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Abstract. Higher-order multiple-scale methods for general multiparameter mechanical systems are studied. The role played by the control and imperfection parameters in deriving the perturbative equations is highlighted. The definition of the codimension of the problem, borrowed from the bifurcation theory, is extended to general systems, excited either externally or parametrically. The concept of a reduced dynamical system is then invoked. Different approaches followed in the literature to deal with reconstituted amplitude equations are discussed, both in the search for steady-state solutions and in the analysis of stability. Four classes of methods are considered, based on the consistency or inconsistency of the approach, and on the completeness or incompleteness of the terms retained in the analysis. The four methods are critically compared and general conclusions drawn. Finally, three examples are illustrated to corroborate the findings and to show the quantitative differences between the various approaches.

Keywords: Perturbation methods, higher-order approximations, dynamical systems, codimension, stability.

1. Introduction

The multiple-scale method [1] is a powerful tool for dealing with nonlinear dynamical problems. It has been widely used in the last few decades to study free and forced oscillatory phenomena [2] and, more recently, to describe nonlinear normal modes [3–6] and postcritical behavior in bifurcation problems [7–10]. It has also been applied to discrete-time dynamical systems [11]. As in other reduction methods, the multiple time-scale method transforms the analysis of the evolution of a multidimensional dynamical system into that of a smaller, equivalent problem (the so-called amplitude modulation equations). Very often, the lower-order approximation of such equations is sufficient to describe steady-state solutions and their stability, both qualitatively and quantitatively. However, recent works have generated considerable interest in second and higher-order approximations of these amplitude equations. The reason for this interest is twofold: first, symbolic manipulators have made it easy to proceed to a higher order level, thus improving the accuracy of the analytical solutions; second, there are problems in which higher-order solutions entail *qualitative changes* to the first-order solutions (see, e.g., [12–18]), so that it is necessary to use them to describe the phenomenon accurately.

Higher-order amplitude equations are obtained by combining the solvability conditions of the perturbative equations at different levels, according to the *reconstitution method* [19]. However, there are alternatives to deal with such equations. Two versions of the method are discussed in key papers by Rahman and Burton [17, 20]. By using an example, they showed that the so-called version I (i.e., the most widely used method in the past [19]) may lead to erroneous results. In particular, they highlighted the existence of spurious solutions responsible for distorting the true solutions. The question was later studied in greater depth by Hassan [21, 22] who showed that the distortion due to spurious solutions in the Rahman and Burton

problem is exacerbated by 'the combined effects of using transformed time $T = \Omega t$ and introducing a detuning parameter in the square of the excitation frequency', possibly leading to nonuniform expansions at large amplitudes. However, the papers by Rahman and Burton have not been fully understood, since other versions of their method have appeared in the literature, incorrectly justified in those papers [23–27]. In particular, a somewhat questionable so-called version II of the method, which involves fewer computations than version I and is increasingly popular among researchers, is erroneously attributed to Rahman and Burton. Therefore, at the moment a broad range of approaches exists making the procedure rather confusing.

The aim of this paper is to classify the alternatives and discuss them critically, referring to general systems rather than particular examples. To this end, the main steps of the multiple-scale method for multiparameter systems are recalled in Section 2, in order to highlight the role of the control and imperfection parameters in the algorithm. Four different methods of analysis are then identified and described in Section 3. They are discussed and critically compared in Section 4, where general conclusions are drawn regarding steady-state solutions and their stability. Finally, three examples are illustrated in Section 5, in which the second approximation entails either qualitative or quantitative changes with respect to the first approximation. Some further details are given in [28].

2. The Multiple Scale Method for Multiparameter Systems

2.1. The Hypotheses

The equations of motion of a general discrete mechanical system read:

$$\ddot{\mathbf{q}} + \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, t; \boldsymbol{\mu}) = \mathbf{0}, \tag{1}$$

where the mass matrix has been assumed to be unitary, $\mathbf{q}$ are the Lagrangian coordinates, $\mathbf{F}$ the force vector, $\boldsymbol{\mu}$ is a vector of parameters and the dot denotes differentiation with respect to time t. The vector $\boldsymbol{\mu}$ contains *small* physical quantities (such as damping coefficients, detunings between natural and/or excitation frequencies, external and parametric resonant excitation amplitudes, geometrical imperfections, and so on) or *small deviations* from the critical values of other parameters (e.g., load factors, flow velocities in fluid-structure interaction problems or angular velocities in rotating systems). Hard nonresonant excitations are not considered.

The parameters $\boldsymbol{\mu}$ play a fundamental role in the perturbative solution to Equation (1). They are chosen in such a way that the following properties hold:

1. $(\mathbf{q}, \boldsymbol{\mu}) = (\mathbf{0}, \mathbf{0})$ is an equilibrium position O.
2. When $\boldsymbol{\mu} \to \mathbf{0}$, Equation (1) tends to an autonomous equation. However, external excitation amplitudes tend to zero more rapidly than parametric excitation amplitudes. This assumption implies that, for example, the square or cubic root of the external excitation amplitudes have to be considered as parameters rather than the amplitudes themselves.
3. By linearizing in $\mathbf{q}$ in Equation (1) at $\boldsymbol{\mu} = \mathbf{0}$, the following *generating equation* is obtained

$$\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}q = \mathbf{0}, \tag{2}$$

where $\mathbf{C} := \partial\mathbf{F}/\partial\dot{\mathbf{q}}|_0$ and $\mathbf{K} := \partial\mathbf{F}/\partial\mathbf{q}|_0$ are the (generally nonsymmetric) damping and stiffness matrices at O, respectively. The eigenvalue problem

$$(\lambda^2\mathbf{E} + \lambda\mathbf{C} + \mathbf{K})\mathbf{u} = \mathbf{0} \tag{3}$$

associated with Equation (2) admits at least one eigenvalue with zero real part. The general case is considered here in which Equation (3) admits a cluster of n_r zero eigenvalues $\lambda_j = 0$ and n_c couples of purely imaginary eigenvalues $\lambda_k = \pm i\omega_k$, associated with $n := n_r + 2n_c$ linearly independent eigenvectors $\mathbf{u}_j$ and $\mathbf{u}_k$. Cases of nilpotent matrices are thus excluded. The n eigenvalues of interest will be referred to as *active* eigenvalues. The remaining ones, which are assumed to have no positive real part, will be considered as *passive* eigenvalues.

4. The n_c active imaginary eigenvalues are involved in r linearly independent resonance conditions, of the type

$$\sum_{k=1}^{n_c} m_{ik}\omega_k + \Omega_i = \sigma_i, \quad i = 1, 2, \dots, r, \tag{4}$$

where Ω_i are excitation frequencies, m_{ik} are small integers and σ_i small detunings included in the vector $\boldsymbol{\mu}$. Internal resonances are also included in Equation (4) if $\Omega_i = 0$.

2.2. The Perturbation Method

Due to the spectral hypothesis, Equation (2) admits a steady-state multi-periodic solution, called a *generating solution*. Nontrivial solutions $\mathbf{q} = \mathbf{q}(t; \boldsymbol{\mu})$ to Equation (1) are sought which asymptotically tend towards the generating solution when $(\mathbf{q}, \boldsymbol{\mu}) \to (\mathbf{0}, \mathbf{0})$. To formalize the procedure, an ordering of the (small) parameters $\boldsymbol{\mu}$ is made:

$$\boldsymbol{\mu} = \varepsilon\widehat{\boldsymbol{\mu}}, \quad \widehat{\boldsymbol{\mu}} = O(1), \tag{5}$$

where ε is a perturbation parameter. Moreover, the Lagrangian coordinates are expanded in series of ε around $\varepsilon = 0$:

$$\mathbf{q} = \varepsilon\mathbf{q}_1 + \varepsilon^2\mathbf{q}_2 + \cdots. \tag{6}$$

Several temporal scales $t_k = \varepsilon^k t$ $(k = 0, 1, \dots)$ are introduced so that $\mathrm{d}/\mathrm{d}t = d_0 + \varepsilon d_1 + \varepsilon^2 d_2 + \cdots$, with $d_k := \partial/\partial t_k$. By substituting Equations (5, 6) in Equation (1), expanding it and separately vanishing terms with the same powers of ε, the perturbative equations are obtained. Up to ε^3-order, they read:

$$\begin{aligned}
\mathcal{L}\mathbf{q}_1 &= \mathbf{0}, \\
\mathcal{L}\mathbf{q}_2 &= \mathcal{F}_2(\mathbf{q}_1; \widehat{\boldsymbol{\mu}}) - 2d_0d_1\mathbf{q}_1, \\
\mathcal{L}\mathbf{q}_3 &= \mathcal{F}_3(\mathbf{q}_1, \mathbf{q}_2; \widehat{\boldsymbol{\mu}}) - [2d_0d_1\mathbf{q}_2 + (d_1^2 + 2d_0d_2)\mathbf{q}_1],
\end{aligned} \tag{7}$$

where $\mathcal{L} := \mathbf{E}d_0^2 + \mathbf{C}d_0 + \mathbf{K}$ and $\mathcal{F}_j$ denotes the jth-order terms in the MacLaurin series expansion of $\mathbf{F}$ in terms of $\mathbf{q}$ and $\boldsymbol{\mu}$. Equation (7_1) admits the generating solution

$$\mathbf{q}_1 = \frac{1}{2}\sum_{j=1}^{n_r} a_j(t_1, t_2, \dots)\mathbf{u}_j + \sum_{k=1}^{n_c} A_k(t_1, t_2, \dots)\mathbf{u}_k\, \mathrm{e}^{i\omega_k t_0} + \text{c.c.}, \tag{8}$$

where a_j and $A_k := 1/2\, a_k \exp(i\vartheta_k)$ are real and complex functions of the slow times, respectively, 'c.c.' stands for complex conjugate and i is the imaginary unit. To solve higher-order

perturbation equations, solvability conditions must be imposed at each step, requiring the resonant terms (i.e., constant and $(2\pi/\omega_k)$-periodic terms) on the right side to be orthogonal to the n left eigenvectors $\mathbf{v}_j, \mathbf{v}_k$ dual of $\mathbf{u}_j, \mathbf{u}_k$. The solvability conditions lead to sets of n nonautonomous first-order differential equations on the scales $t_1, t_2, \dots$ in the n unknowns $(\mathbf{a}_j, \mathbf{a}_k, \boldsymbol{\vartheta}_k)$. To render the systems autonomous, it is necessary to introduce as many new functions $\gamma_i = \gamma_i(t_1, t_2, \dots)$ as there are the resonant conditions, namely

$$\gamma_i := \sum_{k=1}^{n_c} m_{ik}\vartheta_k + \widehat{\sigma}_i t_1, \quad i = 1, 2, \dots, r, \tag{9}$$

where $\widehat{\sigma}_i$ are scaled detunings. Definition (9) implies that if $\dot{\gamma}_i = 0$, then

$$\sum_{k=1}^{n_c} m_{ik}\omega_k^*(\varepsilon) + \Omega_i = 0, \quad i = 1, 2, \dots, r, \tag{10}$$

with $\omega_k^* := \omega_k + \varepsilon d_1\vartheta_k + \varepsilon^2 d_2\vartheta_k + \cdots$ denoting the nonlinear frequencies. Therefore, for finite amplitudes, the resonance conditions (4) are satisfied with zero detunings.

To summarize, the introduction of the phase-combinations γ_i allows the reduction of the solvability conditions to a set of $m := n_r + n_c + r$ autonomous equations in the m unknowns $\mathbf{a} := (\mathbf{a}_j, \mathbf{a}_k, \boldsymbol{\gamma}_i)$, each set governing the evolution of amplitudes and phases on a different time-scale. They assume the following form:

$$\begin{aligned} d_1\mathbf{a} &= \mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}), \\ d_2\mathbf{a} &= \mathbf{f}_2(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}})\, d_1\mathbf{a} + \boldsymbol{\alpha}_2\, d_1^2\mathbf{a} + \boldsymbol{\alpha}_3\, (d_1\mathbf{a})^2, \end{aligned} \tag{11}$$

where $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ are constant matrices and vectors $\mathbf{f}_i$ $(i = 1, 2)$ and matrix $\boldsymbol{\alpha}_1$ are functions of $\mathbf{a}$ and $\widehat{\boldsymbol{\mu}}$.

The integer m, equal to the number of critical eigenvalues (with the complex ones counted in pairs) plus the number of resonance conditions, will be referred to as the *codimension* of the problem as is usual in bifurcation problems [29]. It is equal to the number of the degeneracy conditions of the linear operator of Equations (2) (see, e.g., [30]). Therefore, in the physical parameter space, m is the codimension of the manifold on which the assumed spectral properties are satisfied. It is worth noting that the codimension coincides with the number of amplitudes and phase-combinations $\mathbf{a}$ that govern the nonlinear problem on the slow scales.

2.3. The Reconstitution Method

The solvability equation sets, Equations (11), govern the evolution of the amplitudes and phases $\mathbf{a}$ on the slow time-scales. They should be solved in sequence to determine the dependence of $\mathbf{a} = \mathbf{a}(t_1, t_2, \dots)$ first on t_1, then on t_2, and so on. However, this procedure is quite inapplicable in general cases so that an alternative method has to be followed. Therefore, the solvability equations are combined in a unique equation by returning to the true time t. By accounting for $\mathrm{d}\mathbf{a}/\mathrm{d}t = \varepsilon d_1\mathbf{a} + \varepsilon^2 d_2\mathbf{a} + \cdots$, and using Equation (11_1) in Equation (11_2), it follows that

$$\dot{\mathbf{a}} = \varepsilon \mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \varepsilon^2[\mathbf{f}_2(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}})\mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_2 \mathbf{f}_{1,\mathbf{a}}(\mathbf{a}; \widehat{\boldsymbol{\mu}})\mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_3 \mathbf{f}_1^2(\mathbf{a}; \widehat{\boldsymbol{\mu}})] + \mathrm{O}(\varepsilon^3), \tag{12}$$

where an index after a comma denotes differentiation. Equation (12) can be integrated to evaluate the amplitude modulation on the true time-scale t. It is known as the *reconstituted* amplitude equation, and the method is known as the *reconstitution method* [19].

The reconstituted amplitudes equation (12) is an asymptotic representation of a *reduced dynamical system*

$$\dot{\mathbf{a}} = \mathbf{G}(\mathbf{a}; \boldsymbol{\mu}) \tag{13}$$

able to capture the main aspect of the dynamics of the original system (1). The MSM can therefore be viewed as a reduction method (e.g. as the center manifold method [29]), which allows a reduction of the number of state variables to the codimension m of the problem. However, the reduced dynamical system (13) is not known in closed form, *since only its asymptotic form* (12) *can be built up* through a perturbation method (or through the center manifold procedure). More specifically, the right-hand member of Equation (12) is the ε-expansion of the (unknown) function $\mathbf{G}(\mathbf{a}(\varepsilon), \boldsymbol{\mu}(\varepsilon))$ in which $\mathbf{a}(\varepsilon) = (\varepsilon \mathbf{a}_j, \varepsilon \mathbf{a}_k, \boldsymbol{\gamma}_i)$ (i.e. only the amplitudes but not the phases are assumed to be small) and $\boldsymbol{\mu} = \varepsilon \widehat{\boldsymbol{\mu}}$.

3. Methods of Analysis

The reconstituted amplitudes equation (12) is usually studied in two steps: first, the steady-state solutions $\mathbf{a} = \text{const}$ are evaluated as functions of the parameters $\widehat{\boldsymbol{\mu}}$. Then their stability is analyzed. However, a large number of different methods has been used in literature to perform these steps. Here, they are tentatively classified.

Two main classes, *consistent methods* and *inconsistent methods*, and two sub-classes, *complete methods* and *incomplete methods*, are distinguished. In the consistent approach, the asymptotic nature of the reduced dynamical system is taken into account, consistently with the basic assumptions of the perturbation method, whereas in the inconsistent approach, this feature is ignored. Moreover, in the complete methods, all terms deriving from the solvability conditions are retained in the analysis, while in the incomplete methods, some of them are neglected. By combining the alternatives, four approaches are identified; these are discussed below.

3.1. Complete Inconsistent Method (CIM)

This method has been applied in [12–19]. According to the philosophy of the inconsistent approaches, the reconstituted amplitude equations are dealt with as if they were a *closed-form* representation of the reduced dynamical system rather than an *asymptotic approximation* of the unknown system. Accordingly, the steady-state solutions are found by requiring the right-hand members of Equation (12) to vanish:

$$\varepsilon \mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \varepsilon^2[\mathbf{f}_2(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}})\mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_2 \mathbf{f}_{1,\mathbf{a}}(\mathbf{a}; \widehat{\boldsymbol{\mu}})\mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \boldsymbol{\alpha}_3 \mathbf{f}_1^2(\mathbf{a}; \widehat{\boldsymbol{\mu}})] = \mathbf{0}. \tag{14}$$

For a fixed ε, Equation (14) is a set of m parameter-dependent nonlinear equations in the m unknown amplitudes. By solving them, if necessary through numerical algorithms, several

paths $\mathbf{a}_s = \mathbf{a}_s(\widehat{\boldsymbol{\mu}}; \varepsilon)$ are found. Their stability is analyzed through the variational equation

$$\delta\dot{\mathbf{a}} = \left\{\varepsilon\mathbf{f}_{1,\mathbf{a}} + \varepsilon^2[\mathbf{f}_{2,\mathbf{a}} + \boldsymbol{\alpha}_{1,\mathbf{a}}\mathbf{f}_1 + \boldsymbol{\alpha}_1\mathbf{f}_{1,\mathbf{a}} + \boldsymbol{\alpha}_2(\mathbf{f}_{1,\mathbf{a}a}\mathbf{f}_1 + \mathbf{f}_{1,\mathbf{a}}^2) + 2\boldsymbol{\alpha}_3\mathbf{f}_{1,\mathbf{a}}\mathbf{f}_1]\right\}_{\mathbf{a}=\mathbf{a}_s}\delta\mathbf{a} \tag{15}$$

in which all quantities are evaluated at $\mathbf{a} = \mathbf{a}_s$.

3.2. Incomplete Inconsistent Method (IIM)

This differs from the CIM in that the $d_1\mathbf{a}$ and $d_1^2\mathbf{a}$ terms are omitted from the ε^3-order solvability conditions (see Equation (11_2)). This procedure was introduced by Lee and Perkins [23] and justified by Lee and Lee [26] as follows: 'time derivative terms are nonzero only on their corresponding time scale, e.g., d_1 terms are nonzero on the t_1 scale but vanish on the t_2 scale'. The procedure was subsequently followed by several authors [24, 25, 27].

The reconstituted amplitudes equation (12) simplifies as:

$$\dot{\mathbf{a}} = \varepsilon\mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \varepsilon^2\mathbf{f}_2(\mathbf{a}; \widehat{\boldsymbol{\mu}}). \tag{16}$$

Then, steady-state solutions are drawn from

$$\varepsilon\mathbf{f}_1(\mathbf{a}; \widehat{\boldsymbol{\mu}}) + \varepsilon^2\mathbf{f}_2(\mathbf{a}; \widehat{\boldsymbol{\mu}}) = \mathbf{0} \tag{17}$$

and their stability analyzed through

$$\delta\dot{\mathbf{a}} = [\varepsilon\mathbf{f}_{1,\mathbf{a}} + \varepsilon^2\mathbf{f}_{2,\mathbf{a}}]_{\mathbf{a}=\mathbf{a}_s}\delta\mathbf{a}. \tag{18}$$

In [23–27], a different procedure is used. Namely, parameters $\boldsymbol{\mu}$ are expanded in series as $\boldsymbol{\mu} = \varepsilon\boldsymbol{\mu}_1 + \varepsilon^2\boldsymbol{\mu}_2 + \cdots$, instead of being ordered as in Equation (5). However, the inverse transformation, $\varepsilon\boldsymbol{\mu}_1 + \varepsilon^2\boldsymbol{\mu}_2 \rightarrow \boldsymbol{\mu}$ was later introduced in the reconstituted equations so that the parameters expansion has no role. The procedure illustrated here leads in a more straightforward way to the same results [28].

A more significant difference exists between the method applied in [23–27] (the so-called version II of the MSM, here referred to as IIM-II) and the procedure illustrated here, clearly emerging in the resonant case. Here, complex solvability conditions are first put in real form at each order and phase combination introduced. Reconstitution is then performed. In IIM-II the two operations are exchanged: first reconstitution is carried out on the complex solvability conditions, then the equations are put in real form and phases γ_i introduced. The two procedures obviously lead to the same equations if all the terms are retained in the analysis. On the contrary, if an incomplete method is used, they entail some differences, since terms are omitted at different levels. Specifically, in the first procedure, the t_1-derivatives of the real amplitudes a_k and phases γ_i are omitted (i.e., $d_1a_k = d_1^2a_k = d_1\gamma_i = d_1^2\gamma_i = 0$) whereas in the second procedure, the t_1-derivatives of the complex amplitude A_k are omitted (i.e. $d_1a_k = d_1^2a_k = d_1\vartheta_k = d_1^2\vartheta_k = 0$). By remembering the definition of phases γ_i (Equation (9)), it follows that, in the IIM-II, $d_1\gamma_i = \widehat{\sigma}_i \neq 0$, $d_1^2\gamma_i = 0$. Therefore, some extra terms appear in Equations (17) and (18). If the problem is nonresonant, the amplitude equations in IIM and IIM-II coincide. However, phase modulation equations in the unknowns $\vartheta_k(t)$, which are uncoupled from the former, differ. The nonlinear frequencies in the two versions are therefore expected to be different. This question will be discussed later in the paper.

This was introduced in a systematic way by Rahman and Burton [20] after Luongo et al. [31] had used it in a particular case. In this method, the reconstituted amplitude equations are dealt with as an asymptotic approximation of the reduced dynamical system, corrected up to a certain power of ε. Therefore, the steady-state solutions, as well as the eigenvalues of the variational equation, are consistently sought as series expansions corrected up to the same ε-order. As a first step, parameters $\widehat{\boldsymbol{\mu}}$ in Equation (12) are expanded in series of ε (i.e. $\widehat{\boldsymbol{\mu}}(\varepsilon) = \boldsymbol{\mu}_1 + \varepsilon\boldsymbol{\mu}_2 + \cdots$), so that the reconstituted amplitudes equation reads:

$$\begin{aligned}\dot{\mathbf{a}} \;=\; & \varepsilon\mathbf{f}_1(\mathbf{a};\boldsymbol{\mu}_1) + \varepsilon^2\left[\mathbf{f}_2(\mathbf{a};\boldsymbol{\mu}_1) + \mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a};\boldsymbol{\mu}_1)\boldsymbol{\mu}_2\right. \\ & \left. + \boldsymbol{\alpha}_1(\mathbf{a};\boldsymbol{\mu}_1)\mathbf{f}_1(\mathbf{a};\boldsymbol{\mu}_1) + \boldsymbol{\alpha}_2\mathbf{f}_{1,\mathbf{a}}(\mathbf{a};\boldsymbol{\mu}_1)\mathbf{f}_1(\mathbf{a};\boldsymbol{\mu}_1) + \boldsymbol{\alpha}_3\mathbf{f}_1^2(\mathbf{a};\boldsymbol{\mu}_1)\right]. \end{aligned} \tag{19}$$

By requiring that *the amplitudes* $\mathbf{a}$ *be stationary for any* ε, i.e. $\dot{\mathbf{a}} = \mathbf{0}\ \forall\varepsilon$, the ε and ε^2 terms in Equation (19) must vanish separately, i.e.,

$$\begin{aligned}&\mathbf{f}_1(\mathbf{a};\boldsymbol{\mu}_1) = \mathbf{0},\\ &\mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a};\boldsymbol{\mu}_1)\boldsymbol{\mu}_2 + \mathbf{f}_2(\mathbf{a};\boldsymbol{\mu}_1) = \mathbf{0},\end{aligned} \tag{20}$$

where Equation (20_1) has been accounted for in deriving Equation (20_2). Conditions (20) express the vanishing of the amplitude time-derivatives on the different slow scales, namely $d_1\mathbf{a} = \mathbf{0}$, $d_2\mathbf{a} = \mathbf{0}$. Equation ($20_1$) is a set of *m nonlinear* equations in the amplitudes $\mathbf{a}$ and in the first-order part $\boldsymbol{\mu}_1$ of the parameters $\boldsymbol{\mu}$ (except for bifurcation problems from a known path, for which Equation (20_1) is linear in $\boldsymbol{\mu}_1$). They can be solved with respect to $\boldsymbol{\mu}_1$ for fixed $\mathbf{a}$, to furnish $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_1(\mathbf{a})$. Therefore, *the procedure entails expanding exactly m control parameters, i.e. as many parameters as the codimension of the problem is*. By substituting the solution in Equation (20_2), a set of *linear* equations in the $\boldsymbol{\mu}_2$ unknowns is found, from which $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_2(\mathbf{a})$ is drawn. By moving to higher orders, *linear* equations in $\boldsymbol{\mu}_3$, $\boldsymbol{\mu}_4, \ldots$ are still drawn. Finally,

$$\boldsymbol{\mu} = \varepsilon\boldsymbol{\mu}_1(\mathbf{a}) + \varepsilon^2\boldsymbol{\mu}_2(\mathbf{a}) + \cdots \tag{21}$$

is obtained.

In the method illustrated, steady-state solutions are described asymptotically in the form $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{a})$. However, very often steady-state solutions are sought in the more convenient form $\mathbf{a} = \mathbf{a}(\boldsymbol{\mu})$, so that Equation (21) has to be inverted [28]. As an alternative, steady-state solutions can be found directly in the form $\mathbf{a} = \mathbf{a}(\boldsymbol{\mu})$ simply *by expanding the amplitudes* $\mathbf{a}$, *instead of the parameters* $\widehat{\boldsymbol{\mu}}$, *in the steady version of* Equation (12). Mixed solutions, in which a set of m (dependent) amplitudes and parameters are expressed as functions of the remaining (independent) m quantities, can also be determined. An example of this latter procedure will be shown later (see Section 5.2). In the literature (see, e.g., [20–27, 31]), an alternative method is followed to derive Equation (19), in which the parameters $\boldsymbol{\mu}$ are expanded directly in the equation of motion. However, this procedure involves longer computations and does not allow expansion of the amplitudes.

Difficulties arise in solving Equation (20_2) when the matrix $\mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a}, \boldsymbol{\mu}_1)$ becomes singular at a critical point $C \equiv (\mathbf{a} = \mathbf{a}_c, \boldsymbol{\mu}_1 = \widehat{\boldsymbol{\mu}}_c)$, i.e. at the limit or bifurcation points of the first approximation $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_1(\mathbf{a})$. Since at the limit points, the $m \times 2m$ matrix $\mathbf{D} = [\mathbf{f}_{1,\boldsymbol{\mu}} \mid \mathbf{f}_{1,\mathbf{a}}]$ has maximum rank, the operator can be rendered nonsingular by suitably replacing one of the

unknown parameters by an amplitude, similarly to the method followed in some numerical algorithms to overcome limit points. At bifurcation points, in contrast, matrix **D** has rank less than m, and a special procedure has to be established to solve the problem. This is briefly described in Appendix A, where it is shown that *higher-order approximations possibly destroy first-order bifurcations*. Moreover, a procedure to build up series expansions around bifurcation points, which in some cases calls for the use of the fractional power of ε, is also sketched.

Concerning stability analysis, the variational equation built up in Equation (19), is

$$\delta\dot{\mathbf{a}} = \left\{\varepsilon\mathbf{f}_{1,\mathbf{a}} + \varepsilon^2[\mathbf{f}_{2,\mathbf{a}} - \mathbf{f}_{1,\mu\mathbf{a}}\mathbf{f}_{1,\mu}^{-1}\mathbf{f}_2 + \boldsymbol{\alpha}_1\mathbf{f}_{1,\mathbf{a}} + \boldsymbol{\alpha}_2\mathbf{f}_{1,\mathbf{a}}^2]\right\}_{\mathbf{a}=\mathbf{a}_s}\delta\mathbf{a}, \tag{22}$$

where Equation (20) has been used. Equation (22) is studied in detail in Appendix B. It should be noted that $\boldsymbol{\alpha}_3$ terms play no role.

3.4. Incomplete Consistent Method (ICM)

This has only been applied in [32]. In the spirit of the incomplete methods, the $d_1\mathbf{a}$ and $d_1^2\mathbf{a}$ terms are omitted in the reconstituted amplitudes equation (19), which therefore reads:

$$\dot{\mathbf{a}} = \varepsilon\mathbf{f}_1(\mathbf{a};\boldsymbol{\mu}_1) + \varepsilon^2[\mathbf{f}_2(\mathbf{a};\boldsymbol{\mu}_1) + \mathbf{f}_{1,\mu}(\mathbf{a};\boldsymbol{\mu}_1)\boldsymbol{\mu}_2]. \tag{23}$$

In contrast to the IIM (see Equation (16)), steady-state solutions are required to satisfy Equation (23) for any ε. Therefore, Equations (20) and the same solution of the CCM are recovered. Stability is instead analyzed on the variational equation based on Equation (23):

$$\delta\dot{\mathbf{a}} = \left\{\varepsilon\mathbf{f}_{1,\mathbf{a}} + \varepsilon^2[\mathbf{f}_{2,\mathbf{a}} - \mathbf{f}_{1,\mu\mathbf{a}}\mathbf{f}_{1,\mu}^{-1}\mathbf{f}_2]\right\}_{\mathbf{a}=\mathbf{a}_s}\delta\mathbf{a} \tag{24}$$

which differs from Equation (22) in the absence of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ terms.

4. Discussion

The methods of analysis described in the previous section are now discussed and compared. Steady-state solutions are first considered and their stability is then studied.

4.1. Steady-State Solutions

The above methods lead to three different steady-state solutions, Equation (14) (CIM), Equation (16) (IIM) and Equation (20) (CCM and ICM). In [20], it is pointed out that if an inconsistent method is applied, ε-dependent steady-state amplitudes are found, in spite of the fact that the coefficients $\mathbf{q}_k$ of the series expansion of $\mathbf{q}$ were assumed to be ε-independent. In addition, spurious solutions arise (also discussed in depth in [21, 22]), sometimes causing strong distortion of the true solutions. It has also been shown in [20], with reference to a particular example, that inconsistent solutions entail ordering violation.

Here, two aspects of the problem are dealt with: (a) the ordering violation is studied in more depth, and (b) the role played by terms neglected in the incomplete methods is analyzed.

(a) *Ordering violation*. Equations (14) and (16) contain terms of ε- and ε^2-order. If the equations are solved exactly, solutions $\mathbf{a}_s(\widehat{\boldsymbol{\mu}};\varepsilon)$, generally nonpolynomial in ε, are found. The solutions therefore contain terms up to ε^∞ which are not consistent with the order equation. In particular, whereas the ε and ε^2-order terms are correct, the higher-order contributions *are not* the terms of the series expansion of the exact, unknown solution. On

the contrary, if a consistent method is applied, steady-state solutions are found as correct series expansions truncated at the highest order contained in the equation. If consistent and inconsistent solutions are compared, large differences due to the inconsistent terms can be observed at moderately large values of ε for which ε^3-order terms are significant. Depending on the problem, these terms *may or may not* improve the approximation of the consistent solution. Since the accuracy of the inconsistent solution cannot generally be predicted in advance, the use of inconsistent methods is problematic.

(b) *Influence of the omitted terms*. The amplitude t_1-derivatives $d_1\mathbf{a}$ and $d_1^2\mathbf{a}$, involved in the ε^3-order solvability conditions do not affect consistent steady-state solutions, since they automatically vanish in the procedure (see Equation (20)). However, they appear in the variational equations (22) and (24) and, therefore, influence the stability. On the other hand, in the inconsistent approaches, such derivatives contribute to both steady-state and stability in the CIM (Equations (14) and (15)) but to neither of these in the IIM (Equations (16) and (18)). To highlight the nature of the t_1-derivatives, it is useful to consider the reconstituted amplitude equations as balance equations between kinetic forces, associated with slow amplitude-modulated motions, and forces of a different type (e.g. elastic), the former being proportional to $d_1\mathbf{a}$ and $d_1^2\mathbf{a}$, the latter to $\mathbf{f}_1$ and $\mathbf{f}_2$. Since in steady-state motions $\mathbf{a} = \text{const}$, slow varying kinetic forces vanish. Thus, *steady-state solutions are governed by nonkinetic forces, i.e. they do not depend on* $d_1\mathbf{a}$ *and* $d_2\mathbf{a}$. In contrast, the CIM-solutions erroneously depend on kinetic forces, although these affect only the ε^3 and higher-order part of the solution, as it has been discussed. Paradoxically, if static equilibrium positions of a damped mechanical system are sought via the CIM, *qualitatively wrong* damping-dependent solutions are found, as will be shown by an example (see Section 5.1). The reason for the drawback probably depends on the fact that the kinetic nature of $d_1\mathbf{a}$ and $d_1^2\mathbf{a}$ in the reconstituted amplitudes equation (12) is forgotten, after which the first-order solvability condition (11_1) is used to express such derivatives in terms of $\mathbf{a}$ and $\boldsymbol{\mu}$. On the contrary, in the IIM the t_1-derivatives are neglected and this drawback is avoided. However, incorrect reasons are invoked to omit the terms with the result that stability will be affected, as will be discussed later.

Problems arise if the IIM-II is applied in the resonant case. In fact, as discussed in Section 3.2, the derivatives omitted in the reconstituted equations are $d_1\vartheta_k$ instead of $d_1\gamma_k$, in spite of the fact that, in a steady-motion, the former are different from zero, while the latter vanish. An error similar to that contained in the CIM therefore occurs in steady-state solutions.

4.2. STABILITY

The stability of the steady-state solutions is governed by the eigenvalues of the variational equation. The four methods previously described lead to four different variational equations, namely, Equation (15) (CIM), Equation (18) (IIM), Equation (22) (CCM) and Equation (24) (ICM). Moreover, the variational matrices in these equations are evaluated at different steady-state amplitudes, each associated with the same values of $\boldsymbol{\mu}$, according to the four methods. In examining the alternative approaches, two problems are discussed: (a) the failure of consistent methods for a class of systems, and (b) the error caused by the omission of t_1-derivatives in the incomplete methods.

(a) *Failure of consistent methods*. When a consistent method is applied, the eigenvalues of the variational matrix have to be determined as perturbations of the eigenvalues of the

variational matrix of the first approximation. Consistently, the critical amplitudes for which incipient instability occurs are perturbations of the first approximation critical values. It is necessary to distinguish problems in which the second approximation furnishes a *quantitative* improvement of the first approximation, from problems in which it entails *qualitative* changes. As an example of the latter class of problems, in a subcritical Hopf bifurcation, the second approximation makes it possible to detect the existence of a possible limit point at which stability is regained and which cannot be described by the first approximation. In such cases, consistent methods fail, since, no critical state exists along the bifurcated paths at the first-order (i.e., in the generating solution of the perturbative process). On the contrary, inconsistent methods sometimes give qualitatively correct results. The reason for the failure lies in the fact that in this type of problem an ordering violation occurs, even in steady-state solutions, since, near to the critical state, higher-order terms prevail over lower-order terms, in contrast with the basic principle of asymptotic methods. A better approach would require shifting lower-order terms one step further in the perturbative scheme, so that they appear on the same level as higher-order terms (see, e.g., [33]). However, in spite of this ordering violation, it is customary to 'hope' that the steady-state solutions are fairly accurate. In these cases, it is necessary to resort to an inconsistent stability analysis.

(b) *Influence of the omitted terms*. To analyze the influence on stability of the terms neglected in the incomplete methods, let us consider the acceleration of the system in an unsteady motion. By using Equation (6) and the chain rule, the acceleration reads

$$\ddot{\mathbf{q}} = d_0^2\mathbf{q} + 2\varepsilon d_0 d_1 \mathbf{q}_1 + \varepsilon^2[2d_0 d_1 \mathbf{q}_2 + (d_1^2 + 2d_0 d_2)\mathbf{q}_1]. \tag{25}$$

In Equation (25), $d_0^2\mathbf{q}$ accounts for fast-scale motions, on which amplitudes remain constant, while the ε- and ε^2-terms account for slow-scale motions, on which amplitudes vary. By recalling, from perturbation equations (7), that $\mathbf{q}_1$ is proportional to $\mathbf{a}$, and $\mathbf{q}_2$ is proportional to $\mathbf{a}^2$, $\mathbf{a}\widehat{\boldsymbol{\mu}}$, $\widehat{\boldsymbol{\mu}}^2$ and $d_1\mathbf{a}$, it follows that the ε-order part of $\ddot{\mathbf{q}}$ depends on $d_1\mathbf{a}$, while its ε^2-order part depends on $d_2\mathbf{a}$ as well as $\mathbf{a}d_1\mathbf{a}$, $\widehat{\boldsymbol{\mu}}d_1\mathbf{a}$ and $d_1^2\mathbf{a}$. However, in incomplete methods, the last three terms are omitted, i.e. $d_1\mathbf{q}_2$ *and* $d_1^2\mathbf{q}_1$ *are neglected with respect to* $d_2\mathbf{q}_1$, *although these terms are all of the same order of magnitude*. In particular, the contribution of $\mathbf{q}_2$, which accounts for both higher-order harmonic components (proportional to $\mathbf{a}^2$ and $\mathbf{a}\widehat{\boldsymbol{\mu}}$) and passive coordinates possibly triggered by the active component, is lost. These (arbitrarily) omitted terms could be expected to play some role in nonsteady-state motions around the equilibrium positions, i.e., on the stability of the equilibrium itself. To this end, the complete variational equation (22) is analyzed in Appendix B through a perturbation method. By assuming that steady-state solutions lose stability through a codimension-1 bifurcation, it is shown that *the critical amplitudes depend on slow time derivatives* (in particular on $d_1^2\mathbf{a}$) *if the bifurcation is of a dynamic type*, while *they are independent of slow time derivatives if the bifurcation is of a static type*. It is concluded that, if an incomplete method is applied, the amplitudes at which possible Hopf bifurcations occur are affected by ε^2-order errors.

5. Illustrative Examples

Three examples are developed to corroborate the comments of Section 4. They have been chosen so that the second approximation entails qualitative changes (Example 1) or quantitative changes only (Example 2) with respect to the first approximation. Example 3 is devoted to comparing the two versions of the IIM in a nonresonant case.

5.1. Example 1: A Static Bifurcation Problem of Codimension-1

A one-d.o.f. system of equations

$$\ddot{q} + c\dot{q} + F(q, \mu) = 0, \quad F(0, \mu) = 0 \quad \forall \mu, \tag{26}$$

is considered, where c is the damping coefficient and $F(q, \mu)$ is the nonlinear restoring elastic force, depending on the displacement q and on the parameter μ. Let the system undergo a static bifurcation at $(q, \mu) = (0, 0)$, i.e., $F_q(0, 0) = 0$, $F_{q\mu}(0, 0) \neq 0$. Exact equilibrium positions are solutions of the algebraic equations $F(q, \mu) = 0$. Here, the MSM is applied to obtain asymptotic solutions and to analyze stability. The following perturbation equations are derived from Equation (26):

$$\begin{aligned}
d_0^2 q_1 + c\, d_0 q_1 &= 0, \\
d_0^2 q_2 + c\, d_0 q_2 &= -2 d_0 d_1 q_1 - c d_1 q_1 - \frac{1}{2} F^0_{qq} q_1^2 - F^0_{q\mu} q_1 \widehat{\mu}, \\
d_0^2 q_3 + c\, d_0 q_3 &= -2 d_0 d_1 q_2 - (d_1^2 + 2 d_0 d_2) q_1 - c(d_2 q_1 + d_1 q_2) \\
&\quad - F^0_{qq} q_1 q_2 - F^0_{q\mu} q_2 \widehat{\mu} - \frac{1}{6} F^0_{qqq} q_1^3 - \frac{1}{2} F^0_{qq\mu} q_1^2 \widehat{\mu} - \frac{1}{2} F^0_{q\mu\mu} q_1 \widehat{\mu}^2,
\end{aligned} \tag{27}$$

in which $c = \mathrm{O}(1)$ has been assumed (heavily damped system). From Equation (27), $q_1 = a(t_1, t_2)$ and $q_2 = 0$ are drawn, together with the solvability conditions [28]. By combining them, the reconstituted amplitude equation (12) follows:

$$\begin{aligned}
\dot{a} &= -\frac{1}{c}\left[\varepsilon\left(\frac{1}{2} F^0_{qq} a^2 + F^0_{q\mu} a \widehat{\mu}\right)\right. \\
&\quad \left. + \varepsilon^2\left(\frac{1}{6} F^0_{qqq} a^3 + \frac{1}{2} F^0_{qq\mu} a^2 \widehat{\mu} + \frac{1}{2} F^0_{q\mu\mu} a \widehat{\mu}^2 + d_1^2 a\right)\right]
\end{aligned} \tag{28}$$

in which

$$d_1^2 a = \frac{1}{c^2}\left(F^0_{qq} a + F^0_{q\mu} \widehat{\mu}\right)\left(\frac{1}{2} F^0_{qq} a^2 + F^0_{q\mu} a \widehat{\mu}\right). \tag{29}$$

If the CIM is applied, the r.h.s. of Equation (28) must vanish in one piece. Therefore, because of the presence of the d_1^2 term, steady-state solutions erroneously depend on damping c. Instead, if the IIM is followed, this term is omitted in Equation (28) and the error is avoided. If a consistent method (CCM or ICM) is applied, $\widehat{\mu}$ has to be expanded in series of ε and the steady version of Equation (28) solved for any ε. Therefore, in the IIM, CCM and ICM, the dependence on c of the equilibrium positions disappears. It should be noted that if $F^0_{qq\mu} = F^0_{q\mu\mu} = 0$, which is a quite common case, *the last three methods give the same results*.

To compare the algorithms, a particular force-displacement relationship is chosen:

$$F(q,\mu) := q(1 - \mathrm{e}^{\mu} + k\sin q), \tag{30}$$

where k is an auxiliary parameter. Exact equilibrium positions are then given by $\mu_{\mathrm{EXC}} = \ln(1 + k\sin q)$ while Equation (28) reads:

$$\dot{a} = \frac{1}{c}\left\{\varepsilon(-ka^2 + a\widehat{\mu}) + \varepsilon^2\left[\left(\frac{1}{2} - \frac{1}{c^2}\right)a\widehat{\mu}^2 - \frac{1}{c^2}\left(2k^2a^3 - 3ka^2\widehat{\mu}\right)\right]\right\}. \tag{31}$$

By letting $\dot{a} = 0$, solving with respect to $\widehat{\mu}$ and returning to $\mu = \varepsilon\widehat{\mu}$, two CIM solutions μ_{CIM} are found from Equation (31), one of which is a spurious solution ($\mu \neq 0$ when $a = 0$). If terms proportional to c^{-2} are omitted in Equation (31), two IIM-solutions μ_{IIM} follow. If, on the other hand, the CCM is employed, by expanding $\widehat{\mu}$ up to ε^2-order, Equation (31) reads

$$\dot{a} = \frac{1}{c}\left\{\varepsilon(-ka^2 + \mu_1 a) + \varepsilon^2\left[a\mu_2 + \left(\frac{1}{2} - \frac{1}{c^2}\right)a\mu_1^2 - \frac{1}{c^2}\left(2k^2a^3 - 3ka^2\mu_1\right)\right]\right\}. \tag{32}$$

By requiring that $\dot{a} = 0\ \forall\varepsilon$, $\mu_1 = ka$ and $\mu_2 = -k^2a^2/2$ are found (index s dropped). Therefore, the (unique) solution is (CCM and ICM solutions):

$$\mu_{\mathrm{CCM}} = \varepsilon ka - \frac{\varepsilon^2}{2}k^2a^2. \tag{33}$$

To put in the correct light the ordering violation intrinsic to the inconsistent methods (see point (a) of Section 4.1), the exact solution, the CIM and IIM (true) solutions are expanded in ε series up to cubic terms:

$$\begin{aligned}
\widetilde{\mu}_{\mathrm{EXC}} &= \varepsilon ka - \frac{\varepsilon^2}{2}k^2a^2 + \frac{\varepsilon^3}{3}k(k^2 - 1/2)a^3 + \mathrm{O}(\varepsilon^4),\\
\widetilde{\mu}_{\mathrm{CIM}} &= \varepsilon ka - \frac{\varepsilon^2}{2}k^2a^2 + \frac{\varepsilon^3}{2}k^3(1 + \frac{1}{c^2})a^3 + \mathrm{O}(\varepsilon^4),\\
\widetilde{\mu}_{\mathrm{IIM}} &= \varepsilon ka - \frac{\varepsilon^2}{2}k^2a^2 + \frac{\varepsilon^3}{2}k^3a^3 + \mathrm{O}(\varepsilon^4).
\end{aligned} \tag{34}$$

From Equations (33) and (34), it follows that all the approximated solutions are correct up to ε^2-order. However, while the consistent solution is truncated, inconsistent solutions contain ε^3 and higher-order terms *which differ from the equal-order expansion terms of the exact solution*. This circumstance is due to the fact that terms of an order higher than two were neglected in deriving Equation (31). It is easy to check that if the perturbation algorithms is pursued one step further, while the consistent method furnishes the exact ε^3-order term, the inconsistent methods give nonpolynomial solutions which are correct to ε^3-order but contain incorrect ε^4 and higher-order terms. The influence of the inconsistent terms depends on the auxiliary parameter k. As a special case, if $k = \sqrt{2}/2$ the ε^3-order term of the exact solution vanishes (see Equation (34_1)). Therefore, the consistent solution (33) contain an error of order ε^4, while the inconsistent solutions still contain errors of order ε^3.

The (not expanded nonpolynomial) exact solution μ_{EXC}, the (not expanded nonpolynomial) inconsistent solutions μ_{CIM} and μ_{IIM}, and the consistent solution μ_{CCM} have been plotted in Figure 1 for different values of the auxiliary parameter k. It is seen that in all cases the CIM solution furnishes an unsatisfactory approximation of the exact solution when moderately

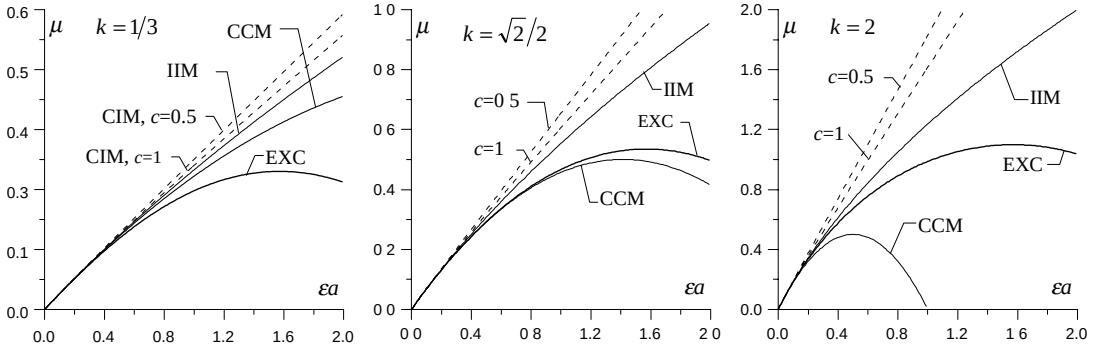

Figure 1. Equilibrium paths for the one-d.o.f. sample system.

large amplitudes are considered. In addition, it is damping-dependent. For k approaching $\sqrt{2}/2$ from below, the consistent solution gives a better representation than the IIM solution, whereas for sufficiently large k, the opposite occurs if one looks at the same interval of the amplitude.

The stability of the bifurcated path is analyzed next. By making the variation of Equation (26), with $F(q, \mu)$ given by Equation (30), and solving it, the following exact critical eigenvalue is found:

$$\lambda_{\mathrm{EXC}} = -\frac{c}{2}\left(1 - \sqrt{1 - \frac{4k}{c^2}\varepsilon a \cos(\varepsilon a)}\right) \tag{35}$$

since $q = \varepsilon a$. Hence, a saddle-node bifurcation occurs at $\varepsilon a_c = \pi/2$. It emerges that the eigenvalue depends on damping c. However, the critical state is independent of c. When an inconsistent method is applied, Equation (31) has to be varied, accounting for (CIM) or neglecting (IIM) the terms proportional to $1/c^2$, and expressing μ_{CIM} or μ_{IIM} as functions of a, respectively. In both cases, approximate expressions for the critical eigenvalue are found which exactly vanish at the limit point of the (approximate) equilibrium curves. Therefore, in the inconsistent methods, the stability analysis entails no further errors with respect to the equilibrium analysis, since instability is of the static type. On the contrary, if a consistent method is applied, Equation (32) has to be varied, accounting for (CCM) or neglecting (ICM) the terms proportional to $1/c^2$, and by expressing μ_1 and μ_2 in terms of a. The following forms for the critical eigenvalue are found:

$$\lambda_{\mathrm{CCM}} = -\varepsilon\,\frac{k}{c}a - \varepsilon^2\frac{k^2}{c^3}a^2 + \mathrm{O}(\varepsilon^3), \quad \lambda_{\mathrm{ICM}} = -\varepsilon\,\frac{k}{c}a + \mathrm{O}(\varepsilon^3). \tag{36}$$

According to the spirit of consistent methods, coefficients of different powers of ε should vanish separately at the criticality. However, by following this approach, no critical states are found in addition to the trivial one. In fact, at the ε-order, the equilibrium path is approximated by its tangent at the origin, which exhibits no limit point. A critical point appears only at the ε^2-order, which therefore entails a *qualitative* change in the first approximation. As observed in point (a) of Section 4.2, this qualitative change causes the failure of the consistent method.

Finally, it should be noted that Equation (36_1) represents the exact ε^2-order expansion of λ_{EXC}, while in Equation (36_2) the ε^2 term is absent, since a contribution to the acceleration has been incorrectly neglected. However, the ε^2 expansion λ_{CCM} represents a poor approximation

of λ_{EXC} near the critical value, since the last occurs for a large value of εa. In fact, if $\lambda_{CCM} = 0$ is solved (i.e., by following a procedure that contrasts with the consistent approach), an incorrect critical amplitude is determined, depending on damping c.

5.2. Example 2: A Resonance Problem of Codimension-2

A Duffing–Van der Pol oscillator in primary resonance with an external excitation is considered, having the equation:

$$\ddot{q} + q - \nu\dot{q} + c_1 q^3 + c_2\dot{q}^3 = p\cos(1+\mu)t, \tag{37}$$

where μ and ν are control parameters. When $\nu = 0$, the homogeneous linear part of Equation (37) admits a couple of purely imaginary eigenvalues. Moreover, when $\mu = 0$, the oscillator is in resonance with the external excitation. Therefore, the problem has codimension $m = 2$. By ordering the control parameters as $\mu = \varepsilon\widehat{\mu}$, $\nu = \varepsilon\widehat{\nu}$, the excitation amplitude (imperfection parameter) as $p = \varepsilon^2\widehat{p}$ and expanding the displacement as $q = \varepsilon^{1/2}q_1 + \varepsilon^{3/2}q_2 + \varepsilon^{5/2}q_3 + \cdots$, the perturbation equations are

$$\begin{aligned}
d_0^2 q_1 + q_1 &= 0,\\
d_0^2 q_2 + q_2 &= \widehat{\nu} d_0 q_1 - c_1 q_1^3 - c_2 (d_0 q_1)^3 - 2 d_0 d_1 q_1 + \widehat{p}\cos(t_0 + \widehat{\mu} t_1),\\
d_0^2 q_3 + q_3 &= \widehat{\nu}(d_0 q_2 + d_1 q_1) - 3 c_1 q_1^2 q_2 - 3 c_2 (d_0 q_1)^2 (d_1 q_1 + d_0 q_2)\\
&\quad - 2 d_0 d_1 q_2 - (d_1^2 + 2 d_0 d_2) q_1.
\end{aligned} \tag{38}$$

By solving Equations (38),

$$q_1 = a\cos(t+\vartheta), \quad q_2 = \frac{a^3}{32}[c_1\cos(3t+3\vartheta) + c_2\sin(3t+3\vartheta)],$$

and the solvability conditions are obtained [28]. The reconstituted amplitudes equations read

$$\begin{aligned}
\dot{a} &= \frac{\varepsilon}{2}\left(a\widehat{\nu} - \frac{3}{4}c_2 a^3 + \widehat{p}\sin\gamma\right)\\
&\quad + \varepsilon^2\left[\frac{3}{64}c_1 c_2 a^5 + \frac{1}{2}(\widehat{\mu} - d_1\gamma)\left(a\widehat{\nu} - \frac{9}{4}c_2 a^3 - 2 d_1 a\right) + \frac{1}{2}a d_1^2\gamma\right],\\
\dot{\gamma} &= \varepsilon\left(\widehat{\mu} - \frac{3}{8}c_1 a^2 + \frac{\widehat{p}}{2a}\cos\gamma\right)\\
&\quad + \varepsilon^2\left[\frac{3}{256}\left(3c_2^2 - c_1^2\right)a^4 - \frac{1}{2}d_1 a\left(\frac{3}{4}c_2 a - \frac{\nu}{a}\right) - \frac{1}{2a}d_1^2 a + \frac{1}{2}(\widehat{\mu} - d_1\gamma)^2\right],
\end{aligned} \tag{39}$$

where $\gamma = \widehat{\mu} t_1 - \vartheta$ is the phase difference between the excitation and the response, and $d_1^2 a$ and $d_1^2\gamma$ can be expressed in terms of a and γ by differentiating the first-order part of Equations (39).

Inconsistent methods are considered first. According to the CIM, by substituting $d_1 a$, $d_1\gamma$, $d_1^2 a$ and $d_1^2\gamma$ in Equations (39) and by requiring $\dot{a} = \dot{\gamma} = 0$ steady-state solutions are found by numerically solving the nonlinear equations. Stability is then analyzed using the variational equation. If, instead, the IIM is applied, all the derivatives $d_1 a$, $d_1\gamma$, $d_1^2 a$ and $d_1^2\gamma$ appearing in the r.h.s. of Equations (39) have to be omitted. As discussed in Section 3.2, an alternative

(IIM-II) exists to the IIM. It calls for the omission of $d_1\vartheta$ and $d_1^2\vartheta$ (instead of $d_1\gamma$ and $d_1^2\gamma$) in Equations (39), together with $d_1 a$ and $d_1^2 a$. Since, $d_1\gamma = \widehat{\mu} - d_1\vartheta$ and $d_1^2\gamma = -d_1^2\vartheta$, according to this method $d_1\gamma = \widehat{\mu}$ and $d_1^2\gamma = 0$ must be posed in Equations (39).

Consistent methods are then applied. As a first step, according to the theory previously developed, the two control parameters $\widehat{\mu}$ and $\widehat{\nu}$ should be expanded in series of ε, in order to obtain steady solutions of the type $\mu = \mu(a, \gamma)$ and $\nu = \nu(a, \gamma)$. However, with the aim of fixing one parameter, e.g., ν, steady solutions are sought in the form of $\mu = \mu(a, \nu)$ and $\gamma = \gamma(a, \nu)$. The scaled parameter $\widehat{\mu}$ and the phase γ are therefore expanded in series of ε:

$$\widehat{\mu} = \mu_1 + \varepsilon\mu_2, \quad \gamma = \gamma_1 + \varepsilon\gamma_2. \tag{40}$$

By substituting Equations (40) in Equations (39) and separately vanishing terms of order ε and ε^2, the following sets of equations are drawn:

$$\begin{aligned} a\widehat{\nu} - \frac{3}{4}c_2 a^3 + \widehat{p}\sin\gamma_1 &= 0, \\ \mu_1 - \frac{3}{8}c_1 a^2 + \frac{\widehat{p}}{2a}\cos\gamma_1 &= 0; \end{aligned} \tag{41}$$

$$\begin{aligned} (\widehat{p}\cos\gamma_1)\gamma_2 &= -\frac{3}{32}c_1 c_2 a^5 - \widehat{\mu}_1\left(a\widehat{\nu} - \frac{9}{4}c_2 a^3\right), \\ \mu_2 - \left(\frac{\widehat{p}}{2a}\sin\gamma_1\right)\gamma_2 &= \frac{3}{256}(c_1^2 - 3c_2^2)a^4 - \frac{1}{2}\widehat{\mu}_1^2, \end{aligned} \tag{42}$$

where, in Equations (42), the t_1-derivatives have been put as equal to zero, because of Equations (41). Equations (41) and (42) furnish steady-state solutions according to both CCM and ICM. Equations (41) are a set of *nonlinear* equations in the unknowns μ_1 and γ_1; they constitute the first approximation of the MSM. Equations (42) are a set of *linear* equations in the unknowns μ_2 and γ_2 that supply the corrections to the first approximation. After having determined steady solutions, stability is analyzed by asymptotically solving the relevant variational equations.

In order to make a quantitative comparison of different solutions, numerical values of the auxiliary parameters c_1 and c_2 and of the excitation amplitude p are chosen, namely $c_1 = 1/30$, $c_2 = 1/60$, $p = 1/5$. Moreover, the control parameter ν is kept fixed at $\nu = 1/40$. The steady-state amplitude a and the phase γ are then plotted vs. μ. Figure 2 shows the first approximation, common to all methods. It is found that the amplitude reaches a limit point at A, saddle-node bifurcations occur at points B and C and a Hopf bifurcation manifests itself at H. The steady solution loses stability at B, regains it at C, and again becomes unstable at H.

The second approximation is considered next according to the methods illustrated above. The results are shown in Figure 3. It is seen that the second approximation entails only quantitative modifications of the curves, i.e. the first approximation captures all the qualitative aspects of the phenomenon. It is remarkable that the IIM-II produces only small modifications with respect to the first approximation. The remaining three methods, in contrast, give curves that are very close to each other and fairly distant from the first approximation curve, above all at higher amplitudes. The CCM expansion is not valid near the limit point A, where the coefficients matrix of the unknowns γ_2 and μ_2 appearing in Equations (42) becomes singular. However, this drawback does not entail errors on points B and C, where instability occurs,

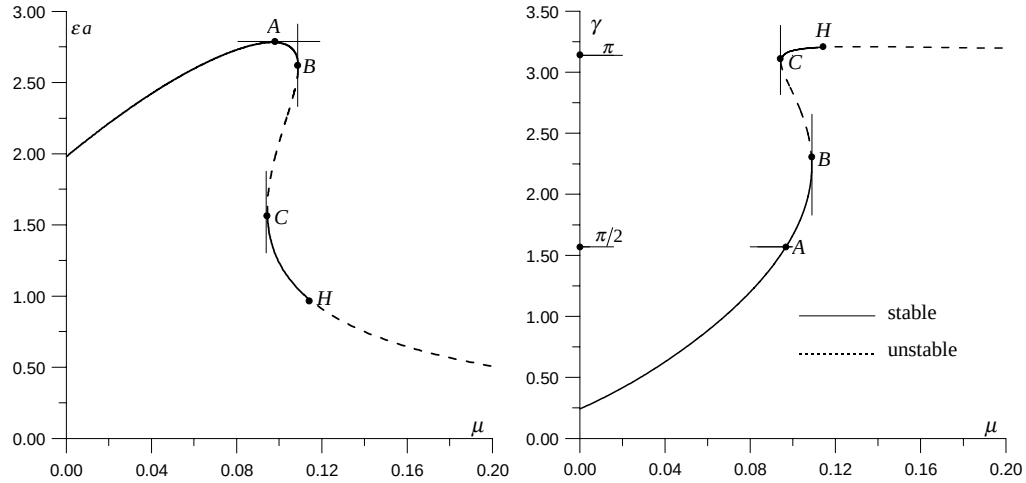

Figure 2. Amplitude and phase response vs. detuning in the first approximation.

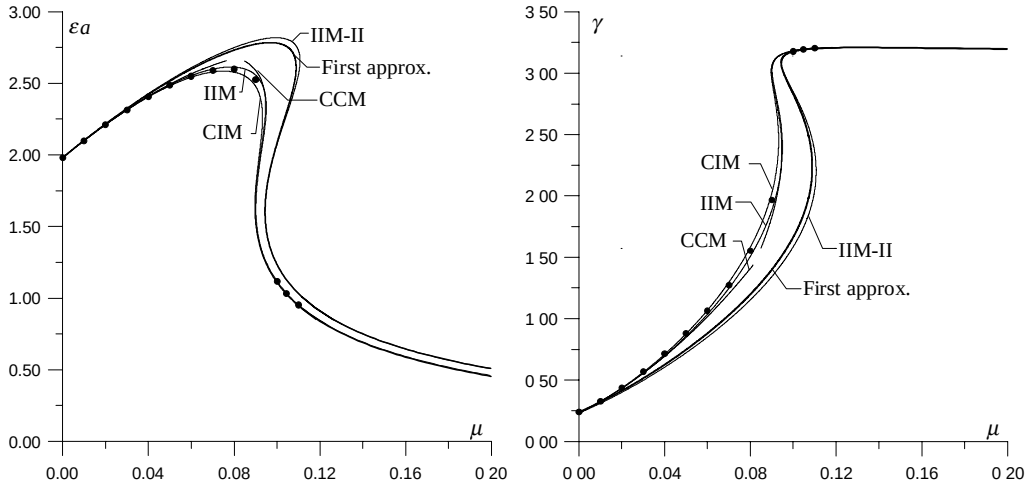

Figure 3. Amplitude and phase response vs. detuning in the second approximation; • numerical results.

although B is fairly close to A. Vice versa, if a and γ, instead of μ and γ, were expanded in series, the method would fail precisely at points B and C. The problem would be completely avoided if ν were expanded in series instead of being kept constant.

The second-order perturbative solutions are compared with exact numerical solutions obtained by directly integrating the equation of motion (37) and performing an FFT of the regime response. The amplitudes of the fundamental harmonic for different μ's in the stable zones are marked in Figure 3. It is found that the CIM and IIM give an excellent approximation of the exact solutions over the whole range, while the small errors of the CCM are due to the presence of the amplitude limit point. The IIM-II, in contrast, is affected by errors at large amplitudes.

An eigenvalue analysis has been performed in [28]. It has been found that CCM and IIM solutions are different, but practically coincident at B and C points and slightly different at H, according to the behavior predicted in Section 4.2 and in Appendix B.

5.3. EXAMPLE 3: A FREE VIBRATION PROBLEM OF CODIMENSION-1

This example shows how the IIM-II leads to wrong results, even in a nonresonant problem. The free vibrations of a weakly nonlinear undamped Duffing oscillator are governed by the equation

$$\ddot{q} + q + \mu q^3 = 0, \tag{43}$$

where the small parameter μ is identified with the coefficient of the cubic part of the restoring force. By applying the Lindstedt–Poincaré method, the following nonlinear frequency is found (see, e.g., [34]):

$$\omega^* = 1 + \frac{3}{8}\mu a^2 - \frac{15}{256}\mu^2 a^4. \tag{44}$$

Here the MSM is used to compare the different versions. By ordering the parameter as $\mu = \varepsilon\widehat{\mu}$ and expanding the displacement as $q = q_1 + \varepsilon q_2 + \varepsilon^2 q_3 + \cdots$, the perturbation equations are:

$$\begin{aligned} d_0^2 q_1 + q_1 &= 0, \\ d_0^2 q_2 + q_2 &= -\widehat{\mu} q_1^3 - 2 d_0 d_1 q_1, \\ d_0^2 q_3 + q_3 &= -3\widehat{\mu} q_1^2 q_2 - 2 d_0 d_1 q_2 - \left(d_1^2 + 2 d_0 d_2\right) q_1. \end{aligned} \tag{45}$$

By solving them, it is found that

$$q_1 = A\,\mathrm{e}^{it_0} + \text{c.c.}, \quad q_2 = \frac{1}{8}\mu A^3\,\mathrm{e}^{3it_0} + \text{c.c.}$$

By combining the solvability conditions [28], the reconstituted complex amplitude equation reads

$$\dot{A} = \varepsilon\frac{3}{2} i\widehat{\mu} A^2 \overline{A} + \frac{\varepsilon^2}{2}\left(\frac{3}{8} i\widehat{\mu}^2 A^3 \overline{A}^2 + i d_1^2 A\right), \tag{46}$$

where, $d_1^2 A = -(9/4)\widehat{\mu}^2 A^3 \overline{A}^2$ and an overbar denotes a complex conjugate.

According to the complete methods (CIM and CCM), by substituting $d_1^2 A$ in Equation (46), by letting $A = a/2\,\mathrm{e}^{i\vartheta}$ and separating the real and imaginary parts, it follows that:

$$\begin{aligned} \dot{a} &= 0, \\ \dot{\vartheta} &= \varepsilon\frac{3}{8}\widehat{\mu} a^2 - \varepsilon^2 \frac{15}{256}\widehat{\mu}^2 a^4. \end{aligned} \tag{47}$$

Since the problem has codimension-1, the unique (real) amplitude equation is Equation (47_1) which furnishes $a = \text{const}$, since the system is conservative. However, Equation (47_2), which is uncoupled from the first, also gives an important result, i.e. the nonlinear frequency. This is exactly the same as Equation (44).

If the IIM is applied, the same correct result is found, since the method calls for the omission of $d_1^2 a$ only in Equation (46), which is in fact zero. However, if the IIM-II is followed, $d_1^2 A$ is instead omitted in Equation (46), from which

$$\begin{aligned} \dot{a} &= 0, \\ \dot{\vartheta} &= \varepsilon\frac{3}{8}\widehat{\mu} a^2 + \varepsilon^2 \frac{3}{256}\widehat{\mu}^2 a^4, \end{aligned} \tag{48}$$

follow, instead of Equations (47). The nonlinear frequency given by the IIM-II is therefore incorrect.

6. Conclusions

Higher-order multiple-scale methods (MSM) for the analysis of general multiparameters mechanical systems have been critically discussed. The following conclusions have been drawn:

1. Parameters $\boldsymbol{\mu}$ play a fundamental role in the perturbation analysis of a system. They must be chosen in such a way that the associated linear system becomes autonomous when $\boldsymbol{\mu} \to \mathbf{0}$, and admits a cluster of eigenvalues with zero real part. By counting the conjugate roots in pairs, the number of eigenvalues possibly involved in resonances (active eigenvalues) plus the number of the resonance conditions themselves is defined as the *codimension m* of the problem.
2. The number of solvability conditions obtained at each step of the algorithm is equal to the codimension of the problem. The equations involve as many amplitudes as there are active eigenvalues, plus as many phase combinations as there are resonance conditions.
3. Solvability conditions at different orders are combined according to the *reconstitution method*. The equations obtained are an approximate representation of a *reduced dynamical system*, able to capture the main aspects of the dynamics of the original system. However, the reduced dynamical system is not known in closed form, only an asymptotic form being furnished by the MSM.
4. There are four procedures for the analysis of the reconstituted amplitude equations. They have been named Complete Inconsistent Method (CIM), Incomplete Inconsistent Method (IIM), Complete Consistent Method (CCM) and Incomplete Consistent Method (ICM). In consistent methods, the asymptotic nature of the reduced dynamical system is consistently taken into account, whereas in inconsistent methods, the reduced dynamical system is dealt with as if it were expressed in closed-form. In the complete methods, all terms deriving from the analysis are retained, while in incomplete methods the amplitude derivatives $d_1\mathbf{a}$ and $d_1^2\mathbf{a}$, appearing in the higher-order solvability conditions, are neglected.
5. If steady-state solutions are sought through an inconsistent method, an ordering violation occurs. In fact, if the nonpolynomial inconsistent solutions are expanded in series, powers of ε greater than the highest power present in the equations are found. These higher-order terms are incorrect, since they do not represent the series expansion of the (unknown) exact solution. On the contrary, consistent methods furnish the correct coefficients of the series expansion, up to the equation order. Indeed, inconsistent terms can sometimes improve the truncated solution. However, when this happens it cannot be predicted in advance.
6. Inconsistent methods are usually applied in the literature by first expanding the parameters and then recombining them in the reconstituted equation. Therefore, the expansion plays no role and it can be avoided by using the illustrated procedure.
7. To apply consistent methods a number of control parameters equal to the codimension of the problem has to be expanded. Steady-state solutions are described as functions of the amplitudes, rather than of the parameters. However, these expressions can easily be inverted or found directly in inverse form. Mixed solutions are also allowed, in which a set of m amplitudes and parameters is expressed in terms of the m remaining ones.

8. If the first approximation exhibits bifurcation points, some difficulties arise in the search for steady-state solutions according to consistent methods. It has been shown that bifurcation persists at higher orders only if higher-order terms satisfy special conditions. The use of local series expansions, possibly entailing fractional powers of ε, has also been discussed.
9. The t_1-derivatives accounted for in the CIM affect steady-state solutions. This result appears to be *qualitatively incorrect*, since the steady amplitudes, which are constant in time, cannot depend on their time-derivatives. Paradoxically, by using the CIM, static equilibrium positions of a damped mechanical system have been found to depend on the damping coefficient. This drawback is avoided in the IIM simply by neglecting $d_1\mathbf{a}$ and $d_1^2\mathbf{a}$. However, this omission is arbitrary and affects stability.
10. The IIM is widely applied in the literature in a slightly different form from that presented here (the so-called version II of the MSM). Specifically, the slow derivatives of the *phases of the complex amplitudes* are neglected instead of the *phase combinations*. This entails errors of ε^2-order in the steady-state solutions (in resonant cases) and in nonlinear frequencies (both in resonant and nonresonant cases).
11. For a particular class of systems, which is encountered fairly frequently in applications [23, 24, 26], the IIM and CCM furnish the same steady-state solutions. This occurs whenever the parameters appear linearly in the first-order part of the reconstituted amplitude equations and are absent in the higher-order parts.
12. The slow-time amplitude derivatives omitted in incomplete methods influence the stability of steady-state solutions, since they describe the contribution to the acceleration of higher-order harmonics as well as of the passive coordinates possibly triggered by the active ones. It has been shown through a consistent approach that omission of these derivatives entails an error of ε^2-order on the critical amplitude if it is associated with a dynamic bifurcation, while no error exists at the ε^2-order if the bifurcation is of a static type. However, the error persists in all noncritical states.
13. Consistent methods sometimes fail in stability analysis. It happens whenever the second approximation is responsible for the occurrence of a critical condition which is absent in the first approximation, i.e. it entails *qualitative* changes. In these cases, an inconsistent approach can instead give correct qualitative information. The failure of the consistent approach depends on the occurrence of an ordering violation caused by higher-order terms prevailing over lower-order terms. This type of problem could be analyzed more easily using a different approach, ordering competitive terms at the same level in the perturbation scheme (see, e.g., [33]).
14. Three examples are shown to illustrate the concepts described: a static codimension-1 bifurcation problem, a simultaneous dynamic bifurcation/external resonance problem with codimension-2 and a free oscillation problem. The numerical results obtained confirm findings obtained theoretically.

Appendix A: Consistent Second-Order Solutions around First-Order Bifurcation Points

According to consistent methods, steady-state solutions $\widehat{\boldsymbol{\mu}} = \widehat{\boldsymbol{\mu}}(\mathbf{a};\varepsilon)$ satisfy the following equation:

$$\varepsilon\mathbf{f}_1(\mathbf{a};\widehat{\boldsymbol{\mu}}) + \varepsilon^2\mathbf{f}_2(\mathbf{a};\widehat{\boldsymbol{\mu}}) + \varepsilon^3\mathbf{f}_3(\mathbf{a};\widehat{\boldsymbol{\mu}}) + \cdots = \mathbf{0} \quad \forall\varepsilon, \tag{49}$$

where terms up to ε^3-order have been considered and the inessential slow-time derivatives neglected. By performing the expansion $\widehat{\boldsymbol{\mu}} = \boldsymbol{\mu}_1 + \varepsilon\boldsymbol{\mu}_2 + \varepsilon^2\boldsymbol{\mu}_3$, the following perturbation equations are drawn:

$$\mathbf{f}_1(\mathbf{a};\boldsymbol{\mu}_1) = \mathbf{0},$$

$$\mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a};\boldsymbol{\mu}_1)\boldsymbol{\mu}_2 + \mathbf{f}_2(\mathbf{a};\boldsymbol{\mu}_1) = \mathbf{0},$$

$$\mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a};\boldsymbol{\mu}_1)\boldsymbol{\mu}_3 + \frac{1}{2}\mathbf{f}_{1,\boldsymbol{\mu}\boldsymbol{\mu}}(\mathbf{a};\boldsymbol{\mu}_1) + \mathbf{f}_{2,\boldsymbol{\mu}}(\mathbf{a};\boldsymbol{\mu}_1)\boldsymbol{\mu}_2 + \mathbf{f}_3(\mathbf{a};\boldsymbol{\mu}_1) = \mathbf{0}. \tag{50}$$

Let us assume that the first approximation $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_1(\mathbf{a})$, obtained by solving Equation (50_1), admits a simple bifurcation point $C \equiv (\mathbf{a} = \mathbf{a}_c, \boldsymbol{\mu}_1 = \widehat{\boldsymbol{\mu}}_c)$. Since $\mathbf{f}_{1,\boldsymbol{\mu}}(\mathbf{a};\boldsymbol{\mu}_1)$ is singular at C, Equations (50) furnish nonuniformly valid solutions, since $\boldsymbol{\mu}_2(\mathbf{a})$, $\boldsymbol{\mu}_3(\mathbf{a})$, ..., and higher-order terms tend to infinity when $\mathbf{a} \to \mathbf{a}_c$. As a special case, Equations (50) does admit uniformly valid solutions if the known terms of the linear equations belong to the range of the singular operator.

To find a uniformly valid expansion around C in the general case and to investigate the geometrical meaning of the special case, it is convenient to expand *both* variables $\mathbf{a}$ and $\widehat{\boldsymbol{\mu}}$ around C. By letting $\dot{\mathbf{a}} = \mathbf{a} - \mathbf{a}_c$ and $\dot{\boldsymbol{\mu}} = \widehat{\boldsymbol{\mu}} - \widehat{\boldsymbol{\mu}}_c$, Equation (49) reads

$$\varepsilon\left[\mathbf{f}^c_{1,\mathbf{a}}\dot{\mathbf{a}} + \mathbf{f}^c_{1,\boldsymbol{\mu}}\dot{\boldsymbol{\mu}} + \frac{1}{2}(\mathbf{f}^c_{1,\mathbf{a}a}\dot{\mathbf{a}}^2 + 2\mathbf{f}^c_{1,\mathbf{a}\boldsymbol{\mu}}\dot{\mathbf{a}}\dot{\boldsymbol{\mu}} + \mathbf{f}^c_{1,\boldsymbol{\mu}\boldsymbol{\mu}}\dot{\boldsymbol{\mu}}^2) + \cdots\right]$$
$$+ \varepsilon^2[\mathbf{f}^c_2 + \mathbf{f}^c_{2,\mathbf{a}}\dot{\mathbf{a}} + \mathbf{f}^c_{2,\boldsymbol{\mu}}\dot{\boldsymbol{\mu}} + \cdots] + \varepsilon^3[\mathbf{f}^c_3 + \cdots] = \mathbf{0} \quad \forall\varepsilon, \tag{51}$$

where index c denotes that the derivatives are evaluated at C.

To gain insight into the problem, let us consider first the one-dimensional case $m = 1$, in which all quantities in Equation (51) are scalar. Since $f^c_{1,a} = 0$, $f^c_{1,\mu} = 0$, two cases arise:

(a) $f^c_2 \neq 0$: in this case, $\dot{a} = \mathrm{O}(\varepsilon^{1/2})$, $\dot{\mu} = \mathrm{O}(\varepsilon^{1/2})$ and the leading terms in Equation (51) are

$$\varepsilon[f^c_{1,aa}\dot{a}^2 + 2f^c_{1,a\mu}\dot{a}\dot{\mu} + f^c_{1,\mu\mu}\dot{\mu}^2] + 2\varepsilon^2 f^c_2 = 0 \quad \forall\varepsilon; \tag{52}$$

(b) $f^c_2 = 0$: in this case, $\dot{a} = \mathrm{O}(\varepsilon)$, $\dot{\mu} = \mathrm{O}(\varepsilon)$ and the leading terms in Equation (51) are

$$\varepsilon[f^c_{1,aa}\dot{a}^2 + 2f^c_{1,a\mu}\dot{a}\dot{\mu} + f^c_{1,\mu\mu}\dot{\mu}^2] + 2\varepsilon^2[f^c_{2,a}\dot{a} + f^c_{2,\mu}\dot{\mu}] + 2\varepsilon^3 f^c_3 = 0 \quad \forall\varepsilon. \tag{53}$$

In case (a), Equation (52) describes a conic curve in the $(\dot{a}, \dot{\mu})$-plane. When $\varepsilon = 0$ it degenerates into two real and distinct straight lines, tangent at the first approximation curve at C, i.e. $f^c_{1,aa}f^c_{1,\mu\mu} - (f^c_{1,a\mu})^2 < 0$. Curve (52) is therefore a hyperbola admitting the two lines as asymptotes. It follows that, if $f^c_2 \neq 0$, *the bifurcation predicted by the first approximation is destroyed by higher-order terms.*

In case (b), Equation (53) still describes a hyperbola. It degenerates into two straight lines if and only if the following determinant vanishes:

$$\Delta_c = \begin{vmatrix} 2\varepsilon^2 f^c_3 & \varepsilon f^c_{2,a} & \varepsilon f^c_{2,\mu} \\ \varepsilon f^c_{2,a} & f^c_{1,aa} & f^c_{1,a\mu} \\ \varepsilon f^c_{2,\mu} & f^c_{1,a\mu} & f^c_{1,\mu\mu} \end{vmatrix} = 0. \tag{54}$$

From Equation (54) a further condition on f^c_3 is found.

If $f_2^c = 0$, Equation (50_2) can be solved and μ_2 determined by applying the L'Hôpital theorem. It is easy to check that if Equation (54) is satisfied, Equation (50_3) can also be solved. It is concluded that *if the known terms of Equations* (50) *satisfy the relevant solvability conditions, the bifurcation existing at the ε-order persists at higher orders, otherwise it is destroyed*.

Previous findings can be extended to the general case $m > 1$, by solving the equation $\mathbf{f}^c_{1,\mathbf{a}}\dot{\mathbf{a}} + \mathbf{f}^c_{1,\boldsymbol{\mu}}\dot{\boldsymbol{\mu}} = \mathbf{0}$ and then projecting Equation (51) on the one-dimensional eigenspace of $\mathbf{f}^c_{1,\mathbf{a}}$. For the sake of brevity, details will not be presented here.

To sum up, if $\mathbf{f}^c_2$ does not belong to the range of $\mathbf{f}^c_{1,\boldsymbol{\mu}}$ the bifurcation is destroyed, and use has to be made of series expansion of *fractional powers* of ε, namely

$$\mathbf{a} = \mathbf{a}_c + \varepsilon^{1/2}\mathbf{a}_1, \quad \widehat{\boldsymbol{\mu}} = \boldsymbol{\mu}_c + \varepsilon^{1/2}\boldsymbol{\mu}_1 + \varepsilon\boldsymbol{\mu}_2 + \cdots, \tag{55}$$

with $\mathbf{a}_1$ assuming the meaning of a detuning. Vice versa, if $\mathbf{f}^c_2$ does belong to the range of $\mathbf{f}^c_{1,\boldsymbol{\mu}}$, series expansions of *integer powers* of ε work well:

$$\mathbf{a} = \mathbf{a}_c + \varepsilon\mathbf{a}_1, \quad \widehat{\boldsymbol{\mu}} = \boldsymbol{\mu}_c + \varepsilon\boldsymbol{\mu}_1 + \varepsilon^2\boldsymbol{\mu}_2 + \cdots. \tag{56}$$

It should be observed that such solutions are of a *local character*, i.e. they are valid only around C, in contrast to the *global character* of the series obtained from Equations (50), which are valid for any $\mathbf{a}$. Indeed, as has been discussed, Equations (50) still hold in the whole interval of $\mathbf{a}$ if the bifurcation persists. However, numerical problems arise around bifurcation points. The question constitutes an interesting matter for future work.

Appendix B: Consistent Perturbation Analysis of the Variational Equation

The CCM variational equation (22) is studied and the contribution of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ terms analyzed. In compact form, it reads

$$\delta\dot{\mathbf{a}} = (\varepsilon\mathbf{L}(\mathbf{a}_s) + \varepsilon^2\mathbf{M}(\mathbf{a}_s))\delta\mathbf{a}, \tag{57}$$

where

$$\begin{aligned} \mathbf{L}(\mathbf{a}_s) &:= \mathbf{f}_{1,\mathbf{a}}|_{\mathbf{a}=\mathbf{a}_s}, \\ \mathbf{M}(\mathbf{a}_s) &:= \mathbf{f}_{2,\mathbf{a}} - \mathbf{f}_{1,\boldsymbol{\mu}\mathbf{a}}\mathbf{f}^{-1}_{1,\boldsymbol{\mu}}\mathbf{f}_2 + \boldsymbol{\alpha}_1\mathbf{L} + \boldsymbol{\alpha}_2\mathbf{L}^2|_{\mathbf{a}=\mathbf{a}_s}, \end{aligned} \tag{58}$$

are the first and second-order parts, respectively, of the $m \times m$ variational matrix evaluated at $\mathbf{a} = \mathbf{a}_s$, $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_1(\mathbf{a}_s)$, $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_2(\mathbf{a}_s)$. By letting $\delta\mathbf{a} = \mathbf{w}\exp(\varepsilon\lambda t)$, an algebraic eigenvalue problem follows from Equation (57):

$$[\mathbf{L}(\mathbf{a}_s) + \varepsilon\mathbf{M}(\mathbf{a}_s)]\mathbf{w} = \lambda\mathbf{w}. \tag{59}$$

By solving it, the eigenpairs $(\lambda, \mathbf{w})$ are determined as functions of the m steady-state amplitudes $\mathbf{a}_s$ and of the perturbation parameter ε. To analyze bifurcations of codimension-1 it is convenient *to keep $m-1$ amplitudes fixed*, e.g., $a_1, a_2, \ldots, a_{m-1}$, and to vary the last one, $a^* := a_m$, which assumes the meaning of *distinguished amplitude*. The eigenpairs of Equation (59) are therefore functions of a^* and ε.

A consistent approach is followed to solve the eigenvalue problem. A monoparametric family of solutions $\lambda = \lambda(\varepsilon)$, $\mathbf{w} = \mathbf{w}(\varepsilon)$, $a^* = a^*(\varepsilon)$, is sought. They are expanded around $\varepsilon = 0$, as

$$\left\{ \begin{array}{c} \lambda \\ \mathbf{w} \\ a^* \end{array} \right\} = \left\{ \begin{array}{c} \lambda_0 \\ \mathbf{w}_0 \\ a_0^* \end{array} \right\} + \varepsilon \left\{ \begin{array}{c} \lambda_1 \\ \mathbf{w}_1 \\ a_1^* \end{array} \right\} + \cdots . \tag{60}$$

Consequently, it follows that

$$\mathbf{L} = \mathbf{L}_0 + \varepsilon \mathbf{L}^0_{,\mathbf{a}} a_1^* + \cdots , \quad \mathbf{M} = \mathbf{M}_0 + \cdots , \tag{61}$$

where $\mathbf{L}_0 = \mathbf{L}(a_0^*)$, $\mathbf{M}_0 = \mathbf{M}(a_0^*)$ and $\mathbf{L}^0_{,\mathbf{a}} = [\partial \mathbf{L}/\partial a^*]_{a_0^*}$. By substituting Equations (60) and (61) in Equation (59), the following perturbation equations are obtained:

$$\begin{aligned} \mathbf{L}_0 \mathbf{w}_0 - \lambda_0 \mathbf{w}_0 &= \mathbf{0}, \\ \mathbf{L}_0 \mathbf{w}_1 - \lambda_0 \mathbf{w}_1 &= -\mathbf{M}_0 \mathbf{w}_0 + \lambda_1 \mathbf{w}_0 - \mathbf{L}^0_{,\mathbf{a}} \mathbf{w}_0 a_1^*. \end{aligned} \tag{62}$$

From Equation (62_1), λ_0 and $\mathbf{w}_0 = \mathbf{u}$ are derived as an eigenpair of $\mathbf{L}_0$. From the solvability condition of Equation (62_2), it follows that

$$\lambda_1 = \mathbf{v}^H (\mathbf{M}_0 \mathbf{u} + \mathbf{L}^0_{,\mathbf{a}} \mathbf{u}\, a_1^*), \tag{63}$$

where $\mathbf{v}$ is the left eigenvector of $\mathbf{L}_0$ associated with λ_0 and $\mathbf{v}^H \mathbf{u} = 1$ has been assumed. Equations (60_1) and (63) provide an asymptotic expression of the eigenvalue λ when a^* is perturbed around a_0^*.

Critical states are now sought in which one eigenvalue λ crosses the imaginary axis from the left, while the remaining eigenvalues still lie on the $\mathrm{Re}\,\lambda < 0$ half-plane. For fixed $a_1, a_2, \ldots, a_{m-1}$, the critical states are identified by the critical values a_c^* of the distinguished amplitude a^*. By varying the amplitudes, an ε-dependent, $(m-1)$-manifold exists in the m-dimensional space, having the equation $a_c^* = a_c^*(a_1, a_2, \ldots, a_{m-1}; \varepsilon)$.

The general case of a Hopf bifurcation, $\mathrm{Re}\,\lambda = 0$, $\mathrm{Im}\,\lambda \neq 0$, is considered. By requiring that $\mathrm{Re}\,\lambda = 0$, $\forall \varepsilon$,

$$\mathrm{Re}\,\lambda_0 = 0, \quad \mathrm{Re}\,\lambda_1 = 0 \tag{64}$$

follow. From Equation (64_1), the zero-order part a_{0c}^* of the critical distinguished amplitude a_c^* is determined (if it exists; see point (a) of Section 4.2). It is, e.g., evaluated by applying the Routh–Hurwitz criterion to the characteristic polynom of the eigenvalue problem (62_1), which reads

$$\mathbf{L}_0 \mathbf{u}_c = i\omega_0 \mathbf{u}_c, \tag{65}$$

where $\omega_0 = \mathrm{Im}\,\lambda_0$ and $\mathbf{u}_c$ is the zero-order critical eigenvector. Then, from Equations (64_2) and (63), the first-order part a_{1c}^* of the critical distinguished amplitude is determined:

$$a_{1c}^* = -\mathrm{Re}\,(\mathbf{v}_c^H \mathbf{L}^0_{,\mathbf{a}} \mathbf{u}_c)^{-1} \left\{ \mathrm{Re}\,[\mathbf{v}_c^H (\mathbf{f}_{1,\boldsymbol{\mu}\mathbf{a}} \mathbf{f}_{1,\boldsymbol{\mu}}^{-1} \mathbf{f}_2 - \mathbf{f}_{2,\mathbf{a}} - \omega_0^2 \boldsymbol{\alpha}_2) \mathbf{u}] \right\}_{\mathbf{a}=\mathbf{a}_s}, \tag{66}$$

where Equations (58_2) and (65) have been used.

It is concluded that a_{1c}^* depends on $\boldsymbol{\alpha}_2$ if $\omega_0 \neq 0$. Therefore, *if a Hopf bifurcation occurs at the criticality, the critical distinguished amplitude depends on* $d_1^2 \mathbf{a}$; *if the bifurcation is of static type* ($\omega_0 = 0$), *the critical distinguished amplitude is independent of* $d_1^2 \mathbf{a}$.

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References

1. Nayfeh, A. H., *Perturbation Methods*, Wiley, New York, 1973.
2. Nayfeh, A. H. and Mook, D.T., *Nonlinear Oscillations*, Wiley, New York, 1979.
3. Nayfeh, A. H. and Nayfeh, S. A., 'On nonlinear modes of continuous systems', *Journal of Vibration and Acoustics* **116**, 1994, 129–136.
4. Nayfeh, A. H. and Nayfeh, S. A., 'Nonlinear normal modes of a continuous system with quadratic nonlinearities', *Journal of Vibration and Acoustics* **117**, 1995, 199–205.
5. Nayfeh, A. H., Chin, C., and Nayfeh, S. A., 'On nonlinear normal modes of systems with internal resonance', *Journal of Vibration and Acoustics* **118**, 1996, 340–345.
6. Nayfeh, A. H., 'On direct methods for constructing nonlinear normal modes of continuous systems', *Journal of Vibration and Control* **1**, 1995, 389–430.
7. Nayfeh, A. H. and Balachandran, B., *Applied Nonlinear Dynamics*, Wiley, New York, 1995.
8. Natsiavas, S., 'Free vibration of two coupled nonlinear oscillators', *Nonlinear Dynamics* **6**, 1994, 69–86.
9. Luongo, A. and Paolone, A., 'Perturbation methods for bifurcation analysis from multiple nonresonant complex eigenvalues', *Nonlinear Dynamics* **14**, 1997, 193–210.
10. Luongo, A. and Paolone, A., 'Multiple scales analysis for divergence-Hopf bifurcation of imperfect symmetric systems', *Journal of Sound and Vibration* **218**, 1998, 527–539.
11. Luongo, A., 'Perturbation methods for nonlinear autonomous discrete-time dynamical systems', *Nonlinear Dynamics* **10**, 1996, 317–331.
12. Nayfeh, A. H., 'The response of single of freedom systems with quadratic and cubic non-linearity to a subharmonic excitation', *Journal of Sound and Vibration* **89**, 1983, 457–470.
13. Nayfeh, A. H., 'Combination resonances in the non-linear response of bowed structures to a harmonic excitation', *Journal of Sound and Vibration* **90**, 1983, 457–470.
14. Nayfeh, A. H., 'Quenching of primary resonance by a superharmonic resonance', *Journal of Sound and Vibration* **92**, 1984, 363–377.
15. Nayfeh, A. H., 'Combination tones in the response of single degree of freedom systems with quadratic and cubic non-linearity', *Journal of Sound and Vibration* **92**, 1984, 379–386.
16. Nayfeh, A. H., 'Quenching of a primary resonance by a combination resonance of the additive or difference type', *Journal of Sound and Vibration* **97**, 1984, 65–73.
17. Rahman, Z. and Burton, T. D., 'Large amplitude primary and superharmonic resonances in the Duffing oscillator', *Journal of Sound and Vibration* **110**, 1986, 363–380.
18. Rega, G. and Benedettini, F., 'Planar non-linear oscillations of elastic cables under subharmonic resonance conditions', *Journal of Sound and Vibration* **132**, 1989, 367–381.
19. Nayfeh, A. H., 'Topical course on nonlinear dynamics', in *Perturbation Methods in Nonlinear Dynamics*, Società Italiana di Fisica, Santa Margherita di Pula, Sardinia, 1985.
20. Rahman, Z. and Burton, T. D., 'On higher order methods of multiple scales in non-linear oscillations-periodic steady state response', *Journal of Sound and Vibration* **133**, 1989, 369–379.
21. Hassan, A., 'Use of transformations with the higher order method of multiple scales to determine the steady state periodic response of harmonically excited non-linear oscillators, Part I: Transformation of derivative', *Journal of Sound and Vibration* **178**, 1994, 21–40.
22. Hassan, A., 'Use of transformations with the higher order method of multiple scales to determine the steady state periodic response of harmonically excited non-linear oscillators, Part II: Transformation of detuning' *Journal of Sound and Vibration* **178**, 1994, 1–19.
23. Lee, C. L. and Perkins, N. C., 'Nonlinear oscillations of suspended cables containing a two-to-one internal resonance', *Nonlinear Dynamics* **3**, 1992, 465–490.

24. Benedettini, F., Rega, G., and Alaggio, R., ‘Non-linear oscillations of a four-degree-of-freedom model of a suspended cable under multiple internal resonance conditions’, *Journal of Sound and Vibration* **182**, 1995, 775–798.
25. Pan, R. and Davies, H. G., ‘Responses of a non-linearly coupled pitch-roll ship model under harmonic excitation’, *Nonlinear Dynamics* **9**, 1996, 349–368.
26. Lee, C. L. and Lee, C. T., ‘A higher order method of multiple scales’, *Journal of Sound and Vibration* **202**, 1997, 284–287.
27. Boyaci, H. and Pakdemirli, M., ‘A comparison of different versions of the method of multiple scales for partial differential equations’, *Journal of Sound and Vibration* **204**, 1997, 595–607.
28. Luongo, A. and Paolone, A., ‘On the reconstitution problem in the multiple time scale method’, *Report Dipartimento di Ingegneria delle Strutture, delle Acque e del Terreno* **4**, 1998.
29. Guckenheimer, J. and Holmes, P. J., *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.
30. Nayfeh, A. H., *Method of Normal Forms*, Wiley, New York, 1993.
31. Luongo, A., Rega, G., and Vestroni, F., ‘On nonlinear dynamics of planar shear indeformable beams’, *Journal of Applied Mechanics* **53**, 1986, 619–624.
32. Benedettini, F. and Rega, G., ‘Non-linear dynamics of an elastic cable under planar excitation’, *International Journal of Non-Linear Mechanics* **22**, 1987, 497–509.
33. Luongo, A. and Pignataro, M., ‘On the perturbation analysis of interactive buckling in nearly symmetric structures’, *International Journal of Solids and Structures* **6**, 1992, 721–733.
34. Luongo, A., Rega, G., and Vestroni, F., ‘ Planar nonlinear free vibrations of an elastic cables’, *International Journal of Non-Linear Mechanics* **19**, 1984, 39–52.