The irregular chromatic index of trees

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Abstract. A graph G is locally irregular if adjacent vertices of G have distinct degrees. An edge colouring of G is locally irregular if each of its colours induces a locally irregular subgraph of G. The irregular chromatic index of G refers to the least number of colours used by a locally irregular edge colouring of G (if any). We propose a linear-time algorithm for determining the irregular chromatic index of any tree.

1 Introduction

Let G be a graph. We say that G is *locally irregular* if adjacent vertices of G have distinct degrees. A k-edge colouring $\phi : E(G) \rightarrow \{1, ..., k\}$ of G is said to be *locally irregular* if each colour of ϕ induces a locally irregular subgraph of G. The least number of colours used by a locally irregular edge colouring of G (if any) is the *irregular chromatic index of* G, denoted $\chi'_{irr}(G)$.

Locally irregular edge colourings were introduced to deal with the famous 1-2-3 and Detection Conjectures. Assume that ϕ is a k-edge colouring of G taking values in $\{1, ..., k\}$. Each vertex v of G may then be assigned a colour $s_\phi(v)$ which is the sum of the colours assigned to the edges incident to v. If the resulting vertex colouring of G is proper, i.e. we have $s_\phi(u) \neq s_\phi(v)$ for every pair $\{u, v\}$ of adjacent vertices in G, then we say that ϕ is *neighbour sum distinguishing*. It was conjectured by Karoński, Łuczak and Thomason that every graph with no isolated edges admits a neighbour sum distinguishing edge colouring using 3 colours [7].

1-2-3 Conjecture. *All graphs without isolated edges admit a neighbour sum distinguishing 3-edge colouring.*

A list version of the 1-2-3 Conjecture was also raised in [2]. In the paper that introduced the 1-2-3 Conjecture, the authors considered another version of the problem where adjacent vertices are distinguished by the multiset of their incident colours rather than by their sum. This led to the following definition. Let $m_\phi(v)$ denote the multiset of colours of the edges incident to v. We say that ϕ is *neighbour multiset distinguishing* if $m_\phi(u) \neq m_\phi(v)$ for every pair $\{u, v\}$ of adjacent vertices in G. Neighbour multiset distinguishing edge colourings (also called *detectable colourings* in the literature) were mainly considered by Addario-Berry et al. in [1]. In particular, they showed that all graphs without isolated edges admit a neighbour multiset distinguishing 4-edge colouring and that only 3 colours suffice for a wide number of graphs. These results agree with the following conjecture.

Detection Conjecture. *All graphs without isolated edges admit a neighbour multiset distinguishing 3-edge colouring.*

Similarly as for neighbour sum and neighbour multiset distinguishing edge colourings, there exist graphs that do not admit any locally irregular edge colouring. We say that these graphs are *non-colourable* (with respect to locally irregular edge colourings). Connected non-colourable graphs are odd length paths, odd length cycles, and some special tree-like graphs with maximum degree at most 3 obtained by connecting an arbitrary number of triangles in a specific way [3]. These exceptions apart, it is believed that graphs have a small irregular chromatic index. The following conjecture was thus raised in [3].

Local Irregularity Conjecture. *Colourable graphs have irregular chromatic index at most 3.*

Clearly, a locally irregular edge colouring is also neighbour multiset distinguishing. Thus, if the Local Irregularity Conjecture were true, then the Detection Conjecture would also be true for colourable graphs. However, there does not seem to be any systematic relationship between locally irregular and neighbour sum distinguishing edge colourings. Please refer to [8] for an in-depth review on the background of the problem of distinguishing adjacent vertices of a graph thanks to an edge colouring.

A negative result exhibited in [4] states that even if the Local Irregularity Conjecture turned out to be true, i.e. even if the irregular chromatic index of colourable graphs only took value in $\{1, 2, 3\}$, it would remain difficult, in general, to determine the irregular chromatic index of a given graph. This result meets similar complexity results on the problems of determining the least number of colours used by a neighbour sum or neighbour multiset distinguishing edge colouring of a graph (see [5] and [6], respectively). Unfortunately, this complexity result is quite general as it is not restricted to any particular class of graphs. In particular, we still do not know whether it is hard, in general, to determine the irregular chromatic index of a bipartite graph. On the other hand, some results on the irregular chromatic index of trees are known. In particular, it was proved, as mentioned above, that odd length paths are the only non-colourable trees, and that colourable trees do not refute the Local Irregularity Conjecture. Examples of trees with irregular chromatic index 3 were also exhibited in [3]

Colourable Trees Theorem. *Colourable trees have irregular chromatic index at most 3.*

In this work, we propose a linear-time algorithm for determining the irregular chromatic index of any tree. A motivation for focusing on trees is that a lot of graphs are known to be decomposable into a few edge-disjoint forests (corresponding to the notion of *arboricity* of graphs), and that this property could lead to natural upper bounds on the irregular chromatic index of such graphs. Suppose indeed that G can be decomposed into a edge-disjoint forests with no odd length paths. Then, since the irregular chromatic index of a tree is at most 3, one can use a new set of 3 colours to colour independently the edges of each of the a forests in a locally irregular way. The a resulting locally irregular 3-edge colourings then perform a locally irregular $3a$-edge colouring of G. We could even expect to use less colours if we knew sufficient conditions for a tree to have irregular chromatic index at most 2.

This paper is organized as follows. We first provide, in Section 2, some terminology and notation that are used in the next sections. We then introduce in Section 3 an algorithm for constructing 'almost' locally irregular 2-edge colourings of a special class of trees called *shrubs*. In Section 4, we show how to get a locally irregular 2-edge colouring of a tree with maximum degree at least 5 by first decomposing it into shrubs and then unifying almost locally irregular 2-edge colourings of these shrubs. We then exhibit, in Section 5, the conditions under which our colouring strategy does not lead to a locally irregular 2-edge colouring when applied to a tree with maximum degree 3 or 4. All these results yield a linear-time algorithm for determining the irregular chromatic index of any tree in concluding Section 6.

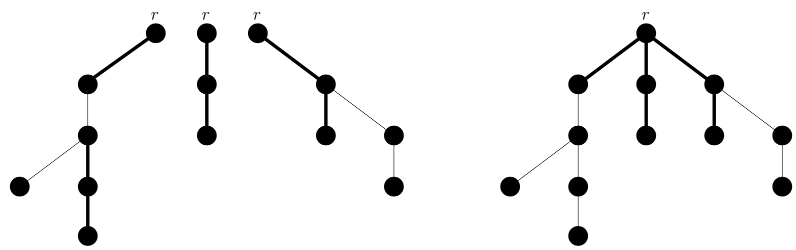
2 Definitions and terminology

By choosing a particular node r of a tree T as the *root* of T, one naturally defines an orientation of T from its root to its leaves. The resulting *rooted tree* is denoted T_r. According to the orientation of T_r, a node u has at most one neighbour, denoted u^-, which is nearer from r than u. This node, if it exists, is referred to as the *father* of u in T_r. In contrast, the other neighbours of u are the *children* of u in T_r. Clearly, the root r has no father and the leaves of T_r have no children. In the special case where u has only one child in T_r, we denote by u^+ this node. We say that T_r is a *shrub* if r^+ is defined, i.e. if r has only one child.

Assuming that u has $p \geq 1$ children $v_1, ..., v_p$ in T_r, for every $i \in \{1, ..., p\}$ we denote by $T_r[u, i]$ the subtree of T_r induced by u and the nodes in the subtree of T_r rooted at v_i. Notice that every such $T_r[u, i]$ is a shrub, with $v_i = u^+$ and $u = v_i^-$. Clearly, T_r is isomorphic to the tree obtained by identifying the roots of the shrubs $T_r[r, 1], ..., T_r[r, d(r)]$.

In the next sections, a locally irregular k-edge colouring is also called a *k-liec* for short. Assuming that a liec ϕ of T_r uses colour a, the *a-subgraph* of T_r refers to the subgraph of T_r induced by the edges with colour a. If u is a node of T_r, then the *a-degree* of u is the degree of u in the a-subgraph of T_r.

Suppose now that T_r is a shrub and that $\phi : E(T_r) \to \{a, b\}$ is a 2-edge colouring of T_r. To make the colour of the edge rr^+ by ϕ explicit, we also denote ϕ by $\phi_{a,b}$ when $\phi(rr^+) = a$, or $\phi_{b,a}$ when $\phi(rr^+) = b$. If $\phi_{a,b}$ is a 2-edge colouring of T_r using colours a and b, and $\{c, d\}$ is a pair of distinct colours, then we can



(a) 2-aliec $\phi^1_{a,b}$, $\phi^2_{a,b}$ and $\phi^3_{a,b}$ of $T_r[r,1]$, $T_r[r,2]$ and $T_r[r,3]$, respectively

(b) A 2-liec $\phi_{a,b}$ of T_r

Fig. 1. Examples of rooted trees, shrubs, 2-liec and 2-aliec. Thick (resp. thin) edges represent a-coloured (resp. b-coloured) edges.

obtain a 2-edge colouring $\phi_{c,d}$ of T_r using colours c and d by *swapping* $\{a,b\}$ and $\{c,d\}$: $\phi_{c,d}(uv) = c$ if $\phi_{a,b}(uv) = a$, or $\phi_{c,d}(uv) = d$ otherwise. A swapping of $\phi_{a,b}$ to $\phi_{b,a}$ is called an *inversion*. Clearly, a node with a-degree p in T_r by $\phi_{a,b}$ has b-degree p by $\phi_{b,a}$. We further say that $\phi_{a,b}$ is 'almost' a 2-liec of T_r (*2-aliec* for short) if either $\phi_{a,b}$ is a 2-liec of T_r, or rr^+ is isolated in the a-subgraph and $\phi_{a,b}$ is a 2-liec of $T_r[r,1]$.

Consider finally a tree T whose edge set $E(T)$ is partitioned into p parts $E_1, ..., E_p$, and let $\phi^1_{a,b}, ..., \phi^p_{a,b}$ be 2-edge colourings of $E_1, ..., E_p$, respectively. The *union* $\phi_{a,b} = \phi^1_{a,b} + ... + \phi^p_{a,b}$ of $\phi^1_{a,b}, ..., \phi^p_{a,b}$ is defined by $\phi_{a,b}(uv) = \phi^i_{a,b}(uv)$ if and only if $uv \in E_i$.

Figure 1 depicts how a 2-liec $\phi_{a,b}$ of a tree T_r with $d(r) = 3$ can be obtained by first decomposing it into shrubs $T_r[r,1]$, $T_r[r,2]$ and $T_r[r,3]$, then computing 2-aliec $\phi^1_{a,b}$, $\phi^2_{a,b}$ and $\phi^3_{a,b}$ of these shrubs, and finally considering the union $\phi^1_{a,b} + \phi^2_{a,b} + \phi^3_{a,b}$ as $\phi_{a,b}$.

3 Constructing 2-aliec of shrubs

Algorithm 1 constructs a 2-aliec $\phi_{a,b}$ of any shrub T_r. In this algorithm, $p \geq 0$ denotes the number of children of r^+. Roughly speaking, the algorithm first inductively constructs 2-aliec $\phi^1_{a,b}, ..., \phi^p_{a,b}$ of $T_r[r^+,1], ..., T_r[r^+,p]$, respectively. It then inverts some of the $\phi^i_{a,b}$'s so that their union is a 2-aliec of T_r when rr^+ is coloured a.

The keystone of Algorithm 1 is Line 7. Let us prove that the 2-aliec $\phi_{a,b}$ of T_r, obtained by inverting some of the $\phi^i_{a,b}$'s, necessarily exists.

Lemma 1. *The 2-aliec $\phi_{a,b}$ of T_r claimed at Line 7 necessarily exists.*

Proof. If $p = 0$, then there is nothing to prove. Thus, r^+ has $p \geq 1$ children $v_1, ..., v_p$ in T_r. We first consider small values of p, i.e. $p \in \{1,2,3\}$, before generalizing our arguments.

```
if p = 0 then
    φ_{a,b}(rr⁺) = a;
else
    foreach i ∈ {1, ..., p} do
        compute a 2-aliec φ^i_{a,b} of T_r[r⁺, i] inductively;
    φ^0_{a,b}(rr⁺) = a;
    choose φ^i_{c_i,c'_i} = φ^i_{a,b} or φ^i_{b,a} for every i ∈ {1, ..., p} so that
    φ_{a,b} = φ^0_{a,b} + φ^1_{c_1,c'_1} + φ^2_{c_2,c'_2} + ... + φ^p_{c_p,c'_p} is a 2-aliec of T_r;
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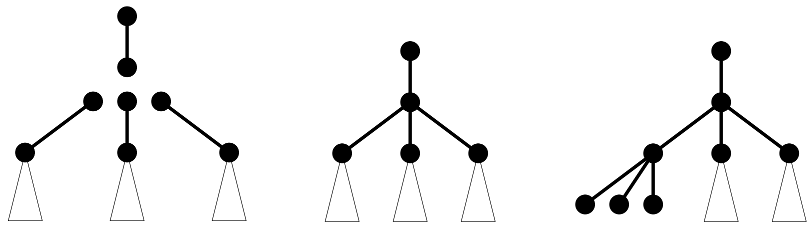
Algorithm 1: Algorithm for constructing 2-aliec $\phi_{a,b}$ of a shrub T_r

- Suppose $p = 1$. If $\phi_{a,b} = \phi^0_{a,b} + \phi^1_{a,b}$ is not a 2-aliec of T_r, then v_1 has a-degree 2 in $T_r[r^+, 1]$ by $\phi^1_{a,b}$. Besides, $\phi^1_{a,b}$ is a 2-liec of $T_r[r^+, 1]$. The colouring $\phi_{a,b} = \phi^0_{a,b} + \phi^1_{b,a}$, obtained by inverting $\phi^1_{a,b}$, is thus clearly a 2-aliec of T_r.
- Suppose $p = 2$. If $\phi_{a,b} = \phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{a,b}$ is not a 2-aliec of T_r, then a child of r^+, say v_1, has a-degree 3 in $T_r[r^+, 1]$ by $\phi^1_{a,b}$, and $\phi^1_{a,b}$ is a 2-liec of $T_r[r^+, 1]$. Now consider $\phi_{a,b} = \phi^0_{a,b} + \phi^1_{b,a} + \phi^2_{a,b}$. If $\phi_{a,b}$ is not a 2-aliec of T_r, then the other child v_2 of r^+ has a-degree 2 in $T_r[r^+, 2]$ by $\phi^2_{a,b}$. Moreover, $\phi^2_{a,b}$ is a 2-liec of $T_r[r^+, 2]$. Thus, $\phi_{a,b} = \phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{b,a}$ is a 2-aliec of T_r.
- Suppose $p = 3$. If $\phi_{a,b} = \phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{a,b} + \phi^3_{a,b}$ is not a 2-aliec of T_r, then a child of r^+, say v_1, has a-degree 4 in $T_r[r^+, 1]$ by $\phi^1_{a,b}$, and $\phi^1_{a,b}$ is a 2-liec of $T_r[r^+, 1]$. Now, if $\phi^0_{a,b} + \phi^1_{b,a} + \phi^2_{a,b} + \phi^3_{a,b}$ is not a 2-aliec of T_r, then another child of r^+, say v_2, has a-degree 3 in $T_r[r^+, 2]$ by $\phi^2_{a,b}$, and $\phi^2_{a,b}$ is a 2-liec of $T_r[r^+, 2]$. Again, the a-degree of the last child v_3 of r^+ in $T_r[r^+, 3]$ by $\phi^3_{a,b}$ is 3 if $\phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{b,a} + \phi^3_{a,b}$ is not a 2-aliec of T_r. Under all these assumptions, we clearly get that $\phi_{a,b} = \phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{b,a} + \phi^3_{b,a}$ is a 2-aliec of T_r.

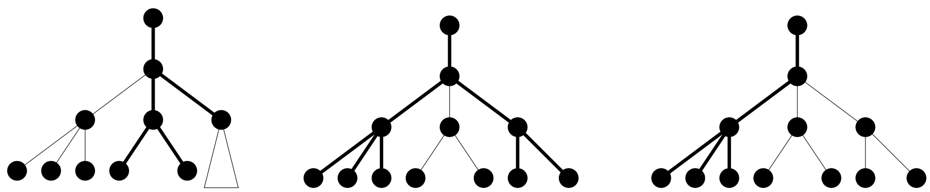
By following the same scheme for $p \geq 4$, i.e. by inverting none of the $\phi^i_{a,b}$'s, then one, two, three, ..., of them, we either find a 2-aliec $\phi_{a,b}$ of T_r or find out what are all of the a-degrees of $v_1, ..., v_p$ in $T_r[r^+, 1], ..., T_r[r^+, p]$ by $\phi^1_{a,b}, ..., \phi^p_{a,b}$, respectively. More precisely, in this last situation, we get that one of these a-degrees is equal to $p + 1$, two of them are equal to p, three of them are equal to $p - 1$ (unless p is not big enough), and so on. Under the assumption that $p \geq 4$, notice that the biggest $\lfloor \frac{p+1}{2} \rfloor$ values of the resulting a-degree sequence are strictly greater than $\lfloor \frac{p+1}{2} \rfloor + 1$, while its other values are strictly greater than $\lceil \frac{p+1}{2} \rceil - 1$. Considering that the a-degrees of $v_1, ..., v_p$ are ordered decreasingly, i.e. v_1 has a-degree $p + 1$, v_2 has a-degree p, ..., the 2-edge colouring $\phi_{a,b} = \phi^0_{a,b} + \phi^1_{a,b} + ... + \phi^{\lfloor \frac{p+1}{2} \rfloor}_{a,b} + \phi^{\lfloor \frac{p+1}{2} \rfloor + 1}_{b,a} + ... + \phi^p_{b,a}$, obtained by inverting the last $(\lceil \frac{p+1}{2} \rceil - 1)$ 2-aliec, is a 2-aliec of T_r since r^+ thus has a- and b-degree $\lfloor \frac{p+1}{2} \rfloor + 1$ and $\lceil \frac{p+1}{2} \rceil - 1$, respectively. □



(a) 2-aliec $\phi^1_{a,b}$, $\phi^2_{a,b}$ and $\phi^3_{a,b}$ of $T_r[r^+,1]$, $T_r[r^+,2]$ and $T_r[r^+,3]$ are computed, and $\phi^0_{a,b}(rr^+) = a$

(b) The 2-edge colouring $\phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{a,b} + \phi^3_{a,b}$

(c) If $\phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{a,b} + \phi^3_{a,b}$ is not a 2-aliec, then v_1 has a-degree 4 in $T_r[r^+,1]$ by $\phi^1_{a,b}$



(d) If $\phi^0_{a,b} + \phi^1_{b,a} + \phi^2_{a,b} + \phi^3_{a,b}$ is not a 2-aliec, then v_2 has a-degree 3 in $T_r[r^+,2]$ by $\phi^2_{a,b}$

(e) If $\phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{b,a} + \phi^3_{a,b}$ is not a 2-aliec, then v_3 has a-degree 3 in $T_r[r^+,3]$ by $\phi^3_{a,b}$

(f) $\phi^0_{a,b} + \phi^1_{a,b} + \phi^2_{b,a} + \phi^3_{b,a}$ is a 2-aliec

Fig. 2. Application of Algorithm 1 on a shrub T_r such that r^+ has 3 children

Figure 2 shows an application of Algorithm 1 on a shrub. Using Algorithm 1, and thanks to Lemma 1, we get:

Theorem 2. *Every shrub admits a 2-aliec.*

4 From shrubs to trees

Consider the following procedure based on Algorithm 1 for possibly computing a 2-liec of any colourable tree T. Let r be a node of T with degree $p \geq 1$. Start by decomposing T_r into the p shrubs $T_r[r,1], ..., T_r[r,p]$ and, then, compute 2-aliec $\phi^1_{a,b}, ..., \phi^p_{a,b}$ of $T_r[r,1], ..., T_r[r,p]$, respectively. These necessarily exist according to Theorem 2. Finally, invert some of the $\phi^i_{a,b}$'s so that their union is a 2-liec of T_r.

The success of this colouring procedure is not guaranteed since, in special cases, inverting the $\phi^i_{a,b}$'s in every possible way does not lead to a 2-liec of T_r. However, the more children r has, the more possible ways for inverting the $\phi^i_{a,b}$'s there are. Hence, the choice of r for rooting T before applying the colouring procedure above is crucial. Because the number of possibilities for inverting the

$\phi^i_{a,b}$'s grows exponentially in front of $d(r)$, this strategy actually leads to a 2-liec of T_r whenever $d(r) \geq 5$.

Theorem 3. *If $\Delta(T) \geq 5$, then $\chi'_{irr}(T) \leq 2$.*

Proof. Let r be a node of T with $p \geq 5$ neighbours $v_1, ..., v_p$. Let $\phi^1_{a,b}, ..., \phi^p_{a,b}$ be 2-aliec of $T_r[r,1], ..., T_r[r,p]$, respectively, which necessarily exist according to Theorem 2. Consider successively the 2-edge colourings $\phi_{a,b}$ of T_r obtained by inverting none, one, two, ..., of the $\phi^i_{a,b}$'s. If, at some step, $\phi_{a,b}$ is a 2-liec, then the claim is true for T. Otherwise, at each step, a conflict arises because, for at least one of the children v_i, the a-degree of v_i in $T_r[r,i]$ by $\phi^i_{a,b}$ is equal to the a-degree of r by $\phi_{a,b}$. In particular, if the 2-edge colouring obtained by inverting none of the $\phi^i_{a,b}$'s is not a 2-liec of T_r, then we reveal that one of the v_i's has a-degree p. Similarly, if the 2-edge colourings obtained by inverting one of the $\phi^i_{a,b}$'s are not 2-liec of T_r, then we reveal that two of the v_i's have a-degree $p-1$. If the 2-edge colourings obtained by inverting two of the $\phi^i_{a,b}$'s are not 2-liec of T_r, then we reveal that three of the v_i's have a-degree $p-2$. And so on. We stop the procedure once all of the a-degrees have been revealed.

Once the procedure has stopped, we get that the a-degree sequence is $(p, p-1, p-1, p-2, p-2, p-2, ...)$, where the element $p-k$ appears exactly $k+1$ times, except maybe in the case where $p-k$ is the last value of the sequence. When $p \geq 5$, each of the a-degrees is strictly greater than $\lfloor \frac{p}{2} \rfloor$. Hence, if the a-degrees of $v_1, ..., v_p$ are ordered decreasingly, then $\phi_{a,b} = \phi^1_{a,b} + ... + \phi^{\lceil \frac{p}{2} \rceil}_{a,b} + \phi^{\lceil \frac{p}{2} \rceil + 1}_{b,a} + ... + \phi^p_{b,a}$, obtained by inverting the last $\lfloor \frac{p}{2} \rfloor$ $\phi^i_{a,b}$'s, is a 2-liec of T_r since the a- and b-degrees of r are then $\lceil \frac{p}{2} \rceil$ and $\lfloor \frac{p}{2} \rfloor$, respectively, which are strictly less than the a- and b-degree of its neighbours in the a- and b-subgraphs, respectively. $\square$

Thanks to Theorems 2 and 3, we can now easily give an alternate proof of the Colourable Trees Theorem (Section 1).

Proof (Colourable Trees Theorem). Let T be a colourable tree. If we have $\Delta(T) \leq 2$, then T is a path with even length and $\chi'_{irr}(T) \leq 2$. If $\Delta(T) \geq 5$, then $\chi'_{irr}(T) \leq 2$ according to Theorem 3. Let us thus suppose that $\Delta(T) \in \{3, 4\}$, and let r be a node of T with degree $p = \Delta(T)$ whose neighbours are denoted by $v_1, ..., v_p$. As in the proof of Theorem 3, let $\phi^1_{a,b}, ..., \phi^p_{a,b}$ be 2-aliec of $T_r[r,1], ..., T_r[r,p]$ (these 2-aliec necessarily exist by Theorem 2), respectively, and try out the inversion procedure. If no 2-liec $\phi_{a,b}$ of T_r can be found, then the revealed a-degree sequence is necessarily $(3,2,2)$ when $p=3$, or $(4,3,3,2)$ when $p=4$. Assuming that the a-degrees of $v_1, ..., v_p$ are ordered decreasingly, then $\phi_{a,b} = \phi^1_{a,b} + \phi^2_{b,a} + \phi^3_{c,a}$, where c is a third colour, is a 3-liec of T_r for $p=3$ since r thus has a-, b- and c-degree 1 while its neighbours have degree 3, 2, and 2 in the a-, b- and c-subgraph, respectively. When $p=4$, a 3-liec of T_r is, for example, $\phi_{a,b} = \phi^1_{a,b} + \phi^2_{b,a} + \phi^3_{b,a} + \phi^4_{c,a}$ since r then has a-, b- and c-degree 1, 2, and 1, respectively, while its neighbours have a-, b- and c-degree 4, 3 and 2, respectively. $\square$

Signature $\mathbf{D_1, D_2}$	**Resulting $\mathbf{D_0}$**	**Signature $\mathbf{D_1, D_2}$**	**Resulting $\mathbf{D_0}$**
$\{1\},\{1\}$	$\{1,3\}$	$\{\mathbf{2}\},\{\mathbf{3}\}$	$\{\mathbf{2}\}$
$\{1\},\{2\}$	$\{2,3\}$	$\{2\},\{4\}$	$\{2,3\}$
$\{1\},\{3\}$	$\{1,2\}$	$\{3\},\{3\}$	$\{1,2\}$
$\{1\},\{4\}$	$\{1,2,3\}$	$\{3\},\{4\}$	$\{1,2\}$
$\{\mathbf{2}\},\{\mathbf{2}\}$	$\{\mathbf{3}\}$	$\{4\},\{4\}$	$\{1,2,3\}$

Table 1. All possible canonical signatures of T_r and resulting D_0 when $p = 2$

5 Trees with irregular chromatic index 3

We now turn our concern to trees with maximum degree at most 4. In our proof of the Colourable Trees Theorem, we have pointed out that the colouring procedure presented in Section 4 does not always provide a 2-liec of a tree T_r. This typically occurs when the inversion procedure of the $\phi^i_{a,b}$'s fails, i.e. every possible inversion of some of the $\phi^i_{a,b}$'s is not a 2-liec. A simple computation shows that the inversion procedure fails if and only if the a-degree sequence of the v_i's in the $T_r[r,i]$'s by the $\phi^i_{a,b}$'s is *bad*, i.e. is (1), (2, 1), (3, 2, 2) or (4, 3, 3, 2) when $p = d(r)$ is 1, 2, 3 or 4, respectively.

Consequently, if there exist 2-aliec $\psi^1_{a,b}, ..., \psi^p_{a,b}$ of $T_r[r,1], ..., T_r[r,p]$, respectively, leading to a a-degree sequence which is not bad, then inverting some of the $\psi^i_{a,b}$'s necessarily leads to a 2-liec of T_r. We thus now focus on the structure of shrubs T_r with maximum degree at most 4 such that r^+ has the same a-degree by all of the possible 2-aliec of T_r. Assuming that r^+ always has a-degree k in this way, with $k \in \{1, ..., 4\}$, we call T_r a k-*bad* shrub.

Suppose r^+ has children $v_1, ..., v_p$ with $p \geq 0$. For each of these nodes v_i, we denote by D_i the set of all possible a-degrees of v_i in $T_r[r^+, i]$ by all of the possible 2-aliec of $T_r[r^+, i]$. The D_i's perform the *signature* of T_r. Analogously, we denote by D_0 the set of all possible a-degrees of r^+ by all of the 2-aliec of T_r. According to our definitions, note that T_r is a k-bad shrub if and only if, regarding its signature, we have $D_0 = \{k\}$, i.e. D_0 is a singleton.

The set D_0 of any shrub T_r can be easily computed thanks to an inductive scheme inspired by Algorithm 1. Roughly speaking, we first compute inductively the set D_0 of each of the p schrubs $T_r[r^+, 1], \ldots, T_r[r^+, p]$. By definition, the set D_0 of $T_r[r^+, i]$ corresponds to the set D_i of T_r. Thanks to the signature $D_1, ..., D_p$ of T_r, which is, in some sense, a compact way for representing the 2-aliec of the $T_r[r^+, i]$'s which are of interest for us, the set D_0 of T_r can finally be deduced. Using this procedure, we are able to identify, in the next result, all k-*bad* signatures of T_r, i.e. signatures making the set D_0 of T_r being $\{k\}$ for every $k \in \{1, ..., 4\}$.

Theorem 4. *All k-bad signatures are those given in Table 3.*

Proof. We consider each possible signature of T_r with regards to $p \leq 3$, the number of children of r^+. For the sake of simplicity, we here only detail the proof for the easy cases, i.e. $p = 0$ and $p = 1$, so that the reader gets an idea

Signature $\mathbf{D_1, D_2, D_3}$	**Resulting $\mathbf{D_0}$**	**Signature $\mathbf{D_1, D_2, D_3}$**	**Resulting $\mathbf{D_0}$**
$\{1\},\{1\},\{1\}$	$\{1,2,4\}$	$\{2\},\{2\},\{2\}$	$\{1,3,4\}$
$\{1\},\{1\},\{2\}$	$\{1,3,4\}$	$\{2\},\{2\},\{3\}$	$\{3,4\}$
$\{1\},\{1\},\{3\}$	$\{2,3,4\}$	$\{2\},\{2\},\{4\}$	$\{1,3\}$
$\{1\},\{1\},\{4\}$	$\{1,2,3\}$	$\mathbf{\{2\},\{3\},\{3\}}$	$\mathbf{\{4\}}$
$\{1\},\{2\},\{2\}$	$\{1,3,4\}$	$\mathbf{\{2\},\{3\},\{4\}}$	$\mathbf{\{3\}}$
$\{1\},\{2\},\{3\}$	$\{3,4\}$	$\{2\},\{4\},\{4\}$	$\{1,3\}$
$\{1\},\{2\},\{4\}$	$\{1,3\}$	$\{3\},\{3\},\{3\}$	$\{2,4\}$
$\{1\},\{3\},\{3\}$	$\{2,4\}$	$\mathbf{\{3\},\{3\},\{4\}}$	$\mathbf{\{2\}}$
$\{1\},\{3\},\{4\}$	$\{2,3\}$	$\{3\},\{4\},\{4\}$	$\{2,3\}$
$\{1\},\{4\},\{4\}$	$\{1,2,3\}$	$\{4\},\{4\},\{4\}$	$\{1,2,3\}$

Table 2. All possible canonical signatures of T_r and resulting D_0 when $p = 3$

$\mathbf{D_0 = \{k\}}$	**p**	**Signature**
$\{1\}$	0	-
	1	$D_1 = \{2\}$
$\{2\}$	1	$D_1 = \{1\}$
	2	$D_1 = \{2\}$, $D_2 = \{3\}$
	3	$D_1 = \{3\}$, $D_2 = \{3\}$, $D_3 = \{4\}$
$\{3\}$	2	$D_1 = \{2\}$, $D_2 = \{2\}$
	3	$D_1 = \{2\}$, $D_2 = \{3\}$, $D_3 = \{4\}$
$\{4\}$	3	$D_1 = \{2\}$, $D_2 = \{3\}$, $D_3 = \{3\}$

Table 3. List of all k-bad signatures

of the technique we use. The remaining cases, i.e. $p = 2$ and $p = 3$, are given in Tables 1 and 2. Signatures in bold are those which are k-bad for some k. All remaining cases that do not appear in these tables do not concern bad signatures and can be deduced from canonical cases thanks to the following two rules. First, if $D_1, ..., D_p$ is not a bad signature of T_r, then $D'_1, ..., D'_p$ is not a bad signature when $D_i \subseteq D'_i$ for every $i \in \{1, ..., p\}$ (inclusion rule). Second, if $D_1, ..., D_i, ..., D_p$ is a k-bad signature and $D_1, ..., D'_i, ..., D_p$ is a k'-bad signature with $k' \neq k$ for some $D'_i \neq D_i$, then $D_1, ..., D_i \cup D'_i, ..., D_p$ is not a bad signature (union rule).

If $p = 0$, then rr^+ has to be coloured a and r^+ thus necessarily has a-degree 1. Therefore, the empty signature is a 1-bad signature. Now suppose that $p = 1$. If $D_1 = \{1\}$, then, in every 2-aliec of $T_r[r^+, 1]$, v_1 has a-degree 1 and we have to colour rr^+ with colour a. Thus $D_0 = \{2\}$, and $D_1 = \{1\}$ is a 2-bad signature. Similarly, if $D_1 = \{2\}$, then every 2-aliec of $T_r[r^+, 1]$ is actually a 2-liec and we have to invert it before colouring rr^+ with colour a. Therefore, $D_0 = \{1\}$, and $D_1 = \{2\}$ is a 1-bad signature. If there exists a 2-aliec $\phi^1_{a,b}$ of $T_r[r^+, 1]$ such that v_1 has a-degree 3 or 4, then we may either colour rr^+ with colour a directly or invert $\phi^1_{a,b}$ before. In the first situation, r^+ has a-degree 2, while it has a-degree 1 in the second one. Therefore, $D_0 = \{1, 2\}$ if 3 or 4 belongs to D_1. Thus, D_1 is not a bad signature whenever it contains 3 or 4. Finally, $D_1 = \{1, 2\}$ is not a bad signature since we get $D_0 = \{1, 2\}$ by the union rule. Every other possibilities for D_1 leads to a D_0 which is not a singleton by the inclusion and union rules. Therefore, $D_1 = \{1\}$ and $D_1 = \{2\}$ are the only bad signatures when $p = 1$. $\square$

Arbitrarily many k-bad shrubs can be constructed thanks to Theorem 4 by connecting "bad pieces" together. First choose a k-bad signature, i.e. let p and $D_1 = \{d_1\}, ..., D_p = \{d_p\}$ be values corresponding to one row of Table 3. Let T_r be a single edge rr^+, and $T_1, ..., T_p$ be d_1-, ..., d_p-bad shrubs, respectively. Then identify the roots of $T_1, ..., T_p$ with r^+. The resulting shrub T_r is clearly k-bad.

Suppose r has $p \geq 1$ neighbours in a colourable tree T. As explained above, if the shrubs $T_r[r, 1], ..., T_r[r, p]$ are k_1-, ..., k_p-bad, respectively, and the sequence $(k_1, ..., k_p)$ is one of the bad a-degree sequences (1), $(2, 1)$, $(3, 2, 2)$ or $(4, 3, 3, 2)$, then we cannot deduce a 2-liec of T_r thanks to the colouring procedure introduced in Section 4. In this situation, we say that r is *bad*. We end up this section by showing that if r is bad, i.e. our colouring procedure does not provide a 2-liec of T_r, then every node $r' \neq r$ of T is also bad. This implies that $\chi'_{irr}(T) = 3$ if and only if any node of T is bad.

First remark, by comparing the bad a-degree sequences and the bad signatures from Table 3, that the following holds.

Observation 5. *If $\{d_1\}, ..., \{d_p\}$ is a d_0-bad signature, then $(d_0, d_1, ..., d_p)$ is a bad sequence. Conversely, if σ is any permutation of $\{d_0, d_1, ..., d_p\}$ and $(d_0, d_1, ..., d_p)$ is a bad sequence, then $\{\sigma(d_1)\}, ..., \{\sigma(d_p)\}$ is a $\sigma(d_0)$-bad signature.*

Theorem 6. *If r is a bad node of T, then so is any other node $r' \neq r$ of T.*

Proof. Note that it suffices to show the claim when r and r' are neighbours in T. Suppose that $p \geq 1$ and $p' \geq 0$ denote the degree of r and r', respectively, and r' (resp. r) is the first child of r (resp. r') in T_r (resp. $T_{r'}$), i.e. $r' = r^+$ (resp. $r = (r')^+$) in $T_r[r, 1]$ (resp. $T_{r'}[r', 1]$).

Because r is bad, the shrubs $T_r[r, 1], ..., T_r[r, p]$ are k_1-, ..., k_p-bad, respectively, and $(k_1, ..., k_p)$ is a bad a-degree sequence. According to Theorem 4, if $T_r[r, 1]$ is k_1-bad, then $T_r[r', 1], ..., T_r[r', p' - 1]$ are ℓ_1-, ..., $\ell_{p'-1}$-bad, respectively, and $\{\ell_1\}, ..., \{\ell_{p'-1}\}$ is a k_1-bad signature. Besides, according to Observation 5, the sequence $(k_1, \ell_1, ..., \ell_{p'-1})$ is bad. Now, because r is bad, it means that $\{k_2\}, ..., \{k_p\}$ is a k_1-bad signature again by Observation 5 and $T_{r'}[r', 1]$ is a k_1-bad shrub. Thus, $T_{r'}[r', 1], T_{r'}[r', 2], ..., T_{r'}[r', p']$ are k_1-, ℓ_1-, ..., $\ell_{p'-1}$-bad shrubs, respectively, and $(k_1, \ell_1, ..., \ell_{p'-1})$ is a bad sequence. Therefore, r' is bad. $\square$

Corollary 7. $\chi'_{irr}(T) = 3$ *if and only if any node of T is bad.*

All trees with irregular chromatic index 3 can be constructed as follows. First choose one of the bad sequences $(d_1, ..., d_p)$, and construct p shrubs $T_1, ..., T_p$ which are d_1-, ..., d_p-bad, respectively. Recall that there are infinitely many such shrubs as pointed out above. Finally identify the roots of $T_1, ..., T_p$. By construction, the node used for the identification is bad, and the obtained tree thus has irregular chromatic index 3 according to Corollary 7.

```
1  if T is an odd length path then
2  └ χ'_irr(T) is undefined;
3  else if T is locally irregular then
4  └ χ'_irr(T) = 1;
5  else if Δ(T) ≤ 2 or Δ(T) ≥ 5 then
6  └ χ'_irr(T) = 2;
7  else
8      choose an arbitrary node r of T with degree p ≥ 1;
9      foreach i ∈ {1, ..., p} do
10         let D_i be the set D_0 of T_r[r, i] computed inductively;
11         if D_i is not a singleton then
12             χ'_irr(T) = 2;
13             exit algorithm;
14     let D_i = {d_i} for every i ∈ {1, ..., p};
15     if (d_1, ..., d_p) is not a bad a-degree sequence then
16     └ χ'_irr(T) = 2;
17     else
18     └ χ'_irr(T) = 3;
```

Algorithm 2: Algorithm for the irregular chromatic index of a tree T

6 Determining the irregular chromatic index of trees

We now propose an algorithm that determines, thanks to our previous results, the irregular chromatic index of an input tree T. Recall that the bad a-degree sequences are (1), $(2,1)$, $(3,2,2)$ and $(4,3,3,2)$.

Theorem 8. *Algorithm 2 determines the irregular chromatic index of any tree T in $O(n)$, where n is the order of T.*

Proof. The correctness of Algorithm 2 follows from the previous results and observations. In particular, the correctness of Lines 5-6 follows from Theorem 3, while the correctness of Lines 11-12 and Lines 15-16 follows from observations raised in Section 5. The correctness of Lines-17-18 follows from Corollary 7. The most costly instruction of Algorithm 2 is Line 10, which is achieved in $O(n)$ by computing the values of D_0 from leaves to root for each shrub as in the proof of Theorem 4. Every other line of the algorithm runs either in $O(1)$ or $O(n)$. Therefore, we get that Algorithm 2 has running time $O(n)$. $\square$

Theorem 3 gives a sufficient condition for a tree to have irregular chromatic index at most 2 that is easy to recognize. As mentioned in Section 5, trees with irregular chromatic index 3 have a predictable structure made up of "bad pieces", i.e. those given in Table 3. By carefully studying how these pieces must be connected, we can find sufficient conditions for a tree to have irregular chromatic index 3. These conditions mainly concern the location of nodes with degree 3 or 4 and the way they are organized in such trees.

Observe, for example, that no bad signature includes $\{1\}$ whenever $p \geq 2$. This means that if a node with degree at least 3 of T is connected to a hanging path with odd length, then T has irregular chromatic index at most 2. Additionally, note that if the colouring procedure from Section 4 fails on T_r, i.e. r is bad, when r has degree $\Delta(T) = 4$, then r necessarily has a neighbour with degree 4 since one of the $T_r[r, i]$'s is a 4-bad shrub. Therefore, if T has a node r' with degree 4 which has no neighbour with degree 4, then r' is not bad and T has irregular chromatic index at most 2 by Corollary 7.

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