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p-CHAOTICITY AND REGULAR ACTION OF ABELIAN C^1-DIFFEOMORPHISMS GROUPS OF $\mathbb{C}^n$ FIXING A POINT

YAHYA N'DAO AND ADLENE AYADI

Abstract. In this paper, we introduce the notion of regular action of any subgroup G of $Diff^1(\mathbb{C}^n)$ on $\mathbb{C}^n$ (i.e. the closure of every orbit of G in some open set is a topological sub-manifold of $\mathbb{C}^n$). We prove that the action of G, can not be p-chaotic for every $0 \leq p \leq n-1$. (i.e. If G has a dense orbit then the set of all regular orbit with order p can not be dense in $\mathbb{C}^n$). Moreover, we prove that the action of any abelian lie subgroup of $Diff^1(\mathbb{C}^n)$, , is regular.

1. Introduction

Denote by $Diff^1(\mathbb{C}^n)$ the group of all C^1-diffemorphisms of $\mathbb{C}^n$. Let G be an abelian subgroup of $Diff^1(\mathbb{C}^n)$ such that $0 \in Fix(G)$ and $dim(vect(L_G)) = n$, where $vect(L_G)$ is the vector space generated by $L_G = \{Df(0),\ f \in G\}$ and $Fix(G) = \{x \in \mathbb{C}^n :\ f(x) = x,\ \forall f \in G\}$ be the global fixed point set of G. We can assume that $0 \in Fix(G)$, leaving to replace G by $T_a \circ G \circ T_{-a}$ for any translation T_a of any vector $a \in Fix(G)$. There is a natural action $G \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$. $(f, x) \longmapsto f(x)$. For a point $x \in \mathbb{C}^n$, denote by $G(x) = \{f(x),\ f \in G\} \subset \mathbb{C}^n$ the orbit of G through x. Denote by $\overline{E}$ (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E. A topological space X is called a topological manifold with dimension $r \geq 0$ over $\mathbb{C}$ if every point has a neighborhood homeomorphic to $\mathbb{C}^r$. This means that the image of any topological manifold by a homeomorphism is a topological manifold with the same dimension. An orbit γ is called *regular* with order $ord(\gamma) = m$ if for every $y \in \gamma$ there exists an neighborhood O of y such that $\overline{\gamma} \cap O$ is a topological sub-manifold of $\mathbb{C}^n$ with dimension m over $\mathbb{C}$. In particular, γ is locally dense in $\mathbb{C}^n$ if and only if $m = n$, and it is discrete if and only if $m = 0$. Notice that, the closure of a regular orbit is not necessary a manifold. We say that the action of G is *regular* on $\mathbb{C}^n$ if every orbit of G is regular. The action of G is called *chaotic* if G has a dense orbit and the union of all periodic orbits is dense in $\mathbb{C}^n$ (cf. [11], [13], [5]). We give a generalization of the chaos as follow: The action of G is called *p-chaotic*, $0 \leq p \leq n-1$, if G has a dense orbit and the union of all orbits with order p is dense in $\mathbb{C}^n$. See that every chaotic action is 0-chaotic. Here, the question to investigate is the following:

The natural action of any subgroup of $Diff(\mathbb{C}^n)$ can be p-chaotic, $0 \leq p \leq n-1$?

The action of any abelian lie subgroup of $Diff(\mathbb{C}^n)$ can be regular?

The notion of regular orbit is a generalization of non exceptional orbit defined for the action of any group of diffeomorphisms on $\mathbb{C}^n$. A nonempty subset $E \subset \mathbb{C}^n$ is a minimal set if for every $y \in E$ the orbit of y is dense in E. An orbit with its closure is a *Cantor* set is called an exceptional orbit. Their dynamics were recently initiated for some classes in different point of view, (see for instance, [3],[4],[5],[6],[7],[9]).

The action of G on $\mathbb{C}^n$ is said *proper* if and only if the pre-image of any compact set by the action map, is compact (i.e. for every two compact subsets K_1 and K_2 of $\mathbb{C}^n$, the subset $\{f \in G,\ \ f(K_1) \cap K_2 \neq \emptyset\}$ of G is compact). It is well known, that if the action of a lie group on $\mathbb{C}^n$ is proper then all the orbits are embedded submanifolds in $\mathbb{C}^n$ (see for instance [12] and [15]). Remark that, a proper action of any lie group is regular, this means that the regular action is a generalization of the proper action.

In [1], A.C. Naolekar and P. Sankaran construct chaotic actions of certain finitely generated abelian groups on even-dimensional spheres, and of finite index subgroups of $SL(n.\mathbb{Z})$ on tori. They also study chaotic group actions via compactly supported homeomorphisms on open manifolds.

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In [10], P.W.Michor and C.Vizman proved that some groups of diffeomorphisms of a manifold M act n-transitively for each finite n (i.e. for any two ordred sets of n different points $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ in M there is a smooth diffeomorphism f in the group such that $f(x_i)=y_i$ for each i) .

In [3], the authors studied the minimality of any abelian diffeomorphisms groups acting on $\mathbb{C}^n$ fixing a point and $dim(L_G)=n$, whose generalize the structure's theorem given in [2] for abelian linear group. This paper can be viewed as a continuation of these works.

Our principal results can be stated as follows:

Theorem 1.1. *Let G be an abelian subgroup of $Diff^1(\mathbb{C}^n)$ such that $0\in Fix(G)$ and $dim(vect(L_G))=n$. If G has a dense orbit then the set of all dense orbit is a G-invariant open set, dense in $\mathbb{C}^n$.*

Corollary 1.2. *The natural action of any abelian subgroup of $Diff^1(\mathbb{C}^n)$ such that $0\in Fix(G)$ and $dim(vect(L_G))=n$, can not be p-chaotic for every $0\leq p\leq n-1$. In particular, it can not be chaotic.*

Theorem 1.3. *The natural action of any abelian lie subgroup of $Diff^1(\mathbb{C}^n)$ on $\mathbb{C}^n$ is regular.*

As a directly consequence of Theorems 1.3 and 1.1, we prove the regularity action of any abelian linear group on $\mathbb{C}^n$.

Corollary 1.4. *The natural action of any abelian subgroup of $GL(n,\mathbb{C})$ on $\mathbb{C}^n$ is regular and not chaotic.*

2. Proof Theorem 1.3 and corollary 1.2

We will cite the definition of the exponential map given in [4].

Denote by:
- g be the lie algebra associated to G.
- The exponential map $exp:\mathrm{g}\longrightarrow G$ is defined in above.

Lemma 2.1. *Let $x\in\mathbb{C}^n$. Then $G(x)$ is regular with order $r\geq 0$ if and only if there exist an open set O_x containing $G(x)$ such that $\overline{G(x)}\cap O_x$ is a manifold with dimension $r\geq 0$.*

Proof. The directly sens is obvious by definition. Conversely, let $x\in G(u)$ and O be an open neighborhood of x such that $\overline{G(u)}\cap O_x$ is a manifold with dimension $r\geq 0$ over $\mathbb{C}$. Let $y\in G(u)$, then $y=f(x)$ for some $f\in G$. So $O'_x=f(O_x)$ is an neighborhood of y and satisfying $\overline{G(u)}\cap O'_x=f(\overline{G(u)}\cap O)$ is a manifold with dimension $r\geq 0$ over $\mathbb{C}$. It follows that $G(u)$ is regular with order r. □

2.1. *Whitney Topology on $C^0(\mathbb{C}^n,\mathbb{C}^n)$***.** We will use the definition of Whitney topology given in [14]. For each open subset $U\subset\mathbb{C}^n\times\mathbb{C}^n$ let $\widetilde{U}\subset\mathcal{C}^0(\mathbb{C}^n,\mathbb{C}^n)$ be the set of continuous functions g, whose graphs $\{(x,g(x))\in\mathbb{C}^n\times\mathbb{C}^n,\ \ x\in\mathbb{C}^n\}$ is contained in U. We want to construct a neighborhood basis of each function $f\in\mathcal{C}^0(\mathbb{C}^n,\mathbb{C}^n)$. Let $K_j=\{x\in\mathbb{C}^n,\ \ \|x\|\leq j\}$ be a countable family of compact sets (closed balls with center 0) covering $\mathbb{C}^n$ such that K_j is contained in the interior of K_{j+1}. Consider then the compact subsets $L_j=K_j\backslash\overbrace{K_{j-1}}^{\circ}$, which are compact sets, too. Let $\epsilon=(\varepsilon_j)_j$ be a sequence of positive numbers and then define $V_{(f;\epsilon)}=\{f\in\mathcal{C}^0(\mathbb{C}^n,\mathbb{C}^n)\ :\ \|f(x)-g(x)\|<\varepsilon_j,\ \text{for any } x\in L_j,\ \forall j\}$. We claim this is a neighborhood system of the function f in $\mathcal{C}^0(\mathbb{C}^n,\mathbb{C}^n)$. Since L_i is compact, the set $U=\{(x,y)\in\mathbb{C}^n\times\mathbb{C}^n\ :\ \|f(x)-y\|<\varepsilon_j,\ if\ x\in L_j\}$ is open. Thus, $V_{(f;\epsilon)}=\widetilde{U}$ is an open neighborhood of f. On the other hand, if O is an open subset of $\mathbb{C}^n\times\mathbb{C}^n$ which contains the graph of f, then since L_j is compact, it follows that there exists $\varepsilon_j>0$ such that if $x\in L_j$ and $\|y-f(x)\|<\varepsilon_j$, then $(x;y)\in O$. Thus, taking $\widetilde{\epsilon}=(\varepsilon_j)_j$ we have $V_{(f;\widetilde{\epsilon})}\subset\widetilde{O}$, so we have obtained the family $V_{(f;\epsilon)}$ is a neighborhood system of f. Moreover, for each given $\epsilon=(\varepsilon_j)_j$, we can find a C^∞-function $\epsilon:\mathbb{C}^n\longrightarrow\mathbb{R}_+$, such that $\epsilon(x)<\varepsilon_j$ for any $x\in L_j$. It follows that the family $V_{(f;\epsilon)}=\{g\in\mathcal{C}^0(\mathbb{C}^n,\mathbb{C}^n)\ :\ \|f(x)-g(x)\|<\epsilon(x),\ \text{for every } x\in\mathbb{C}^n\}$ is also a neighborhood system.

2.2. *Linear map***.** For a subset $E\subset\mathbb{C}^n$, denote by $vect(E)$ the vector subspace of $\mathbb{C}^n$ generated by all elements of E. E is called G-invariant if $f(E)\subset E$ for any $f\in G$; that is E is a union of orbits. Set $\mathcal{A}(G)$ be the algebra generated by G. For a fixed vector $x\in\mathbb{C}^n\backslash\{0\}$, denote by:
- $\Phi_x:\mathcal{A}(G)\longrightarrow\Phi_x(\mathcal{A}(G))\subset\mathbb{C}^n$ the linear map given by $\Phi_x(f)=f(x)$.
- $E(x)=\Phi_x(\mathcal{A}(G))$.

Lemma 2.2. *The linear map $\Phi_x:\mathcal{A}(G)\longrightarrow E(x)$ is continuous.*

Proof. Firstly, we take the restriction of the Whitney topology to $\mathcal{A}(G)$. Secondly, let $f \in \mathcal{A}(G)$ and $\varepsilon > 0$. Then for $\epsilon = (\varepsilon_j)_j$ with $\varepsilon_j = \varepsilon$ and for $V_{(f;\epsilon)}$ be a neighborhood system of f, we obtain: for every $g \in V_{(f;\epsilon)} \cap \mathcal{A}(G)$ and for every $y \in L_j$, $\|f(y) - g(y)\| < \varepsilon$, $\forall j$. In particular for $j = j_0$ in which $x \in L_{j_0}$, we have $\|f(x) - g(x)\| < \varepsilon$, so $\|\Phi_x(f) - \Phi_x(g)\| < \varepsilon$. It follows that Φ_x is continuous. □

Denote by:
- $r(x) = dim(E(x))$.
- $U_j = \{y \in \mathbb{C}^n, \ \ r(y) \geq j\}$.

Proposition 2.3. *Let G be a subgroup of $Diff^1(\mathbb{C}^n)$. Suppose that G has a dense orbit. Then U_n is a G-invariant open subset of $\mathbb{C}^n$.*

Proof. Let $x \in \mathbb{C}^n$ such that $\overline{G(x)} = \mathbb{C}^n$, then $x \in U_n$ and so $U_n \neq \emptyset$. Let $y \in U_n$, then $E(y) = \mathbb{C}^n$ and so, there exist $f_1, \dots, f_n \in F_y$ such that the n vectors $f_1(y), \dots, f_n(y)$ are linearly independent in $\mathbb{C}^n$. For all $z \in \mathbb{C}^n$, we consider the Gram's determinant

$$\Delta(z) = det\left(\langle f_i(z) \mid f_j(z)\rangle\right)_{1\leq i,j\leq n}$$

of the vectors $f_1(z), \dots, f_n(z)$ where $\langle .|.\rangle$ denotes the scalar product in $\mathbb{C}^n$. It is well known that these vectors are independent if and only if $\Delta(z) \neq 0$, in particular $\Delta(y) \neq 0$. Let

$$V_y = \{z \in \mathbb{C}^n, \ \ \Delta(z) \neq 0\}$$

The set V_y is open in $\mathbb{C}^n$, because the map $z \longmapsto \Delta(z)$ is continuous. Now $\Delta(y) \neq 0$, and so $y \in V_y \subset U_n$. The proof is completed. □

The construction of the open U given in [3], is the same of U_n if G has a dense orbit.

Lemma 2.4. ([3], *Corollary 1.2*) *Let G be an abelian subgroup of $Diff^1(\mathbb{C}^n)$, such that $0 \in Fix(G)$ and $dim(vect(L_G)) = n$. If G has a dense orbit then every orbit in U_n is dense in $\mathbb{C}^n$.*

Proof of Theorem 1.1. Suppose that the group G has a dense orbit denoted by $G(x)$, $x \in \mathbb{C}^n$. Let $\mathcal{L}$ be the set of all dense orbits, so $\mathcal{L} \neq \emptyset$ since $x \in \mathcal{L}$. let $y \in \mathcal{L}$ then $y \in U_n$ and $U_n \subset \overline{G(y)}$. By Lemma 2.4, for each $z \in U_n$, one has $\overline{G(z)} = \mathbb{C}^n$, so $y \in U_n \subset \mathcal{L}$. By Proposition 2.3, U_n is a non empty open subset of $\mathbb{C}^n$. This completes the proof. □

Remark 2.5. By the proof of Theorem 1.1, $U_n = \mathcal{L}$.

Proof of Corollary 1.2. Suppose that the action of the group G is p-chaotic, then G has a dense orbit denoted by $G(x)$, $x \in \mathbb{C}^n$. By Theorem 1.1, the set $\mathcal{L}$ of all dense orbit is a dense open set in $\mathbb{C}^n$. This means that if $\mathcal{P}$ is the union of all regular orbits with order p, then $\mathcal{L} \cap \mathcal{P} = \emptyset$, so $\mathcal{P}$ can not be dense in $\mathbb{C}^n$. The proof is completed. □

3. Regular action of abelian lie subgroups of $Diff^1(\mathbb{C}^n)$

We will cite the definition of the exponential map given in [4].

3.1. *Exponential map.* In this section, we illustrate the theory developed of the group $Diff(\mathbb{C}^n)$ of diffeomorphisms of $\mathbb{C}^n$. For simplicity, throughout this section we only consider the case of $\mathbb{C} = \mathbb{R}$; however, all results also hold for complexes case. The group $Diff(\mathbb{R}^n)$ is not a Lie group (it is infinite-dimensional), but in many way it is similar to Lie groups. For example, it easy to define what a smooth map from some Lie group G to $Diff(\mathbb{R}^n)$ is: it is the same as an action of G on $\mathbb{R}^n$ by diffeomorphisms. Ignoring the technical problem with infinite-dimensionality for now, let us try to see what is the natural analog of the Lie algebra g for the group G. It should be the tangent space at the identity; thus, its elements are derivatives of one-parameter families of diffeomorphisms.
Let $\varphi^t : G \longrightarrow G$ be one-parameter family of diffeomorphisms. Then, for every point $a \in G$, $\varphi^t(a)$ is a curve in G and thus $\frac{\partial}{\partial t}\varphi^t(a)_{/t=0} = \xi(a) \in T_aG$ is a tangent vector to G at a. In other words, $\frac{\partial}{\partial t}\varphi^t$ is a vector field on G.

The exponential map exp : g $\longrightarrow G$ is defined by $exp(x) = \gamma_x(1)$ where $\gamma_x(t)$ is the one-parameter subgroup with tangent vector at 1 equal to x.
If $\xi \in$ g is a vectorfield, then $exp(t\xi)$ should be one-parameter family of diffeomorphisms whose derivative is vector field ξ. So this is the solution of differential equation

$$\frac{\partial}{\partial t}\varphi^t(a)_{/t=0} = \xi(a).$$

In other words, φ^t is the time t flow of the vector field. Thus, it is natural to define the Lie algebra of G to be the space g of all smooth vector ξ fields on $\mathbb{R}^n$ such that $exp(t\xi) \in G$ for every $t \in \mathbb{R}$.

Proposition 3.1. ([4], *Theorem 3.29*) *Let G be a Lie group acting on $\mathbb{C}^n$ with lie algebra* g *and let $x\in\mathbb{C}^n$.*
(i) The stabilizer $G_x=\{f\in G:\ \ f(x)=x\}$ is a closed Lie subgroup in G, with Lie algebra $\mathfrak{h}_x=\{f\in \mathrm{g}:\ f(x)=0\}$.
(ii) The map $G_{/G_x}\longrightarrow\mathbb{C}^n$ given by $f.G_x\longmapsto f(x)$ is an immersion. Thus, the orbit $G(u)$ is an immersed submanifold in $\mathbb{C}^n$. In particular $\dim(G(x))=\dim(\mathrm{g})-\dim(\mathfrak{h}_x)$.

Denote by $p=dim(\mathrm{g})$. Since G is abelian so is g. Set $f_1,\dots,f_p\in \mathrm{g}$ be the generators of g. We let:
- $exp:\mathrm{g}\longrightarrow G$ the lie exponential map associated to G.
- G_0 be the connected component of G containing the identity map id. So G_0 is generated by $exp(\mathrm{g})$ and it is an abelian lie subgroup of G. Since g is abelian, $G_0=exp(\mathrm{g})$.

For a fixed point $x\in\mathbb{C}^n$, denote by:
- $G_x=\{f\in G_0,\ \ f(x)=x\}$ the stabilizer of G_0 on the point x. It is a lie subgroup of G_0.

Denote by:
- H be the algebra associated to G_x and F_x is the supplement of H_x in g (i.e. $F_x\oplus H_x=\mathrm{g}$). By Proposition 3.1, we have $H_x=\{f\in \mathrm{g},\ \ f(x)=0\}$ and
$$G_0=exp(F_x)\circ exp(H_x).$$
In particular $G_0(x)=\Phi_x(exp(F_x))$.
- $V=\{exp(t_1f_1+\dots+t_pf_p),\ \ \ |t_k|<1\}$.

Proposition 3.2. *Let G be an abelian subgroup of $Diff^1(\mathbb{C}^n)$, and $x\in\mathbb{C}^n$. Then:*
(i) $G_0(x)$ is the connected component of $G(x)$ containing x.
(ii) The restriction $\Phi_x^{(1)}:exp(F_x)\cap V\longrightarrow \Phi_x(exp(F_x)\cap V)\subset G_0(x)$ of Φ_x to $exp(F_x)\cap V$ is an homeomorphism.

Proof. (i) By Lemma 2.2, the map $\Phi_x:\mathcal{A}(G)\longrightarrow E(x)\subset\mathbb{C}^n$ is a continuous surjective linear map. The proof follows then from the fact that $G_0(x)=\Phi_x(G_0)$ and G_0 is connected.

(ii) By Lemma 2.2, the map $\Phi_x^{(1)}$ is continuous, surjective.

It is injective: Indded, if $f,g\in exp(F_x)\cap V$ such that $\Phi_x^{(1)}(f)=\Phi_x^{(1)}(g)$, then $f(x)=g(x)$, so $g^{-1}\circ f(x)=x$. Hence $g^{-1}\circ f\in G_x\cap exp(F_x)=\{id\}$. It follows that $f=g$.

$(\Phi_x^{(1)})^{-1}:\Phi_x(exp(F_x)\cap V)\longrightarrow exp(F_x)\cap V$ is continuous; inded, let $y=exp(f)(x)\in\Phi_x(exp(F_x)\cap V)$, $f\in F_x$ and $(y_m)_m$ be a sequence in $\Phi_x(exp(F_x)\cap V)$ tending to y. Let $(f_1,\dots,f_q)$ be a basis of F_x and set $y_m=exp(t_{1,m}f_1+\dots+t_{q,m}f_q)(x)$ and $y=exp(t_1f_1+\dots+t_qf_q)(x)$, with $|t_k|<1$ and $|t_{k,m}|<1$. We can assume (leaving to take a subsequence) that $\underset{m\to+\infty}{lim}t_{k,m}=s_k$, with $|s_k|\leq 1$ for every $k=1,\dots,q$. Write $g=exp(s_1f_1+\dots+s_qf_q)$ and $g_m=exp(t_1f_1+\dots+t_qf_q)$. By continuity of the exponential map we have $(g_m)_m$ tends to g when $m\to+\infty$. By continuity of Φ_x (Lemma 2.2) we obtain $y_m=\Phi_x(g_m)$ tends to $y=\Phi_x(g)$, so $s_k=t_k$ for every $k=1,\dots,p$. As $g=(\Phi_x^{(1)})^{-1}(y)$ and $g_m=(\Phi_x^{(1)})^{-1}(y_m)$, it follows that $(\Phi_x^{(1)})^{-1}(y_m)$ tends to $(\Phi_x^{(1)})^{-1}(y)$. This completes the proof. □

3.2. *Wedge, Lie wedge and almost abelian notions***.** We will use the notion of wedge and Lie wedge given by K.H. Hofmann in [7] and [8]:
- A *wedge* or a closed convex cone in a finite dimensional vector g is a topologically subset ω with $\omega+\omega=\omega$ and $\lambda.\omega\subset\omega$ for every $\lambda\geq 0$. In particular, any vector subspace of g is a wedge in g.
- $h(\omega)=(-\omega)\cap\omega$ is called the edge of the wedge.
- A *Lie wedge* ω in a Lie algebra g is a wedge such that
$$exp(ad(x))\omega=\omega,\quad \text{for all } x\in h(\omega).$$
In particular, any subalgebra of g is a Lie wedge in g.
- A Lie algebra η is called *almost abelian* if there is a linear form $\alpha:\eta\longrightarrow\mathbb{R}$ such that the bracket is given by
$$[X,Y]=\alpha(X)Y-\alpha(Y)X.$$
In particular, any abelian Lie algebra is almost abelian for $\alpha=0$. If $\alpha\neq 0$ the η is called *truly almost abelian.*

Lemma 3.3. ([8], *Theorem 4.3*) *Let* g *be a Lie algebra, then the following are equivalent:*
(i) g *is almost abelian.*
(ii) Every wedge is a Lie wedge.
(iii) For every Lie wedge ω, we have $\overline{\langle exp(\omega)\rangle}=exp(\omega)$, where $\langle exp(\omega)\rangle$ is the group generated by $exp(\omega)$.

As a consequence of above Lemma, we obtain:
Corollary 3.4. *We have $exp(F_x)$ is a lie subgroup of G_0.*

Proof. By definition, we have F_x is a wedge and by Lemma 3.3, (ii), it is a Lie wedge because g is almost abelian (since it is abelian). Then by Lemma 3.3, (iii), we have $\overline{\langle exp(F_x)\rangle}=exp(F_x)$, so

$exp(F_x)$ is closed subgroup of G_0. It follows that F_x is a Lie group. □

Corollary 3.5. (*Under notations of Proposition 3.2*) *The set $B(x) = \Phi_x(exp(F_x) \cap V)$ is a topological submanifold of $\mathbb{C}^n$ containing x. Moreover, there exists an open subset W of $\mathbb{C}^n$ such that $W \cap G(x) = B(x)$.*

Proof. By Corollary 3.4, $exp(F_x)$ is a lie subgroup of G_0, so it is a topological manifold. By Proposition 3.2, $B(x)$ is homoeomorphic to $exp(F_x) \cap V$ wich is an open subset of $exp(F_x)$. Then $B(x)$ is a topological manifold with dimension equal to $\dim(exp(F_x))$. On the other hand, by (i), $G_0(x) = \Phi_x(exp(F_x))$ is a connected component of $G(x)$ containing x, then there exists an open subset O of $\mathbb{C}^n$ such that $O \cap G(x) = G_0(x)$. Since the exponential map exp is a locally diffeomorphism on a neighborhood of 0 then $\dim(exp(F_x)) = \dim(F_x)$, so $\dim(B(x)) = \dim(exp(F_x)) = \dim(F_x)$. By Proposition 3.1, $G_0(x)$ is an immersed submanifold of $\mathbb{C}^n$ with dimension $\dim(F_x) = \dim(\mathrm{g}) - \dim(H_x)$ because g is also the lie algebra of G_0. Therefore $\dim(B(x)) = \dim(G_0(x))$, so $B(x)$ is an open subset of $G_0(x)$. Then there exists an open subset W of $\mathbb{C}^n$ containing x and contained in O such that $G_0(x) \cap W = B(x)$. It follows that $W \cap G(x) = G_0(x) \cap W = B(x)$. The proof is completed. □

Lemma 3.6. *For every neighborhood W of a point $x \in \mathbb{C}^n$, we have $\overline{G(x)} \cap W = \overline{G(x) \cap W} \cap W$.*

Proof. It is clear that $\overline{G(x) \cap W} \cap W \subset \overline{G(x)} \cap W$. Now, let $y \in \overline{G(x)} \cap W$ then there exists a sequence $(y_m)_m$ in $G(x)$ tending to y. So $y_m \in W$ from some row m_0. Thus $y \in \overline{G(x) \cap W} \cap W$. □

Proof of Theorem 1.3. Let G be an abelian subgroup of $Diff^1(\mathbb{C}^n)$. By Corollary 3.5, there exists an open subset W of $\mathbb{C}^n$ such that $W \cap G(x) = B(x)$ is a submanifold of $\mathbb{C}^n$. So $B(x)$ is locally closed, we can assume that $\overline{B(x)} \cap W = B(x)$. Therefore, by Lemma 3.6 we have $\overline{G(x)} \cap W = \overline{G(x) \cap W} \cap W$, so $B(x) \subset \overline{G(x)} \cap W = \overline{G(x) \cap W} \cap W = \overline{B(x)} \cap W = B(x)$. Hence $\overline{G(x)} \cap W = B(x)$ is a topological manifold. By Lemma 2.1, it follows that $G(x)$ is regular. We conclude that the action of G is regular. □

Let $M_n(\mathbb{C})$ be the set of all square matrix over $\mathbb{C}$ with order n and $GL(n, \mathbb{C})$ be the group of all reversible matrix of $M_n(\mathbb{C})$. Let L be an abelian subgroup of $GL(n, \mathbb{C})$, denote by:
- $\widetilde{L} = \overline{L} \cap GL(n, \mathbb{C})$, where $\overline{L}$ is the closure of L in $M_n(\mathbb{C})$. It is clear that $\widetilde{L}$ is a lie subgroup of $GL(n, \mathbb{C})$.
- $\overline{L}(x) = \{Ax, \quad A \in \overline{L}\}$.

We will use the following lemma to prove Corollary 1.4.

Lemma 3.7. *For every $x \in \mathbb{C}^n$. We have $\overline{\widetilde{L}(x)} = \overline{L(x)}$.*

Proof. We have $\overline{L(x)} \subset \overline{\widetilde{L}(x)}$. Let $y \in \overline{\widetilde{L}(x)}$, so $y = \underset{m \to +\infty}{lim} A_m(x)$ for some sequence $(A_m)_{m \in \mathbb{N}}$ in $\widetilde{G}$. Therefore, for every $m \in \mathbb{N}$, there exists a sequence $(A_{m,k})_{k \in \mathbb{N}}$ in $G_{/E(x)}$ tending to A_m. Then $\underset{k \to +\infty}{lim} A_{m,k}x = A_m x$. Thus for every $\varepsilon > 0$, there exists $M > 0$ and for every $m \geq M$, there exists $k_m > 0$, such that for every $k \geq k_m$, we have $\|A_m x - y\| < \frac{\varepsilon}{2}$ and $\|A_{m,k}x - A_m x\| < \frac{\varepsilon}{2}$. Then, for every $m > M$,

$$\|A_{m,k_m}x - y\| \leq \|A_{m,k_m}x - A_m x\| + \|A_m x - y\| < \varepsilon.$$

Hence $\underset{m \to +\infty}{lim} A_{m,k_m}x = y$, so $y \in \overline{G(x)}$. It follows that $\overline{\widetilde{G}(u)} \subset \overline{G(u)}$. The proof is completed. □

Proof of Corollary 1.4. By Theorem 1.3, the action of $\widetilde{L}$ is regular. Then for each $x \in \mathbb{C}^n$, there exists an open subset O of $\mathbb{C}^n$ such that $\overline{\widetilde{L}(x)} \cap O$ is a topological sub-manifold of $\mathbb{C}^n$. It follows by Lemma 3.7, that $\overline{L(x)} \cap O$ is a topological submanifold of $\mathbb{C}^n$. The proof is completed. □

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