, for any $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$:*

1. *$\emptyset \tilde{\circ} d = \emptyset$.*

2. *$(x_0 c)\tilde{\circ} d = x_0(c\tilde{\circ} d)$.*

3. *$(x_1 c)\tilde{\circ} d = x_1(c\tilde{\circ} d) + x_0(d \sqcup\!\sqcup (c\tilde{\circ} d))$.*

Proof. 1. $\emptyset \tilde{\circ} d = \varphi_d(\emptyset)(\emptyset) = Id(\emptyset) = \emptyset$.

2. $(x_0 c)\tilde{\circ} d = \varphi_d(x_0 c)(\emptyset) = \varphi_d(x_0) \circ \varphi_d(c)(\emptyset) = \varphi_d(x_0)(c\tilde{\circ} d) = x_0(c\tilde{\circ} d)$.

3. $(x_1 c)\tilde{\circ} d = \varphi_d(x_1 c)(\emptyset) = \varphi_d(x_1) \circ \varphi_d(c)(\emptyset) = \varphi_d(x_1)(c\tilde{\circ} d) = x_1(c\tilde{\circ} d) + x_0(d \sqcup\!\sqcup (c\tilde{\circ} d))$. $\square$

1.2 Dual Hopf algebra

We here give an inductive description of the Hopf algebra of the coordinates of the group $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ of [5].

For any word c, let us consider the map $X_c \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle^*$, such that for any $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, $X_c(d)$ is the coefficient of c in d. We denote by V the subspace of A^* generated by these maps. Let $H = S(V)$, or equivalently the free commutative algebra generated by the X_c's. The elements of

H are seen as polynomial functions on $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle$; the elements of $H \otimes H$ are seen as polynomial functions on $\mathbb{K}\langle\langle x_0, x_1\rangle\rangle \times \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$. Then H is given a multiplicative coproduct defined in the following way: for any word c, for any $f, g \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$,

$$\Delta(X_c)(f,g) = X_c(f \circ g).$$

As $\circ$ is associative, Δ is coassociative, so H is a bialgebra.

Notations.

1. The space of words is a commutative algebra for the shuffle product $\sqcup\!\sqcup$. Dually, the space V inherits a coassociative, cocommutative coproduct, denoted by $\Delta_{\sqcup\!\sqcup}$. For example, if $a, b, c \in \{x_0, x_1\}$:

$$\begin{array}{rcl} \Delta_{\sqcup\!\sqcup}(X_\emptyset) &=& X_\emptyset \otimes X_\emptyset, \\ \Delta_{\sqcup\!\sqcup}(X_a) &=& X_a \otimes X_\emptyset + X_\emptyset \otimes X_a, \\ \Delta_{\sqcup\!\sqcup}(X_{ab}) &=& X_{ab} \otimes X_\emptyset + X_a \otimes X_b + X_b \otimes X_a + X_\emptyset \otimes X_{ab}, \\ \Delta_{\sqcup\!\sqcup}(X_{abc}) &=& X_{abc} \otimes X_\emptyset + X_a \otimes X_{bc} + X_b \otimes X_{ac} + X_c \otimes X_{ab} \\ && + X_{ab} \otimes X_c + X_{ac} \otimes X_b + X_{bc} \otimes X_a + X_\emptyset \otimes X_{abc}. \end{array}$$

2. We define two linear endomorphisms θ_0, θ_1 of V by $\theta_i(X_c) = X_{x_i c}$ for any word c.

The following proposition allows to compute $\Delta(X_c)$ for any word c by induction on the length of c.

Proposition 3 *For all $x \in V$, we put $\tilde{\Delta}(x) = \Delta(x) - 1 \otimes x$.*

1. $\tilde{\Delta}(X_\emptyset) = X_\emptyset \otimes 1$.
2. $\tilde{\Delta} \circ \theta_0 = (\theta_0 \otimes Id) \circ \tilde{\Delta} + (\theta_1 \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\sqcup\!\sqcup}$.
3. $\tilde{\Delta} \circ \theta_1 = (\theta_1 \otimes Id) \circ \tilde{\Delta}$.

Proof. For any word c, for any $f, g \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$:

$$\tilde{\Delta}(X_c)(f,g) = \Delta(X_c)(f,g) - (1 \otimes X_c)(f,g) = X_c(f \circ g) - X_c(g) = X_c(f \otimes g - g) = X_c(f \tilde{\circ} g).$$

As $\tilde{\circ}$ is linear on the left, $\tilde{\Delta}(X_c) \in V \otimes H$, so formulas in 2. and 3. make sense.

Let $f \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$. It can be uniquely written as $f = x_0 f_0 + x_1 f_1 + \lambda\emptyset$, with $f_0, f_1 \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, $\lambda \in K$. For all $g \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$:

$$\begin{array}{rcl} f \tilde{\circ} g &=& (x_0 f_0)\tilde{\circ} g + (x_1 f_1)\tilde{\circ} g + \lambda\emptyset\tilde{\circ} g \\ &=& x_0(f_0 \tilde{\circ} g + g \sqcup\!\sqcup (f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda\emptyset. \end{array}$$

1. We obtain:

$$\tilde{\Delta}(X_\emptyset)(f,g) = X_\emptyset(x_0(f_0 \tilde{\circ} g + g \sqcup\!\sqcup (f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda\emptyset) = 0 + 0 + \lambda = (X_\emptyset \otimes 1)(f,g).$$

So $\Delta(X_\emptyset) = X_\emptyset \otimes 1$.

2. Let c be a word.

$$\begin{array}{rcl}
\tilde{\Delta}\circ\theta_0(X_c)(f,g) &=& \tilde{\Delta}(X_{x_0c})(f,g)\\
&=& X_{x_0c}(x_0(f_0\tilde{\circ}g+g\sqcup\!\sqcup(f_1\tilde{\circ}g))+x_1(f_1\tilde{\circ}g)+\lambda\emptyset)\\
&=& X_c(f_0\tilde{\circ}g+g\sqcup\!\sqcup(f_1\tilde{\circ}g))+0+0\\
&=& X_c(f_0\tilde{\circ}g+(f_1\tilde{\circ}g)\sqcup\!\sqcup g)+0+0\\
&=& \tilde{\Delta}(X_c)(f_0,g)+(\tilde{\Delta}\otimes Id)\circ\Delta_{\sqcup\!\sqcup}(X_c)(f_1,g,g)\\
&=& \tilde{\Delta}(X_c)(f_0,g)+(Id\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{\sqcup\!\sqcup}(X_c)(f_1,g)\\
&=& (\theta_0\otimes Id)\circ\tilde{\Delta}(X_c)(f,g)+(\theta_1\otimes Id)\circ(Id\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{\sqcup\!\sqcup}(X_c)(f,g),
\end{array}$$

so $\tilde{\Delta}\circ\theta_0(X_c)=(\theta_0\otimes Id)\circ\tilde{\Delta}(X_c)+(\theta_1\otimes Id)\circ(Id\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{\sqcup\!\sqcup}(X_c)$.

3. Let c be a word.

$$\begin{array}{rcl}
\tilde{\Delta}\circ\theta_1(X_c)(f,g) &=& \tilde{\Delta}(X_{x_0c})(f,g)\\
&=& X_{x_1c}(x_0(f_0\tilde{\circ}g+g\sqcup\!\sqcup(f_1\tilde{\circ}g))+x_1(f_1\tilde{\circ}g)+\lambda\emptyset)\\
&=& 0+X_c(f_1\tilde{\circ}g)+0\\
&=& \tilde{\Delta}(X_c)(f_1,g)\\
&=& (\theta_1\otimes Id)\circ\tilde{\Delta}(X_c)(f,g),
\end{array}$$

so $\tilde{\Delta}\circ\theta_1(X_c)=(\theta_1\otimes Id)\circ\tilde{\Delta}(X_c)$. □

Examples.

$$\begin{array}{rcl}
\Delta(X_{x_0}) &=& X_{x_0}\otimes 1+1\otimes X_{x_0}+X_{x_1}\otimes X_\emptyset,\\
\Delta(X_{x_0^2}) &=& X_{x_0^2}\otimes 1+1\otimes X_{x_0^2}+X_{x_0x_1}\otimes X_\emptyset+X_{x_1x_0}\otimes X_\emptyset+X_{x_1x_1}\otimes X_\emptyset^2+X_{x_1}\otimes X_{x_0},\\
\Delta(X_{x_0x_1}) &=& X_{x_0x_1}\otimes 1+1\otimes X_{x_0x_1}+X_{x_1x_1}\otimes X_\emptyset+X_{x_1}\otimes X_{x_1},\\
\Delta(X_{x_1x_0}) &=& X_{x_1x_0}\otimes 1+1\otimes X_{x_1x_0}+X_{x_1x_1}\otimes X_\emptyset.
\end{array}$$

Corollary 4 *For all $n\geq 1$, $\tilde{\Delta}(X_{x_1^n})=X_{x_1^n}\otimes 1$ and $\Delta(X_{x_1^n})=X_{x_1^n}\otimes 1+1\otimes X_{x_1^n}$.*

Proof. By induction on n. □

1.3 gradation

It is proved in [5] that the Hopf algebra H is graded by the length of words, but this gradation is not connected, that is to say that the homogeneous component of degree 0 is not (0), as it contains $X_\emptyset$. We here define another gradation, which is connected.

Definition 5 *Let $c=c_1\dots c_k$ be a word. We put:*

$$deg(c)=lg(c)+1+\sharp\{i\mid c_i=x_0\}.$$

For all $k\geq 1$, we put:

$$V_k=Vect(X_c\mid deg(x)=k).$$

This define a connected gradation of V, that is to say:

$$V=\bigoplus_{k\geq 1}V_k.$$

This gradation induces a connected gradation of the algebra H:

$$H=\bigoplus_{k\geq 0}H_k,\ \text{and } H_0=\mathbb{K};$$

Proposition 6 *If c is a word of degree n, then:*

$$\tilde{\Delta}(X_c) \in \bigoplus_{i+j=n} V_i \otimes H_j.$$

So the gradation $(V_k)_{k\geq 1}$ is a gradation of the Hopf algebra H.

Proof. Let us start by the following observations:

1. Let c be a word of degree k. Then $x_0 c$ is a word of degree $k+2$. Hence, θ_0 is homogeneous of degree 2 on V.
2. Let c be a word of degree k. Then $x_1 c$ is a word of degree $k+1$. Hence, θ_1 is homogeneous of degree 1 on V.
3. Let c and d be two words of respective degrees k and l. Then any word obtained by shuffling c and d is of degree $k+l-1$: its length is the sum of the length of c and d, and the number of x_0 in it is the sum of the numbers of x_0 in c and d. Hence, the coproduct $\Delta_{ш}$ is homogeneous of degree 1 from V to $V \otimes V$.

Let us prove the result by induction on the length k of c. If $k=0$, then $c=\emptyset$ so $n=1$, and $\tilde{\Delta}(X_c) = X_c \otimes 1 \in V_1 \otimes H_0$. Let us assume the result for all words of length $< k-1$. Two cases can occur.

1. If $c = x_0 d$, then $deg(d) = n-2$. we put $\Delta_{ш}(X_d) = \sum x_i' \otimes x_i''$. By the preceding third observation, we can assume that for all i, x_i', x_i'' are homogeneous elements of V, with $deg(x_i') + deg(x_i') = n-2+1 = n-1$. Then:

   $$\tilde{\Delta}(X_c) = (\theta_0 \otimes Id) \circ \tilde{\Delta}(X_d) + \sum_i (\theta_1 \otimes m) \circ (\tilde{\Delta}(x_i') \otimes x_i'').$$

   By the induction hypothesis, $\tilde{\Delta}(X_d) \in (V \otimes H)_{n-1}$. By the second observation, $(\theta_0 \otimes Id) \circ \tilde{\Delta}(X_d) \in (V \otimes H)_n$. By the induction hypothesis applied to x_i', for all i, $(\tilde{\Delta}(x_i') \otimes x_i'') \in (V \otimes H \otimes V)_{n-1}$, so by the first observation, $(\theta_1 \otimes m) \circ (\tilde{\Delta}(x_i') \otimes x_i'') \in (V \otimes H)_{n-1+1} \subseteq (V \otimes H)_n$. So $\Delta(X_c) \in (V \otimes H)_n$.
2. $c = x_1 d$, then $deg(d) = n-1$. Moreover, $\tilde{\Delta}(X_c) = (\theta_1 \otimes Id) \circ \tilde{\Delta}(X_d)$. By the induction hypothesis, $\tilde{\Delta}(X_d) \in (V \otimes H)_{n-1}$. By the second observation, $\tilde{\Delta}(X_c) \in (V \otimes H)_n$.

So the result holds for any word c. □

Corollary 7 *For all $n \geq 0$:*

$$\Delta(H_n) \subseteq \bigoplus_{i+j=n} H_i \otimes H_j.$$

Proof. The firs assertion comes from the multiplicativity of Δ. As H is a graded, connected bialgebra, it is a Hopf algebra. □

Let us now study the formal series of V and H.

Proposition 8 *1. For all k, let us put $p_k = dim(V_k)$ and $F_V = \sum_{k=1}^{\infty} p_k X^k$. Then:*

$$F_V = \frac{X}{1-X-X^2},$$

and for all $k \geq 1$:

$$p_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right).$$

This is the Fibonacci sequence (A000045 in [14]).

2. *We put* $F_H=\sum_{k=0}^{\infty} dim(H_k)X^k$. *Then:*

$$F_H=\prod_{k=1}^{\infty}\frac{1}{(1-X^k)^{p_k}}.$$

Proof. Let us consider the formal series:

$$F(X_0,X_1)=\sum_{i,j\geq 0}\sharp\{\text{words in } x_0,x_1 \text{ with } i\ x_0 \text{ and } j\ x_1\}X_0^iX_0^j.$$

Then $F(X_0,X_1)=\dfrac{1}{1-X_0-X_1}$. Moreover, by definition of the degree of a word:

$$F_V=XF(X^2,X)=\frac{X}{1-X-X^2}.$$

As H is the symmetric algebra generated by V, its formal series is given by the second point. □

Examples. We obtain:

k	0	1	2	3	4	5	6	7	8	9	10
$dim(V_k)$	0	1	1	2	3	5	8	13	21	34	55
$dim(H_k)$	1	1	2	4	8	15	30	56	108	203	384

The third row is sequence A166861 of [14].

Remark. Consequently, the space V inherits a bigradation:

$$V_{k,n}=Vect(X_c\mid deg(c)=k \text{ and } lg(c)=n).$$

If c is a word of length n and of degree k, denoting by a the number of its letters equal to x_0 and by b the number of its letters equal to x_1, then:

$$\left\{\begin{array}{rcl} a+b&=&n,\\ 2a+b+1&=&k,\end{array}\right.$$

so $a=k-n-1$. Hence:

$$dim(V_{k,n})=\binom{n}{k-n-1},$$

and the formal series of this bigradation is:

$$\sum_{k,n\geq 0} dim(V_{k,n})X^kY^n=\frac{X}{1-XY-X^2Y}.$$

2 Pre-Lie structure on $\mathbb{K}\langle x_0,x_1\rangle$

2.1 pre-Lie coproduct on V

As the composition $\circ$ is linear on the left, the dual coproduct satisfies $\tilde{\Delta}(V)\subseteq V\otimes H$, so H is a commutative right-sided Hopf algebra in the sense of [9], and V inherits a right pre-Lie coproduct: if π is the canonical projection from $H=S(V)$ onto V,

$$\delta=(\pi\otimes\pi)\circ\Delta=(Id\otimes\pi)\circ\tilde{\Delta}.$$

It satisfies the right pre-Lie coalgebra axiom:

$$(23).((\delta\otimes Id)\circ\delta-(Id\otimes\delta)\circ\delta)=0.$$

The following proposition allows to compute $\delta(X_c)$ by induction on the length of c.

Proposition 9 *1.* $\delta(X_\emptyset)=0$.

2. $\delta\circ\theta_0=(\theta_0\otimes Id)\circ\delta+(\theta_1\otimes Id)\circ\Delta_{ш}$.

3. $\delta\circ\theta_1=(\theta_1\otimes Id)\circ\delta$.

Proof. The first point comes from $\Delta(X_\emptyset)=X_\emptyset\otimes 1+1\otimes X_\emptyset$. Let $x\in V$. We put $\Delta_{ш}(x)=x'\otimes x''\in V\otimes V$. For any $y\in V$, we put $\tilde{\Delta}(y)-y\otimes 1=y^{(1)}\otimes y^{(2)}\in V\otimes H_+$. Then:

$$\begin{aligned}(\theta_1\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{ш}(x) &= (\theta_1\otimes m)(x'\otimes 1\otimes x''+x'^{(1)}\otimes x'^{(2)}\otimes x''\\ &= \theta_1(x')\otimes\underbrace{x''}_{\in V}+x'^{(1)}\otimes\underbrace{x'^{(2)}x''}_{\in Ker(\pi)}.\end{aligned}$$

Applying $Id\otimes\pi$, it remains:

$$(Id\otimes\pi)\circ(\theta_1\otimes m)\circ(\tilde{\Delta}\otimes Id)\circ\Delta_{ш}(x)=(\theta_1\otimes Id)\circ\Delta_{ш}(x).$$

Let $i=0$ or 1. Then:

$$(Id\otimes\pi)\circ(\theta_i\otimes Id)\circ\tilde{\Delta}=(\theta_i\otimes Id)\circ(Id\otimes\pi)\circ\tilde{\Delta}=(\theta_i\otimes Id)\circ\delta.$$

The result is induced by these remarks, combined with proposition 3. $\square$

Examples.

$$\begin{aligned}\delta(X_{x_0}) &= X_{x_1}\otimes X_\emptyset,\\ \delta(X_{x_0^2}) &= X_{x_0x_1}\otimes X_\emptyset+X_{x_1x_0}\otimes X_\emptyset+X_{x_1}\otimes X_{x_0},\\ \delta(X_{x_0x_1}) &= X_{x_1x_1}\otimes X_\emptyset+X_{x_1}\otimes X_{x_1},\\ \delta(X_{x_1x_0}) &= X_{x_1x_1}\otimes X_\emptyset.\end{aligned}$$

Proposition 10 $Ker(\delta)=Vect(X_{x_1^n},n\geq 0)$.

Proof. The inclusion $\supseteq$ is trivial by corollary 4. Let us prove the other inclusion.

First step. Let us prove the following property: if $x\in V_k$ is such that

$$\delta(x)=\lambda\sum_{i+j=k-2}\frac{(k-2)!}{i!j!}X_{x_1^i}\otimes X_{x_1^j},$$

then there exists $\mu\in\mathbb{K}$ such that $x=\mu x_1^{k-1}$. It is obvious if $k=1$, as then $x=\mu\emptyset$. Let us assume the result at all ranks $<k$. We put $x=x_1^\alpha(x_0f_0+x_1f_1)$, where $\alpha\geq 0$, f_0 is homogeneous of degree $k-2-\alpha$ and f_1 is homogeneous of degree $k-1-\alpha$.

$$\delta(x)=(\theta_1^\alpha\otimes Id)\left((\theta_0\otimes Id)\circ\delta(f_0)+(\theta_1\otimes Id)\circ\delta(f_1)+(\theta_1\otimes Id)\circ\Delta_{ш}(f_0)\right).$$

Let us consider the terms in this expression of the form $X_\emptyset\otimes X_c$, with c a word. This gives:

$$\lambda X_\emptyset\otimes X_{x_1^{k-2}}=0,$$

so $\lambda=0$ and $\delta(x)=0$. Let us now consider the terms of the form $X_{x_1^\alpha x_0c}\otimes X_d$, with c,d words. We obtain:

$$0=(\theta_1^\alpha\circ\theta_0\otimes Id)\circ\delta(f_0).$$

As both θ_0 and θ_1 are injective, we obtain $\delta(f_0)=0$. By the induction hypothesis, $f_0=\nu X_1x_1^l$, with $l=k-2-\alpha<k$. Hence:

$$\Delta_{ш}(f_0)=\nu\sum_{i+j=l}\frac{l!}{i!j!}X_{x_1^i}\otimes X_{x_1^j},$$

and:

$$(\theta_1^{\alpha+1} \otimes Id) \left(\delta(f_1) + \nu \sum_{i+j=l} \frac{l!}{i!j!} X_{x_1^i} \otimes X_{x_1^j} \right) = 0.$$

As θ_1 is injective, we obtain with the induction hypothesis that $f_1 = \mu X_{x_1^{k-2-\alpha}}$, so:

$$x = \mu X_{x_1^{k-1}} + \nu X_{x_1^\alpha x_0 x_1^{k-\alpha-2}}.$$

This gives:

$$\begin{array}{rcl} \delta(x) & = & \nu(\theta_1^{\alpha+1} \otimes Id) \left(\displaystyle\sum_{i+j=k-\alpha-2} \frac{(k-\alpha-2)!}{i!j!} X_{x_1^i} \otimes X_{x_1^j} \right) \\ & = & \nu \displaystyle\sum_{i+j=k-\alpha-2} \frac{(k-\alpha-2)!}{i!j!} X_{x_1^{i+\alpha}} \otimes X_{x_1^j} \\ & = & 0, \end{array}$$

so necessarily $\nu = 0$ and $x = \mu X_{x_1^{k-1}}$.

Second step. Let $x \in Ker(\delta)$. As δ is homogeneous of degree 0, the homogeneous components of x are in $Ker(\delta)$. By the first step, with $\lambda = 0$, these homogeneous components, hence x, belong to $Vect(X_{x_1^k}, k \geq 0)$. □

2.2 Dual pre-Lie algebra

As V is a graded pre-Lie coalgebra, its graded dual is a pre-Lie algebra. We identify this graded dual with $\mathbb{K}\langle x_0, x_1\rangle \subseteq \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$; for any words c, d, $X_c(d) = \delta_{c,d}$. The pre-Lie product of $\mathbb{K}\langle x_0, x_1\rangle$ is denoted by $\bullet$. Dualizing proposition 9, we obtain:

Proposition 11 *1. For all word c, $\emptyset \bullet c = 0$.*

2. For all words c, d, $(x_0 c) \bullet d = x_0(c \bullet d)$.

3. For all words c, d, $(x_1 c) \bullet d = x_1(c \bullet d) + x_0(c ш d)$.

Proof. Let u, v, w be words. Then $X_w(u \bullet v) = \delta(X_w)(u \otimes v)$. Hence, if d is a word:

$$\begin{array}{rcl} X_\emptyset(u \bullet v) & = & 0, \\ X_{x_0 d}(u \bullet v) & = & (\theta_0 \otimes Id) \circ \delta(X_d)(u \otimes v) + (\theta_1 \otimes Id) \circ \Delta_ш(X_d)(u \otimes v) \\ & = & X_d(\theta_0^*(u) \bullet v + \theta_1^*(u) ш v), \\ X_{x_1 d}(u \bullet v) & = & (\theta_1 \otimes Id) \otimes \delta(X_d)(u \otimes v) \\ & = & X_d(\theta_1^*(u) \bullet v). \end{array}$$

Moreover, for all word c:

$$\begin{array}{rclcrclcrcl} \theta_0^*(\emptyset) & = & 0, & & \theta_0^*(x_0 c) & = & c, & & \theta_0^*(x_1 c) & = & 0 \\ \theta_1^*(\emptyset) & = & 0, & & \theta_1^*(x_0 c) & = & 0, & & \theta_1^*(x_1 c) & = & c. \end{array}$$

Hence, for any words c, d:

$$\begin{array}{rcl} X_{x_0d}(x_0c\bullet v) &=& X_d(c\bullet v)\\ &=& X_{x_0d}(x_0(x\bullet v)),\\ X_{x_1d}(x_0c\bullet v) &=& 0\\ &=& X_{x_1d}(x_0(x\bullet v));\\ \\ X_{x_0d}(x_1c\bullet v) &=& X_d(c⧢v)\\ &=& X_{x_0d}(x_1(c\bullet v)+x_0(c⧢v)),\\ X_{x_1d}(x_1c\bullet v) &=& X_d(c\bullet v)\\ &=& X_{x_1d}(x_1(c\bullet v)+x_0(c⧢v)). \end{array}$$

Hence, for any w, $X_w(x_0c\bullet v)=X_w(x_0(x\bullet v))$ and $X_w(x_1c\bullet v)=X_w((x_1(c\bullet v)+x_0(c⧢v))$. $\Box$

Examples.

$$\begin{array}{rcl|rcl|rcl|} x_0\bullet x_0 &=& 0 & x_0\bullet x_0x_0 &=& 0 & x_1\bullet x_0x_0 &=& x_0x_0x_0\\ x_0\bullet x_1 &=& 0 & x_0\bullet x_0x_1 &=& 0 & x_1\bullet x_0x_1 &=& x_0x_0x_1\\ x_1\bullet x_0 &=& x_0x_0 & x_0\bullet x_1x_0 &=& 0 & x_1\bullet x_1x_0 &=& x_0x_1x_0\\ x_1\bullet x_1 &=& x_0x_1 & x_0\bullet x_1x_1 &=& 0 & x_1\bullet x_1x_1 &=& x_0x_1x_1 \end{array}$$

$$\begin{array}{rcl|rcl} x_0x_0\bullet x_0 &=& 0 & x_0x_0\bullet x_1 &=& 0\\ x_0x_1\bullet x_0 &=& x_0x_0x_0 & x_0x_1\bullet x_1 &=& x_0x_0x_1\\ x_1x_0\bullet x_0 &=& 2x_0x_0x_0 & x_1x_0\bullet x_1 &=& x_0x_0x_1+x_0x_1x_0\\ x_1x_1\bullet x_0 &=& x_1x_0x_0+x_0x_1x_0+x_0x_0x_1 & x_1x_1\bullet x_1 &=& x_1x_0x_1+2x_0x_1x_1 \end{array}$$

Dualizing proposition 10:

Theorem 12 $\mathbb{K}\langle x_0,x_1\rangle = Vect(x_1^n, n\geq 0)\oplus(\mathbb{K}\langle x_0,x_1\rangle\bullet\mathbb{K}\langle x_0,x_1\rangle)$. *Hence, $(x_1^n)_{n\geq 0}$ is a minimal system of generators of the pre-Lie algebra $\mathbb{K}\langle x_0,x_1\rangle$.*

Proof. As $\bullet=\delta^*$, $Im(\bullet)=Ker(\delta)^\perp = Vect(X_{x_1^n}, n\geq 0)^\perp$. The first assertion is then immediate. As $\mathbb{K}\langle\langle x_0,x_1\rangle\rangle$ is a graded, connected pre-Lie coalgebra, $\mathbb{K}\langle x_0,x_1\rangle$ is a graded, connected pre-Lie algebra. The result then comes from the next lemma. $\Box$

Lemma 13 *Let A be a graded, connected pre-Lie algebra, and V be a graded subspace of A.*

1. *V generates A if, and only if, $A=V+A\bullet A$.*

2. *V is a minimal subspace of generators of A if, and only if, $A=V\oplus A\bullet A$.*

Proof. 1. $\Longrightarrow$. Let $x\in A$. Then it can be written as an element of the pre-Lie subalgebra generated by v, so as the sum of an element of V and of iterated pre-Lie products of elements of V. Hence, $x\in V+A\bullet A$. Note that we did not use the gradation of A to prove this.

1. $\Longleftarrow$. Let B be the pre-Lie subalgebra generated by V. Let $x\in A_n$, let us prove that $x\in B$ by induction on n. As $A_0=(0)$, it is obvious if $n=0$. Let us assume the result at all ranks $<n$. We obtain, by the gradation:

$$A_n=V_n\oplus\sum_{i=1}^{n-1}A_i\bullet A_{n-i}.$$

So we can write:

$$x=\lambda x_1^{n-1}+\sum x_i\bullet y_i,$$

where x_i, y_i are homogeneous of degree $< n$. By the induction hypothesis, these elements belong to B, so $x \in B$.

2. $\Longrightarrow$. By 1. $\Longrightarrow$, $A = V + A \bullet A$. If $V \cap A \bullet A \neq (0)$, we can choose a graded subspace $W \subsetneq V$, such that $A = W \oplus A \bullet A$. By 1. $\Longleftarrow$, W generates A, so V is not a minimal system of generators of A: contradiction. So $A = V \oplus A \bullet A$.

2. $\Longleftarrow$. By 1. $\Longleftarrow$, V is a space of generators of A. If $W \subsetneq V$, then $W \oplus A \bullet A \subsetneq A$. By 1. $\Longrightarrow$, W does not generate V. So V is a minimal subspace of generators. $\Box$

Proposition 14 *For all $x, y, z \in \mathbb{K}\langle x_0, x_1\rangle$,*

$$(x ⧢ y) \bullet z = (x \bullet z) ⧢ y + x ⧢ (y \bullet z).$$

Proof. We prove it if x, y, z are words. If $x = \emptyset$, then:

$$(\emptyset ⧢ y) \bullet z = y \bullet z = (\emptyset \bullet z) ⧢ y + \emptyset ⧢ (u \bullet z).$$

If $y = \emptyset$, the result is also true, using the commutativity of ⧢. We can now consider that x, y are nonempty words.

Let us proceed by induction on $k = lg(x) + lg(y)$. If $k = 0$ or 1, there is nothing to prove. Let us assume the result at all rank $< k$. Four cases can occur.

First case. $x = x_0 a$ and $y = x_0 b$. Then:

$$\begin{aligned}
(x ⧢ y) \bullet z &= (x_0(a ⧢ x_0 b) \bullet z + (x_0(x_0 a ⧢ b)) \bullet z \\
&= x_0((a ⧢ x_0 b) \bullet z) + x_0((x_0 a ⧢ b) \bullet z) \\
&= x_0((a \bullet z) ⧢ x_0 b) + x_0(a ⧢ ((x_0 b) \bullet z)) + x_0(((x_0 a) \bullet z) ⧢ b) + x_0(x_0 a ⧢ (b \bullet z)) \\
&= x_0((a \bullet z) ⧢ x_0 b) + x_0(a ⧢ (x_0(b \bullet z)) + x_0((x_0(a \bullet z)) ⧢ b) + x_0(x_0 a ⧢ (b \bullet z)) \\
&= x_0(a \bullet z) ⧢ x_0 b + x_0 a ⧢ x_0(b \bullet z) \\
&= (x \bullet z) ⧢ y + x ⧢ (y \bullet z).
\end{aligned}$$

Second case. $x = x_1 a$ and $y = x_0 b$. This gives:

$$\begin{aligned}
(x ⧢ y) \bullet z &= (x_1(a ⧢ x_0 b)) \bullet z + (x_0(x_1 a ⧢ b)) \bullet z \\
&= x_1((a \bullet z) ⧢ x_0 b) + x_1(a ⧢ x_0(b \bullet z)) \\
&\quad + x_0(a ⧢ x_0 b ⧢ z) + x_0(((x_1 a) \bullet z) ⧢ b) + x_0(x_1 a ⧢ (b \bullet z)) \\
&= x_1((a \bullet z) ⧢ x_0 b) + x_1(a ⧢ x_0(b \bullet z)) \\
&\quad + x_0(a ⧢ x_0 b ⧢ z) + x_0((x_1(a \bullet z)) ⧢ b) + x_0((x_0(a ⧢ z)) ⧢ b) + x_0(x_1 a ⧢ (b \bullet z)),
\end{aligned}$$

$$\begin{aligned}
(x \bullet z) ⧢ y &= (x_1(a \bullet z)) ⧢ x_0 b + (x_0(a ⧢ z)) ⧢ (x_0 b) \\
&= x_1((a \bullet z) ⧢ (x_0 b)) + x_0(x_1(a \bullet z) ⧢ b) \\
&\quad + x_0(a ⧢ z ⧢ x_0 b) + x_0((x_0(a ⧢ z)) ⧢ b),
\end{aligned}$$

$$\begin{aligned}
x ⧢ (y \bullet z) &= x_1 a ⧢ x_0(b \bullet z) \\
&= x_1(a ⧢ x_0(b \bullet z)) + x_0(x_1 a ⧢ (b \bullet z)).
\end{aligned}$$

These computations imply the required equality.

Third case. $x = x_0 a$ and $y = x_1 b$. This is a consequence of the second case, using the commutativity of ⧢.

Last case. $x = x_1 a$ and $y = x_1 b$. Similar computations give:

$$\begin{aligned}
(x ш y) \bullet z &= x_1((a \bullet z) ш x_1 b) + x_1(a ш x_1(b \bullet w)) + x_1(a ш x_0(b ш z)) + x_0(a ш x_1 b ш z) \\
&\quad + x_1(x_1 a ш (b \bullet z)) + x_1((x_1(a \bullet z)) ш b) + x_1((x_0(a ш z)) ш b) + x_0(a ш x_1 b ш z),
\end{aligned}$$

$$(x \bullet z) ш y = x_1((a \bullet z) ш x_1 b) + x_1((x_1(a \bullet z)) ш b) + x_0(a ш x_1 b ш z) + x_1((x_0(a ш z)) ш b),$$

$$x ш (y \bullet z) = x_1(a ш x_1(b \bullet w)) + x_1(a ш x_0(b ш z)) + x_1(x_1 a ш (b \bullet z)) + x_0(a ш x_1 b ш z).$$

So the result holds in all cases. $\square$

3 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a Com-pre-Lie algebra

Proposition 14 motivates the following definition:

Definition 15 *An* Com-pre-Lie algebra *is a triple* $(V, \bullet, ш)$*, such that:*

1. $(V, \bullet)$ *is a pre-Lie algebra.*
2. $(V, ш)$ *is a commutative, associative algebra (non necessarily unitary).*
3. *For all* $a, b, c \in V$, $(a ш b) \bullet c = (a \bullet c) ш b + a ш (b \bullet c)$.

For example, $\mathbb{K}\langle x_0, x_1 \rangle$ is a Com-pre-Lie algebra.

3.1 Free Com-pre-Lie algebras

Definition 16 1. *A* partitioned forest *is a pair* (F, I) *such that:*

   (a) F *is a rooted forest (the edges of* F *being oriented from the leaves to the roots).*

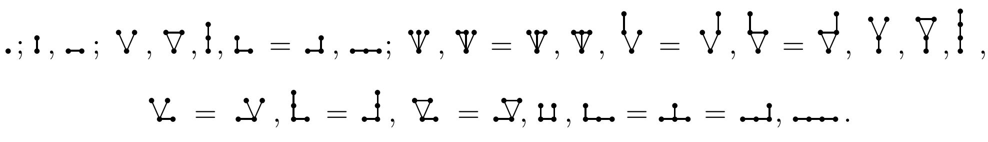
   (b) I *is a partition of the vertices of* F *with the following condition: if* x, y *are two vertices of* F *which are in the same part of* I*, then either they are both roots, or they have the same direct descendant.*

2. *We shall say that a partitioned forest is a* partitioned tree *if all the roots are in the same part of the partition.*

3. *Let* $\mathcal{D}$ *be a set. A* partitioned tree decorated by $\mathcal{D}$ *is a pair* (t, d)*, where* t *is a partitioned tree and* d *is a map from the set of vertices of* t *into* $\mathcal{D}$*. For any vertex* x *of* t*,* $d(x)$ *is called the* decoration *of* x*.*

4. *The set of isoclasses of partitioned trees will be denoted by* $\mathcal{PT}$*. For any set* $\mathcal{D}$*, the set of isoclasses of partitioned trees decorated by* $\mathcal{D}$ *will be denoted by* $\mathcal{PT}(\mathcal{D})$*.*

Examples. We represent partitioned trees by the Hasse graph of the underlying rooted forest, the partition being represented by horizontal edges. Here are partitioned trees with ≤ 4 vertices:



Definition 17 *Let* $t = (t, I)$ *and* $t' = (t', J) \in \mathcal{PT}$*.*

1. *Let* s *be a vertex of* t'*. The partitioned tree* $t \bullet_s t'$ *is defined as follows:*

(a) *As a rooted forest, $t \bullet_s t'$ is obtained by grafting all the roots of t' on the vertex s of t.*

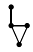
(b) *We put $I = \{I_1, \ldots, I_k\}$ and $J = \{J_1, \ldots, J_l\}$. The partition of the vertices of this rooted forest is $I \sqcup J = \{I_1, \ldots, I_k, J_1, \ldots, J_l\}$.*

2. *The partitioned tree $t \amalg t'$ is defined as follows:*

(a) *As a rooted forest, $t \bullet_s t'$ is tt'.*

(b) *We put $I = \{I_1, \ldots, I_k\}$ and $J = \{J_1, \ldots, J_l\}$ and we assume that the set of roots of t is I_1 and the set of roots of t' is J_1. The partition of the vertices of $t \bullet t'$ $\{I_1 \sqcup J_1, I_2, \ldots, I_k, J_1, \ldots, J_l\}$.*

Examples.

1. Here are the three possible graftings  $\bullet_s$ $\centerdot$: ,  and .

2. Here are the two possible graftings $\bullet_s$:  and .

These operations can also be defined for decorated partitioned trees.

Proposition 18 *Let $\mathcal{D}$ be a set. We denote by $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ the vector space generated by $\mathcal{PT}(\mathcal{D})$. We extend $\amalg$ by bilinearity on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ and we define a second product $\bullet$ on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ in the following way: if $t, t' \in \mathcal{PT}(\mathcal{D})$,*

$$t \bullet t' = \sum_{s \in V(t)} t \bullet_s t'.$$

Then $(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}, \bullet, \amalg)$ is a Com-pre-Lie algebra.

Proof. Let t, t', t'' be three partitioned trees.

If s', s'' are two vertices of t, we define by $t \bullet_{s,s'} (t', t'')$ the partitioned trees obtained by grafting the roots of t' on s', the roots of t'' on s'', the partition of the vertices of the obtained rootes forest being the union of the partitions of t, t' and t''. Then:

$$\begin{aligned}
(t \bullet t') \bullet t'' &= \sum_{s' \in V(t)} (t \bullet_{s'} t') \bullet t'' \\
&= \sum_{s', s'' \in V(t)} (t \bullet_{s'} t') \bullet_{s''} t'' + \sum_{s' \in V(t), s'' \in V(t')} (t \bullet_{s'} t') \bullet_{s''} t'' \\
&= \sum_{s', s'' \in V(t)} t \bullet_{s's''} (t', t'') + \sum_{s' \in V(t), s'' \in V(t')} t \bullet_{s'} (t' \bullet_{s''} t'') \\
&= \sum_{s', s'' \in V(t)} t \bullet_{s's''} (t', t'') + t \bullet (t' \bullet t'').
\end{aligned}$$

So $(t \bullet t') \bullet t'' - t \bullet (t' \bullet t'')$ is clearly symmetric in t and t', and $\bullet$ is pre-Lie.

Moreover, $(t \amalg t') \amalg t'' = t \amalg (t' \amalg t'')$ is the rooted forest $tt't''$, the partition being $\{I_1 \cup J_1 \cup K_1, I_2, \ldots, I_k, J_2, \ldots, J_l, K_2, \ldots, K_m\}$, with immediate notations; $t \amalg t' = t' \amalg t$ is the rooted forest tt', the partition being $\{I_1 \cup J_1, I_2, \ldots, I_k, J_2, \ldots, J_l\}$. So $\amalg$ is an associative, commutative product.

Finally:

$$\begin{aligned}
(t \amalg t') \bullet t'' &= \sum_{s \in V(t)} (t \amalg t') \bullet_s t'' + \sum_{s' \in V(t')} (t \amalg t') \bullet_{s'} t'' \\
&= \sum_{s \in V(t)} (t \bullet_s t'') \amalg t' + \sum_{s' \in V(t')} t \amalg (t' \bullet_{s'} t'') \\
&= (t \bullet t') \amalg t'' + t \amalg (t' \bullet t'').
\end{aligned}$$

So $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is Com-pre-Lie. $\square$

In particular, $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is pre-Lie. Let us use the extension of the pre-Lie product $\bullet$ to $S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$ defined by Oudom and Guin [11, 12]:

1. If $t_1, \ldots, t_k \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $t_1 \ldots t_k \bullet 1 = t_1 \ldots t_k$.
2. If $t, t_1, \ldots, t_k \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $t \bullet t_1 \ldots t_k = (t \bullet t_1 \ldots t_{k-1}) \bullet t_k - t \bullet (t_1 \ldots t_{k-1} \bullet t_k)$.
3. If $a, b, c \in S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$, $ab \bullet c = (a \bullet c^{(1)})(b \bullet c^{(2)})$, where $\Delta(c) = c^{(1)} \otimes c^{(2)}$ is the usual coproduct of $S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$. In particular, if $t_1, \ldots, t_k, t \in \mathcal{PT}(\mathcal{D})$:

$$t_1 \ldots t_k \bullet t = \sum_{i=1}^{k} t_1 \ldots (t_i \bullet t) \ldots t_k.$$

Lemma 19 *Let $t = (t, I), t_1 = (t_1, I^{(1)}), \ldots, t_k = (t_k, I^{(k)})$ be partitioned trees ($k \geq 1$). Let $s_1, \ldots, s_k \in V(t)$. The partitioned tree $t \bullet_{s_1,\ldots,s_k} (t_1, \ldots, t_k)$ is obtained by grafting the roots of t_i on s_i for all i, the partition being $I \sqcup I^{(1)} \sqcup \ldots \sqcup I^{(k)}$. Then:*

$$t \bullet t_1 \ldots t_k = \sum_{s_1,\ldots,s_k \in V(t)} t \bullet_{s_1,\ldots,s_k} (t_1, \ldots, t_k).$$

Proof. By induction on k. This is obvious if $k = 1$. Let us assume the result at rank k.

$$\begin{aligned}
t \bullet t_1 \ldots t_{k+1} &= (t \bullet t_1 \ldots t_k) \bullet t_{k+1} - \sum_{i=1}^{k} t \bullet (t_1 \ldots (t_i \bullet t_{k+1}) \ldots t_k) \\
&= \sum_{s_1,\ldots,s_k \in V(t)} (t \bullet_{s_1,\ldots,s_k} (t_1, \ldots, t_k)) \bullet t_{k+1} - \sum_{i=1}^{k} \sum_{s \in V(t_i)} t \bullet (t_1 \ldots (t_i \bullet_s t_{k+1}) \ldots t_i) \\
&= \sum_{s_1,\ldots,s_{k+1} \in V(t)} (t \bullet_{s_1,\ldots,s_k} (t_1, \ldots, t_k)) \bullet_{s_{k+1}} t_{k+1} \\
&\quad + \sum_{i=1}^{k} \sum_{s \in V(t_i)} (t \bullet_{s_1,\ldots,s_k} (t_1, \ldots, t_k)) \bullet_s t_{k+1} \\
&\quad - \sum_{i=1}^{k} \sum_{s_1,\ldots,s_k \in V(t)} \sum_{s \in V(t_i)} t \bullet_{s_1,\ldots,s_k} (t_1, \ldots, t_i \bullet_s t_{k+1}, \ldots, t_i) \\
&= \sum_{s_1,\ldots,s_{k+1} \in V(t)} t \bullet_{s_1,\ldots,s_{k+1}} (t_1, \ldots, t_{k+1}).
\end{aligned}$$

Hence, the result holds for all k. $\square$

Theorem 20 *Let $\mathcal{D}$ be a set, let A be a Com-pre-Lie algebra, and let $a_d \in A$ for all $d \in \mathcal{D}$. There exists a unique morphism of Com-pre-Lie algebra $\phi : \mathfrak{g}_{\mathcal{PT}(\mathcal{D})} \longrightarrow A$, such that $\phi(\bullet_d) = a_d$ for all $d \in \mathcal{D}$. In other words, $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is the free Com-pre-Lie algebra generated by $\mathcal{D}$.*

Proof. *Unicity.* Let $t \in \mathcal{T}^d$. We denote by $r_1, \ldots, r_n$ its roots. For all $1 \leq i \leq n$, let $t_{i,1}, \ldots, t_{i,k_i}$ be the partitioned trees born from r_i and let d_i be the decoration of r_i. Then:

$$t = (\bullet_{d_1} \bullet t_{1,1} \ldots t_{1,k_1}) Ш \ldots Ш (\bullet_{d_n} \bullet t_{n,1} \ldots t_{n,k_n}).$$

So ϕ is inductively defined by:

$$\phi(t) = (a_{d_1} \bullet \phi(t_{1,1}) \ldots \phi(t_{1,k_1})) Ш \ldots Ш (a_{d_n} \bullet \phi(t_{n,1}) \ldots \phi(t_{n,k_n})). \tag{1}$$

Existence. As the product $ш$ of A is commutative and associative, (1) defines inductively a morphism ϕ from $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ to A. By definition, it is compatible with the product $ш$. Let us prove the compatibility with the product $\bullet$. Let t, t' be two partitioned trees, let us prove that $\phi(t \bullet t') = \phi(t) \bullet \phi(t')$ by induction on the number N of vertices of t. If $N = 1$, then $t = \bullet_d$ and:

$$\phi(t \bullet t') = a_d \bullet \phi(t') = \phi(t) \bullet \phi(t'),$$

by definition of t'. If $N > 1$, two cases are possible.

First case. If t has only one roots, then $t = \bullet_d \bullet t_1 \dots t_k$, and:

$$t \bullet t' = \bullet_d \bullet t_1 \dots t_k t' + \sum_{i=1}^{k} \bullet_d \bullet t_1 \dots t_i \circ t' \bullet t_k.$$

Using the induction hypothesis on $t_1, \dots, t_k$:

$$\begin{aligned}
\phi(t \bullet t') &= a_d \bullet \phi(t_1) \dots \phi(t_k)\phi(t') + \sum_{i=1}^{k} a_d \bullet \phi(t_1) \dots \phi(t_1 \circ t') \dots \phi(t_k) \\
&= a_d \bullet \phi(t_1) \dots \phi(t_k)\phi(t') + \sum_{i=1}^{k} a_d \bullet (\phi(t_1) \dots \phi(t_1) \circ \phi(t') \dots \phi(t_k)) \\
&= (a_d \bullet \phi(t_1) \dots \phi(t_k)) \bullet \phi(t') \\
&= \phi(t) \bullet \phi(t').
\end{aligned}$$

Second case. If t has $k > 1$ roots, we put $t = t_1 ш \dots ш t_k$. The induction hypothesis holds for $t_1, \dots, t_k$, so:

$$\begin{aligned}
\phi(t \circ t') &= \sum_{i=1}^{k} \phi(t_1 ш t_i \bullet t' ш \dots ш t_k) \\
&= \sum_{i=1}^{k} \phi(t_1) ш \phi(t_i \bullet t') ш \dots ш \phi(t_k) \\
&= \sum_{i=1}^{k} \phi(t_1) ш \phi(t_i) \bullet \phi(t') ш \dots ш \phi(t_k) \\
&= (\phi(t_1) ш \dots ш \phi(t_k)) \bullet \phi(t') \\
&= \phi(t) \bullet \phi(t').
\end{aligned}$$

Hence, ϕ is a morphism of Com-pre-Lie algebras. □

3.2 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a Com-pre-Lie algebra

Proposition 21 *As a Com-pre-Lie algebra, $\mathbb{K}\langle x_0, x_1 \rangle$ is generated by $\emptyset$ and x_1.*

Proof. Let A be the Com-pre-Lie subalgebra of $\mathbb{K}\langle x_0, x_1 \rangle$ generated by $\emptyset$ and x_1. For all $n \geq 1$, it contains $x_1^{ш n} = n! x_1^n$, so it contains x_1^n for all $n \geq 0$. As $\mathbb{K}\langle x_0, x_1 \rangle$ is generated by these elements as a pre-Lie algebra, $A = \mathbb{K}\langle x_0, x_1 \rangle$. □

We denote by $\phi_{APL} : \mathfrak{g}_{\mathcal{PT}(\{1,2\})} \longrightarrow \mathbb{K}\langle x_0, x_1 \rangle$ the unique morphism of Com-pre-Lie algebras which sends $\bullet_1$ to $\emptyset$ and $\bullet_2$ to $\bullet_2$. By proposition 21, it is surjective.

Lemma 22 *Let $t_1, \dots, t_k \in \mathcal{PT}(\{1, 2\})$.*

1. $\phi_{APL}(\bullet_1 \bullet t_1 \dots t_k) = 0$ *if* $k \geq 1$.

2. $\phi_{APL}(\bullet_2 \bullet t_1 \ldots t_k) = 0$ *if* $k \geq 2$.

3. *If* $t \in \mathcal{PT}(\{1,2\})$, $\phi_{APL}(\bullet_2 \bullet t) = x_0 \phi_{APL}(t)$.

Proof. We proceed by induction on k. If $k = 1$:

$$\begin{array}{rcl} \phi_{APL}(\bullet_1 \bullet t) & = & \emptyset \bullet \phi_{APL}(t) \\ & = & 0, \\ \phi_{APL}(\bullet_2 \bullet t) & = & x_1 \bullet \phi_{APL}(t) \\ & = & x_0 \phi_{APL}(t). \end{array}$$

Let us assume the result at rank $k-1 \geq 1$. Then:

$$\begin{array}{rcl} \phi_{APL}(\bullet_1 \bullet t_1 \ldots t_k) & = & \emptyset \bullet \phi_{APL}(t_1) \ldots \phi_{APL}(t_k) \\ & = & (\emptyset \bullet \phi_{APL}(t_1) \ldots \phi_{APL}(t_{k-1})) \bullet \phi_{APL}(t_k) \\ & & -\displaystyle\sum_{i=1}^{k} \emptyset \bullet \phi_{APL}(t_1) \ldots \phi_{APL}(t_i \bullet t_k) \ldots \phi_{APL}(t_{k-1}) \\ & = & 0 - 0, \\ \phi_{APL}(\bullet_2 \bullet t_1 \ldots t_k) & = & x_1 \bullet \phi_{APL}(t_1) \ldots \phi_{APL}(t_k) \\ & = & (x_1 \bullet \phi_{APL}(t_1) \ldots \phi_{APL}(t_{k-1})) \bullet \phi_{APL}(t_k) \\ & & -\displaystyle\sum_{i=1}^{k} x_1 \bullet \phi_{APL}(t_1) \ldots \phi_{APL}(t_i \bullet t_k) \ldots \phi_{APL}(t_{k-1}). \end{array}$$

If $k \geq 3$, the induction hypothesis immediately allows to conclude that $\phi_{APL}(\bullet_2 \bullet t_1 \ldots t_k) = 0 - 0 = 0$. If $k = 2$, this gives:

$$\begin{array}{rcl} \phi_{APL}(\bullet_2 \bullet t_1 t_2) & = & (x_1 \bullet \phi_{APL}(t_1)) \bullet \phi_{APL}(t_2) - x_1 \bullet \phi_{APL}(t_1 \bullet t_2) \\ & = & (x_0 \phi_{APL}(t_1)) \bullet \phi_{APL}(t_2) - x_0 \phi_{APL}(t_1 \bullet t_2) \\ & = & x_0(\phi_{APL}(t_1) \bullet \phi_{APL}(t_2)) - x_0 \phi_{APL}(t_1 \bullet t_2) \\ & = & 0. \end{array}$$

Hence, the result holds for all $k \geq 1$. $\square$

Lemma 23 *For all* $t \in \mathcal{PT}(\{1,2\})$, $\phi_{APL}(t)$ *is a linear span of words of length the number of vertices of* t *decorated by* 2.

Proof. By induction on the number of vertices N of t. If $N = 1$, then $t = \bullet_1$ or $\bullet_2$ and the result is obvious. Let us assume the result at all rank $< N$.

First case. If t has only one root, we put $t = \bullet_i \bullet t_1 \ldots t_k$. By the preceding lemma, we can assume that $i = 2$ and $k = 1$. Then $\phi_{APL}(t) = x_0 \phi_{APL}(t_1)$ and the result is obvious.

Second case. If t has $k > 1$ roots, we put $t = t_1 ⧢ \ldots ⧢ t_k$. Then $\phi_{APL}(t_1)$ is equal to $\phi_{APL}(t_1) ⧢ \ldots ⧢ \phi_{APL}(t_k)$ and the result is immediate. $\square$

Lemma 24 *We define inductively a family* F *of elements of* $\mathcal{PT}(\{1,2\})$ *by:*

1. $F(1) = \{\bullet_1, \bullet_2\}$.

2. $F(n+1) = (\bullet_2 \bullet F(n)) \cup \displaystyle\bigcup_{i=1}^{n} (F(i) ⧢ F(n+1-i))$.

3. $F = \displaystyle\bigcup_{n \geq 1} F(n)$.

Let $t \in \mathcal{PT}(\{1,2\})$. Then $\phi_{APL}(t) \neq 0$ if, and only if, $t \in F$.

Proof. $\Longrightarrow$. We proceed by induction on the number N of vertices of t. This is obvious if $N = 1$. Let us assume the result at all rank $< N$.

First case. If N has only one root, we put $N = \bullet_i \bullet t_1 \ldots t_k$. By lemma 22, $i = 2$ and $k = 1$. Then $\phi_{APL}(t) = x_0 \phi_{APL}(t_1)$. By the induction hypothesis, $t_1 \in F$, so $t \in F$.

Second case. If N has $k > N$ roots, we put $t = t_1 ш \ldots ш t_k$. Then:

$$\phi_{APL}(t) = \phi_{APL}(t_1) ш \phi_{APL}(t_2 ш \ldots ш t_k) \neq 0,$$

so by the induction hypothesis, t_1 and $t_2 ш \ldots ш t_k \in F$, and $t \in F$.

$\Longleftarrow$. Let $t \in T(n)$. We proceed by induction on n. It $n = 1$, this is obvious. If $n > 1$ then $t = \bullet_2 \bullet t'$, with $t' \in F(n-1)$, or $t = t' ш t''$, with $t' \in F(i)$, $t'' \in F(n-i)$. In the first case, by the induction hypothesis, $\phi_{APL}(t') \neq 0$ and $\phi_{APL}(t) = x_0 \phi_{APL}(t') \neq 0$. In the second case, $\phi_{APL}(t'), \phi_{APL}(t'') \neq 0$ by the induction hypothesis, so $\phi_{APL}(t) = \phi_{APL}(t') ш \phi_{APL}(t'') \neq 0$. $\Box$

We define a second family of elements of $\mathcal{PT}(\{1,2\})$ in the following way:

1. $F'(1) = \{\bullet_1, \bullet_2\}$.
2. $F'(2) = \{\bullet_2 ш \bullet_2, \mathfrak{l}^{2}_{2}, \mathfrak{l}^{1}_{2}\}$.
3. $F'(n+1) = (\bullet_2 \bullet F'(n)) \cup \displaystyle\bigcup_{i=2}^{n-1} \big(F'(i) ш F'(n+1-i)\big) \cup \big(\bullet_2 ш F'(n)\big)$ if $n \geq 2$.
4. $F' = \displaystyle\bigcup_{n \geq 1} F'(n)$.

We define a map π from F to $\mathcal{PT}(\{1,2\})$ in the following way:

1. $\pi(\bullet_i) = \bullet_i$ if $i = 1, 2$.
2. $\pi(\bullet_1 ш \ldots ш \bullet_1) = \bullet_1$.
3. If $t = \bullet_1 ш \ldots ш \bullet_1 ш t_1 ш \ldots ш t_k$, $k \geq 1$, with $t_1, \ldots, t_k \neq \bullet_1$, then $\pi(t) = \pi(t_1) ш \ldots ш \pi(t_k)$.
4. If $t = \bullet_2 \bullet t_1 \ldots t_k$, then $\pi(t) = \bullet_2 \bullet \pi(t_1) \ldots \pi(t_k)$.

Lemma 25 *π is a projection on F' and $\phi_{APL} \circ \pi = \phi_{APL|F}$.*

Proof. Let $t \in F$. Let us prove by induction on the number N of vertices of t that:

1. $\pi(t) \in F'$.
2. If $t \in F'$, $\pi(t) = t$.
3. $\phi_{APL} \circ \pi(t) = \phi_{APL}(t)$.
4. If $\pi(t) = \bullet_1$, then $t = \bullet_1^{ш n}$ for a particular n.

All these points are immediate if $N = 1$. Let us assume the result at all rank $< N$, $N \geq 2$. We put $t = \bullet_1 ш \ldots ш \bullet_1 ш t_1 ш \ldots ш t_k$, $k \geq 0$, with $t_1, \ldots, t_k \neq \bullet_1$.

First case. If $k \geq 2$, then $\pi(t) = \pi(t_1) ⧢ \ldots ⧢ \pi(t_k)$. By the induction hypothesis, $\pi(t_1), \ldots, \pi(t_k) \in F'$ and are not equal to $\cdot_1$, so $\pi(t) \in F'$. By the induction hypothesis, $\pi(t_1) \neq \cdot_1$, so $\pi(t) \neq \cdot_1$. Moreover:

$$\begin{aligned} \phi_{APL}(t) &= \phi_{APL}(\cdot_1) ⧢ \ldots ⧢ \phi_{APL}(\cdot_1) ⧢ \phi_{APL}(t_1) ⧢ \ldots ⧢ \phi_{APL}(t_k) \\ &= \emptyset ⧢ \ldots ⧢ \emptyset ⧢ \phi_{APL} \circ \pi(t_1) ⧢ \ldots ⧢ \phi_{APL} \circ \pi(t_k) \\ &= \phi_{APL}(\pi(t_1) ⧢ \ldots ⧢ \pi(t_k)) \\ &= \phi_{APL} \circ \pi(t). \end{aligned}$$

If $t \in F'$, necessarily $t = t_1 ⧢ \ldots ⧢ t_k$, and $t_1, \ldots, t_k \in F'$. By the induction hypothesis, $\pi(t_1) = t_1, \ldots, \pi(t_k) = t_k$, so $\pi(t) = t$.

Second case. If $k = 1$, as $t_1 \in F$, we put $t_1 = \cdot_2 \bullet s$. Then $\pi(t) = \cdot_2 \bullet \pi(s)$. By the induction hypothesis, $\pi(s) \in F'$, so $\pi(t) = F'$. Moreover:

$$\begin{aligned} \phi_{APL}(t) &= \phi_{APL}(\cdot_1) ⧢ \ldots ⧢ \phi_{APL}(\cdot_1) ⧢ (\phi_{APL}(\cdot_2) \bullet \phi_{APL}(s)) \\ &= \emptyset ⧢ \ldots ⧢ \emptyset ⧢ (\phi_{APL}(\cdot_2) \bullet \phi_{APL}(s)) \\ &= \phi_{APL} \circ \pi(\cdot_2) \bullet \phi_{APL} \circ \pi(s) \\ &= \phi_{APL} \circ \pi(t). \end{aligned}$$

If $t' \in F'$, then $s \in F'$, and $t = \cdot_2 \bullet s$. Then $\pi(t) = \cdot_2 \bullet \pi(s) = \cdot_2 \bullet s = t$.

Last case. If $k = 0$, all the results are obvious. □

Lemma 26 *Let $t, t' \in \mathcal{PT}(\{1,2\})$. Then:*

$$\phi_{APL}\left((\cdot_2 \bullet t) ⧢ (\cdot_2 \bullet t')\right) = \phi_{APL}\left(\cdot_2 \bullet ((\cdot_2 \bullet t) ⧢ t' + t ⧢ (\cdot_2 \bullet t'))\right).$$

Proof. Indeed, putting $w = \phi_{APL}(t)$ and $w' = \phi_{APL}(t')$:

$$\begin{aligned} \phi_{APL}\left((\cdot_2 \bullet t) ⧢ (\cdot_2 \bullet t')\right) &= x_0 w ⧢ x_0 w' \\ &= x_0(w ⧢ x_0 w') + x_0(x_0 w ⧢ w') \\ &= \phi_{APL}\left(\cdot_2 \bullet ((\cdot_2 \bullet t) ⧢ t' + t ⧢ (\cdot_2 \bullet t'))\right). \end{aligned}$$

We used lemma 22 for the first and third equalities. □

Theorem 27 *The kernel of ϕ_{APL} is the Com-pre-Lie ideal generated by the elements:*

1. $\cdot_1 \bullet t_1 \ldots t_k$, *where* $k \geq 1$, $t_1, \ldots, t_k \in \mathcal{PT}(\{1,2\})$.
2. $\cdot_2 \bullet t_1 \ldots t_k$, *where* $k \geq 2$, $t_1, \ldots, t_k \in \mathcal{PT}(\{1,2\})$.
3. $\cdot_1 ⧢ t - t$, *where* $t \in \mathcal{PT}(\{1,2\})$.
4. $(\cdot_2 \bullet t) ⧢ (\cdot_2 \bullet t') - \cdot_2 \bullet ((\cdot_2 \bullet t) ⧢ t' - t ⧢ (\cdot_2 \bullet t'))$, *where* $t, t' \in \mathcal{PT}(\{1,2\})$.

Proof. Let I be the ideal generated by these elements. Lemmas 22 and 26 prove that the elements 1., 2. and 4. belong to $Ker(\phi_{APL})$. Moreover, for all $t \in \mathcal{PT}(\{1,2\})$, $\pi(\cdot_1 ⧢ t) = \pi(t)$. For all $t \in \mathcal{PT}(\{1,2\})$:

$$\phi_{APL}(\cdot_1 ⧢ t) = \emptyset ⧢ \phi_{APL}(t) = \phi_{APL}(t),$$

so elements 3. also belong to $Ker(\phi_{APL})$. Hence, $I \subseteq Ker(\phi_{APL})$.

Let $h = \mathfrak{g}_{\mathcal{PT}(\{1,2\})}/I$. As the elements 1. and 2. belong to I, h is linearly spanned by the elements $\overline{t}$, $t \in F$. As the elements 3. belong to I, for all $t \in F$, $\overline{\pi(t)} = \overline{t}$. As π is a projection on F', h is linearly spanned by the elements $\overline{t}$, $t \in F'$.

We now define inductively two families of partitionned trees in the following way:

1. $T''(1) = \{\cdot_2\}$ and $F''(1) = \{\cdot_1, \cdot_2\}$.
2. $\mathcal{T}''(n+1) = \cdot_2 \bullet F''(n)$.
3. $F''(n+1) = \displaystyle\bigcup_{i=1}^{n+1} T''(i) \sqcup\!\sqcup \cdot_2{}^{\sqcup\!\sqcup(n+1-i)}$.
4. $F'' = \displaystyle\bigcup_{n\geq 1} F''(n)$.

Let us prove that for all $t \in F'$, there exists $t' \in Vect(F'')$ such that $\overline{t} = \overline{t'}$. We proceed by induction on the number N of vertices of t. If $N = 1$, then $t = \cdot_1$ or $\cdot_2$ and we take $t' = t$. Let us assume the result at all rank $< N$. We put $t = t_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup t_k \sqcup\!\sqcup \cdot_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \cdot_2$, with $t_i = \cdot_2 \bullet s_i$, $s_i \neq 1$, for all $1 \leq i \leq k$. We proceed by induction on k. If $k = 0$, we take $t' = t = \cdot_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \cdot_2$. If $k = 1$, then, by the induction hypothesis on N applied to s_1:

$$\overline{t} = (\overline{\cdot_2} \bullet \overline{s_1}) \sqcup\!\sqcup \overline{\cdot_2} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \overline{\cdot_2} = (\overline{\cdot_2} \bullet \overline{s'_1}) \sqcup\!\sqcup \overline{\cdot_2} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \overline{\cdot_2} = \overline{(\cdot_2 \bullet s'_1) \sqcup\!\sqcup \cdot_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \cdot_2}.$$

We take $t' = (\cdot_2 \bullet s'_1) \sqcup\!\sqcup \cdot_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \cdot_2$, which clearly belongs to $Vect(F'')$, as $s'_1 \in Vect(F'')$. Let us assume the result at all rank $< k$. Then, as the elements 4. belong to I:

$$\overline{t_1 \sqcup\!\sqcup t_2} = \overline{\underbrace{\cdot_2 \bullet (t_1 \sqcup\!\sqcup s_2)}_{t'_1}} + \overline{\underbrace{\cdot_2 \bullet (s_1 \bullet t_2)}_{t''_1}},$$

so:

$$\overline{t} = \overline{t'_1 \sqcup\!\sqcup t_3 \sqcup\!\sqcup \ldots \sqcup\!\sqcup t_k \sqcup\!\sqcup \cdot_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \cdot_2} + \overline{t''_1 \sqcup\!\sqcup t_3 \sqcup\!\sqcup \ldots \sqcup\!\sqcup t_k \sqcup\!\sqcup \cdot_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \cdot_2}.$$

By the induction hypothesis on k applied to these two partitionned trees, there exists x'_1 and $x''_1 \in Vect(F'')$, such that $\overline{t} = \overline{x'_1} + \overline{x''_1}$. We take $t' = x'_1 + x''_1$. Consequently, the elements $\overline{t}$, $t \in F''$, linearly span h.

Let $t \in F''(n)$. Then it has n vertices, and at most one of them is decorated by 1. We denote by $F''_1(n)$ the set of elements of $F''(n)$ with one vertex decorated by 1, and we put $F''_2(n) = F''(n) \setminus F''_1(n)$. Let us prove that for all $n \geq 1$, $|F''_1(n+1)| \leq 2^{n-1}$ and $|F''_2(n)| \leq 2^{n-1}$. For $n = 0$, as $FF'_1(2) = \{\mathfrak{l}^{1}_{2}\}$ and $F''_2(1) = \{\cdot_2\}$, this is immediate. Let us assume the result at all rank $\leq n$. Then:

$$\begin{array}{rcl} F''_2(n+1) & = & \displaystyle\bigcup_{i=1}^{n+1} \cdot_2{}^{\sqcup\!\sqcup(n+1-i)} \sqcup\!\sqcup T''(i) \cap F''_2(i) \\ & = & \{\cdot_2{}^{\sqcup\!\sqcup(n+1)}\} \cup \displaystyle\bigcup_{i=1}^{n} \cdot_2{}^{\sqcup\!\sqcup(n+1-i)} \sqcup\!\sqcup \cdot_2 \bullet F''_2(i). \end{array}$$

Hence, $|F''_2(n+1)| \leq 1 + 1 + 2 + \ldots + 2^{n-1} = 2^n$.

$$\begin{array}{rcl} F''_1(n+2) & = & \displaystyle\bigcup_{i=1}^{n+2} \cdot_2{}^{\sqcup\!\sqcup(n+2-i)} \sqcup\!\sqcup T''(i) \cap F''_1(i) \\ & = & \displaystyle\bigcup_{i=2}^{n+2} \cdot_2{}^{\sqcup\!\sqcup(n+2-i)} \sqcup\!\sqcup \cdot_2 \bullet F''_1(i-1) \end{array}$$

Hence, $|F''_1(n+2)| \leq +1 + 1 + \ldots + 2^{n-1} = 2^n$.

Let $\overline{\phi}_{APL}$ be the linear map induced by ϕ_{APL} on h. If $t \in F_1''(n)$, by lemma 23, $\overline{\phi}_{APL}(\overline{t})$ is a linear span of word of length $n-1$. If $t \in F_2''(n)$, by lemma 23, $\overline{\phi}_{APL}(\overline{t})$ is a linear span of word of length n. Hence, for all $n \geq 0$:

$$\overline{\phi}_{APL}(Vect(F_2''(n)) + Vect(F_1''(n+1))) \subseteq Vect(\text{words of length } n).$$

As ϕ_{APL} is surjective, we obtain:

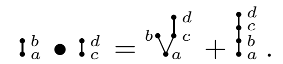
$$\overline{\phi}_{APL}(Vect(F_2''(n)) + Vect(F_1''(n+1))) = Vect(\text{words of length } n).$$

Moreover, as $dim(Vect(\text{words of length } n)) = 2^n$ and $dim(Vect(F_2''(n)) + Vect(F_1''(n+1))) \leq |F_2''(n)| + |F_1''(n)| \leq 2^{n-1} + 2^{n-1} = 2^n$, the restriction of $\overline{\phi}_{APL}$ to $Vect(F_2''(n)) + Vect(F_1''(n+1))$ is injective. Finally, $\overline{\phi}_{APL}$ is injective, so $Ker(\phi_{APL}) = I$. $\Box$

4 Presentation of $\mathbb{K}\langle x_0, x_1\rangle$ as a pre-Lie algebra

4.1 A surjective morphism

Let $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ be the free pre-Lie algebra generated by $\mathbb{N}^*$, as described in [1]. It can be seen as the subspace of $\mathfrak{g}_{\mathcal{PT}(\mathbb{N}^*)}$ generated by rooted trees (which are seen as partitioned trees such that any part of the partition is a singleton), with the restriction of the pre-Lie product $\bullet$ defined by graftings. For example, in $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, if $a, b, c, d > 0$:



This pre-Lie algebra is graded, the degree of a tree being the sum of its decorations.

By theorem 12, there exists a unique surjective map of pre-Lie algebras $\Phi_{PL} : \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} \longrightarrow \mathbb{K}\langle x_0, x_1\rangle$, sending $\bullet_n$ to x_1^{n-1} for all $n \geq 1$. As x_1^{i-1} is homogeneous of degree i for all i, this morphism is homogeneous of degree 0.

Notation. If $t_1 \dots t_k \in \mathcal{T}(\mathbb{N}^*)$ and $n \in \mathbb{N}^*$, we put:

$$B_n(t_1 \dots t_k) = \bullet_n \bullet t_1 \dots t_k.$$

This is the tree obtained by grafting $t_1, \dots, t_k$ on a common root decorated by n.

Proposition 28 *Let $t = B_n(t_1 \dots t_k) \in \mathcal{T}(\mathbb{N}^*)$. We put $\phi_{PL}(t_i) = w_i$ for all $1 \leq i \leq k$. Then:*

$$\phi_{PL}(t) = \begin{cases} x_0 w_1 ⧢ \dots ⧢ x_0 w_k ⧢ x_1^{n-1-k} & \text{if } k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As $\mathfrak{g}_{\mathcal{PT}(\{1,2\})}$ is pre-Lie, there exists a unique morphism of pre-Lie algebras:

$$\psi : \begin{cases} \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} & \longrightarrow & \mathfrak{g}_{\mathcal{PT}(\{1,2\})} \\ \bullet_n & \longrightarrow & \frac{1}{(n-1)!} \bullet_2^{⧢(n-1)}. \end{cases}$$

Then $\phi_{APL} \circ \psi$ is a pre-Lie algebra morphism sending $\bullet_n$ to $\frac{1}{(n-1)!} x_1^{⧢(n-1)} = x_1^{n-1}$ for all $n \geq 1$, so $\phi_{APL} \circ \psi = \phi_{PL}$. We obtain, by lemma 19:

$$\begin{aligned} \psi(\bullet_n \bullet t_1 \dots t_k) &= \frac{1}{(n-1)!} \bullet_2^{⧢(n-1)} \bullet (\psi(t_1) \dots \psi(t_k)) \\ &= \frac{1}{(n-1)!} \sum_{I_1 \sqcup \dots \sqcup I_n = \{1,\dots,k\}} \bullet_2 \bullet \left(\prod_{i \in I_1} t_i\right) ⧢ \dots ⧢ \bullet_2 \bullet \left(\prod_{i \in I_k} t_i\right) \end{aligned}$$

Let us apply ϕ_{APL} to this expression. If $|I_j| \geq 2$, by theorem 27:

$$\phi_{APL}(\bullet_2 \bullet \left(\prod_{i\in I_j} t_i\right) = 0.$$

Consequently, if $k \geq n$, at least one of the I_j contains two elements, so $\phi_{APL} \circ \psi(t) = \phi_{PL}(t) = 0$. Let us assume that $k < n$. Hence, using the commutativity of $\sqcup\!\sqcup$:

$$\begin{aligned}
\phi_{PL}(\bullet_n \bullet t_1 \dots t_k) &= \frac{1}{(n-1)!} \sum_{I_1 \sqcup \dots \sqcup I_n = \{1,\dots,k\},\, |I_j| \leq 1} x_1 \bullet \left(\prod_{i \in I_1} w_i\right) \sqcup\!\sqcup \dots \sqcup\!\sqcup x_1 \bullet \left(\prod_{i \in I_k} w_i\right)\\
&= \frac{1}{(n-1)!} \sum_{\iota:\{1,\dots,k\} \longrightarrow \{1,\dots,n-1\},\text{injective}} x_1 \bullet w_1 \sqcup\!\sqcup \dots x_1 \bullet w_k \sqcup\!\sqcup x_1^{\sqcup\!\sqcup (n-1-k)}\\
&= \frac{1}{(n-1)!} \sum_{\iota:\{1,\dots,k\} \longrightarrow \{1,\dots,n-1\},\text{injective}} x_0 w_1 \sqcup\!\sqcup \dots x_0 w_k \sqcup\!\sqcup x_1^{\sqcup\!\sqcup (n-1-k)}\\
&= \frac{(n-1)\dots(n-k)}{(n-1)!} x_0 w_1 \sqcup\!\sqcup \dots x_0 w_k \sqcup\!\sqcup x_1^{\sqcup\!\sqcup (n-1-k)}\\
&= \frac{(n-1)\dots(n-k)(n-1-k)!}{(n-1)!} x_0 w_1 \sqcup\!\sqcup \dots x_0 w_k \sqcup\!\sqcup x_1^{n-1-k}\\
&= x_0 w_1 \sqcup\!\sqcup \dots x_0 w_k \sqcup\!\sqcup x_1^{n-1-k}.
\end{aligned}$$

$\square$

Corollary 29 *Let $s_1, \dots, s_k, t_1, \dots, t_l \in \mathcal{T}(\{N^*), k, l \geq 0$. For all $i, j, n \geq 1$:*

$$\begin{aligned}
& \phi_{PL}\left(B_{n+1}((B_i(s_1 \dots s_k) B_j(t_1 \dots t_l))\right)\\
= \; & \phi_{PL}\left(B_n(B_{i+1}(s_1 \dots s_k B_j(t_1 \dots t_l))\right) + \phi_{PL}\left(B_n(B_{j+1}(B_i(s_1 \dots s_k) t_1 \dots t_l)\right).
\end{aligned}$$

Proof. We note:

$$\begin{aligned}
T_1 &= B_{n+1}((B_i(s_1 \dots s_k) B_j(t_1 \dots t_l))\\
&= \bullet_{n+1} \bullet ((\bullet_i \bullet s_1 \dots s_k)(\bullet_j \bullet t_1 \dots t_l))\\
T_2 &= B_n(B_{i+1}(s_1 \dots s_k B_j(t_1 \dots t_l))\\
&= \bullet_n \bullet (\bullet_{i+1} \bullet (s_1 \dots s_k(\bullet_j \bullet t_1 \dots t_l)))\\
T_3 &= B_n(B_{j+1}(B_i(s_1 \dots s_k) t_1 \dots t_l)\\
&= \bullet_n \bullet (\bullet_{j+1} \bullet ((\bullet_i \bullet s_1 \dots s_k) t_1 \dots t_l)).
\end{aligned}$$

If $k \geq i$, or $l \geq j$, or $n = 1$, all these elements are sent to zero by ϕ_{PL} by proposition 28. Let us assume now that $k < i$, $l < j$, $n < 1$. We put $v_i = \phi_{PL}(s_i)$ and $w_i = \phi_{PL}(t_i)$. Then:

$$\begin{aligned}
\phi_{PL}(T_1) &= x_0(\underbrace{x_0 v_1 \sqcup\!\sqcup \dots \sqcup\!\sqcup x_0 v_k \sqcup\!\sqcup x_1^{i-1-k}}_{X}) \sqcup\!\sqcup x_0(\underbrace{x_0 w_1 \sqcup\!\sqcup \dots \sqcup\!\sqcup x_0 w_l \sqcup\!\sqcup x_1^{j-1_l}}_{Y}) \sqcup\!\sqcup x_1^{n-2}\\
&= x_0 X \sqcup\!\sqcup x_0 Y \sqcup\!\sqcup x_1^{n-2},\\
\phi_{PL}(T_2) &= x_0(x_0 v_1 \sqcup\!\sqcup \dots \sqcup\!\sqcup x_0(x_0 w_1 \sqcup\!\sqcup \dots \sqcup\!\sqcup x_0 w_l \sqcup\!\sqcup x_1^{j-1-l}) \sqcup\!\sqcup x_1^{i-1-k}) \sqcup\!\sqcup x_1^{n-2}\\
&= x_0(X \sqcup\!\sqcup x_0 Y) \sqcup\!\sqcup x_1^{n-2},\\
\phi_{PL}(T_3) &= x_0(x_0(x_0 v_1 \sqcup\!\sqcup \dots \sqcup\!\sqcup x_0 v_k \sqcup\!\sqcup x_1^{i-1-k}) \sqcup\!\sqcup x_0 w_1 \sqcup\!\sqcup x_0 w_l \sqcup\!\sqcup x_1^{j-1-l}) \sqcup\!\sqcup x_1^{n-2}\\
&= x_0(x_0 X \sqcup\!\sqcup Y) \sqcup\!\sqcup x_1^{n-2}.
\end{aligned}$$

As $x_0 X \sqcup\!\sqcup x_0 Y = x_0(X \sqcup\!\sqcup x_0 Y) + x_0(x_0 X \sqcup\!\sqcup Y)$, we obtain the result. $\square$

Theorem 30 *The kernel of ϕ_{PL} is the pre-Lie ideal generated by:*

1. $B_1(t_1 \dots t_k)$, *where* $k \geq 1$, $t_1, \dots, t_k \in \mathcal{T}(\mathbb{N}^*)$.
2. $B_{n+1}(B_i(s_1 \dots s_k)B_j(t_1 \dots t_l)) - B_n(B_{i+1}(s_1 \dots s_k B_j(t_1 \dots t_l)) - B_{j+1}(B_i(s_1 \dots s_k)t_1 \dots t_l))$, *where* $k, l \geq 0$, $s_1, \dots, s_k, t_1, \dots, t_l \in \mathcal{T}(\mathbb{N}^*)$.

Proof. Let I be the ideal generated by these elements. By proposition 28 and corollary 29, $I \subseteq Ker(\phi_{PL})$. We put $h = \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}/I$. Applying repeatedly the relation given by elements 2., it is not difficult to prove that for any $t \in \mathcal{T}(\mathbb{N}^*)$, there exists a linear span of ladders t' such that $\overline{t} = \overline{t'}$ in h. Moreover, by the relation given by elements 1., if one of the vertices of a ladder t which is not the leaf is decorated by 1, then $\overline{t} = 0$. Let us denote by $L(n)$ the set of ladders decorated by $\mathbb{N}^*$, of weight n, such that all the vertices which are not the leaf are decorated by integer > 1. It turns out that h is generated by the elements $\overline{t}$, $t \in L = \bigcup L(n)$.

Let $\overline{\phi_{PL}}$ be the morphism form h to $\mathbb{K}\langle x_0, x_1 \rangle$ induced by ϕ_{PL}. By homogeneity, as ϕ_{PL} is surjective, for all $n \geq 1$:

$$\overline{\phi}_{PL}(Vect(L(n))) = Vect(\text{words of degree } n).$$

In order to prove that $I = Ker(\phi_{PL})$, it is enough to prove that $\overline{\phi}_{PL}$ is injective. By homogeneity, it is enough to prove that $\overline{\phi}_{|Vect(L(n))}$ is injective for all $n \geq 1$. Hence, it is enough to prove that for all $n \geq 1$,

$$|L(n)| = dim(Vect(\text{words of degree } n)) = p_n,$$

where the p_n are the integers defined in proposition 8. Let $l_n = |L(n)|$ and q_n be the number of $t \in L(n)$ with no vertex decorated by 1. Then for all $n \geq 2$, $l_n = q_n + q_{n-1}$, and $l_1 = 1$. We put:

$$L = \sum_{n=1}^{\infty} l_n X^n, \; Q = \sum_{n=1}^{\infty} q_n X^n.$$

We obtain $P = X + Q + XQ$. Moreover:

$$Q = \frac{1}{1 - \sum\limits_{i \geq 2} X^i} - 1 = \frac{1}{1 - \frac{X^2}{1-X}} - 1 = \frac{X^2}{1 - X - X^2},$$

Finally:

$$L = \frac{X}{1 - X - X^2} = F.$$

So, for all $n \geq 1$, $|L(n)| = p_n$. $\Box$

As an immediate corollary, a basis of h is given by the classes of the elements of L. Turning to $\mathbb{K}\langle x_0, x_1 \rangle$, we obtain:

Corollary 31 *Let $w = a_1 \dots a_k$ be a word with letters in $\mathbb{N}^*$.*

1. *We put:*
$$m_w = x_1^{a_1 - 1} \bullet (x_1^{a_1 - 1} \bullet (\dots (x_1^{a_{k-1} - 1} \bullet x_1^{a_k}) \dots).$$
2. *We shall say that w is* admissible *if $a_1, \dots, a_{k-1} > 1$. The set of admissible words is denoted by $\mathcal{A}dm$.*

Then $(m_w)_{w \in \mathcal{A}dm}$ is a basis of $\mathbb{K}\langle x_0, x_1 \rangle$.

Remark. If w is not admissible, that is to say if there exists $1 \leq i < k$, such that $a_i = 1$, then $m_w = 0$ by proposition 28.

We extend the map $w \longrightarrow m_w$ by linearity.

4.2 Pre-Lie product in the basis of admissible words

Notations.

1. For all k, l, we denote by $Sh(k, l)$ the set of (k, l)- shuffles, that is to say $k+l$-permutations ζ such that $\zeta(1) < \ldots < \zeta(k)$, $\zeta(k+1) < \ldots < \zeta(k+l)$.
2. For all k, l we denote by $Sh_{\prec}(k, l)$ the set of (k, l)-shuffles ζ such that $\zeta^{-1}(k+l) = k$.
3. For all k, l we denote by $Sh_{\succ}(k, l)$ the set of (k, l)-shuffles ζ such that $\zeta^{-1}(k+l) = k+l$.
4. The symmetric group $\mathfrak{S}_n$ acts on the set of words with letters in $\mathbb{N}^*$ of length n by permutation of the letters:
$$\sigma.(a_1 \ldots a_n) = a_{\sigma^{-1}(1)} \ldots a_{\sigma^{-1}(n)}.$$

Proposition 32 *Let $\mathbb{K}\langle \mathbb{N}^* \rangle$ be the space generated by words with letters in $\mathbb{N}^*$. We define a dendriform structure on this space by:*

$$\begin{aligned}
(a_1 \ldots a_k) \prec (b_1 \ldots b_l) &= \sum_{\zeta \in Sh_{\prec}(k,l)} \zeta.a_1 \ldots a_k b_1 \ldots b_{k-1}(b_k+1) \\
(a_1 \ldots a_k) \succ (b_1 \ldots b_l) &= \sum_{\zeta \in Sh_{\succ}(k,l)} \zeta.a_1 \ldots a_{k-1}(a_k+1) b_1 \ldots b_k.
\end{aligned}$$

The associative product $\prec + \succ$ is denoted by $\star$.

Proof. We denote by $Sh(k, l, m)$ the set of $k+l+m$-permutations such that $\zeta(1) < \ldots < \zeta(k)$, $\zeta(k+1) < \ldots < \zeta(k+l)$, $\zeta(k+l+1) < \ldots \zeta(k+l+m)$. Then:

$$\begin{aligned}
& (a_1 \ldots a_k \prec b_1 \ldots b_l) \prec c_1 \ldots c_m = a_1 \ldots a_k \prec (b_1 \ldots b_l \star c_1 \ldots c_m) \\
= & \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k} \zeta.a_1 \ldots a_k b_1 \ldots (b_l+1) c_1 \ldots (c_m+1); \\
& (a_1 \ldots a_k \succ b_1 \ldots b_l) \prec c_1 \ldots c_m = a_1 \ldots a_k \succ (b_1 \ldots b_l \prec c_1 \ldots c_m) \\
= & \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k+l} \zeta.a_1 \ldots (a_k+1) b_1 \ldots b_l c_1 \ldots (c_m+1); \\
& (a_1 \ldots a_k \star b_1 \ldots b_l) \succ c_1 \ldots c_m = a_1 \ldots a_k \succ (b_1 \ldots b_l \succ c_1 \ldots c_m) \\
= & \sum_{\zeta \in Sh(k,l,m), \zeta^{-1}(k+l+m)=k+l+m} \zeta.a_1 \ldots (a_k+1) b_1 \ldots (b_l+1) c_1 \ldots c_m.
\end{aligned}$$

So $\mathbb{K}\langle\langle \mathbb{N}^* \rangle\rangle$ is a dendriform algebra. $\square$

We postpone the study of this dendriform algebra to section 5.2.

Notations. For all $a_1, \ldots, a_k \in \mathbb{N}^*$, we denote by $l(a_1 \ldots a_k) = B_{a_1} \circ \ldots \circ B_{a_k}(1)$ the ladder decorated from the root to the leaf by $a_1, \ldots, a_k$. Note that $m_{a_1 \ldots a_k} = \phi_{PL}(l(a_1 \ldots a_k))$.

Lemma 33 *Let $k, l \geq 1$ and let $a_1, \ldots, a_l, b_1, \ldots, b_l \in \mathbb{N}^*$. Then:*

$$\phi_{PL}(B_{a_1+1}(l(a_2 \ldots a_k) l(b_1 \ldots b_l)) + B_{b_1+1}(l(a_1 \ldots a_k) l(b_2 \ldots b_l))) = m_{a_1 \ldots a_k \star b_1 \ldots b_l}.$$

Proof. By induction on $k+l$. If $k = l = 1$, then:

$$\phi_{PL}(\mathfrak{l}^{b_1}_{a_1+1} + \mathfrak{l}^{a_1}_{b_1+1}) = m_{(a_1+1)b_1 + (b_1+1)a_1} = m_{a_1 \star b_1}.$$

Let us assume the result at all ranks $< k+l$. If $k=1$, then:

$$\begin{array}{rcl} & = & \phi_{PL}(B_{a_1+1}(l(b_2\ldots b_l))+B_{b_1+1}(l(a_1)l(b_2\ldots b_l))\\ & = & \phi_{PL}(\bullet_{a_1+1}\bullet l(b_2\ldots b_l)+\bullet_{b_1+1}\bullet(l(a_1)l(b_2\ldots b_l)))\\ & = & \phi_{PL}(l((a_1+1)b_2\ldots b_l))+\phi_{PL}(\bullet_{b_1}\bullet(l((a_1+1)b_2\ldots b_l)+\bullet_{b_2+1}\bullet(l(a_1)l(b_3\ldots b_l)))\\ & = & m_{(a_1+1)b_2\ldots b_l}+m_{b_1(a_1\star b_2\ldots b_l)}\\ & = & m_{(a_1+1)b_2\ldots b_l}+\displaystyle\sum_{i=1}^{l-1} m_{b_1\ldots b_i(a_1+1)\ldots b_l}+m_{b_1\ldots(b_l+1)a_1}\\ & = & m_{a_1\star b_1\ldots b_l}.\end{array}$$

If $l=1$, a similar computation, permuting the a_i's and the b_j's, proves the result. If $k,l>1$, then:

$$\begin{array}{rl} & \phi_{PL}(B_{a_1+1}(l(a_2\ldots a_k)l(b_1\ldots b_l))+B_{b_1+1}(l(a_1\ldots a_k)l(b_2\ldots b_l))\\ = & \phi_{PL}(\bullet_{a_1}\bullet(\bullet_{a_2+1}\bullet l(a_3\ldots a_k)l(b_1\ldots b_l))+\bullet_{b_1+1}\bullet l(a_1\ldots a_k)l(b_2\ldots b_l)))\\ & +\phi_{PL}(\bullet_{b_1}\bullet(\bullet_{a_1+1}\bullet l(a_2\ldots a_k)l(b_2\ldots b_l))+\bullet_{b_2+1}\bullet l(a_1\ldots a_k)l(b_3\ldots b_l)))\\ = & m_{a_1(a_2\ldots a_k\star b_1\ldots b_l)+b_1(a_1\ldots a_k\star b_2\ldots b_l)}\\ = & m_{a_1\ldots a_k\star b_1\ldots b_l}.\end{array}$$

Hence, the result holds for all $k,l\geq 1$. □

Theorem 34 *For all $a_1,\ldots,a_k,b_1,\ldots,b_l\in\mathbb{N}^*$:*

$$m_{a_1\ldots a_k}\bullet m_{b_1\ldots b_l}=\sum_{i=1}^{k-1} m_{a_1\ldots a_{i-1}(a_i-1)(a_{i+1}\ldots a_k\star b_1\ldots b_l)}+m_{a_1\ldots a_k b_1\ldots b_l}.$$

Proof. By definition of $m_{a_1b_1\ldots b_l}$, if $k=1$, $m_{a_1}\bullet m_{b_1\ldots b_l}=m_{a_1b_1\ldots b_l}$. So the result holds if $k=1$. Let us assume that $k\geq 2$. In $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, we have:

$$l(a_1\ldots a_k)\bullet l(b_1\ldots b_l)=\bullet_{a_1}\bullet(l(a_2\ldots a_k)\bullet l(b_1\ldots b_l))+\bullet_{a_1}\bullet l(a_2\ldots a_k)l(b_1\ldots b_l).$$

Applying ϕ_{PL}:

$$\begin{array}{rcl} m_{a_1\ldots a_k}\bullet m_{b_1\ldots b_l} & = & m_{a_1(a_2\ldots a_k)\bullet(b_1\ldots b_l)}\\ & & +\phi_{PL}(\bullet_{a_1-1}\bullet(\bullet_{a_2+1}l(a_3\ldots a_k)l(b_1\ldots b_l))+\bullet_{b_1+1}\bullet l(a_1\ldots a_k)l(b_2\ldots b_l)))\\ & = & m_{a_1(a_2\ldots a_k)\bullet(b_1\ldots b_l)}+m_{(a_1-1)(a_2\ldots a_k\star b_1\ldots b_l)},\end{array}$$

by the preceding lemma. The result follows from an easy induction. □

Remark. In particular, $m_1\circ m_{b_1\ldots b_l}=0$.

Corollary 35 *Let $a_1\ldots a_k,b_1\ldots b_l$ be two words with letters in $\mathbb{N}^*$. Then $m_{a_1\ldots a_k}\bullet m_{b_1\ldots b_l}$ is a span of m_w, where w is a word with $k+l$ letters and of weight $a_1+\ldots+a_k+b_1+\ldots+b_l$.*

Hence, $\mathbb{K}\langle x_0,x_1\rangle$ is a bigraded pre-Lie algebra, with:

$$\mathbb{K}\langle x_0,x_1\rangle_{n,k}=Vect(m_{a_1\ldots a_k}\mid a_1+\ldots+a_k=n).$$

We put:

$$G=\sum_{k,n\geq 0} dim(\mathbb{K}\langle x_0,x_1\rangle_{n,k})X^nY^k.$$

Proposition 36 $G = \dfrac{XY}{1-X-X^2Y} = \displaystyle\sum_{k=1}^{\infty}\sum_{l=2k-1}^{\infty}\binom{l-k}{k-1}X^lY^k.$

Proof. Note that $dim(\mathbb{K}\langle x_0,x_1\rangle_{n,k})$ is the number of words $a_1\ldots a_k$ of length k, such that $a_1,\ldots,a_{k-1}\geq 2$, and $a_1+\ldots+a_k=n$. Hence:

$$\begin{aligned}
G &= \sum_{k=1}^{\infty}\left(\frac{X^2Y}{1-X}\right)^{k-1}\frac{XY}{1-X}\\
&= \frac{XY}{1-X}\frac{1}{1-\frac{X^2Y}{1-X}}\\
&= \frac{XY}{1-X-X^2Y},\\
&= \sum_{k=1}^{\infty}\frac{X^{2k-1}Y^k}{(1-X)^n}\\
&= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\binom{k+n-1}{k-1}X^{2k+n-1}Y^k\\
&= \sum_{k=1}\sum_{l=2k-1}^{\infty}\binom{l-k}{k-1}X^lY^k.
\end{aligned}$$

□

4.3 An associative product on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$

We now define an associative product on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, in such a way that ϕ_{PL} becomes a morphism of Com-pre-Lie algebras.

Proposition 37 *We define a product* $⧢$ *on* $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ *by:*

$$B_p(s_1\ldots s_k)⧢B_q(t_1\ldots t_l)=\binom{p+q-k-l-2}{p-k-1}B_{p+q-1}(s_1\ldots s_kt_1\ldots t_l).$$

Then $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ *is a Com-pre-Lie algebra and* ϕ_{PL} *is a morphism of Com-pre-Lie algebras.*

Proof. As $\binom{p+q-k-l-2}{p-k-1}=\binom{p+q-k-l-2}{q-l-1}$, $⧢$ is commutative. Let $t=B_p(s_1\ldots s_k)$, $t'=B_q(\bullet t_1\ldots t_l)$ and $t''=B_r(u_1\ldots u_m)$. Then:

$$t⧢(t'⧢t'')=\underbrace{\binom{q+r-l-m-2}{q-l-1}\binom{p+q+r-k-l-m-3}{q+r-l-m-2}}_{A}B_{p+q+r-2}(s_1\ldots s_kt_1\ldots t_lu_1\ldots u_m),$$

$$(t⧢t')⧢t''=\underbrace{\binom{p+q-k-l-2}{p-k-1}\binom{p+q+r-k-l-m-3}{p+q-k-l-2}}_{B}B_{p+q+r-2}(s_1\ldots s_kt_1\ldots t_lu_1\ldots u_m).$$

If $p\leq k$ or $q\leq l$ or $r\leq m$, then $A=B=0$. If $p>k$ and $q>l$ and $r>m$, then:

$$A=B=\frac{(p+q+r-k-l-m-3)!}{(p-k-1)!(q-l-1)!(r-m-1)!}.$$

So $⧢$ is associative.

Let $t_1 = B_p(s_1 \dots s_k)$, $t_2 = B_q(t_1 \dots t_l)$ and $t \in \mathcal{T}(\mathbb{N}^*)$. Then:

$$\begin{aligned}
(t_1 ш t_2) \circ T &= \binom{p+q-k-l-2}{m-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t) \\
&\quad + \sum_{i=1}^{k} \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots (s_i \bullet t) \dots s_k t_1 \dots t_l) \\
&\quad + \sum_{j=1}^{l} \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots (t_j \bullet t) \dots t_l), \\
(t_1 \bullet t) ш t_2 &= \left(\sum_{i=1}^{k} B_p(s_1 \dots (s_i \bullet t) \dots s_k) + B_p(s_1 \dots s_k t) \right) ш t_2 \\
&= \sum_{i=1}^{k} \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots (s_i \bullet t) \dots s_k t_1 \dots t_l) \\
&\quad + \binom{p+q-k-l-3}{p-k-2} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t), \\
t_1 ш (t_2 \bullet t) &= t_1 ш \left(\sum_{j=1}^{l} B_q(t_1 \dots (t_j \bullet t) \dots t_l) + B_q(t_1 \dots t_j t) \right) \\
&= \sum_{j=1}^{l} \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots (t_j \bullet t) \dots t_l) \\
&\quad + \binom{p+q-k-l-3}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t).
\end{aligned}$$

As $\binom{p+q-k-l-3}{p-k-2} + \binom{p+q-k-l-3}{p-k-1} = \binom{p+q-k-l-2}{p-k-1}$, $(t_1 ш t_2) \bullet t = (t_1 \bullet t) ш t_2 + t_1 ш (t_2 \bullet t)$. So $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ is Com-pre-Lie.

Let $t_1 = B_p(s_1 \dots s_k)$ and $t_2 = B_q(t_1 \dots t_l)$. If $k \geq p$, then $\binom{p+q-k-l-2}{p-k-1} = 0$, so $t_1 ш t_2 = 0$. By proposition 28, $\phi_{PL}(t_1) = 0$, so $\phi_{PL}(t_1 ш t_2) = \phi_{PL}(t_1) ш \phi_{PL}(t_2) = 0$. Similarly, if $l \geq q$, $\phi_{PL}(t_1 ш t_2) = \phi_{PL}(t_1) ш \phi_{PL}(t_2) = 0$. If $k < p$ and $l < q$, we put $w_i = \phi_{PL}(s_i)$ and $w'_j = \phi_{PL}(t_j)$. Then:

$$\begin{aligned}
\phi_{PL}(t_1) ш \phi_{PL}(t_2) &= x_0 w_1 ш \dots ш x_0 w_k ш x_1^{p-1-k} ш x_0 w'_1 ш \dots ш x_0 w'_l ш x_1^{q-1-l} \\
&= \binom{p+q-k-l-2}{p-k-1} x_0 w_1 ш \dots x_0 w'_l ш x_1^{p+q-k-l-2} \\
&= \binom{p+q-k-l-2}{p-k-1} \phi_{PL}(B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l)) \\
&= \phi_{PL}(t_1 ш t_2).
\end{aligned}$$

So ϕ_{PL} is a Com-pre-Lie algebra morphism. $\square$

Remark. ψ is not compatible with $\sqcup\!\sqcup$. Indeed:

$$\begin{array}{rcl}
\psi(\mathfrak{l}^1_2) & = & \psi(\bullet_2)\bullet\psi(\bullet_1)\\
& = & \mathfrak{l}^1_2,\\
\psi(\mathfrak{l}^1_2)\sqcup\!\sqcup\psi(\mathfrak{l}^1_2) & = & \mathfrak{l}^1_2 \sqcup\!\sqcup \mathfrak{l}^1_2\\
& = & {}^1_2\mathbb{V}^1_2;
\end{array}$$

$$\begin{array}{rcl}
\mathfrak{l}^1_2 \sqcup\!\sqcup \mathfrak{l}^1_2 & = & {}^1\mathbb{V}_3{}^1,\\
\psi(\mathfrak{l}^1_2 \sqcup\!\sqcup \mathfrak{l}^1_2) & = & \psi(\bullet_3)\bullet\psi(\bullet_1)\psi(\bullet_1)\\
& = & \frac{1}{2}{}_2\!\bullet\!\!-\!\!\bullet_2 \bullet \bullet_1\bullet_1\\
& = & {}^1_2\mathfrak{l}\mathfrak{l}^1_2 + {}^1_2\mathbb{V}^1_2.
\end{array}$$

5 Appendix

5.1 Enumeration of partitioned trees

Let $d \geq 1$. For all $n \geq 1$, let f_n be the number of partitioned trees decorated by $\{1,\ldots,d\}$ with n vertices and let t_n be the number of partitioned trees decorated by $\{1,\ldots,d\}$ with n vertices and one root. By convention, $f_0 = 1$. We put:

$$T = \sum_{n=1}^{\infty} t_n X^n,\ F = \sum_{n=0}^{\infty} f_n X^n.$$

Let V_T be the vector space generated by the set of partitioned trees decorated by $\{1,,\ldots,d\}$ and V_F be the vector space generated by the set of partitioned trees decorated by $\{1,,\ldots,d\}$ with only one root. There is a bijection:

$$\left\{\begin{array}{rcl}
S(V_T) & \longrightarrow & V_F\\
t_1\ldots t_k & \longrightarrow & t_1\sqcup\!\sqcup\ldots\sqcup\!\sqcup t_k.
\end{array}\right.$$

Hence:

$$F = \prod_{i=1}^{\infty}\frac{1}{(1-X^k)^{t_k}}. \tag{2}$$

There is a bijection:

$$\left\{\begin{array}{rcl}
\displaystyle\bigoplus_{i=1}^{d} S(V_F) & \longrightarrow & V_T\\
(F_{1,1}\ldots,F_{1,k_1},\ldots,F_{d,1}\ldots F_{d,k_d}) & \longrightarrow & \displaystyle\sum_{i=1}^{d}\bullet_i\bullet(F_{i,1}\ldots F_{i,k_i}).
\end{array}\right.$$

This gives:

$$T = dX\prod_{i=1}^{\infty}\frac{1}{(1-X^k)^{f_{k-1}}}. \tag{3}$$

Formulas (2) and (3) allow to compute inductively f_k and t_k for all $k \geq 1$. This gives for example:

$$\left\{\begin{array}{rcl}
f_1 & = & d\\
f_2 & = & \dfrac{d(3d+1)}{2}\\
f_3 & = & \dfrac{d(19d^2+9d+2)}{6}\\
f_4 & = & \dfrac{d(63d^2+34d^2+13d+2)}{8}\\
f_5 & = & \dfrac{d(644d^4+400d^3+175d^2+35d+6)}{30}
\end{array}\right.$$

Here are examples of f_n for $d = 1$ or 2:

n	1	2	3	4	5	6	7	8	9	10
$d=1$	1	2	5	14	42	134	444	1518	5318	18989
$d=2$	2	7	32	167	952	5759	36340	236498	1576156	10702333

The row $d = 1$ is sequence $A035052$ of [14].

5.2 Study of the dendriform structure on admissible words

We here study the dendriform algebra $\mathbb{K}\langle\mathbb{N}^*\rangle$ of proposition 32. It is clearly commutative, via the bijection from $Sh_\prec(k,l)$ to $Sh_\succ(l,k)$ given by the composition (on the left) by the permutation $(l+1\ldots l+k\,1\ldots l)$.

Let V be a vector space. The shuffle dendriform algebra $Sh(V)$ is $T_+(V)$, with the produts given by:

$$\begin{array}{rcl}(a_1\ldots a_k)\prec(b_1\ldots b_l)&=&\displaystyle\sum_{\zeta\in Sh_\prec(k,l)}\zeta.a_1\ldots a_kb_1\ldots b_{k-1}b_k\\(a_1\ldots a_k)\succ(b_1\ldots b_l)&=&\displaystyle\sum_{\zeta\in Sh_\succ(k,l)}\zeta.a_1\ldots a_{k-1}a_kb_1\ldots b_k.\end{array}$$

Moreover, this is the free commutative dendriform algebra generated by V, that is to say if A is a commutative dendriform algebra and $f : V \longrightarrow A$ is any linear map, there exists a morphism of dendriform algebras $\phi : Sh(V) \longrightarrow A$ such that $\phi_{|}V = f$. As $a_1\ldots a_k \succ b = a_1\ldots a_kb$ in $Sh(V)$ for all $a_1,\ldots,a_k,b \in V$, this morphism ϕ is defined by:

$$\phi(a_1\ldots a_k) = (\ldots(a_1 \succ a_2) \succ a_3)\ldots) \succ a_k.$$

Proposition 38 *1. Let V be the space generated by the words 1^ki, $k \in \mathbb{N}$, $i \geq 1$. Then $K\langle\mathbb{N}^*\rangle$ is isomorphic, as a dendriform algebra, to $Sh(V)$.*

2. Let A be the subspace of $K\langle\mathbb{N}^*\rangle$ generated by admissible words. Then it is a dendriform subalgebra of $K\langle\mathbb{N}^*\rangle$. Moreover, if W is the space generated by the letters i, $i \geq 1$, then A is isomorphic, as a dendriform algebra, to $Sh(W)$.

Proof. Let $w = a_1\ldots a_k$ be a word with letters in $\mathbb{N}^*$. We denote by $o(w)$ the sequence of indices $j \in \{1,\ldots,k-1\}$ such that $a_j \neq 1$. This sequences are totally ordered in this way: $(j_1,\ldots,j_k) < (j'_1,\ldots,j'_l)$ if there exists a p such that $j_k = j'_l$, $j_{k-1} = j'_{l-1}$, …, $j_{k-p+1} = j'_{l-p+1}$, $j_{k-p} < j'_{l-p}$, with the convention $j_0 = j_{-1} = \ldots = j'_0 = j'_{-1} = \ldots = 0$.

Let $\phi : Sh(V) \longrightarrow K\langle\mathbb{N}^*\rangle$ be the unique morphism of dendriform algebras which extends the identity of V. Then:

$$\begin{array}{rcl}\phi((1^{k_1-1}a_1)\ldots(1^{k_n-1}a_n))&=&=1^{k_1-1}(a_1+1)\ldots 1^{k_{n-1}-1}(a_{n-1}+1)1^{k_n-1}a_n\\&&+\text{words } w' \text{ such that } o(w') > (k_1,\ldots,k_{n-1}).\end{array}$$

By thriangularity, ϕ is an isomorphism. Moreover, for all $a_1,\ldots,a_n \geq 1$:

$$\phi(a_1\ldots a_n) = (a_1+1)\ldots(a_{n-1}+1)a_n.$$

Consequently, $\phi(Sh(W)) = A$, so A is a dendriform subalgebra of $K\langle\mathbb{N}^*\rangle$ and is isomorphic to $Sh(W)$. □

5.3 Freeness of the pre-Lie algebra $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$

Notations. Let $k \geq 1$, $d_1, \ldots, d_k \in \mathcal{D}$ and let $F_1, \ldots, F_k$ be decorated partitioned forests. We put:

$$B_{d_1,\ldots,d_k}(F_1, \ldots F_k) = (\bullet_{d_1} \bullet F_1) ш \ldots ш (\bullet_{d_k} \bullet F_k).$$

Note that any partitioned tree can be written under the form $B_{d_1,\ldots,d_k}(F_1, \ldots F_k)$. This writing is unique up to a permutation of the d_i's and the F_i's.

Proposition 39 *We define a coproduct δ on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ in the following way: for any decorated partitioned tree $t = B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{k,1} \ldots t_{k,n_k})$,*

$$\delta(t) = \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{n_i} B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,j-1} t_{i,j+1} \ldots t_{i,n_i}, \ldots, t_{k,1} \ldots t_{k,n_k}) \otimes t_{i,j}.$$

1. *For all $x \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $(\delta \otimes Id) \circ \delta(x) = (23)(\delta \otimes Id) \circ \delta(x)$.*
2. *For all $x, y \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $\delta(x \bullet y) = x \otimes y + \delta(x) \bullet y$.*

Proof. 1. Let $t = B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{k,1} \ldots t_{k,n_k})$. For all i, j, we put:

$$t/t_{i,j} = B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,j-1} t_{i,j+1} \ldots t_{i,n_i}, \ldots, t_{k,1} \ldots t_{k,n_k}).$$

Then:

$$\delta(t) = \frac{1}{k} \sum_{i,j} t/t_{i,j} \otimes t_{i,j}.$$

Hence:

$$(\delta \otimes Id) \circ \delta(t) = \sum_{(i,j) \neq (i',j')} (t/t_{i,j})/t_{i',j'} \otimes t_{i',j'} \otimes t_{i,j}$$

As $(t/t_{i,j})/t_{i',j'}$ and $(t/t_{i',j'})/t_{i,j}$ are both the partitioned tree obtained by cutting $t_{i,j}$ and $t_{i',j'}$ in t, they are equal, so $(\delta \otimes Id) \circ \delta(t)$ is invariant under the action of (23).

2. Let t' be a decorated partitioned tree.

$$\begin{aligned}
\delta(t \bullet t') &= \sum_{i=1}^{k} \delta(B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,n_i} t', \ldots, t_{k,1} \ldots t_{k,n_k})) \\
&\quad + \sum_{i,j} \delta(B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,j} \bullet t' \ldots t_{i,n_i}, \ldots, t_{k,1} \ldots t_{k,n_k})) \\
&= \frac{1}{k} kt \otimes t' + \frac{1}{k} \sum_{i} \sum_{i',j'} B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,n_i} t', \ldots, t_{k,1} \ldots t_{k,n_k})/t_{i',j'} \otimes t_{i',j'} \\
&\quad + \frac{1}{k} \sum_{(i,j) \neq (i',j')} B_{d_1,\ldots,d_k}(t_{1,1} \ldots t_{1,n_1}, \ldots, t_{i,1} \ldots t_{i,j} \bullet t' \ldots t_{i,n_i}, \ldots, t_{k,1} \ldots t_{k,n_k})/t_{i',j'} \otimes t_{i',j'} \\
&\quad + \frac{1}{k} \sum_{i,j} t/t_{i,k} \otimes t_{i,j} \bullet t' \\
&= t \otimes t' + \sum t^{(1)} \otimes t^{(2)} \bullet t' + \sum t^{(1)} \otimes t^{(2)} \bullet t'.
\end{aligned}$$

So $\delta(t \bullet t') = t \otimes t' + \delta(t) \bullet t'$. $\square$

By Muriel Livernet's pre-Lie rigidity theorem [7]:

Corollary 40 *The pre-Lie algebra $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is freely generated by $Ker(\delta)$.*

Remarks.

1. It is not difficult to prove that for any $x, y \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$:

$$\delta(x \sqcup\!\sqcup y) = \sum x^{(1)} \otimes x^{(2)} \sqcup\!\sqcup y + \sum y^{(1)} \otimes x \sqcup\!\sqcup y^{(2)}.$$

   Hence, $Ker(\delta)$ is an algebra for the product $\sqcup\!\sqcup$.

2. Here are elements of $Ker(\delta)$ in the non decorated case. Let t_1, t_2, t_3, t_4 be partitioned trees.

$$\begin{array}{rcl} X & = & B(t_1t_2, 1) - B(t_1, t_2), \\ Y & = & B(t_1t_2t_3, 1, 1) - B(t_1t_2, t_3, 1) - B(t_1t_3, t_2, 1) - B(t_2t_3, t_1, 1) + 2B(t_1, t_2, t_3), \\ Z & = & B(t_1t_2t_3t_4, 1) - B(t_1t_2t_3, t_4) - B(t_1t_2t_4, t_3) - B(t_1t_3t_4, t_2) - B(t_2t_3t_4, t_1) \\ & & +B(t_1t_2, t_3t_4) + B(t_1t_3, t_2t_4) + b(t_1t_4, t_2t_3), \\ T & = & B(t_1t_2, t_3t_4, 1, 1) + B(t_1t_3, t_2t_4, 1, 1) + B(t_1t_4, t_2t_3, 1, 1) - B(t_1t_2, t_3, t_4, 1) \\ & & -B(t_1t_3, t_2, t_4, 1) - B(t_1t_4, t_2, t_3, 1) - B(t_2t_3, t_1, t_4, 1) - B(t_2t_4, t_1, t_3, 1) \\ & & -B(t_3t_4, t_1, t_2, 1) + 3B(t_1, t_2, t_3, t_4). \end{array}$$

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