K-STABILITY AND PARABOLIC STABILITY

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Abstract. Parabolic structures with rational weights encode certain iterated blowups of geometrically ruled surfaces. In this paper, we show that the three notions of parabolic polystability, K-polystability and existence of constant scalar curvature Kähler metrics on the iterated blowup are equivalent, for certain polarizations close to the boundary of the Kähler cone.

1. Introduction

The Calabi program is concerned with finding canonical metrics on Kähler manifolds. The idea is look for critical points of the *Calabi functional*, i.e. the L^2-norm of the scalar curvature, within a prescribed Kähler class. Such metrics are called *extremal metrics*. The existence problem for extremal metrics is open, even for complex surfaces. The Donaldson-Tian-Yau conjecture roughly says that the existence of extremal metrics with integral Kähler class should be equivalent to some algebro-geometric notion of stability of the corresponding polarized complex manifold.

The Euler-Lagrange equation for an extremal metric g is equivalent to the fact that $\partial^\sharp s$ – the $(1,0)$-component of the gradient of the scalar curvature of g – is a holomorphic vector field. If the complex manifold do not carry any nontrivial holomorphic vector field, a Kähler metric is extremal if and only if it has constant scalar curvature. It seems reasonable, at first, to focus on this "generic" case, thus limiting our study to constant scalar curvature Kähler metrics (we shall use the acronym CSCK as a shorthand).

Ruled surfaces are an excellent probing playground for the Donaldson-Tian-Yau conjecture. In this paper, we study iterated blowups of ruled surfaces encoded by a parabolic structure. Our main result, stated below, shows that the Donaldson-Tian-Yau conjecture holds for this class of surfaces and certain polarizations. In addition, we prove that stability for parabolic bundles plays an important role in this picture. The relevant definitions are explained in the rest of this section.

Theorem A. *Let $\mathcal{X} \to \Sigma$ be a parabolic geometrically ruled surface with rational weights and $\widehat{\mathcal{X}} \to \mathcal{X}$ the iterated blowup encoded by the parabolic structure.*

If $\widehat{\mathcal{X}}$ has no nontrivial holomorphic vector fields, the following properties are equivalent:

1. *$\widehat{\mathcal{X}}$ is basically CSCK,*
2. *$\widehat{\mathcal{X}}$ is basically K-stable,*
3. *$\mathcal{X} \to \Sigma$ is parabolically stable.*

1.1. Parabolic ruled surfaces.

A ruled surface $\widehat{\mathcal{X}}$ can be described as a (possibly iterated) blowup $\pi_{\mathcal{X}} : \widehat{\mathcal{X}} \to \mathcal{X}$ of a geometrically ruled surface $\pi_\Sigma : \mathcal{X} \to \Sigma$. By definition, a geometrically ruled surface is obtained as the projectivization

$\mathcal{X} = \mathbb{P}(E)$ of some holomorphic complex vector bundle of rank 2 over a closed Riemann surface $E \to \Sigma$.

A*parabolic structure* on a geometrically ruled surface $\pi_\Sigma : \mathcal{X} \to \Sigma$ consists of the following data:

- A finite sets of distinct marked points $y_1, \cdots, y_m \in \Sigma$;
- marked points $x_1, \cdots, x_m \in \mathcal{X}$ such that $\pi_\Sigma(x_j) = y_j$;
- real numbers $\alpha_1, \cdots, \alpha_m \in (0,1)$ associated to each marked points and called the weights of the marked points.

The ruled surface together with its parabolic structure is simply called a *parabolic ruled surface*.

We consider smooth holomorphic curves $S \subset \mathcal{X}$ such that $\pi_\Sigma|_S : S \to \Sigma$ has degree 1, in other words, *holomorphic sections* of $\mathcal{X} \to \Sigma$. The *parabolic slope* of S is defined by the formula

$$\mathrm{par}\mu(S) = [S]^2 + \sum_{x_j \notin S} \alpha_j - \sum_{x_j \in S} \alpha_j,$$

where $[S] \in H_2(\mathcal{X}, \mathbb{Z})$ is the homology class of S and $[S]^2$ its self-intersection. In the rest of this text, the homology class of a curve S will be denoted S as well, without using the brackets.

A parabolic ruled surface is *stable* if $\mathrm{par}\mu(S) > 0$ for every holomorphic section S. More generally, we say that a parabolic ruled surface $\mathcal{X} \to \Sigma$ is *polystable*, if it is either stable, or there are two non-intersecting holomorphic sections S_- and S_+ with vanishing parabolic slope (i.e. sections such that $S_+ \cdot S_- = 0$ and $\mathrm{par}\mu(S_\pm) = 0$).

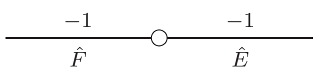
Remark 1.1.1. A parabolic structure on $\mathcal{X} = \mathbb{P}(E) \to \Sigma$ gives a line $x_j \subset E_{y_j}$ together with a weight $\beta_j = \alpha_j/2$ over each point y_j. Such a data defines a parabolic structure on the vector bundle $E \to \Sigma$ in the sense of Mehta-Seshadri [6]. With our conventions we have $\mathrm{par}\mu(S) = \mathrm{par}\deg(E) - 2\mathrm{par}\deg(L)$, where $\mathrm{par}\deg$ is the parabolic degree of a parabolic bundle in the sense of Mehta-Seshadri and L is the line sub-bundle corresponding to S.

By definition, the notions of parabolic stability for a parabolic ruled surface $\mathcal{X} \to \Sigma$ are equivalent to the various notions of parabolic stability in the sense of Mehta-Seshadri for the underlying parabolic vector bundle $E \to \Sigma$ (cf. [7] for more details).

1.2. **Iterated blowups of a parabolic ruled surface with rational weights.** Our main results focus on parabolic structures with rational weights. We shall use the conventions $\alpha_j = \frac{p_j}{q_j}$ where $0 < p_j < q_j$ with p_j and q_j coprime integers.

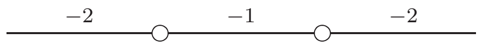
In such situation, the marked points and rational weights define an iterated blowup of $\widehat{\mathcal{X}} \to \mathcal{X}$ introduced in [7]. We recall the construction as it is an essential ingredient of this paper. In order to simplify the notations, we pretend that the parabolic structure on $\mathcal{X}$ is reduced to a single point $y \in \Sigma$; let x be the corresponding point in $F = \pi^{-1}(y)$ and let $\alpha = \frac{p}{q}$ be the weight.

The first step is to blowup the point x, to get a diagram of the form



Here the edges represent rational curves, the number above each edge is the self-intersection of the curve and the hollow dots represent transverse intersections with intersection number $+1$. The curve $\hat{F}$ is the proper transform of F, whereas the other component $\hat{E}$ is the exceptional divisor of the blowup at x.

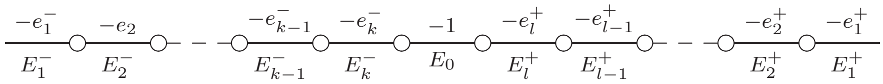
By blowing the intersection point of $\hat{F}$ and $\hat{E}$ we get the diagram



The -1-curve above has exactly two intersection points with the rest of the string. We can decide to blowup either one of them and we carry on with this iterative procedure. After a finite number of blowups we obtain an iterated blowup

$$\pi_{\mathcal{X}} : \widehat{\mathcal{X}} \to \mathcal{X}$$

with $\pi_{\mathcal{X}}^{-1}(F)$ given by the configuration of curves below:



where the curves $E_j^{\pm}$ have self-intersection $-e_j^{\pm} \leq -2$ and E_1^- is the proper transform of the fiber $F = \pi_\Sigma^{-1}(y) \subset \mathcal{X}$. It turns out that there is exactly one way to perform the iterated blowup so that the integers e_j^- are given by the continuous fraction expansion of α:

$$\alpha = \frac{p}{q} = \cfrac{1}{e_1^- - \cfrac{1}{e_2^- - \cdots \cfrac{1}{e_k^-}}}; \tag{1.2.1}$$

Then the e_j^+'s are given by the continuous fraction

$$1 - \alpha = \frac{q-p}{q} = \cfrac{1}{e_1^+ - \cfrac{1}{e_2^+ - \cdots \cfrac{1}{e_l^+}}}. \tag{1.2.2}$$

Note that these expansions are unique since we are assuming $e_j^{\pm} \geq 2$.

1.3. **From parabolic to orbifold ruled surfaces.** Contracting the strings of $E_j^{\pm}$-curves in $\widehat{\mathcal{X}}$ gives an orbifold surface $\overline{\mathcal{X}}$ and $\pi_{\overline{\mathcal{X}}} : \widehat{\mathcal{X}} \to \overline{\mathcal{X}}$ is the minimal resolution. Replacing the marked points y_j of Σ with orbifold singularities of order q_j, we obtain an orbifold Riemann surface $\overline{\Sigma}$. It turns out that there is a holomorphic map of orbifolds

$$\pi_{\overline{\Sigma}} : \overline{\mathcal{X}} \to \overline{\Sigma}$$

which gives $\overline{\mathcal{X}}$ the structure of a geometrically ruled orbifold surface. All these facts are detailed in [7] where the structure of the orbifold singularities is studied precisely.

1.4. **Near a boundary ray of the Kähler cone.** The positive ray

$$\mathscr{R}(\overline{\Sigma}) = \{\gamma \in H^2_{orb}(\overline{\Sigma},\mathbb{R}),\ \gamma\cdot[\overline{\Sigma}]>0\}$$

is by definition the entire Kähler cone of the orbifold Riemann surface $\overline{\Sigma}$. For practical reasons, the image of $\mathscr{R}(\overline{\Sigma})$ under the canonical injective maps

$$H^2_{orb}(\overline{\Sigma},\mathbb{R}) \overset{\pi^*_{\overline{\Sigma}}}{\hookrightarrow} H^2_{orb}(\overline{\mathcal{X}},\mathbb{R}) \overset{\pi^*_{\overline{\mathcal{X}}}}{\hookrightarrow} H^2(\widehat{\mathcal{X}},\mathbb{R})$$

shall be denoted by $\mathscr{R}(\overline{\Sigma})$ as well. So, depending on the context, $\mathscr{R}(\overline{\Sigma})$ will represent a ray in $H^2_{orb}(\overline{\Sigma},\mathbb{R})$, $H^2_{orb}(\overline{X},\mathbb{R})$ or $H^2(\widehat{\mathcal{X}},\mathbb{R})$.

Remark 1.4.1. The notation H^k_{orb} stands for the orbifold De Rham cohomology. Here, we emphasize the fact that we like to represent cohomology classes by closed differential forms which are smooth in the orbifold sense. However, the notation is unimportant, as there is a canonical isomorphism $H^k_{orb}(\overline{\mathcal{X}},\mathbb{R})\simeq H^k(\overline{\mathcal{X}},\mathbb{R})$ with the standard singular cohomology. The proof of this property boils down to the fact that there is a local Poincaré lemma in the context of orbifold De Rham cohomology.

Let $\mathscr{K}(\overline{\mathcal{X}})$ and $\mathscr{K}(\widehat{\mathcal{X}})$ be the Kähler cones of the orbifold $\overline{\mathcal{X}}$ and of $\widehat{\mathcal{X}}$. It is well known that $\widehat{\mathcal{X}}$ and $\overline{\mathcal{X}}$ are of Kähler type. These cones are therefore nonempty. The following lemma is more precise, and concerns the Kähler classes that will be relevant for our results:

Lemma 1.4.2. *The ray $\mathscr{R}(\overline{\Sigma})$ is contained in the closure of $\mathscr{K}(\widehat{\mathcal{X}})\cap H^2(\widehat{\mathcal{X}},\mathbb{Q})$ in $H^2(\widehat{\mathcal{X}},\mathbb{R})$. In other words, for every open cone $U\subset H^2(\widehat{\mathcal{X}},\mathbb{R})$ such that $\mathscr{R}(\overline{\Sigma})\subset U$, the cone $\mathscr{K}(\widehat{\mathcal{X}})\cap H^2(\widehat{\mathcal{X}},\mathbb{Q})\cap U$ is nonempty.*

The fiberwise hyperplane section of $\overline{\mathcal{X}}\to\overline{\Sigma}$ defines a holomorphic orbifold line bundle denoted $\mathcal{O}_{\overline{\mathcal{X}}}(1)\to\overline{X}$ (the construction is completely similar to the case of smooth geometrically ruled surfaces).

Like in the smooth case, one can construct a Hermitian metric h on $\mathcal{O}_{\overline{\mathcal{X}}}(1)\to\overline{\mathcal{X}}$ with curvature F_h, such the closed $(1,1)$ form $\omega_h=\frac{i}{2\pi}F_h$ restricted to any fiber of $\overline{\mathcal{X}}\to\overline{\Sigma}$ is a Kähler form. We may even assume that the restriction of ω_h to the fibers agrees with the Fubini-Study metric on $\mathbb{CP}^1$. Notice that with our conventions

$$[\omega_h]=c_1^{orb}(\mathcal{O}_{\overline{\mathcal{X}}}(1)),$$

where $c_1^{orb}\in H^2_{orb}(\overline{\mathcal{X}},\mathbb{R})$ denotes the first (orbifold) Chern class of an orbifold complex line bundle. In fact, modulo the isomorphism $H^2_{orb}(\overline{\mathcal{X}},\mathbb{R})\simeq H^2(\overline{\mathcal{X}},\mathbb{R})$ between DeRham orbifold cohomology and singular cohomology, one can show that orbifold Chern classes are in fact rational.

Let $\Omega_{\overline{\Sigma}}\in\mathscr{R}(\overline{\Sigma})$ be a Kähler class on $\overline{\Sigma}$ represented by a Kähler metric with Kähler form $\omega_{\overline{\Sigma}}$. We shall assume that $\Omega_{\overline{\Sigma}}$ is integral, which is always possible, up to multiplication by a positive constant. It is easy to check that for every constant $c>0$ sufficiently small, the closed $(1,1)$-form

$$\omega_c^{orb}=\pi^*_{\overline{\Sigma}}\,\omega_{\overline{\Sigma}}+c\,\omega_h \tag{1.4.3}$$

is definite positive on $\overline{\mathcal{X}}$. Thus ω_c^{orb} defines a Kähler orbifold metric on $\overline{\mathcal{X}}$ with Kähler class

$$\Omega_c^{orb}=\bar\pi^*\Omega_{\overline{\Sigma}}+c\cdot c_1^{orb}(\mathcal{O}_{\overline{\mathcal{X}}}(1)). \tag{1.4.4}$$

Form this point we consider $(1,1)$-cohomology class on $\widehat{\mathscr{X}}$ given by

$$\Omega = \pi^*_{\overline{\mathscr{X}}}\Omega^{orb}_c + \sum_{j=1}^{k} c_j^-[E_j^-] + \sum_{j=1}^{l} c_j^+[E_j^+], \tag{1.4.5}$$

where $c_j^\pm \in \mathbb{R}$. It is easy to check that the constant $c_j^\pm$ are uniquely determined by the values of $\Omega \cdot [E_j^\pm]$ (since the intersection matrix between the $E_j^\pm$-curves is invertible). Then we have the following result:

Lemma 1.4.6. *Given $c > 0$, there exists $\varepsilon > 0$, such that every cohomology class Ω given by (1.4.5) and satisfying $0 < \Omega \cdot [E_j^\pm] < \varepsilon$ is a Kähler class.*

Proof. The proof is closely related to the argument of Kodaira that shows that the blowup of a Kähler manifold is still Kähler.

Following [2] and using the scalar-flat ALE metrics of Calderbank-Singer [4], one can construct a Kähler metric ω on $\widehat{\mathscr{X}}$ by gluing $\pi^*_{\overline{X}}\omega_c^{\mathrm{orb}}$ and a small copy of one to the Calderbank-Singer metrics. Any Kähler classes such that the areas $\Omega \cdot [E_j^\pm]$ are sufficiently small are obtained by this method. □

Proof of Lemma 1.4.2. An element of $\mathscr{R}(\overline{\Sigma})$ is represented by a Kähler class $\Omega_{\overline{\Sigma}}$. The constant c and $\Omega \cdot [E_j^\pm]$ that appear in the above discussion can be chosen to be rational and we may assume that we have a Kähler class $\Omega \in H^2(\widehat{\mathscr{X}}, \mathbb{Q})$. It is now obvious that Ω is arbitrarily close to the pullback of $\Omega_{\overline{\Sigma}}$ for c and $\Omega \cdot [E_j^\pm]$ sufficiently small. The result follows. □

Definition 1.4.7. If there exists an open cone U in $H^2(\widehat{\mathscr{X}}, \mathbb{R})$, containing the ray $\mathscr{R}(\overline{\Sigma})$, with the property that any Kähler class in $U \cap \mathscr{K}(\widehat{\mathscr{X}})$ can be represented by a CSCK metric, we say that the iterated blowup $\widehat{\mathscr{X}}$ is *basically CSCK*.

If there exists an open cone U in $H^2(\widehat{\mathscr{X}}, \mathbb{R})$, containing the ray $\mathscr{R}(\overline{\Sigma})$, with the property that any rational Kähler class in $U \cap \mathscr{K}(\widehat{\mathscr{X}})$ is K-stable, we say that the iterated blowup $\widehat{X}$ is *basically K-stable*.

More generally, any property $\mathscr{P}(\Omega)$ depending on the choice of a cohomology class $\Omega \in H^2(\widehat{\mathscr{X}}, \mathbb{R})$ is said to be *basically satisfied*, if it holds for every Ω contained in a sufficiently small cone about the ray $\mathscr{R}(\overline{\Sigma})$. In other words, if $\mathscr{P}$ holds for every Ω sufficiently close to a *basic class*. This explains my choice of terminology, but I am sure there are better choices and I am open to suggestions.

Remark 1.4.8. The condition of K-stability may be defined for varieties polarized by a rational Kähler class. From a more down to earth point of view, a rational class becomes an integral Kähler class Ω after multiplication by a suitable positive integer. The class Ω defines an ample holomorphic line bundle $L_\Omega \to \widehat{\mathscr{X}}$ with $c_1(L_\Omega) = \Omega$ and the condition of K-stability for the original rational polarization is equivalent to the usual notion of K-stability for $(\widehat{\mathscr{X}}, L_\Omega)$ (cf. §3.2 for more details).

1.5. **Comment on Theorem A and its proof.** Theorem A is very far from a general result for ruled surfaces. Indeed, the complex structures of iterated blowups $\widehat{\mathscr{X}} \to \mathscr{X}$ encoded by parabolic structures are highly non-generic. Moreover, the Kähler classes under consideration are also very special since they must be close to the boundary ray $\mathscr{R}(\overline{\Sigma})$ of the Kähler cone.

We should point out that when the parabolic structure is empty, i.e. when $\widehat{\mathcal{X}} = \mathcal{X} = \mathbb{P}(E)$ the problem is completely understood [1]. In this case, $E \to \Sigma$ is a polystable holomorphic bundle if and only if $\mathbb{P}(E)$ is CSCK, for *any* Kähler class.

It seems tempting to believe that our result could be used as the very first step toward a proof of the general Donaldson-Tian-Yau conjecture for ruled surfaces. Here some kind of deformation theory and continuity method is needed. However, important progress ought to be made, especially for dealing with the delicate compactness issue of the relevant moduli spaces.

The proof of (1) $\Rightarrow$ (2) in Theorem A is an immediate corollary of Stoppa's result [11]. The proof (3) $\Rightarrow$ (1) is essentially contained in the joint work of the author with Michael Singer [7, 9, 8] with some slight improvements explained at §2.

The proof of (2) $\Rightarrow$ (3) was the missing piece of the puzzle. We shall prove that if $\mathcal{X} \to \Sigma$ is not parabolically stable, one can construct a destabilizing test configuration (cf. Corollary 4.0.4). This requires a delicate computation for the Futaki invariant at §4.6.

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2. Extremal ruled surfaces and gluing theory

2.1. **Application of the Mehta-Seshadri theorem.** Let $\mathcal{X} \to \Sigma$ be a parabolic geometrically ruled surface with rational weights. If $\mathcal{X} \to \Sigma$ is parabolically polystable, it is a flat $\mathbb{CP}^1$ bundle on the complement of the fibers $\pi_\Sigma^{-1}(y_j)$ by Mehta-Seshadri theorem [6] and the monodromy of the flat connection is given by a morphism $\rho : \pi_1(\Sigma \setminus \{y_j\}) \to \mathrm{SU}_2/\mathbb{Z}_2$. In addition, if l_j is the homotopy class of a loop in $\Sigma \setminus \{y_j\}$ winding once around y_j, then $\rho(l_j)$ is given by the matrix

$$\begin{pmatrix} e^{i\pi\alpha_j} & 0 \\ 0 & e^{-i\pi\alpha_j} \end{pmatrix}$$

up to conjugation. In particular $\rho(l_j)$ has order q_j and the morphism descends to

$$\rho : \pi_1^{orb}(\overline{\Sigma}) \to \mathrm{SU}_2/\mathbb{Z}_2,$$

as π_1^{orb} is just deduced from π_1 by adding the relation $l_j^{q_j} = 1$.

Now, the orbifold Riemann surface $\overline{\Sigma}$ admits an orbifold metric $g_{\overline{\Sigma}}$ of constant curvature in its conformal class, unless it is a "bad" orbifold in the sense of Thurston. That is if $\overline{\Sigma}$ is a *teardrop* or a *football*[1] with two singularities of distinct orders.

Remark 2.1.1. In fact $\overline{\Sigma}$ cannot be bad if $\mathcal{X} \to \Sigma$ is polystable. Indeed, assume that $\Sigma \simeq \mathbb{CP}^1$ and that the parabolic structure has exactly one marked point with weight $\alpha = p/q$. Let l be the homotopy class of a loop winding once around the parabolic point of $\mathbb{CP}^1$. By the Mehta-Seshadri theorem, $\rho(l)$ has order q. But l is trivial, since $\mathbb{CP}^1$ with one puncture is contractible. It follows that $q = 1$ which is impossible. A similar argument shows that $\overline{\Sigma}$ cannot be a football with two singularities of distinct orders.

[1]Using the more politically correct term *northern-American-football* may be a safer option.

Since the monodromy acts isometrically on $\mathbb{CP}^1$ endowed with the Fubini-Study metric, the twisted product $\overline{\Sigma}\times_\rho \mathbb{CP}^1$ carries a local product deduced $g_{\overline{\Sigma}}$ and g_{FS}. Adjusting the metrics on each factor by a constant. In conclusion of the above discussion and Remark 2.1.1, we get the following lemma:

Lemma 2.1.2. *If $\mathcal{X} \to \Sigma$ is parabolically polystable, $\overline{\Sigma}$ and $\overline{\mathcal{X}}$ admit orbifold CSCK metrics in every Kähler classes.*

This construction was the key argument used in [7, 9, 8] together with the Arezzo-Pacard gluing theory [2], for producing CSCK metrics on the desingularization $\widehat{\mathcal{X}}$ of $\overline{\mathcal{X}}$. The point is that the local resolution of isolated singularity that occur in $\overline{\mathcal{X}}$ admit scalar-flat Kähler metrics deduced from the Calderbank-Singer ALE scalar-flat Kähler metrics [4].

The gluing theorem that we shall use can be stated as follows.

Theorem 2.1.3. *Let $\overline{\mathcal{X}}$ be a CSCK orbifold surface with Kähler class Ω^{orb} and isolated singularities z_i modelled on $\mathbb{C}^2/\Gamma_i$, where Γ_i is a finite cyclic subgroup of $U(2)$. Let $\pi_{\overline{\mathcal{X}}} : \widehat{\mathcal{X}} \to \overline{\mathcal{X}}$ be the minimal resolution. Then $\widehat{\mathcal{X}}$ admits extremal metrics in every Kähler class sufficiently close to $\pi^*_{\overline{\mathcal{X}}}\Omega^{orb}$.*

Proof. In the case where $\widehat{\mathcal{X}}$ has no nontrivial holomorphic vector field, the result is essentially an application of Arezzo-Pacard gluing theorem [2] to $\overline{X}$ and the Calderbank-Singer metrics [4].

Arezzo-Pacard gluing theorem actually provides only a one parameter family of CSCK metrics, by gluing in a copy of one particular Calderbank-Singer metric with scale ε. One can improve the Arezzo-Pacard gluing theory, working uniformly with all (a finite dimensional smoothly varying family) Calderbank-Singer metrics and the result follows.

In the case where $\widehat{\mathcal{X}}$ admits non trivial holomorphic vector fields, one can prove the same result working with the equation of extremal metrics instead. It suffices to work modulo a maximal compact torus of the isometry group of $\overline{\mathcal{X}}$. This approach has been successfully implemented by Tipler [12]. Again one has to be extra careful to get a uniform result, not only a one parameter family. □

We deduce the following theorem:

Theorem 2.1.4. *Suppose that $\mathcal{X} \to \Sigma$ is a parabolic ruled surface and Ω^{orb} be an orbifold Kähler class on $\overline{\mathcal{X}}$.*

If $\mathcal{X} \to \Sigma$ is parabolically polystable, then every Kähler class of $\widehat{\mathcal{X}}$ sufficiently close to $\pi^*_{\overline{\mathcal{X}}}\Omega^{orb}$ contains an extremal metric.

If $\mathcal{X} \to \Sigma$ is parabolically stable then $\widehat{\mathcal{X}}$ has no nontrivial holomorphic vector fields and the extremal metric must be CSCK.

Proof. If $\mathcal{X} \to \Sigma$ is polystable, $\overline{X}$ admits a CSCK metric with Kähler class Ω^{orb} by Lemma 2.1.2. The existence of extremal metrics for Kähler classes Ω sufficiently close to Ω^{orb} on $\widehat{\mathcal{X}}$ follows from Theorem 2.1.3.

If $\mathcal{X} \to \Sigma$ is parabolically stable, $\pi_1^{orb}(\overline{\Sigma})$ acts with no fixed points on $\mathbb{CP}^1$ via the morphism ρ (this is a part of the Mehta-Seshadri theorem [6]). This implies that Σ is not $\mathbb{CP}^1$ with two marked points, otherwise, $\mathcal{X} \to \mathbb{CP}^1$ would be at best polystable. In particular, $\overline{\Sigma}$ has no nontrivial holomorphic vector field. Following [9], we deduce that $\widehat{\mathcal{X}}$ has no nontrivial holomorphic vector field either. In conclusion, every extremal metric on $\widehat{\mathcal{X}}$ must be CSCK. □

In [8], it was proved that (under some mild technical assumptions), one can always find Kähler class close to $\pi_{\mathscr{X}}^*\Omega^{orb}$ which are represented by CSCK metrics, even when $\mathscr{X} \to \Sigma$ is polystable but not stable. The technique is based on a refinment of Arezzo-Pacard gluing theory in presence of obstructions [3].

A computation of the Futaki invariant (cf. §4) allows to deduce this result from Theorem 2.1.4 in a simpler way. Indeed, an extremal metric is CSCK if and only if its Futaki invariant vanishes. The following theorem also shows that there are always Kähler classes near $\pi_{\mathscr{X}}^*\Omega_c^{orb}$ which are represented by non CSCK extremal metrics, when the parabolic structure is non trivial:

Theorem 2.1.5. *Suppose that $\mathscr{X} \to \Sigma$ is a parabolically polystable ruled surface which in not parabolically stable. We are also assuming that the parabolic structure is not trivial and that $\overline{\Sigma}$ is not a football.*

Then the Lie algebra of holomorphic vector field of $\widehat{\mathscr{X}}$ has dimension 1, spanned by some vector field Ξ. In addition, there exists an open cone $U \subset H^2(\widehat{\mathscr{X}}, \mathbb{R})$ such that $\mathscr{R}(\overline{\Sigma}) \subset U$ with the property that the equation $\mathfrak{F}(\Xi, \cdot) = 0$ cuts $U \cap \mathscr{K}(\widehat{X})$ along a non empty regular hypersurface containing $\mathscr{R}(\overline{\Sigma})$ in its closure.

Proof. The condition of stability implies that $\overline{\Sigma}$ cannot be a teardrop or a football with two singularities of distinct orders (cf. Remark 2.1.1). If $\overline{\Sigma}$ is a footbal with two singularities of the same order, $\overline{X}$ and $\widehat{X}$ are actually toric and the Lie algebra of holomorphic vector fields is two dimensional. In all the other cases, $\overline{\Sigma}$ has no nontrivial holomorphic vector fields and it follows that le dimension of the Lie algebra is exactly 1 (cf. [8]). The property of the Futaki invariant then follows from Lemma 4.6.3. □

Proof of Theorem A, (3) ⇒ (1). It is an immediate consequence of (1) in Theorem 2.1.4. □

3. Unstable parabolic ruled surfaces and test configurations

The aim of this section of to prove the statement (2) ⇒ (3) of Theorem A. Let $\mathscr{X} \to \Sigma$ be a geometrically ruled surface with parabolic structure and rational weights as in Theorem A. In this section, we shall assume that $\widehat{\mathscr{X}}$ carries no nontrivial holomorphic vector fields and that $\mathscr{X} \to \Sigma$ is parabolically unstable. Then, there is a holomorphic section of $\mathscr{X} \to \Sigma$, denoted S, such that $\mathrm{par}\mu(S) \leq 0$.

3.1. **Holomorphic sections and Extensions.** By definition of a geometrically ruled surface, $\mathscr{X} = \mathbb{P}(E)$, where $E \to \Sigma$ is a rank 2 holomorphic vector bundle. The section S corresponds to a holomorphic line bundle $L_+ \subset E$ and we have an exact sequence of holomorphic vector bundles

$$\begin{tikzcd} 0 \arrow[r] & L_+ \arrow[r] \arrow[dr] & E \arrow[r] \arrow[d] & L_- \arrow[r] \arrow[dl] & 0 \\ & & \Sigma & & \end{tikzcd}$$

where $L_- = E/L_+$. The vector bundle E must be an extension bundle; more precisely, E is defined by a element $\tau \in H^1(\Sigma, L_-^* \otimes L_+)$. Such an extension will be denoted $E = E_\tau$.

Let U_j be an open cover of Σ and a cocycle $\tau_{ij} : U_i \cap U_j \to L_-^* \otimes L_+$ defining τ. Let $\mathscr{L}_\pm \to \mathbb{C} \times \Sigma$ be the holomorphic line bundles obtained as the pullback of

$L_\pm$ via the canonical projection $\mathbb{C}\times\Sigma\to\Sigma$. We introduce the extension bundle $\mathscr{E}\to\mathbb{C}\times\Sigma$ defined as follows: the restriction of $\mathscr{E}$ to any open set $\mathbb{C}\times U_j$ is isomorphic to $(\mathscr{L}_+\oplus\mathscr{L}_-)|_{\mathbb{C}\times U_j}$ and the transition maps on $\mathbb{C}\times(U_i\cap U_j)$ are given by

$$(\lambda, z, l_+, l_-)\mapsto(\lambda, z, l_+ + \lambda\tau_{ij}(z)\cdot l_-, l_-),$$

where $\lambda\in\mathbb{C}$, $z\in U_i\cap U_j$ and $l_\pm$ belong to the fiber of $L_\pm\to\Sigma$ over z.

The restriction o $\mathscr{E}$ over $\{1\}\times\Sigma$ is canonically identified to $E_\tau\simeq E\to\Sigma$ and the bundle $\mathscr{E}$ sits in an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{L}_+ \arrow[r] \arrow[dr] & \mathscr{E} \arrow[r] \arrow[d] & \mathscr{L}_- \arrow[r] \arrow[dl] & 0 \\ & & \mathbb{C}\times\Sigma & & \end{tikzcd}$$

There is an obvious $\mathbb{C}^*$-action defined on open sets, and given by $u\cdot(\lambda, z, l_+, l_-) = (u\lambda, z, ul_+, l_-)$. This action lifts as an linear $\mathbb{C}^*$-action on $\mathscr{E}\to\mathbb{C}\times\Sigma$. Its restriction to $\mathscr{E}_{\{0\}\times\Sigma}$ identified to $L_+\oplus L_-\to\Sigma$ is the induced action on the fibers given by $u\cdot(l_+, l_-) = (ul_+, l_-)$.

Passing to the projectivization $\mathcal{M}=\mathbb{P}(\mathscr{E})$, we obtain a ruled manifold $\mathcal{M}\to\mathbb{C}\times\Sigma$. In particular, the line bundle $\mathscr{L}_+\subset\mathscr{E}$ defines a divisor $\mathscr{S}\subset\mathcal{M}$ with the property that $\mathscr{S}\cap\mathcal{M}_1$ is identified to $S\subset\mathcal{X}$ whereas $\mathscr{S}\cap\mathcal{M}_0$ is identified to $S_+\subset\mathbb{P}(L_+\oplus L_-)$. We summarize our observations in the following lemma:

Lemma 3.1.1. *Given a geometrically ruled surface $\mathcal{X}\to\Sigma$ and a holomorphic section S, there exists a complex manifold $\mathcal{M}$ endowed with a $\mathbb{C}^*$-action and a $\mathbb{C}^*$-equivariant submersive holomorphic map $\pi_\mathbb{C}:\mathcal{M}\to\mathbb{C}$, with respect to the standard $\mathbb{C}^*$-action on $\mathbb{C}$, such that:*

- *$\mathcal{M}_1=\pi_\mathbb{C}^{-1}(1)$ is isomorphic to $\mathcal{X}\simeq\mathbb{P}(E)$;*
- *$\mathcal{M}_0=\pi_\mathbb{C}^{-1}(0)$ is isomorphic to $\mathbb{P}(L_+\oplus L_-)$ with the above notations.*
- *In $\mathcal{M}_0$, the corresponding divisors S_+ and S_- are respectively the attractive and repulsive sets of fixed points in $\mathcal{M}$ under the $\mathbb{C}^*$-action.*
- *There exists a $\mathbb{C}^*$-invariant section $\mathscr{S}$ of $\mathcal{M}\to\mathbb{C}\times\Sigma$ such that $\mathscr{S}\cap\mathcal{M}_0 = S_+$ and $\mathscr{S}\cap\mathcal{M}_1=S$.*

3.2. **Test configurations.** The definition of K-stability involves general *test configurations*. However *regular* test configuration will be sufficient for our purpose, that is:

- a complex manifold $\mathcal{M}$ with a holomorphic $\mathbb{Q}$-line bundle $\mathcal{P}\to\mathcal{M}$,
- a $\mathbb{C}^*$-action on $\mathcal{M}$ that lifts to a linear $\mathbb{C}^*$-action on $\mathcal{P}\to\mathcal{M}$.
- $\mathbb{C}^*$-equivariant submersive holomorphic map $\pi_\mathbb{C}:\mathcal{M}\to\mathbb{C}$

such that $\mathcal{P}\to\mathcal{M}$ is a fiberwise polarization. In other words, the restriction $\mathcal{P}$ restricted to $\mathcal{M}_\lambda=\pi_\mathbb{C}^{-1}(\lambda)$ is an ample $\mathbb{Q}$-line bundle for every $\lambda\in\mathbb{C}$ (i.e. a *polarization*).

In Lemma 3.1.1, we already produces a regular $\mathbb{C}^*$-equivariant family of deformation $\mathcal{M}\to\mathbb{C}$ of the ruled surface $\mathcal{M}_1\simeq\mathbb{P}(\mathscr{E})=\mathcal{M}\to\mathbb{C}$ with the property that $\mathcal{M}_0\simeq\mathbb{P}(L_+\oplus L_-)$. Relying on this result, we can easily construct test configurations for the iterated blowup $\widehat{\mathcal{X}}$.

The central fibers $\mathcal{M}_0$ has two divisors $S_\pm$ determined by the line bundles $L_\pm$. The manifold $\mathcal{M}$ also has a divisor $\mathscr{S}_+$ defined by the subbundle $\mathscr{L}_+\subset\mathscr{E}$ with the property thatt $S_+=\mathscr{S}_+\cap\mathcal{M}_0$.

The parabolic structure on $\mathcal{M}_1 \simeq \mathbb{P}(E) = \mathcal{X}$ consists a finite set of marked points x_j in distinct fibers of $\mathcal{X} \to \Sigma$. Let X_j be the closure in $\mathcal{M}$ of the orbit of the points x_j under the $\mathbb{C}^*$-action. The points $x_j^\lambda = X_j \cap \mathcal{M}_\lambda$ and weights α_j define a parabolic structure on each fiber of $\mathcal{M}_t$. Notice that all the points of the parabolic structure induced on $\mathcal{M}_0 \to \Sigma$ must belong to $S_\pm$.

Following the algorithm described at §1.2, one can make a iterated blowup of every deformation $\mathcal{M}_t$ simultaneously. This boils down to perform an iterated blowup of the curves X_j in $\mathcal{M}$. Thus, we obtain a blowup $\widehat{\mathcal{M}} \to \mathcal{M}$ with the property that $\widehat{\mathcal{M}}_1 \simeq \widehat{\mathcal{X}}$. The $\mathbb{C}^*$-action lifts to a $\mathbb{C}^*$-action on $\widehat{\mathcal{M}}$ and we actually have a $\mathbb{C}^*$-equivariant family of deformations $\widehat{\mathcal{M}} \to \mathbb{C}$.

For a ruled surface $h^{2,0} = h^{0,2} = 0$, hence any class in H^2 is of type $(1,1)$. The fibration $\widehat{\mathcal{M}} \to \mathbb{C}$ is smoothly trivial. So the cohomology spaces $H^2(\widehat{\mathcal{M}}_\lambda, \mathbb{R})$ are all identified canonically to $H^2(\widehat{\mathcal{X}}, \mathbb{R})$. We consider the cohomology class Ω_c^{orb} as a class in $H^2(\widehat{\mathcal{X}}, \mathbb{R})$. We saw that for $c > 0$ sufficiently small, the class Ω_c^{orb} may be perturbed to give a Kähler class on $\widehat{\mathcal{X}}$ (cf. §1.4). The same result applies to Ω_c^{orb} understood as a cohomology class on $\widehat{\mathcal{M}}_0$ and we have the following result:

Lemma 3.2.1. *For $c > 0$ and $\varepsilon > 0$ sufficiently small, the cohomology classes Ω in Lemma 1.4.6 define Kähler classes on $\widehat{\mathcal{M}}_\lambda$ for every $\lambda \in \mathbb{C}$.*

In particular, if the constants c and $c_j^\pm$ are all chosen rational, the cohomology class Ω is rational and it defines a $\mathbb{Q}$-line bundle $\mathcal{P} \to \widehat{\mathcal{M}}$ with the property that $c_1(\mathcal{P}) = \Omega$.

We summarize our construction in the following proposition:

Proposition 3.2.2. *Let $\mathcal{X} \to \Sigma$ be a parabolic ruled surface with rational weights and S a holomorphic section. There exists a sufficiently small open cone $U \subset H^2(\widehat{\mathcal{X}}, \mathbb{R})$ that contains $\mathscr{R}(\overline{\Sigma})$ with the property that for every rational Kähler class $\Omega \in \mathscr{K}(\widehat{\mathcal{X}}) \cap U$, we can define a test configuration $\widehat{\mathcal{M}} \to \mathbb{C}$, polarized by a $\mathbb{Q}$-line bundle $\mathcal{P} \to \mathcal{M}$ with the following property:*

1. *$\widehat{\mathcal{M}} \to \mathbb{C} \times \Sigma$ is an iterated blowup encoded by the parabolic structure of the ruled manifold $\mathcal{M} \to \mathbb{C} \times \Sigma$ given by Lemma 3.1.1, endowed with the induced $\mathbb{C}^*$-action.*
2. *The retriction $\mathcal{P}|_{\widehat{\mathcal{M}}_1} \to \widehat{\mathcal{M}}_1$ is identified to $\widehat{\mathcal{X}}$ endowed with a $\mathbb{Q}$-line bundle of first Chern class Ω,*
3. *$\widehat{\mathcal{M}}_0$ is an iterated blowup of $\mathcal{M}_0 \simeq \mathbb{P}(L_+ \oplus L_-)$ encoded by the induced parabolic. All the parabolic points of $\mathcal{M}_0$ where the blowups occur are located on the sections $S_\pm$ corresponding to $L_\pm$. structure.*

4. On the Futaki invariant of blownup ruled surfaces

In this section, we shall prove the following proposition.

Proposition 4.0.3. *Let $\mathcal{X} \to \Sigma$ be a parabolic ruled surface with rational weights. Let S be a holomorphic section such that $\operatorname{par}\mu(S) \leq 0$. Then for every open cone $U \subset H^2(\widehat{\mathcal{X}}, \mathbb{R})$ such that Proposition 3.2.2 holds, there exists a rational Kähler class $\Omega \in \mathscr{K}(\widehat{\mathcal{X}}) \cap U$ such that corresponding test configuration $\mathcal{P} \to \widehat{\mathcal{M}} \to \mathbb{C}$ has non-positive Donaldson-Futaki weight.*

Since $\widehat{\mathcal{X}}$ has no nontrivial holomorphic vector field and $\widehat{\mathcal{M}}_0$ does, the test configuration must be non-trivial. Thus we get the following corollary which proves the statement (2) $\Rightarrow$ (3) of Theorem A:

Corollary 4.0.4. *If $\mathcal{X} \to \Sigma$ is not parabolically stable and $\widehat{\mathcal{X}}$ has no nontrivial holomorphic vector fields, then $\widehat{\mathcal{X}}$ is not basically K-stable.*

Proof of Proposition 4.0.3. Since the test configuration $\mathcal{P} \to \widehat{\mathcal{M}} \to \mathbb{C}$ is regular, its Donaldson-Futaki invariant is actually given by the usual Futaki invariant of the central fiber. Computing its Futaki invariant is the goal of the rest of this section. In particular, the proposition follows from the Lemmas 4.6.1 and 4.6.3. □

4.1. **Geometrically ruled surfaces with circle symmetry.** From Proposition 3.2.2, we have $\mathcal{M}_0 \simeq \mathbb{P}(L_+ \oplus L_-)$. In addition, this geometrically ruled surface is endowed with a parabolic structure deduced from the parabolic structure on $\mathcal{M}_1$ as explained at §3.2. In more concrete terms, we pass from a parabolic structure on $\mathcal{M}_1$ to a parabolic structure on $\mathcal{M}_0$ as follows: let x_j^1 be a parabolic point in $\mathcal{M}_1 \simeq \mathcal{X}$ such that $\pi_\Sigma(x_j^1) = y_j$. Then, there is a parabolic point $x_j^0 \in \mathcal{M}_0$ in the fiber of $y_j \in \Sigma$, such that $x_j^0 \in S_+$ if $x_j^1 \in S$, and, $x_j^0 \in S_-$ otherwise. Eventually, the parabolic weight attached to x_j^0 is given by the weight of x_j^1. The central fiber $\widehat{\mathcal{M}}_0$ of the test configuration given by Proposition 3.2.2 is the iterated blowup of the parabolic ruled surface $\mathcal{M}_0 \to \Sigma$. Similarly to $\overline{\mathcal{X}} \to \overline{\Sigma}$, we obtain a complex geometrically ruled orbifold surface $\overline{\mathcal{M}}_0 \to \overline{\Sigma}$ by contracting the $E_j^\pm$-curves in $\widehat{\mathcal{M}}_0$.

By construction $\mathcal{M}_0$ is endowed with a $\mathbb{C}^*$-action coming from the $\mathbb{C}^*$-action on the complex manifold $\mathcal{M}$. In fact, this action is determined by the following properties:

- the action is free on a dense open subset of $\mathcal{M}_0$,
- it preserves the fibers of the ruling $\mathcal{M}_0 \to \Sigma$,
- the sections $S_\pm$ are the fixed points of the action,
- the points of S_- are repulsive, and the points of S_+ are attractive.

As all the parabolic points of $\mathcal{M}_0 \to \Sigma$ belong to $S_- \cup S_+$, the $\mathbb{C}^*$-action lifts to $\widehat{\mathcal{M}}_0 \to \mathcal{M}$. Let $\hat{S}_-$ and $\hat{S}_+$ be the proper transforms of S_- and S_+ in $\widehat{\mathcal{M}}_0$. Notice that the $\mathbb{C}^*$-action also descends via the canonical projection $\mathcal{M}_0 \to \overline{\mathcal{M}}_0$, since it must preserve holomorphic spheres of negative self-intersection.

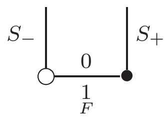
4.2. **Cremona transformations.** As we noticed, any parabolic point $x \in F = \pi_\Sigma^{-1}(y) \subset \mathcal{M}_0$ belongs to S_+ or S_-. Assume $x \in S_-$. Let $\mathcal{M}_0'' \to \mathcal{M}_0$ be the blowup at x and $\hat{F} \subset \mathcal{M}_0''$ be the proper transform of F. Since $\hat{F}$ has self-intersection -1 it can be contracted back to a point x'. The contraction is denoted $\mathcal{M}_0'' \to \mathcal{M}_0'$. Such an operation (blowing up, then contracting) is called a *Cremona transformation*. Notice that the proper transform $S_+' \subset \mathcal{M}_0'$ of S_+ contains the point x'.

Furthermore, the ruled surface $\mathcal{M}_0' \to \Sigma$ has a natural parabolic structure induced by the parabolic structure of $\mathcal{M}_0 \to \Sigma$ with the convention that x has been replaced by x' and the corresponding weight α is now replaced by $\alpha' = 1 - \alpha$. It is an easy exercise to show that the notions of parabolic stability are invariant under such Cremona transformation. In addition the iterated blowup encoded by either parabolic ruled surfaces are both $\widehat{\mathcal{M}}_0$.

Therefore, we may assume that all the parabolic points of $\mathcal{M}_0$ belong to S_+ after performing a finite number of Cremona transformations. The condition of stability is unchanged provided the weights are modified according to the above convention.

4.3. **Weights of the $\mathbb{C}^*$-action along special fibers.** The fibers of $\widehat{\mathcal{M}}_0 \to \Sigma$ are preserved by the $\mathbb{C}^*$-action. Generic fibers are identified to $\mathbb{CP}^1$ endowed with a $\mathbb{C}^*$-action of weight 1, and the two fixed points correspond to the intersections of the fiber with $\hat{S}_\pm$. In contrast, blownup fibers have more complicated $\mathbb{C}^*$-action. We start from $\pi_\Sigma : \mathcal{M}_0 \to \Sigma$ and assume that there is only one parabolic point x for simplicity. By §4.2 we may also assume that $x \in S_+$. The fiber F containing x is represented by the configuration of curves

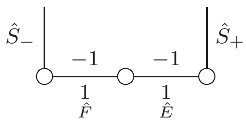
(4.3.1)



Here, the black dot represents the point x in the fiber $F = \pi_\Sigma^{-1}(y)$ of self-intersection 0. The integer 1 represents the weight of the $\mathbb{C}^*$-action induced on F.

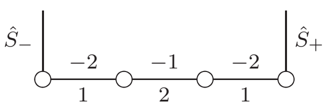
Then we blowup the point x and get a configuration

(4.3.2)



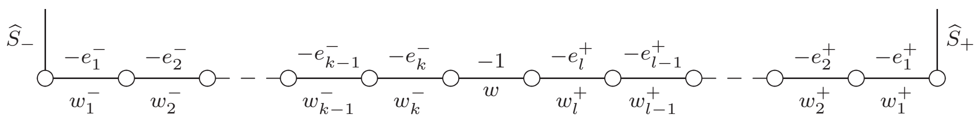
In the above diagram, the integer 1 represent the weights of the induced $\mathbb{C}^*$-actions on the proper transform $\hat{F}$ of F and on $\hat{E}$ the exceptional divisor of the blowup. Using the same notation, we blowup the intersection of the -1 curve and obtain

(4.3.3)



The we iterate our blowup procedure in order to get a diagram of the form

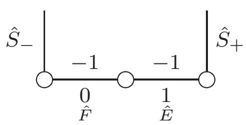
(4.3.4)



The weight of the $\mathbb{C}^*$-action induced on the -1-curve is computed by induction, using the simple formula $w = w_k^- + w_l^+$. We shall also use the notation $E_j^\pm$ for the curve of self-intersection $-e_j^\pm$ and E_0 for the -1-curve.

Instead of starting with the configuration (4.3.5), we can formally replace the weights with the new configuration

(4.3.5)



Using the same induction as for $w_j^{\pm}$, we construct a weight system

$$v_1^-, \cdots, v_k^-, v, v_l^+, \cdots, v_1^+.$$

By definition of the weights and the ajunction, we have

$$F = wE_0 + \sum_{n=1}^{k} w_n^- E_n^- + \sum_{n=1}^{l} w_n^+ E_n^+$$

and

$$\hat{E} = vE_0 + \sum_{n=1}^{k} v_n^- E_n^- + \sum_{n=1}^{l} v_n^+ E_n^+,$$

where $\hat{E}$ and F denote the pullback the homology classes to $\widehat{\mathcal{M}}_0$.

We gather the relevant results in the following lemma:

Lemma 4.3.6. *Using the convention $w = w_{k+1}^- = w_{l+1}^+$, we have*

$$\sum_{n=1}^{k} \frac{1}{w_n^- w_{n+1}^-} = \alpha, \quad \sum_{n=1}^{l} \frac{1}{w_n^+ w_{n+1}^+} = 1 - \alpha.$$

Furthermore the weights introduced above satisfy

$$w = q, \quad v = p, \quad w_1^{\pm} = 1, \quad v_1^+ = 1 \quad \text{and } v_1^- = 0.$$

Proof. The proof by induction is straightforward and left as an exercice for the interested reader. □

4.4. **Futaki invariant.** LeBrun and Simanca computed the Futaki invariant of a ruled surface endowed with a semi-free $\mathbb{C}^*$-action. We are going to point out what should be modified in their formula in the case of the surface $\widehat{\mathcal{M}}_0$. The reader is strongly advised to refer to [5, Section 3.3] as we are following closely their notations.

Let Ξ be the $(1,0)$-holomorphic vector field on $\widehat{\mathcal{M}}_0$ defined as the Euler vector field that generates the $\mathbb{C}^*$-action. We define $\xi = -2\mathrm{Im}\Xi$ as the (real) vector field that spans the underlying circle action. Let ω be a circle-invariant Kähler form on $\widehat{\mathcal{M}}_0$ with Kähler class $\Omega = [\omega]$. Since ξ is a Killing field vanishing at some point, it is automatically Hamiltonian (cf. [5]). In other words, there exists a smooth Hamiltonian function $t : \widehat{\mathcal{M}}_0 \to \mathbb{R}$ such that $dt = -\iota_\xi \omega$. Then t admits a minimum along $\hat{S}_-$ and a maximum along $\hat{S}_+$. Up to adding a suitable constant, we may assume that $t : \widehat{\mathcal{M}}_0 \to [-a, +a]$ is a surjective map for some $a > 0$ and $\hat{S}_\pm = t^{-1}(\pm a)$.

Again, we are assuming that $\mathcal{M}_0$ has only one parabolic point x to keep notations simple. Up to a Cremona tranformation, we may even assume that $x \in S_+$ (cf. §4.2). The set of critical points of the function t on $\widehat{\mathcal{M}}_0$ consists of the divisors $\hat{S}_\pm$ where t is extremal, and isolated sadle points. The latter are given by the intersections of the $E_j^\pm$ and E_0-curves. These points represented by the hollow dots in Diagram (4.3.4). It will be convenient to label them $f_0, \cdots, f_k, f_{k+1}, \cdots, f_{k+l+1}$ from the left to the right. In the same spirit we shall use a notation $w_1 = w_1^-, \cdots, w_k = w_k^-, w_{k+1} = w, w_{k+1} = w_l^+, \cdots, w_{k+l+1} = w_1^+$.

Let $\mathscr{Y} = \widehat{\mathcal{M}}_0 \setminus (S_+ \cup S_- \cup \{f_j\})$, the regular locus of t. By definition $\mathscr{Y}$ is a Seifert manifold. Any point $z \in \mathscr{Y}$ has trivial stabilizer, unless z belongs to E_0 or

$E_j^\pm$ where the stabilizer is the cyclic group respectively of order w and $w_j^\pm$. Hence the quotient $N = \mathscr{Y}/S^1$ has an orbifold structure and $\varpi : \mathscr{Y} \to N$ is an orbifold circle bundle. This is the main difference with the case of a semi-free action, where $\mathscr{Y} \to N$ is a smooth circle bundle.

The function t is invariant under the circle action, hence the fibers $\mathscr{Y}_c$ of $t : \mathscr{Y} \to (-a,a)$ are endowed with a circle action and the map descends to $t : N \to (-a,a)$. If c is a regular value of t, $N_c = t^{-1}(c) \subset Y$ is a compact orbifold. Moreover, N_c has a natural Kähler structure since it is a Kähler moment map reduction of $(\widehat{\mathcal{M}}_0, \omega)$ by the Hamiltonian action of the circle. Furthermore N_c is isomorphic to N_d if there are no critical value in the interval $[c,d]$.

Let $t_j = t(f_j)$ be the t-coordinate of the fixed point f_j. Then the following facts hold, by definition:

(1) If $t_j < c < t_{j+1}$ for some $0 \le j \le k+l$, the Riemann surface N_c is isomorphic to Σ, where the marked point y of the parabolic structure has been replaced by an orbifold point of order w_{j+1}. In particular $N_c \simeq \Sigma$ is $j = 0$ or $k+l$ and $N_c \simeq \overline{\Sigma}$ if $j = k$.

(2) Let $S_j \subset N$ be a small sphere (with orbifold singularities) centered at a point $\varpi(f_j)$, for $1 \le j \le k+l$. Then the orbicircle bundle $\mathscr{Y}|_{S_j} \to S_j$ has orbifold degree

$$c_1(\mathscr{Y}) \cdot [S_j] = \frac{1}{w_j w_{j+1}}. \tag{4.4.1}$$

This readily seen as $\mathscr{Y} \to S_j$ admits a $w_j w_{j+1}$-fold ramified cover by the Hopf fibration $S^3 \to S^2$. This is also an essential difference with the case of a semi-free circle action [5, top of the page 315].

It is also convenient to use a rescaled Kähler class in comparison with §1.4. Here we shall assume that the Kähler class Ω satisfies the identity $\Omega \cdot F = 1$. Adapting carefully the computation of [5, p. 318-319] to this orbifold context, and relying on facts (1) and (2) above, we get the identity

$$\int_{\widehat{\mathcal{M}}_0} t d\mu = \frac{1}{96\pi}\left(\hat{S}_-^2 - \hat{S}_+^2 + 6\Omega \cdot \left(\hat{S}_+ - \hat{S}_-\right) - 64\pi^3 \sum_{j=1}^{k+l} \frac{t_j^3}{w_j w_{j+1}}\right) \tag{4.4.2}$$

where $d\mu$ is the volume form of the Kähler metric ω. We also have the modified formula

$$\int s t d\mu = \Omega \cdot \left(\hat{S}_+ - \hat{S}_-\right) + 4\pi^2 \sum_{j=1}^{k+l} (w_j^{-1} - 1)(t_{j+1}^2 - t_j^2) \tag{4.4.3}$$

where s is the scalar curvature of the metric. By definition of the Futaki invariant $\mathfrak{F}(\Xi, \Omega) = \int (s_0^\Omega - s) t d\mu$ and we end up with the formula

$$\begin{aligned}\mathfrak{F}(\Xi,\Omega) =& \Omega \cdot (\hat{S}_- - \hat{S}_+) - 4\pi^2 \sum_{j=1}^{k+l} (w_j^{-1} - 1)(t_{j+1}^2 - t_j^2) \\ &+ \frac{s_0^\Omega}{96\pi}\left(\hat{S}_-^2 - \hat{S}_+^2 + 6\Omega \cdot (\hat{S}_+ - \hat{S}_-) - 64\pi^3 \sum_{j=1}^{k+l} \frac{t_j^3}{w_j w_{j+1}}\right)\end{aligned}$$

where

$$s_0^\Omega = \int s d\mu = 8\pi \frac{c_1(\widehat{\mathcal{M}}_0)\cdot\Omega}{\Omega^2}.$$

4.5. **A computation in the degenerate case.** Notice that if we let Ω degenerates toward (the pullback of) an orbifold Kähler class Ω^{orb} on $\overline{\mathcal{M}}_0$, we have $t_1 = \cdots = t_k = -a$ and $t_{k+1} = \cdots = t_{k+l} = a$. Therefore, using Lemma 4.3.6 and the fact that $4\pi a = \Omega \cdot F = 1$ (cf. [5, bottom of p. 315]), we obtain

$$\lim_{\Omega\to\Omega^{orb}} \int t d\mu = \frac{1}{96\pi}\left(\hat{S}_-^2 - \hat{S}_+^2 + 6\Omega\cdot(\hat{S}_+ - \hat{S}_-) + \alpha - (1-\alpha)\right)$$

On the other hand $[\hat{S}_+]^2 = [S_+]^2 - 1$ and $[\hat{S}_-]^2 = [S_-]^2$ since the first blowup occurred at $x \in S_+$. It follows that

$$[\hat{S}_-]^2 + \alpha = \mathrm{par}\mu(S_-), \quad \text{and} \quad \hat{S}_+^2 + 1 - \alpha = \mathrm{par}\mu(S_+).$$

Finally $\Omega\cdot(\hat{S}_+ - \hat{S}_-) = \Omega^{orb}\cdot(\bar{S}_+ - \bar{S}_-)$ for a orbifold Kähler class, where $\bar{S}_\pm = \pi_{\overline{\mathcal{M}}_0}(\hat{S}_\pm)$. The holomorphic sections $\bar{S}_\pm$ of $\overline{\mathcal{M}}_0 \to \overline{\Sigma}$ corresponds to orbifold line bundle $\bar{L}_\pm \to \overline{\Sigma}$. In this context, it is well known that (cf. [10, Theorem 3.8])

$$c_1^{orb}(\mathcal{O}_{\overline{\mathcal{M}}_0}(1))\cdot\bar{S}_\pm = \mathrm{orb}\deg \bar{L}_\pm = \mathrm{par}\deg L_\pm,$$

where orb deg is the natural notion of degree for an orbifold line bundle. Hence $\Omega\cdot(\hat{S}_+ - \hat{S}_-) = \mathrm{par}\deg L_+ - \mathrm{par}\deg L_- = (\mathrm{par}\deg L_+ + \mathrm{par}\deg L_-) - 2\mathrm{par}\deg L_- = \mathrm{par}\mu(S_-) = -\mathrm{par}\mu(S_+)$

In conclusion, we have the following result:

Lemma 4.5.1. *Let $\Omega \in \mathscr{K}(\widehat{\mathcal{M}}_0)$ and $\Omega^{orb} \in \mathscr{K}(\overline{\mathcal{M}}_0)$ considered as a class on $\widehat{\mathcal{M}}_0$ as well. Then*

$$\lim_{\Omega\to\Omega^{orb}} \int_{\widehat{\mathcal{M}}_0} t d\mu = -\frac{\mathrm{par}\mu(S_+)}{12\pi}, \quad \lim_{\Omega\to\Omega^{orb}} \int_{\widehat{\mathcal{M}}_0} st d\mu = -\mathrm{par}\mu(S_+).$$

4.6. **Sign of the Futaki invariant.** We deduce the following lemma, which will be crucial for the proof of Proposition 4.0.3.

Lemma 4.6.1. *Suppose that $\mathrm{par}\mu(S_+) \neq 0$. There exists a sufficiently small open cone $U \subset H^2(\widehat{\mathcal{M}}_0, \mathbb{R})$ containing the ray $\mathscr{R}(\overline{\Sigma})$, such that for every Kähler class $\Omega \in U \cap \mathscr{K}(\widehat{\mathcal{M}}_0)$, the Futaki invariant $\mathfrak{F}(\Xi, \Omega)$ does not vanish and has the same sign as $\mathrm{par}\mu(S_+)$.*

Proof. We start with an orbifold Kähler class $\Omega_C^{orb} = c_1^{orb}(\mathcal{O}_{\overline{\mathcal{M}}_0}(1)) + CF$ on $\overline{\mathcal{M}}_0$, where $C > 0$ is chosen very large. We use a generalization of the classical result for smooth geometrically ruled surfaces:

$$c_1^{orb}(K_{\overline{\mathcal{M}}_0}) = -2c_1^{orb}(\mathcal{O}_{\overline{\mathcal{M}}_0}(1)) + (\mathrm{par}\deg(E) - \chi^{orb}(\overline{\Sigma}))F,$$

where $K_{\overline{\mathcal{M}}_0}$ is the (orbifold) canonical line bundle of the orbifold $\overline{\mathcal{M}}_0$.

It follows that $s_0^{\Omega_C^{orb}} = 8\pi \frac{\mathrm{par}\deg(E) + \chi^{orb}(\overline{\Sigma}) + 2C}{\mathrm{par}\deg(E) + 2C}$. In particular, we see that

$$\lim_{C\to+\infty} s_0^{\Omega_C^{orb}} = 8\pi.$$

Using the fact that

$$\lim_{\Omega\to\Omega_c^{orb}} s_0^\Omega = s_0^{\Omega_C^{orb}} \quad \text{and} \quad \lim_{\Omega\to\Omega_C^{orb}} \Omega\cdot\hat{S}_\pm = \Omega_C^{orb}\cdot\bar{S}_\pm.$$

and that the corresponding values of t_j converge in the following way

$$t_j \to -a \text{ for } j \leq k, \text{ and } \quad t_j \to a \text{ for } j \geq k+1$$

as $\Omega \to \Omega^{orb}$, we see deduce that

$$\lim_{\Omega \to \Omega_C^{orb}} \mathfrak{F}(\Xi, \Omega) = \left(1 - \frac{s_0^{\Omega_C^{orb}}}{12\pi}\right) \mathrm{par}\mu(S_+). \tag{4.6.2}$$

For C sufficiently large, the coefficient in front of $\mathrm{par}\mu(S_+)$ is positive since

$$\lim_{C \to +\infty} \left(1 - \frac{s_0^{\Omega_C^{orb}}}{12\pi}\right) = \frac{1}{3}$$

and the lemma follows. $\square$

Eventually, we deal with the case where the section has vanishing slope.

Lemma 4.6.3. *With the above notations, suppose that* $\mathrm{par}\mu(S_+) = 0$. *Then for every open cone U in $H^2(\widehat{\mathcal{M}}_0, \mathbb{R})$ such that $\mathscr{R}(\overline{\Sigma}) \subset U$, there are rational Kähler classes $\Omega \in U \cap \mathscr{K}(\widehat{\mathcal{M}}_0)$ such that $\mathfrak{F}(\Xi, \Omega) > 0$ and such that $\mathfrak{F}(\Xi, \Omega) < 0$.*

In addition the equation $\mathfrak{F}(\Xi, \cdot) = 0$ cuts $U \cap \mathscr{K}(\widehat{\mathcal{M}}_0)$ along a nonempty regular hypersurface.

Proof. By (4.6.2), we have $\lim_{\Omega \to \Omega_C^{orb}} \mathfrak{F}(\Xi, \Omega) = 0$. This correspond to the limiting value of the Futaki invariant when taking the parameters $a = t_j^+ = -t_j^-$ for $j \geq 1$. The idea to prove the Lemma is to compute the partial derivatives of the Futaki invariant at Ω_C^{orb}. In fact it will suffice to consider variations of the Kähler class corresponding to the parameters $t_j^- = \frac{\tau^-}{4\pi} - a$ and $t_j^+ = a - \frac{\tau^+}{4\pi}$ for $j \geq 1$ for $\tau^\pm > 0$ sufficiently small. By definition the corresponding (orbifold) class Ω satisfies $\Omega \cdot E_1^\pm = \tau^\pm$ and $\Omega \cdot E_j^\pm = 0$ for $j > 1$. Using Lemma 4.3.6 and the fact that $\Omega \cdot F = 1$, we find $\Omega \cdot E_0 = \frac{1-(\tau^+ + \tau^-)}{q}$.

By the adjunction formula $\hat{S}_+ = S_+ - \hat{E}$. Furthermore, in terms of Poincaré dual, $S_\pm = c_1(\mathcal{O}_{\mathcal{M}_0}(1)) - \deg(L_\pm)F$. Thus $\hat{S}_+ - \hat{S}_- = (\deg(L_-) - \deg(L_+))F - \hat{E}$ and it follows that $\Omega \cdot (\hat{S}_+ - \hat{S}_-) = \deg L_- - \deg L_+ - \tau^+ - \alpha(1 - (\tau^+ + \tau^-)) = \mathrm{par}\deg L_- - \mathrm{par}\deg L_+ + \alpha\tau^- + (\alpha - 1)\tau^+$. By assumption L_+ and L_- have the same parabolic degree, therefore

$$\Omega \cdot (\hat{S}_+ - \hat{S}_-) = \alpha\tau^- + (\alpha - 1)\tau^+.$$

Finally, the formula for the Futaki invariant for a variation $\tau^\pm$ can be written

$$\mathfrak{F}(\tau^\pm) = (1 - \frac{6}{96\pi} s_0^\Omega)((1-\alpha)\tau^+ - \alpha\tau^-) \\ - 64\pi^3 \frac{s_0^\Omega}{96\pi} \left(\sum_{j=1}^{k} \frac{(t_j^-)^3 + a^3}{w_j^- w_{j+1}^-} + \sum_{j=1}^{l} \frac{(t_j^+)^3 - a^3}{w_j^- w_{j+1}^-} \right)$$

If Ω is a sufficiently small perturbation of Ω_C^{orb} and $C > 0$ is large enough, then s_0^Ω is very close to 8π. It follows that the Futaki invariant is of the form

$\mathfrak{F}(\tau^\pm) = C_1 f_1 + C_2 f_1$ where $C_1, C_2 > 0$, $f_1 = (1-\alpha)\tau^+ - \alpha\tau^-$ and

$$f_2 = -\sum_{j=1}^{k} \frac{(t_j^-)^3 + a^3}{w_j^- w_{j+1}^-} - \sum_{j=1}^{l} \frac{(t_j^+)^3 - a^3}{w_j^- w_{j+1}^-}$$

The differential are easily computed at $\tau^\pm = 0$ and they are positive multiples of

$$(1-\alpha)d\tau^+ - \alpha d\tau^-.$$

It follows that the variation of $\frac{\partial f_j}{\partial \tau^-} < 0$. In particular, f_j are negative for certain arbitrarily small values of $\tau^\pm > 0$. It follows that $\mathfrak{F}(\tau^\pm)$ must be negative as well. Similarly $\frac{\partial f_j}{\partial \tau^+} > 0$ and the Futaki invariant also take positive values for certain values of $\tau^\pm > 0$ arbitrarily small.

By density, we can always assume that the parameters C and $\tau^\pm$ are chosen suitably so that Ω is rational. We can also perturb the cohomology class Ω by higher order terms so that it is a Kähler class on $\widehat{\mathcal{M}}_0$ (and not just an orbifold Kähler class). The first part of the lemma follows.

Notice that our computation shows that the Futaki invariant is a submersion vanishing at Ω_C^{orb}. So the surface given by the vanishing of the Futaki invariant is regular near Ω_C^{orb}. The fact that the partial derivatives $\frac{\partial \mathcal{F}}{\partial \tau^+}$ have opposite signs insures that the surface $\mathfrak{F}(\Xi, \cdot) = 0$ cuts the quadrant $\tau^\pm > 0$ along a nonempty set, hence cuts the Kähler cone and the lemma follows. □

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