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Abstract. We prove that a word hyperbolic group with planar boundary different from the sphere is virtually a convex-cocompact Kleinian group provided its Ahlfors regular conformal dimension is strictly less than 2 or if it acts geometrically on a CAT(0) cube complex.

1. Introduction

Conjecturally, word hyperbolic groups with planar boundary are virtually convex-cocompact Kleinian groups. The aim of this paper is to provide supporting evidence to this picture. It is known to hold when the boundary is a simple closed curve [CJ, Gab]. When the boundary is the 2-sphere, this is the content of the so-called Cannon conjecture [Can].

Recall that the Sierpiński carpet is the metric space obtained by starting with the unit square, subdividing it into nine squares, removing the middle square, repeating this procedure ad infinitum with the remaining squares, and taking the decreasing intersection. Kapovich and Kleiner have conjectured that any *carpet group* i.e., a hyperbolic group with a boundary homeomorphic to the Sierpiński carpet, is virtually a convex-cocompact Kleinian group [KK, Conjecture 6].

A homeomorphism $h : X \to Y$ between metric spaces is called *quasisymmetric* provided there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that $d_X(x,a) \le t d_X(x,b)$ implies $d_Y(f(x), f(a)) \le \eta(t) d_Y(f(x), f(b))$ for all triples of points $x, a, b \in X$ and all $t \ge 0$ [Hei]. The boundary ∂G of a hyperbolic group G is endowed with an *Ahlfors regular conformal gauge* $\mathcal{G}(G)$ i.e., a family of metrics which are pairwise quasisymmetrically equivalent and which are *Ahlfors regular*: this means that there is a Radon measure μ such that for any $x \in X$ and $r \in (0, \operatorname{diam} \partial G]$, $\mu(B(x,r)) \asymp r^Q$ for some given $Q > 0$ [Mat]. The infimum over $\mathcal{G}(G)$ of every dimension Q is the so-called *Ahlfors regular conformal dimension* $\mathrm{confdim}_{AR} G$ of G [MT, Car, Haï1]. This is a numerical invariant of the quasi-isometry class of G.

The conformal dimension provides us with a first characterization of convex-cocompact Kleinian groups:

Theorem 1.1. *Let G be a non-elementary hyperbolic group with planar boundary different from the sphere. Then G is virtually Kleinian if and only if* $\mathrm{confdim}_{AR}(G) < 2$.

The necessity part of Theorem 1.1 is due to Sullivan, see [Sul1]. When the boundary is the whole sphere, Bonk and Kleiner proved that the group is Kleinian if and only if there is a distance in its Ahlfors-regular conformal gauge of minimal dimension [BK3].

An induction argument enables us to weaken the assumption on the conformal dimension provided G has no elements of order two:

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Theorem 1.2. *Let G be a non-elementary hyperbolic group with planar boundary different from the sphere and with no elements of order two. The following conditions are equivalent.*

(1) *G is virtually Kleinian.*
(2) $\mathrm{confdim}_{AR}(G) < 2$.
(3) $\mathrm{confdim}_{AR}(H) < 2$ *for all quasiconvex carpet subgroups H of G.*

The implication of (2) $\Rightarrow$ (3) follows for instance from [MT, Prop. 2.2.11].

The following statement is an immediate consequence of Theorem 1.2.

Corollary 1.3. *Every hyperbolic group G with a one-dimensional planar boundary and no elements of order two is virtually Kleinian if and only if every carpet group is virtually Kleinian. In particular, if G has no carpet subgroup, then G is virtually Kleinian.*

Theorem 1.2 and Corollary 1.3 reduce the dynamical characterization of (convex) cocompact Kleinian groups to both Cannon and Kapovich-Kleiner conjectures. Let us recall that the Kapovich-Kleiner conjecture is implied by the Cannon conjecture [KK, Thm. 5, Cor. 13]. Thus, the dynamical characterization of convex-cocompact Kleinian groups with no 2-torsion would follow from the Cannon conjecture. On the other hand, the Cannon conjecture does not imply Theorems 1.2 and 1.2 since it is not known that a word hyperbolic group with planar boundary is virtually a quasiconvex subgroup of a word hyperbolic group with the sphere as boundary.

Our second characterization relies on the recent progress made on cubulated groups:

Theorem 1.4. *Let G be a hyperbolic group with planar boundary. Then G is virtually Kleinian if and only if G acts geometrically and cellularly on a CAT(0) cube complex.*

Here, we admit groups with boundary the whole sphere, recovering Markovic's criterion of Cannon's conjecture:

Corollary 1.5 (Markovic). *Let G be hyperbolic group with boundary homeomorphic to the sphere and which has a faithful and orientation preserving action on its boundary. Then G is isomorphic to a cocompact Kleinian group if and only if G acts geometrically and cellularly on a CAT(0) cube complex.*

Remark 1.6. The original proof consists in extending the action on the two-sphere to the unit ball as a free convergence action so that the quotient is a Haken manifold [Mak2]. Here, we use the action to split the group to obtain groups with one-dimensional boundary and apply our previous results.

All these results will rely on a particular case for which we know that the action is already planar in the following sense. We shall say that a hyperbolic group has a *planar action* if its boundary admits a topological embedding into the two-sphere such that the action of every element of the group can be extended to a global homeomorphism.

Theorem 1.7. *Let G be a one-ended hyperbolic group with a planar action and with boundary different from the sphere. Then G is virtually a convex-cocompact Kleinian group if and only if* $\mathrm{confdim}_{AR} G < 2$.

Remark 1.8. *If, in the above theorem, the action is faithful and orientation preserving on its boundary, then the proof shows that the group is isomorphic to a convex-cocompact Kleinian group.*

Corollary 1.9 (Bonk and Kleiner). *A carpet group is virtually a convex-cocompact Kleinian group if and only if* $\mathrm{confdim}_{AR} G < 2$.

PROOF. Carpets have essentially one embedding in the sphere up to homeomorphisms of the sphere. Since boundary components do not separate the carpet, the group always has a planar action. So Theorem 1.7 applies. ■

Remark 1.10. This corollary was announced by Bonk and Kleiner in 2006 [Bon1] where a sketch of the proof was given. If the first step is similar (filling-in the "holes" to reconstruct the Riemann sphere), the other steps are different. The originality of the proof presented here is in the use of Hinkkanen-Markovic's characterization of Möbius groups of the circle to extend the action to the whole sphere. Moreover, we rely on the techniques developed in [Haï2] to prove that the carpet embeds quasisymmetrically in $\widehat{\mathbb{C}}$, see Corollary 3.5.

Remark 1.11. *Under the assumptions of Theorem 1.7, the group G is quasi-isometric to a convex subset of $\mathbb{H}^3$ with geodesic boundary iff its boundary is homeomorphic to the Sierpiński carpet (see § 6.2 for a proof).*

Working a little more, we may obtain the following corollary from Theorem 1.7:

Corollary 1.12. *Let G be a torsion-free non-elementary hyperbolic group acting by homeomorphisms on S^2 as a convergence action. Let us assume that the restriction on its limit set $\Lambda_G(\neq S^2)$ is uniform. If $\operatorname{confdim}_{AR} G < 2$, then the action of G is conjugate to that of a discrete group of Möbius transformations.*

Theorem 1.4 and Corollary 1.9 provide us with the following equivalent statements of the Kapovich-Kleiner conjecture.

Corollary 1.13. *Let G be a carpet group with a faithful and orientation preserving action on its boundary. The following are equivalent:*

- *the group G is isomorphic to a convex-cocompact Kleinian group;*
- *the group G acts cellularly and geometrically on a CAT(0) cube complex;*
- *the Ahlfors regular conformal dimension of G is strictly less than two.*

Outline of the proofs. The proof of Theorem 1.7 is proved by filling-in the holes to reconstruct a metric sphere and then to extend the action of the given group G to the whole sphere as a uniformly quasi-Möbius group action. Bonk and Kleiner's characterization of the round sphere enables to conjugate the action on the Euclidean sphere and to apply Sullivan's straightening theorem to conclude.

In general, there is no reason why the group G should admit a planar action, and, it does not even need to be isomorphic to a Kleinian group, see [KK, § 8]. The idea will be to use the algebraic structure of the group to construct a finite index subgroup isomorphic to the fundamental group of a compact Haken 3-manifold. Theorem 1.1 will then follow from Thurston's uniformization theorem. The key point is to prove that the stabilizers of rigid type vertices appearing in the JSJ-decomposition of one-ended hyperbolic groups with planar boundary have a planar action (Prop. 6.7). This has two major consequences. It implies that G is QCERF when its conformal dimension is less than two. This follows essentially from the work of Wise and Agol applied to our specific context (Proposition 6.17). This property is used to find a finite index subgroup such that the JSJ-decomposition of one-ended quasiconvex subgroups are "regular" (Theorem 6.18). This regularity property enables us to see each stabilizer as the fundamental group of a pared compact 3-manifold —-in particular for the rigid vertices— which can be glued together into a larger one.

Assuming there are no 2-torsion provides us with a finite hierarchy so that the resulting subgroups can not be split over elementary groups. We may then prove Theorem 1.2 along the same lines as above by induction on the length of the hierarchy.

By the previous results, the proof of Theorem 1.4 reduces to the case of carpet groups and groups with boundary homeomorphic to the sphere. So we may assume that we are given a group with a planar action. We first show that we may define for such a group a special action on a CAT(0) cube complex such that the stabilizers of hyperplanes are isomorphic to convex-cocompact Fuchsian groups. Splitting inductively along those hyperplanes, one will obtain hyperbolic manifolds endowed with subsurfaces on their boundary which can be glued together to prove that G is virtually the fundamental group of a compact Haken manifold. The proof ends as above.

Outline of the paper. In the next section, we make a systematic analysis of gluing together countably many continua to a fixed continuum and study the properties which are inherited. The proofs are routine and detailed but have the advantage to be checkable. The main results are Theorem 2.3 and Theorem 2.6. These results are specialized in § 3 to planar sets. Section 4 is a continuation of section 2: it is explained how maps can be extended when enlarging the space while keeping a control on their geometric properties. The last two sections are concerned with word hyperbolic groups. In Section 5, we first recall basic facts concerning quasiconvex subgroups of hyperbolic groups, and then we show how to extend convergence actions to larger spaces. The last section 6 is devoted to groups with planar boundaries per se. After recalling basic facts, we prove Theorem 1.7. We then analyse the vertices of rigid type arising in Bowditch's JSJ decomposition. We recall the main results on cubulated groups which will be needed. The QCERF property is then established in order to prove the main theorems.

Remark 1.14. A similar statement holds for topologically cxc maps with planar repellors, see [HP] for their definition and basic properties. But the complexity of the topology of the repellors and the lack of algebraic structure of such maps require to develop other ingredients, so it will be explained elsewhere.

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2. Sewing infinitely many continua

2.1. **Topological sewing.** Let X be a *continuum* i.e., a Hausdorff non-degenerate connected compact space. We assume that we are given a *null-sequence* i.e., an at most countable family $\mathcal{P}$ of subcontinua with the following property: for any finite cover $\mathcal{U}$ of X, for all but finitely many elements K of $\mathcal{P}$, there exists $U \in \mathcal{U}$ with $K \subset U$. We call $\mathcal{P}$ an *admissible collection of boundary components of* X.

For each $K \in \mathcal{P}$, we assume that we are given a continuum L_K together with an injective mapping $\psi_K : K \to L_K$.

Set

$$\Sigma = X \sqcup (\cup_{K\in\mathcal{P}} L_K)/\sim$$

where, for all $K \in \mathcal{P}$, $z \in K$ is identified with $\psi_K(z)$; note that a point z may belong to several boundary components. We define a topology on Σ as follows: a basis of open sets of Σ consists of those sets U such that

(T1) $U \cap X$ is open in X,
(T2) $U \cap L_K$ is open in L_K for all $K \in \mathcal{P}$,
(T3) for all but finitely many components $K \in \mathcal{P}$ with $K \subset U$, one also has $L_K \subset U$.

One may check that these sets are stable under finite intersection, so we have indeed a basis for a topology.

Proposition 2.1. *With the notation above, the topological space Σ is Hausdorff and compact, and each embedding $X \hookrightarrow \Sigma$ and $L_K \hookrightarrow \Sigma$ is continuous. The connected components of $\Sigma \setminus X$ are in bijection with $\{$the connected components of $L_K \setminus \psi_K(K)$, $K \in \mathcal{P}\}$.*

Proof. The continuity of the embeddings follows from (T1) and (T2).

We now construct a collection of open neighborhoods for each point of Σ.

Let us first consider $x \in X$. If U_x is an open neighborhood of x in X and W_x is the interior of a compact neighborhood of ∂U_x disjoint from x, then the collection $\mathcal{F}_x$ of boundary components which intersect ∂U_x and is not contained in W_x is finite since $\mathcal{P}$ is admissible. Let U'_x be the complement in U_x of the union of boundary components $K \notin \mathcal{F}_x$ with $K \cap \partial U_x \neq \emptyset$; the set U'_x is an open neighborhood of x in X: indeed, if $u \in U'_x$ and $N \subset U_x$ is a compact neighborhood of u, then only finitely many components $K \in \mathcal{P}$ intersect both ∂U_x and N, so their complement in N is a neighborhood of u in U'_x.

For each $K \in \mathcal{F}_x$, $\psi_K(U_x \cap K)$ is open in $\psi_K(K)$, so there exists an open set $U_K \subset L_K$ such that $\psi_K(U_x \cap K) = U_K \cap \psi_K(K)$. For each $K \in \mathcal{P} \setminus \mathcal{F}_x$ such that $K \cap U'_x \neq \emptyset$, we let $U_K = L_K$. For the other components, set $U_K = \emptyset$. It follows that

$$V_x = U'_x \cup (\cup_{K\in\mathcal{P}} U_K)$$

is an open neighborhood of x in Σ.

If $x \notin X$, then there exists $K \in \mathcal{P}$ with $x \in L_K \setminus \psi_K(K)$. Let U_x be a neighborhood of x in L_K, then $U_x \setminus \psi_K(K)$ is an open neighborhood of x in Σ.

It follows easily that Σ is Hausdorff.

Let us now consider a covering of Σ by open sets. We may as well assume that each element satisfies (T1), (T2) and (T3). Since X is compact, we may extract a finite cover $\mathcal{U}_0$ of X. The admissibility of $\mathcal{P}$ and condition (T3) imply that the set $Y = \Sigma \setminus \cup_{U\in\mathcal{U}_0} U$ intersects only finitely many elements L_K, each of which is compact by assumption. Therefore, one may extract a finite cover for each of these sets and obtain a finite cover of Σ.

The last statement follows easily since the sets $L_K \setminus \psi_K(K)$, $K \in \mathcal{P}$, are pairwise disjoint and their union forms $\Sigma \setminus X$. ■

2.2. **Geometric sewing.** The basic distortion bound for quasisymmetric maps is given by the following lemma [Hei, Prop. 10.8]:

Lemma 2.2. *Let $h : X \to Y$ be an η-quasisymmetric map between compact metric spaces. For all $A, B \subset X$ with $A \subset B$ and $\operatorname{diam} B < \infty$, we have $\operatorname{diam} h(B) < \infty$ and*

$$\frac{1}{2\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right)} \le \frac{\operatorname{diam} h(A)}{\operatorname{diam} h(B)} \le \eta\left(2\frac{\operatorname{diam} A}{\operatorname{diam} B}\right).$$

We prove a metric version of Proposition 2.1:

Theorem 2.3. *Let X be a metric continuum endowed with an admissible collection of boundary components $\mathcal{P}$. We assume the existence of $\Delta_0 \ge 1$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $K \in \mathcal{P}$, we are given a metric continuum L_K and an η-quasisymmetric embedding $\psi_K : K \to L_K$ such that $\operatorname{diam} L_K \le \Delta_0 \operatorname{diam} \psi_K(K)$.*

Then there exist a metric d_Σ on Σ compatible with its topology and a constant $\Delta > 0$ such that (X, d_Σ) is bi-Lipschitz to X, and, for all $K \in \mathcal{P}$, (L_K, d_Σ) is uniformly quasisymmetric to L_K, $\operatorname{diam}_\Sigma L_K \le \Delta \operatorname{diam}_\Sigma K$ and there is a constant $c > 0$ such that, for all $y \in X$, $z \in L_K$, $z' \in L_{K'}$, $K \ne K'$,

$$\begin{aligned} d_\Sigma(z, z') &\ge c \inf\{d_\Sigma(z, x) + d_\Sigma(x, x') + d_\Sigma(x', z'),\ x \in L_K, x' \in L_{K'}\} \\ d_\Sigma(z, y) &\ge c \inf\{d_\Sigma(z, x) + d_\Sigma(x, y),\ x \in L_K\}. \end{aligned} \tag{2.1}$$

For the proof, we will use the following Ahlfors-Beurling type theorem [Haï2, Thm 2]:

Proposition 2.4. *Let (X, d_X) be a proper metric space containing at least two points and (Y, d_Y) a connected compact metric space. Let us assume that there is an η-quasisymmetric embedding $f : Y \to X$ with $\operatorname{diam}_Y Y = \operatorname{diam}_X f(Y)$. Then there is a metric $\widehat{d}$ on X such that*

1. *$Id : (X, d_X) \to (X, \widehat{d})$ is $\widehat{\eta}$-quasisymmetric;*
2. *$Id : (X \setminus f(Y), d_X) \to (X \setminus f(Y), \widehat{d})$ is locally quasisimilar: there is a finite constant $C \ge 1$ such that, for any $x \in X \setminus f(Y)$ and any $y, z \in B_X(x, d_X(x, f(Y))/2)$,*
$$\frac{1}{C} \le \frac{\widehat{d}(y, z)}{d_X(y, z)} \cdot \frac{d_X(x, f(Y))}{\widehat{d}(x, f(Y))} \le C;$$
3. *$f : (Y, d_Y) \to (X, \widehat{d})$ is bi-Lipschitz onto its image: there exists $L \ge 1$ such that, for all $y_1, y_2 \in Y$,*
$$\frac{1}{L} d_Y(y_1, y_2) \le \widehat{d}(f(y_1), f(y_2)) \le d_Y(y_1, y_2);$$
4. *there is a constant $\Delta \ge 1$ such that*
$$\frac{1}{\Delta} \operatorname{diam}(X, d_X) \le \operatorname{diam}\left(X, \widehat{d}\right) \le \Delta \operatorname{diam}(X, d_X).$$

All the constants involved and $\widehat{\eta}$ only depend on η.

The proof of Theorem 2.3 consists in defining the metric from the necessary conditions given by the conclusion of the statement and then to check it fulfills the requirements.

Proof. (Thm 2.3) For $K \in \mathcal{P}$, we rescale the metric on L_K so that $\operatorname{diam} K = \operatorname{diam} \psi_K(K)$. We apply Proposition 2.4 to $L_K (= X)$, $K (= Y)$ and $\psi_K : K \to L_K$. Let $d_K (= \widehat{d})$ be the metric thus obtained. Note that the collection of maps $\{L_K \xrightarrow{Id} (L_K, d_K)\}_{K \in \mathcal{P}}$ is uniformly quasisymmetric with distortion function $\widehat{\eta}$. Moreover, Lemma 2.2 implies that, for all $K \in \mathcal{P}$,

$$\operatorname{diam}(L_K, d_K) \le \Delta \operatorname{diam}(\psi_K(K), d_K), \tag{2.2}$$

where $\Delta = 2\widehat{\eta}(\Delta_0)$, and, there exists $L \ge 1$ such that, for all $K \in \mathcal{P}$ and for all $x, y \in K$,

$$\frac{1}{L} d_X(x,y) \le d_K(\psi_K(x), \psi_K(y)) \le d_X(x,y)\,. \tag{2.3}$$

In the sequel, we will omit ψ_K when it leads to no confusion.

Let us define a quasimetric on Σ as follows:

- if $x, y \in X$, set $q(x,y) = d_X(x,y)$;
- if $x, y \in L_K \setminus K$, for some $K \in \mathcal{P}$, set $q(x,y) = d_K(x,y)$;
- if $x \in X$ and $y \in L_K \setminus K$ for some $K \in \mathcal{P}$, set
$$q(x,y) = q(y,x) = \inf_{z \in K} \{d_X(x,z) + d_K(z,y)\}\,;$$
- if $x \in L_{K_1} \setminus K_1$, $y \in L_{K_2} \setminus K_2$ for some $K_1 \ne K_2 \in \mathcal{P}$, set
$$q(x,y) = \inf_{(z_1,z_2) \in K_1 \times K_2} \{d_{K_1}(x,z_1) + d_X(z_1,z_2) + d_{K_2}(z_2,y)\}\,.$$

Set finally

$$d_\Sigma(x,y) = \inf \sum_{i=0}^{N-1} q(x_i, x_{i+1})$$

over all finite chains $x_0, , \ldots x_N$ in Σ with $x_0 = x$, $x_N = y$. We claim that d_Σ is a metric comparable to q which is compatible with the topology of Σ.

Let $x_0, \ldots, x_N$ be a chain. Inserting finitely many points if necessary in the chain using the definition of q, we may assume that, for each $j \in \{0, \ldots, N-1\}$, either there exists $K \in \mathcal{P}$ such that $q(x_j, x_{j+1}) = d_K(x_j, x_{j+1})$, or $q(x_j, x_{j+1}) = d_X(x_j, x_{j+1})$. Using the triangle inequality, we may assume that if $x_j \notin X$, $j < N$, then $x_{j+1} \in X$; if furthermore $j > 0$, then there is some $K \in \mathcal{P}$ with $x_{j\pm1} \in K$ and so, with (2.3),

$$q(x_{j-1}, x_j) + q(x_j, x_{j+1}) \ge d_K(x_{j-1}, x_{j+1}) \ge \frac{1}{L} d_X(x_{j-1}, x_{j+1})\,.$$

Therefore, one can extract a subchain $(y_j)_{0 \le j \le M}$ with $y_0 = x_0$ and $y_M = x_N$ such that

(a) if $x_0, x_N \in X$, then
$$\sum_{j=0}^{N-1} q(x_j, x_{j+1}) \ge \frac{1}{L} \sum_{j=0}^{M-1} d_X(y_j, y_{j+1}) \ge \frac{1}{L} d_X(y_0, y_M) = \frac{1}{L} q(x_0, x_N)\,;$$

(b) if $x_0 \in L_K \setminus K$ for some $K \in \mathcal{P}$ and $x_N \in X$, then $y_1 \in K$ and
$$\begin{aligned}\sum_{j=0}^{N-1} q(x_j, x_{j+1}) &\ge d_K(y_0, y_1) + \frac{1}{L} \sum_{j=1}^{M-1} d_X(y_j, y_{j+1}) \\ &\ge d_K(y_0, y_1) + \frac{1}{L} d_X(y_1, y_M) \ge \frac{1}{L} q(x_0, x_N)\,.\end{aligned}$$

(c) if $x_0, x_N \notin X$, then
$$\begin{aligned}\sum_{j=0}^{N-1} q(x_j, x_{j+1}) &\ge q(y_0, y_1) + \frac{1}{L} \sum_{j=1}^{M-2} d_X(y_j, y_{j+1}) + q(y_{M-1}, y_M) \\ &\ge q(y_0, y_1) + \frac{1}{L} d_X(y_1, y_{M-1}) + q(y_{M-1}, y_M)\,.\end{aligned}$$

If there is some $K \in \mathcal{P}$ such that $y_0, y_M \in L_K \setminus K$, then $y_1, y_{M-1} \in K$ and it follows from (2.3) that

$$\begin{aligned} \sum_{j=0}^{N-1} q(x_j, x_{j+1}) &\geq d_K(y_0, y_1) + \frac{1}{L} d_X(y_1, y_{M-1}) + d_K(y_{M-1}, y_M) \\ &\geq \frac{1}{L} d_K(y_0, y_M) = \frac{1}{L} q(x_0, x_N) \,. \end{aligned}$$

If not, then

$$\begin{aligned} \sum_{j=0}^{N-1} q(x_j, x_{j+1}) &\geq \frac{1}{L} \left(q(y_0, y_1) + d_X(y_1, y_{M-1}) + q(y_{M-1}, y_M) \right) \\ &\geq \frac{1}{L} q(y_0, y_M) = \frac{1}{L} q(x_0, x_N) \,. \end{aligned}$$

In either case, we have shown that

$$q(x_0, x_N) \geq d_\Sigma(x_0, x_N) \geq \frac{1}{L} q(x_0, x_N) \,. \tag{2.4}$$

This proves (2.1). Since L_K embeds in Σ uniformly quasisymmetrically, Lemma 2.2 implies the existence of $\Delta > 0$ such that $\mathrm{diam}_\Sigma L_K \leq \Delta \mathrm{diam}_\Sigma K$ for all $K \in \mathcal{P}$.

Since $\mathcal{P}$ is a null-sequence and (2.2) holds, it follows that d_Σ defines the topology of Σ. ■

Remark 2.5. *Note that if $x \in L_K \setminus \psi_K(K)$ for some $K \in \mathcal{P}$, then $d_\Sigma(x, X)$ is realized by a point $y \in K$.*

2.3. **Geometric properties.** We establish a series of properties of Σ —obtained by Theorem 2.3— inherited from the sets X and L_K, $K \in \mathcal{P}$. The terms in the next statement will be defined below.

Theorem 2.6. *Under the assumptions of Theorem 2.3, let (Σ, d_Σ) be the given metric space. Then the following hold.*

1. *If (X, d_X) and all L_K, $K \in \mathcal{P}$, satisfy the bounded turning property uniformly, then it also holds for (Σ, d_Σ).*
2. *If (X, d_X) and all L_K, $K \in \mathcal{P}$, are uniformly LLC, then (Σ, d_Σ) is LLC quantitatively as well.*
3. *If X is doubling and relatively doubling with respect to $\mathcal{P}$ and if $\{L_K,\ K \in \mathcal{P}\}$ is uniformly doubling, then (Σ, d_Σ) is doubling quantitatively.*
4. *If X is relatively porous with respect to $\mathcal{P}$ and if, for all $K \in \mathcal{P}$, K is uniformly porous in L_K, then X is porous in (Σ, d_Σ) quantitatively.*
5. *We assume that X is doubling, doubling relative to $\mathcal{P}$ and porous relative to $\mathcal{P}$. We also assume that every K is uniformly porous in L_K. If each L_K, $K \in \mathcal{P}$, is Q-Ahlfors-regular with uniform constants for some $Q > 1$, and if X is Ahlfors regular of dimension strictly less than Q, then (Σ, d_Σ) is Q-Ahlfors regular.*

We will first see how to extend local properties to global ones. Theorem 2.6 will follow at once.

Given a metric space Z and a constant $c > 0$, we say that $(X, \{L_\alpha\}_{\alpha \in A})$ is a *c-separating structure* of Z if $X \subset Z$ is closed, $\{L_\alpha\}_{\alpha \in A}$ is a possibly infinite collection of compact subsets of Z which satisfies the following properties:

(S1) Setting $K_\alpha = L_\alpha \cap X$ and $\Omega_\alpha = L_\alpha \setminus K_\alpha$ for $\alpha \in A$, the collection $\{\Omega_\alpha\}_{\alpha\in A}$ forms a partition of $Z \setminus X$ by open sets;

(S2) The following flatness condition holds: for all $y \in X$, $z \in \Omega_\alpha$, $z' \in \Omega_{\alpha'}$, $\alpha \neq \alpha'$,

$$\begin{cases} d(z,z') \geq c\inf\{d(z,x)+d(x,x')+d(x',z'),\ x\in K_\alpha, x'\in K_{\alpha'}\}; \\ d(z,y) \geq c\inf\{d(z,x)+d(x,y),\ x\in K_\alpha\}\,. \end{cases} \tag{2.5}$$

Unless explicitly stated, we assume throughout this section that we are given a metric space Z together with a separating structure $(X, \{L_\alpha\}_{\alpha\in A})$. The collection $\mathcal{P} = \{K_\alpha,\ \alpha \in A\}$ denotes the boundary components of X. We also assume the existence of $\Delta > 0$ such that $\operatorname{diam} L_\alpha \leq \Delta \operatorname{diam} K_\alpha$ for all $\alpha \in A$.

Note that if $z \in \Omega_\alpha$ and $\delta(z) = d(z, X)$, then

$$B(z, c\delta(z)) \subset \Omega_\alpha. \tag{2.6}$$

2.3.1. *Connectivity properties.* Recall that a metric space Z satisfies the *bounded turning property* λ-(BT), for some $\lambda \geq 1$, if any pair of points $x, y \in Z$ is contained in a continuum E with $\operatorname{diam} E \leq \lambda d_Z(x,y)$.

Proposition 2.7. *Let $\lambda \geq 1$ be fixed. If X satifies the λ-(BT) condition and every subset L_α satisfies the λ-(BT) condition, then Z satisfies the (λ/c)-(BT) condition.*

Proof. Let $x, y \in Z$. If they both belong to either X or to the same L_α, then the λ-(BT) property implies at once the existence of a continuum E containing x, y with $\operatorname{diam} E \leq \lambda d(x,y)$.

Let $x \in X$ and $y \in \Omega_\alpha$ for some $\alpha \in A$. By the separation condition (2.5), there is some $z \in L_\alpha$ such that $d(x,y) \geq c(d(x,z) + d(z,y))$. There exist continua $E_1 \subset X$ and $E_2 \subset L_\alpha$ which contain x, z and z, y respectively and such that $\operatorname{diam} E_1 \leq \lambda d(x,z)$ and $\operatorname{diam} E_2 \leq \lambda d(z,y)$. It follows that $E = E_1 \cup E_2$ is a continuum containing x and y and

$$\begin{aligned} \operatorname{diam} E &\leq \operatorname{diam} E_1 + \operatorname{diam} E_2 \\ &\leq \lambda d(x,z) + \lambda d(y,z) \\ &\leq (\lambda/c) d(x,y)\,. \end{aligned}$$

Assume $x \in \Omega_\alpha$ and $y \in \Omega_\beta$ with $\alpha \neq \beta$. Let $z_\alpha \in L_\alpha \cap X$ and $z_\beta \in L_\beta \cap X$ satisfy $d(x,y) \geq c(d(x,z_\alpha)+d(z_\alpha,z_\beta)+d(z_\alpha,y))$. There exist continua $E_0 \subset X$, $E_\alpha \subset L_\alpha$ and $E_\beta \subset L_\beta$ which contain $\{z_\alpha, z_\beta\}$, $\{x, z_\alpha\}$ and $\{z_\beta, y\}$ respectively and such that $\operatorname{diam} E_0 \leq \lambda d(z_\alpha, z_\beta)$, $\operatorname{diam} E_\alpha \leq \lambda d(x, z_\alpha)$ and $\operatorname{diam} E_\beta \leq \lambda d(z_\beta, y)$. It follows that $E = E_0 \cup E_\alpha \cup E_\beta$ is a continuum containing x and y and

$$\begin{aligned} \operatorname{diam} E &\leq \operatorname{diam} E_0 + \operatorname{diam} E_\alpha + \operatorname{diam} E_\beta \\ &\leq \lambda d(z_\alpha, z_\beta) + \lambda d(x, z_\alpha) + \lambda d(y, z_\beta) \\ &\leq (\lambda/c) d(x,y)\,. \end{aligned}$$

■

Recall that a metric space Z is *linearly locally connected* (λ-LLC for some $\lambda \geq 1$) if, for all $x \in X$ and $R > 0$,

(LLC1) for all $y, z \in B(x,R)$ there is a continuum $E \subset B(x, \lambda R)$ which contains $\{y, z\}$;

(LLC2) for all $y, z \notin B(x,R)$, there is a continuum $E \subset Z \setminus B(x, (1/\lambda)R)$ which contains $\{y, z\}$.

Proposition 2.8. *Let $\lambda \geq 1$ be fixed. If Z is λ-(LLC) at every point of X and every subset L_α satisfies the λ-(LLC) condition, then Z is LLC quantitatively as well.*

Proof. By Proposition 2.7, the space Z is λ-(BT) so (LLC1) holds. Let us now focus on (LLC2). Let $z \in Z \setminus X$ and $r > 0$. Let Ω_α be the component containing z.

If $d(z, X) > \frac{1}{2}\frac{r}{1+\lambda}$, then $B(z, \frac{1}{2}\frac{r}{\lambda(1+\lambda)})$ is disjoint from X and the (BT)-property implies that this ball is contained in Ω_α, hence in L_α. Therefore, if $z_1, z_2 \notin B(z, r)$, then we may find a continuum K disjoint from $B\left(z, \frac{1}{2}\frac{r}{\lambda^2(1+\lambda)}\right)$ which joins z_1 and z_2.

If $d(z, X) \leq \frac{1}{2}\frac{r}{1+\lambda}$, let $x \in X$ satisfy $d(z, X) = d(z, x)$. Note that

$$B\left(x, \left(1 - \frac{1}{2}\frac{1}{1+\lambda}\right) r\right) \subset B(z, r)$$

and

$$1 - \frac{1}{2}\frac{1}{1+\lambda} = \frac{1+2\lambda}{2(1+\lambda)}.$$

If $z_1, z_2 \notin B(z, r)$, then we may find a continuum K disjoint from $B(x, \frac{r}{\lambda}\frac{1+2\lambda}{2(1+\lambda)})$ which joins z_1 and z_2. Now, let us observe that

$$B\left(z, \left(\frac{1}{\lambda}\frac{1+2\lambda}{2(1+\lambda)} - \frac{1}{2}\frac{1}{1+\lambda}\right) r\right) \subset B\left(x, \frac{r}{\lambda}\frac{1+2\lambda}{2(1+\lambda)}\right)$$

so that K is disjoint from $B(z, r/(2\lambda))$. ■

Corollary 2.9. *Let $\lambda \geq 1$ be fixed. If X and every subset L_α satisfies the λ-(LLC) condition, then Z is LLC quantitatively as well.*

Proof. In order to apply Proposition 2.8, it is enough to prove the following claim: *there exists a constant $\lambda' \geq 1$ such that, for any $x \in X$, for any $r \in (0, \operatorname{diam} Z)$, any point $z \in (Z \setminus B(x, r))$ can be joined by a continuum $E \subset (Z \setminus B(x, r/\lambda'))$ to X.*

The claim and the LLC-property of X imply that Z is LLC at every point of X quantitatively, so that Proposition 2.8 applies. We now prove the claim.

Let $x \in X$, $r > 0$ and $z \notin B(x, r)$; there is some $\alpha \in A$ with $z \in \Omega_\alpha$. Set

$$\kappa = \frac{1}{2(1 + 2\Delta\lambda)}\,.$$

If $d(x, K_\alpha) \geq \kappa r$ then $L_\alpha \cap B(x, c\kappa r) = \emptyset$ according to (2.5) and L_α is a continuum which joins z to a point of $K_\alpha \subset X$. If $d(x, K_\alpha) < \kappa r$, let $x' \in K_\alpha$ be such that $d(x, x') = d(x, K_\alpha)$.

It follows that $B(x', (1 - \kappa)r) \subset B(x, r)$ so that $z \notin B(x', (1 - \kappa)r)$. Moreover,

$$\operatorname{diam} K_\alpha \geq \frac{1}{\Delta}\operatorname{diam} L_\alpha \geq \frac{d(x', z)}{\Delta} \geq \frac{(1-\kappa)r}{\Delta}\,.$$

Therefore, we may find $y \in K_\alpha$ such that $d(x', y) \geq \frac{(1-\kappa)r}{2\Delta}$.

Since L_α is LLC, there is a continuum

$$E \subset \left(Z \setminus B\left(x', \frac{(1-\kappa)r}{2\lambda\Delta}\right)\right)$$

which joins z to y. Now, if $w \in E$, then $d(x, w) \geq d(w, x') - d(x', x)$ so that

$$d(x, w) \geq \frac{(1-\kappa)r}{2\lambda\Delta} - \kappa r \geq \frac{1}{4\lambda\Delta} r\,.$$

Therefore, the claim follows with

$$\lambda' = \max\left\{\frac{2(1+2\Delta\lambda)}{c}, 4\lambda\Delta\right\}.$$

■

2.3.2. *Size properties.* A metric space Z is *doubling* if there exists an integer N such that any set of finite diameter can be covered by at most N sets of half its diameter. This implies that, for all $\varepsilon > 0$, there exists N_ε such that any set E of finite diameter can be covered by N_ε sets of diameter bounded by $\varepsilon \mathrm{diam} E$. We propose a relative notion of doubling:

Definition 2.10 (Relative doubling condition)**.** *Let X be a metric continuum with boundary components $\mathcal{P}$. Then X is* doubling relative to $\mathcal{P}$ *if, for any $\varepsilon > 0$, there is some N_ε and there exists $r_0 > 0$ such that, for any $x \in X$ and $r \in (0, r_0)$, there are at most N_ε components $K \in \mathcal{P}$ such that $B(x,r) \cap K \neq \emptyset$ and* $\mathrm{diam}(K \cap B(x,r)) \geq \varepsilon r$.

Proposition 2.11. *If X is doubling and relatively doubling with respect to $\mathcal{P}$ and if $\{L_\alpha,\ \alpha \in A\}$ is uniformly doubling, then Z is doubling quantitatively.*

Proof. Let us first consider $x \in X$, $r \in (0, \mathrm{diam}\, Z)$ and $\varepsilon \in (0,1)$. We may cover $B(x,r) \cap X$ by M_1 balls $\{B_j\}$ of radius $(\varepsilon r/6)$, where M_1 only depends on the doubling condition of (X, d_X) and on ε. For each ball B_j, we add all the components L_α, $\alpha \in A$, such that $K_\alpha \cap B_j \neq \emptyset$ and $\mathrm{diam}\, K_\alpha \leq (\varepsilon r)/(6\Delta)$; we let B_j' be the resulting set. It follows that $\mathrm{diam}\, B_j' \leq \varepsilon r$.

We are left with the components $K_\alpha \in \mathcal{P}$ with $K_\alpha \cap B(x,r) \neq \emptyset$ and $\mathrm{diam}\, K \geq (\varepsilon r)/(6\Delta)$: the relative doubling condition implies that there are at most N such sets. Each of these sets can be covered by M_2 sets of diameter at most εr by the uniform doubling condition.

Therefore, for any $x \in X$, we may cover $B(x,r)$ by at most $M_1 + NM_2$ sets of diameter at most εr.

If $x \notin X$, then either $B(x,r) \subset L_\alpha$ for some $\alpha \in A$, and then we may use the doubling condition of L_α, or there is some $y \in X$ such that $B(x,r) \subset B(y, 2r/c)$, cf. (2.6). Using $\varepsilon = (c/4)$ above, we may cover $B(y, 2r/c)$, hence $B(x,r)$ by a uniform number of sets of diameter at most $(r/2)$. ■

A subset Y of a metric space Z is said to be *porous* if there exists a constant $p > 0$ such that any ball centered at a point of Y of radius $r \in (0, \mathrm{diam} Z]$ contains a ball of radius pr disjoint from Y. We propose a relative notion of porosity:

Definition 2.12 (Relative porosity)**.** *Let X be a metric continuum with boundary components $\mathcal{P}$. Then X is* porous relative to $\mathcal{P}$ *if there exist a constant $p_X > 0$ and a maximal size $r_0 > 0$ such that, for any $x \in X$ and $r \in (0, r_0)$, there is at least one subcontinuum K' of a boundary component $K \in \mathcal{P}$ such that $K' \subset B_X(x,r)$, $K' \cap B_X(x, r/2) \neq \emptyset$ and* $\mathrm{diam}_X K' \geq p_X r$.

Proposition 2.13. *If X is relatively porous with respect to $\mathcal{P}$ and if, for all $\alpha \in A$, K_α is uniformly porous in L_α, then X is porous in Z quantitatively.*

Proof. Let $x \in X$ and $r \in (0, r_0)$. There exist $\alpha \in A$ and $y \in K_\alpha$ such that $d(x,y) \leq r/2$, and, if K' denotes the component of $K_\alpha \cap B(x,r)$ which contains y, then $\mathrm{diam}\, K' \geq p_X r$.

We have $\mathrm{diam}\, L_\alpha \geq \mathrm{diam}\, K_\alpha \geq p_X r$ and $B(x,r) \supset B(y, p_X r/2)$. It follows from the porosity of K_α in L_α that there is some $z \in \Omega_\alpha$ such that $B(z, cpp_X r/2) \subset \Omega_\alpha \cap B(y, p_X r/2)$, so that

$$B(z, cpp_X r/2) \subset B(x,r) \setminus X.$$

■

Proposition 2.14. *Let $Q > 0$. Let us assume that Z is a doubling metric space and that X is a porous Ahlfors regular compact subset of dimension strictly smaller than Q. If every subspace L_α is Q-Ahlfors regular with uniform constants and K_α is uniformly porous in L_α, then Z is Q-Ahlfors regular.*

The proof is exactly the same as [Haï2, Prop. 2.18]. We repeat it for completeness. Instead of asking X to be Ahlfors-regular, it would have been enough to require that its Assouad dimension be strictly less than Q.

Proof. Let us denote by μ the Q-Hausdorff measure in Z. If $x \notin X$, then there is some $\alpha \in A$ such that $x \in \Omega_\alpha$; we let $\delta(x) = d(x, X)$.

By assumption and (2.6), we have $\mu(B(x, cr\delta(x))) \asymp (r\delta(x))^Q$ for all $r \in (0, 1)$.

Let us fix a point $x \in X$ and $r > 0$. Since X is porous in Z, a constant $p > 0$ exists such that $B(x, r)$ contains a ball $B(y, pr)$ disjoint from X. Therefore $pr \le \delta(y)$ so

$$\mu(B(y, pr)) \ge \mu(B(y.cpr)) \gtrsim r^Q \quad \text{and} \quad \mu(B(x, r)) \ge \mu(B(y, pr)) \gtrsim r^Q\,.$$

For the converse inequality, we first note that since the dimension of X is strictly less than Q, we have $\mu(X) = 0$. Thus it is enough to bound $\mu(B(x, r) \setminus X)$. We cover $B(x, r) \setminus X$ by balls $B(z, c\delta(z)/10)$. We extract an at most countable subfamily $B(z_j, c\delta_j/10)$ of pairwise disjoint balls such that $B(x, r) \setminus X \subset \cup B(z_j, c\delta_j/2)$ [Hei, Thm 1.2].

Denote by A_n the set of centers (z_j) such that $re^{-(n+1)} < \delta(z_j) \le re^{-n}$. It follows that if $z_j \in A_n$, then $\mu(B(z_j, c\delta_j/2)) \asymp \delta(z_j)^Q \asymp r^Q e^{-Qn}$.

For each z_j, choose a point $x_j \in X$ such that $\delta(z_j) = d(z_j, x_j)$. Since Z is doubling and the balls $\{B(z_j, c\delta_j/10)\}_j$ are disjoint, the nerve of the family of balls $\{B(x_j, \delta(z_j)),\ z_j \in A_n\}$ has uniformly bounded valence V (independent from n). Therefore, we may split this family of balls into $V + 1$ families of pairwise disjoint balls. Since $\dim X < Q$, the number of balls involved in A_n is bounded by $e^{Qn}\theta^n$, up to a factor (which depends on V), for some $\theta \in (0, 1)$. Thus

$$\sum_{A_n} \mu(B(z_j, c\delta_j/2)) \lesssim e^{Qn}\theta^n \left(e^{-n}r\right)^Q \lesssim \theta^n r^Q.$$

Therefore

$$\mu(B(y, r)) \le \sum_{n\ge 0}\sum_{A_n} \mu(B(z_j, \delta_j/2)) \lesssim \sum_{n\ge 0} \theta^n r^Q \lesssim r^Q.$$

Let us consider a point $z \in Z \setminus X$, and let $x \in X$ be such that $\delta(z) = d(z, x)$. If $r \in [c\delta(z), 2\delta(z)]$, then

$$\mu(B(z, r)) \ge \mu(B(z, c\delta(z))) \gtrsim \delta(z)^Q \gtrsim r^Q.$$

On the other hand,

$$\mu(B(z, r)) \le \mu(B(x, 2r)) \lesssim r^Q.$$

If $r \ge 2\delta(z)$, then $B(x, r - \delta(z)) \subset B(z, r) \subset B(x, r + \delta(z))$ with $r + \delta(z) \le (3/2)r$ and $r - \delta(z) \ge r/2$, so $\mu(B(z, r)) \asymp r^Q$. ■

2.3.3. *Geometric properties of the sewn space.* We prove Theorem 2.6. Theorem 2.3 implies that $(X, \{L_K,\ K \in \mathcal{P}\})$ is a separating structure for Σ and that there is some $\Delta > 0$ such that $\mathrm{diam}_\Sigma L_K \le \Delta \mathrm{diam}_\Sigma K$ for all $K \in \mathcal{P}$.

1. It follows from Lemma 2.2 that we may assume that (X, d_X) and all (L_K, d_K), $K \in \mathcal{P}$, satisfy λ-(BT) for a fixed $\lambda \ge 1$. Therefore, Proposition 2.7 applies and Σ has bounded turning. ■

2. We recall that the LLC property is preserved quantitatively under quasisymmetric mappings [Hei, Chap. 15], so that we may assume that $\{(L_K, d_\Sigma),\ K \in \mathcal{P}\}$ and (X, d_Σ) are λ-LLC. Therefore, Corollary 2.9 applies and Σ is LLC as well. ■

3. We recall that the doubling condition is preserved quantitatively under quasisymmetric mappings [Hei, Chap. 15] so that $\{(L_K, d_\Sigma),\ K \in \mathcal{P}\}$ is uniformly doubling. Since (X, d_X) is bi-Lipschitz to (X, d_Σ), it is also doubling and relatively doubling with respect to $\mathcal{P}$. Proposition 2.11 applies and we may conclude that Σ is doubling. ■

4. We recall that the porosity of a subset is preserved quantitatively under quasisymmetric mappings so that the sets K, $K \in \mathcal{P}$, are uniformly porous in (L_K, d_Σ) and (X, d_Σ) is also relatively porous to $\mathcal{P}$. Therefore we may apply Proposition 2.13 and conclude that X is porous in Σ. ■

5. We know from above that the space (Σ, d_Σ) is doubling and that X is also porous in Σ. Since (X, d_X) is bi-Lipschitz to (X, d_Σ), we get that (X, d_Σ) is also Ahlfors regular of dimension strictly less than Q. Finally, [Haï2, Prop. 2.18] implies that (L_K, d_Σ), $K \in \mathcal{P}$, are uniformly Q-Ahlfors regular. Therefore, Proposition 2.14 applies. ■

3. Planar continua

Let X be a planar locally connected continuum, and let $\varphi : X \to S^2$ be a topological embedding. Note that there may be embeddings which are not compatible in the sense that if $\varphi_1, \varphi_2 : X \to S^2$ are two embeddings, then $\varphi_2 \circ \varphi_1^{-1}$ might not be the restriction of a selfhomeomorphism of the sphere.

We let $\mathcal{P}$ denote those subcontinua $K \subset X$ such that $\varphi(K)$ bounds a connected component of $S^2 \setminus \varphi(X)$.

The main result of this section is the following:

Theorem 3.1. *Let X be a locally connected planar metric continuum with boundary components $\mathcal{P}$ provided by an embedding $\varphi : X \to S^2$. We assume that X is LLC, Ahlfors regular of dimension $Q < 2$, relatively doubling and porous with respect to $\mathcal{P}$, and that we are given uniformly quasisymmetric gluing maps $\psi_K : K \to L_K$ for all $K \in \mathcal{P}$, where the sets L_K are continua such that (a) $L_K \setminus \psi_K(K)$ is homeomorphic to an open disk; (b) all the sets L_K are uniformly Ahlfors regular of dimension 2; (c) $\psi_K(K)$ is uniformly porous in L_K; and (d) $\mathrm{diam} L_K \le \Delta_0 \mathrm{diam}\, \psi_K(K)$ for some universal constant $\Delta_0 \ge 1$.*

Then the space Σ given by Proposition 2.1 and Theorem 2.3 is quasisymmetric to $\widehat{\mathbb{C}}$ and there exists a quasisymmetric embedding $\phi : X \to \widehat{\mathbb{C}}$ compatible with φ.

3.1. **Topological uniformization.** Let X be a locally connected planar continuum with boundary components $\mathcal{P}$ provided by an embedding $\varphi : X \to S^2$. For each $K \in \mathcal{P}$, we assume that we are given topological embeddings $\psi_K : K \to L_K$ for all $K \in \mathcal{P}$, where the sets L_K are continua such that $L_K \setminus \psi_K(K)$ is homeomorphic to an open disk.

For each $K \in \mathcal{P}$, we denote by U_K the component of $S^2 \setminus \varphi(X)$ with $\partial U_K = \varphi(K)$. It follows from the Torhorst theorem [Why2, Thm. VI.2.2] that K is itself locally connected. Since $\varphi(X)$ is connected, it follows that U_K is homeomorphic to an open disk so there exists a homeomorphism $\varphi_K : L_K \to \overline{U_K}$ such that $\psi_K|_K = (\varphi_K^{-1} \circ \varphi)|_K$.

Proposition 3.2. *The topological space Σ defined as above is homeomorphic to S^2.*

Proof. We first note that since $\varphi(X)$ is a locally connected continuum of the sphere, [Why2, Thm. VI.4.4] implies that $\mathcal{P}$ is an admissible collection of boundary components. Hence Σ is compact by Proposition 2.1.

Define $\phi : \Sigma \to S^2$ as follows: on X, set $\phi = \varphi$; on L_K, $K \in \mathcal{P}$, set $\phi = \varphi_K$. Note that, if $z \in K$ for some $K \in \mathcal{P}$, then

$$\phi(z) = \varphi(z) = \varphi_K \circ (\varphi_K^{-1} \circ \varphi)(z) = \phi(\psi_K(z))$$

so that ϕ is well-defined.

Let us prove that ϕ is a homeomorphism. First, ϕ is a bijection by construction. Let us now prove that ϕ is continuous: let U be an open subset of S^2 and let $V = \phi^{-1}(U)$. On the one hand, one has

$$V \cap X = \phi^{-1}(U) \cap \phi^{-1}(\varphi(X)) = \phi^{-1}(U \cap \varphi(X)) = \varphi^{-1}(U)$$

so that $V \cap X$ is open, hence (T1) is true. On the other hand, let $K \in \mathcal{P}$, then

$$V \cap L_K = \phi^{-1}(U) \cap \varphi_K^{-1}(\overline{U_K}) = \varphi_K^{-1}(U \cap \overline{U_K})$$

so $V \cap L_K$ is also open, so (T2) holds. By [Why2, Thm VI.4.4], for all $\delta > 0$, there are only finitely many components of $S^2 \setminus \varphi(X)$ with diameter at least δ; therefore, we may assume that only finitely many components of $S^2 \setminus \varphi(X)$ intersect ∂U, since any open set can be described as an at most countable union of such open sets. Hence V satisfies (T3) as well. Therefore ϕ is continuous. Since Σ is compact, this implies that ϕ is a homeomorphism. ■

3.2. **Quasisymmetric embedding.** The embedding will be obtained thanks to [BK1, Thm 1.1]:

Theorem 3.3 (Bonk and Kleiner)**.** *A metric 2-sphere is quasisymmetrically equivalent to the Riemann sphere if it is LLC and 2-Ahlfors-regular.*

Proof. (Theorem 3.1) By the assumption (a), Proposition 3.2 implies that Σ is homeomorphic to S^2. The assumption (d) implies that we may endow Σ with a metric d_Σ which enjoys the properties given by Theorem 2.3. The assumption (c), the LLC-assumptions and the Ahlfors-regularity assumptions enable us to apply Theorem 2.6 to conclude that (Σ, d_Σ) is an LLC 2-Ahlfors regular sphere.

It now follows from Theorem 3.3 that Σ is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$, so X embeds quasisymmetrically into $\widehat{\mathbb{C}}$. This ends the proof of Theorem 3.1. ■

3.3. **Carpets.** Define a *carpet* as a planar, one-dimensional, connected, locally connected compact space with no local cut point; any such space is homeomorphic to the Sierpiński carpet and admits a unique embedding up to postcomposition by a selfhomeomorphism of the sphere [Why1]. It follows that the collection of boundary components is canonically defined, and that they are pairwise disjoint simple closed curves. We call them *peripheral* *circles*, and we say that they are *uniform quasicircles* if they are the images of the unit circle by a quasisymmetry under a uniform distortion function. By extension, a *degenerate carpet* will be a one-dimensional, connected locally connected compact space homeomorphic to the complement of a union of disjoint open disks (their closures may intersect). In this case, there may be several non-equivalent embeddings in the sphere.

Lemma 3.4. *Let X be a one-dimensional, connected locally connected planar compact space with no global cut point. Then any embedding of X in S^2 is a degenerate carpet.*

Proof. Let us assume that X is already embedded into S^2, that we identify with the Riemann sphere. Since it is one-dimensional, it has no interior in S^2. We now prove that the boundary of any component of Ω of $S^2 \setminus X$ is a Jordan curve. This will establish that X is a degenerate carpet. We consider a conformal map $h : \mathbb{D} \to \Omega$. Since $\partial\Omega$ is contained in a locally connected compact set (disjoint from Ω), Carathéodory's theorem implies that h admits a continuous and surjective extension $h : \overline{\mathbb{D}} \to \overline{\Omega}$. If Ω is not a Jordan domain, then there are two rays in $\mathbb{D}$ which are mapped to a Jordan curve in $\overline{\Omega}$ which separates $\partial\Omega$, hence X. But X has no (global) cut point. Therefore, Ω is a Jordan domain and X is a degenerate carpet. ■

Corollary 3.5. *Let X be a metric degenerate carpet with* $\mathrm{confdim}_{AR} X < 2$ *endowed with boundary components which are assumed to be uniform quasicircles. We assume that X is LLC, relatively doubling and porous with respect to the boundary components. Then Σ is quasisymmetric to $\widehat{\mathbb{C}}$ and there exists a quasisymmetric embedding $\phi : X \to \widehat{\mathbb{C}}$ compatible with the boundary components.*

Proof. We may choose an Ahlfors regular metric in the gauge of X of dimension $Q < 2$. For each $K \in \mathcal{P}$, there exists a uniform quasisymmetric homeomorphism $\psi_K : K \to \mathbb{S}^1$ $(\subset \overline{\mathbb{D}})$. Note that $\overline{\mathbb{D}}$ is 2-Ahlfors regular and LLC, that $\mathbb{S}^1$ is porous in $\overline{\mathbb{D}}$ and that $\operatorname{diam}\mathbb{D} \le \operatorname{diam}\mathbb{S}^1$. Therefore, Theorem 3.1 applies. ■

4. Extension of maps

In this section, we show how homeomorphisms between the different sets can be glued together to yield a global homemorphism of Σ.

Let X be a continuum endowed with an admissible collection of boundary components $\mathcal{P}$. For each $K \in \mathcal{P}$, we assume that we are given a continuum L_K together with an injective mapping $\psi_K : K \to L_K$; and we consider as above

$$\Sigma = X \sqcup (\cup_{K\in\mathcal{P}} L_K)/\sim$$

4.1. **Global homeomorphisms.** The starting point is a collection of homeomorphisms $(h_K)_{K\in\mathcal{P}}$ and a homeomorphism $h_X : X \to X$ such that, for all $K \in \mathcal{P}$, $h_X(K), h_X^{-1}(K) \in \mathcal{P}$. We assume that the following compatibility condition holds: for all $K \in \mathcal{P}$, $h_K(\psi_K(K)) = \psi_{h_X(K)}(h_X(K))$ and

$$h_X|_K = (\psi_{h_X(K)}^{-1} \circ h_K \circ \psi_K)|_K\,.$$

Lemma 4.1. *The map $h : \Sigma \to \Sigma$ defined by $h(x) = h_X(x)$ if $x \in X$ and $h(x) = h_K(x)$ if $x \in L_K$ is a well-defined homeomorphism.*

Proof. We let the reader check that the compatibility condition implies that $h : \Sigma \to \Sigma$ is a well-defined bijection.

Let $U \subset \Sigma$ be an open set which satisfies (T1), (T2) and (T3). Then $U \cap X$ is open in X so $h^{-1}(U \cap X) = h_X^{-1}(U \cap X)$ is open in X since h_X is continuous. Similarly, for any $K \in \mathcal{P}$, $U \cap L_K$ is open in L_K so $h^{-1}(U \cap L_K) = h^{-1}_{h_X^{-1}(K)}(U \cap L_K)$ is open in $L_{h_X^{-1}(K)}$ since $h_{h_X^{-1}(K)}$ is continuous.

We note that $K \subset U$ if and only if $h_X^{-1}(K) \subset h_X^{-1}(U \cap X)$, and $L_K \subset U$ if and only if $h^{-1}(U \cap L_K) \subset h^{-1}(U)$. So, if for all but finitely many components $K \in \mathcal{P}$ with the property that $K \subset U$ one has $L_K \subset U$, then the same is true for boundary components $K \in \mathcal{P}$ with $K \subset h^{-1}(U)$. So $h^{-1}(U)$ is open, and h is continuous.

Since h is also a bijection and Σ is compact according to Proposition 2.1, it follows that h is a homeomorphism as well. ■

4.2. **Quasi-Möbius maps.** Following Väisälä [Väi], a homeomorphism $f : Z \to Z'$ between metric spaces is *quasi-Möbius* if there exists a homeomorphism $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any distinct points $x_1, x_2, x_3, x_4 \in Z$,

$$[f(x_1) : f(x_2) : f(x_3) : f(x_4)] \leq \theta([x_1 : x_2 : x_3 : x_4])$$

where

$$[x_1 : x_2 : x_3 : x_4] = \frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_3| \cdot |x_2 - x_4|}\,.$$

We record the following relationships with quasisymmetric mappings, see [Väi] for a proof.

Proposition 4.2. *Let $f : Z \to Z'$ be a homeomorphism between proper metric spaces.*

(i) *If f is η-quasisymmetric then f is also θ-quasi-Möbius, where θ only depends on η.*
(ii) *If f is θ-quasi-Möbius, then f is locally η-quasisymmetric, where η only depends on θ.*
(iii) *Let us assume that f is θ-quasi-Möbius. If Z and Z' are unbounded, then f is θ-quasisymmetric. If Z and Z' are compact, then assume that there are three points $z_1, z_2, z_3 \in Z$, such that $|z_i - z_j| \geq \operatorname{diam} Z/\lambda$ and $|f(z_i) - f(z_j)| \geq \operatorname{diam} Z'/\lambda$ for some $\lambda > 0$, then f is η-quasisymmetric, where η only depends on θ and λ.*

Corollary 4.3. *Let $f : Z \to Z'$ be a θ-quasi-Möbius homeomorphism between proper metric spaces. If there is a closed and connected subset $Y \subset Z$ with at least three points such that $\operatorname{diam} Y \geq \Delta \operatorname{diam} Z$, $\operatorname{diam} f(Y) \geq \Delta \operatorname{diam} Z'$ and such that $f|_Y$ is η-quasisymmetric, then f is $\widehat{\eta}$-quasisymmetric on Z where $\widehat{\eta}$ only depends on Δ, η and θ.*

Proof. If Z is unbounded, then there is nothing to prove. If not, we may pick three points $z_1, z_2, z_3 \in Y$, such that $|z_i - z_j| \geq \operatorname{diam} Y/3$; by Lemma 2.2, we also have $|f(z_i) - f(z_j)| \geq \operatorname{diam} f(Y)/\lambda$ for some constant λ which only depends on η. But the assumption on the embeddings $Y \hookrightarrow Z$ and $f(Y) \hookrightarrow Z'$ implies that $|z_i - z_j| \geq \Delta \operatorname{diam} Z/3$ and $|f(z_i) - f(z_j)| \geq \Delta \operatorname{diam} Z'/\lambda$ for all $i \neq j$. Therefore, Proposition 4.2 applies. ■

4.3. **Gluing quasi-Möbius maps together.** We assume that X is now a metric continuum and we suppose the existence of $\Delta_0 \geq 1$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $K \in \mathcal{P}$, we are given a metric continuum L_K and an η-quasisymmetric homeomorphism $\psi_K : K \to L_K$ such that $\operatorname{diam} L_K \leq \Delta_0 \operatorname{diam} \psi_K(K)$. We let d_Σ be the metric given by Theorem 2.3 on Σ.

In this section, we prove

Theorem 4.4. *Given a homeomorphism $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, there exists another homeomorphism $\theta' : \mathbb{R}_+ \to \mathbb{R}_+$ such that, if each h_K and h are θ-quasi-Möbius maps, then $h : (\Sigma, d_\Sigma) \to (\Sigma, d_\Sigma)$ is θ'-quasi-Möbius.*

It is convenient to see Σ as an unbounded space in order to transform quasi-Möbius maps into quasisymmetric ones. The following lemma makes the job. It may be considered as a converse construction of Bonk and Kleiner given in [BK2, Lma 2.2]. We use the same arguments for the proof.

Lemma 4.5. *Let (X, d, w) be a marked complete metric space. There exists a complete metric $\widehat{d}$ on $X \setminus \{w\}$ such that $Id : (X \setminus \{w\}, d) \to (X \setminus \{w\}, \widehat{d})$ is θ-quasi-Möbius with $\theta(t) = 16t$. and*

$$\frac{1}{4}\frac{d(x,y)}{\delta(x)\delta(y)} \le \widehat{d}(x,y) \le \frac{d(x,y)}{\delta(x)\delta(y)}$$

where $\delta(x) = d(x,w)$.

Proof. Set, for $x, y \in X \setminus \{w\}$,

$$q(x,y) = \frac{d(x,y)}{\delta(x)\delta(y)}$$

and let

$$\widehat{d}(x,y) = \inf \sum_{i=0}^{N-1} q(x_i, x_{i+1})$$

over all finite chains $x_0, \dots, x_N$ in $X \setminus \{w\}$ with $x_0 = x$, $x_N = y$. By definition, $\widehat{d}(x,y) \le q(x,y)$ holds.

Without loss of generality, we may assume that $\delta(x) \le \delta(y)$; let us fix a chain $x_0, , \dots x_N$ with $x_0 = x$, $x_N = y$. We consider two cases.

If $\delta(x_j) \le 2\delta(x)$ for all $j \in \{0, \dots, N\}$, then

$$\begin{aligned}\sum_{i=0}^{N-1} q(x_i, x_{i+1}) &\ge \frac{1}{4\delta(x)^2}\sum_{i=0}^{N} d(x_i, x_{i+1}) \\ &\ge \frac{1}{4\delta(x)^2} d(x,y) \ge \frac{1}{4} q(x,y)\,.\end{aligned}$$

We now assume that there is some $j > 0$ with $\delta(x_j) \ge 2\delta(x)$. Let us observe that

$$q(x,y) \le \frac{\delta(x)+\delta(y)}{\delta(x)\delta(y)} \le \frac{2}{\delta(x)}$$

and that, for $u, v \in X \setminus \{w\}$,

$$\left|\frac{1}{\delta(u)} - \frac{1}{\delta(v)}\right| = \frac{|\delta(v)-\delta(u)|}{\delta(u)\delta(v)} \le q(u,v)\,.$$

Therefore,

$$\begin{aligned}\sum_{i=0}^{N-1} q(x_i, x_{i+1}) &\ge \sum_{i=0}^{j-1}\left|\frac{1}{\delta(x_i)} - \frac{1}{\delta(x_{i+1})}\right| \\ &\ge \left|\frac{1}{\delta(x)} - \frac{1}{\delta(x_j)}\right| \\ &\ge \frac{1}{2\delta(x)} \ge \frac{1}{4} q(x,y)\,.\end{aligned}$$

This proves that $\widehat{d}$ is indeed a complete metric. The fact that the identity is quasi-Möbius follows at once from the formulae. ■

We will need to control the relative diameters of the sets L_K and K in this new metric:

Lemma 4.6. *Let X be a complete metric space, $w \in X$ and $K \subset L \subset X \setminus \{w\}$ be such that* $\operatorname{diam} L \le \Delta \operatorname{diam} K$ *and there is a constant $c > 0$ such that, for all $x \in L$,*

$$d(x,w) \ge c \inf\{d(x,y)+d(y,w),\ y \in K\}\,;$$

then there is some constant $\widehat{\Delta} > 0$ which only depends on Δ and c such that

$$\operatorname{diam}(L, \widehat{d}) \le \widehat{\Delta} \operatorname{diam}(K, \widehat{d})$$

where $\widehat{d}$ is the metric on $X \setminus \{w\}$ given by Lemma 4.5.

Proof. We set $\delta(x) = d(x,w)$ and $q(x,y) = d(x,y)/(\delta(x)\delta(y))$ as above, and let $\delta(K) = d(w,K)$.

Let us first estimate $\operatorname{diam}_q K = \sup\{q(x,y),\ x,y \in K\}$; pick $x \in K$ such that $\delta(x) = \delta(K)$ and let $y \in K$ be such that $d(x,y) \ge (1/2)\operatorname{diam} K$. It follows that $\delta(y) \le \operatorname{diam} K + \delta(K)$ so that

$$\begin{aligned}
\operatorname{diam}_q K \ge q(x,y) &\ge \frac{1}{2}\frac{\operatorname{diam} K}{\delta(K)\cdot(\delta(K)+\operatorname{diam} K)}\\
&\ge \frac{1}{4}\frac{\operatorname{diam} K}{\delta(K)\cdot \max\{\delta(K), \operatorname{diam} K\}}\\
&\ge \frac{1}{4}\min\left\{\frac{\operatorname{diam} K}{\delta(K)^2}, \frac{1}{\delta(K)}\right\}.
\end{aligned}$$

We note that the assumptions imply that, for all $x \in L$, $\delta(x) \ge c \cdot \delta(K)$.

On the one hand, we have

$$\operatorname{diam}_q L \le \frac{\operatorname{diam} L}{c^2 \delta(K)^2} \le \frac{\Delta}{c^2}\frac{\operatorname{diam} K}{\delta(K)^2}\,.$$

On the other hand, if $x, y \in L$, then $d(x,y) \le \delta(x) + \delta(y) \le 2\max\{\delta(x), \delta(y)\}$ so that

$$q(x,y) \le \frac{2}{\min\{\delta(x), \delta(y)\}} \le \frac{2}{c\delta(K)}$$

hence

$$\begin{aligned}
\operatorname{diam}_q L &\le \min\left\{\frac{\Delta}{c^2}\frac{\operatorname{diam} K}{\delta(K)^2}, \frac{2}{c\delta(K)}\right\}\\
&\le \frac{2\Delta}{c^2}\min\left\{\frac{\operatorname{diam} K}{\delta(K)^2}, \frac{1}{\delta(K)}\right\}\\
&\le \frac{8\Delta}{c^2}\operatorname{diam}_q K\,.
\end{aligned}$$

Therefore,

$$\operatorname{diam}(L, \widehat{d}) \le \frac{32\Delta}{c^2}\cdot \operatorname{diam}(K, \widehat{d})\,.$$

■

We now introduce the last ingredient for the proof of Theorem 4.4. Let X_1 and X_2 be two closed subsets of a metric space X such that $X_1 \cap X_2 \ne \emptyset$. The *seam* is by definition the closed set $Y = X_1 \cap X_2$. Following Agard and Gehring, the *angle* $\angle(X_1, X_2)$ between X_1 and X_2 is by definition the supremum over all $c > 0$ such that, for any $(x_1, x_2) \in X_1 \times X_2$,

$$|x_1 - x_2| \ge c \cdot \inf_{y \in Y}\{|x_1 - y| + |y - x_2|\}\,.$$

We recall [Haï2, Thm 3.1]:

Proposition 4.7. *Let $X = X_1 \cup X_2$ and $X' = X_1' \cup X_2'$ be metric spaces with positive angles. Let us assume that $Y = X_1 \cap X_2$ and $Y' = X_1' \cap X_2'$ are λ-uniformly perfect subspaces such that* $\operatorname{diam} Y \geq \mu \operatorname{diam} X_1$ *for some $\mu \in (0,1)$.*

If $f : X \to X'$ is a homeomorphism such that $f|_{X_j}$ is η-quasisymmetric and $f(X_j) = X_j'$, then f is globally $\widehat{\eta}$-quasisymmetric quantitatively.

We now need to check that angles remain positive when applying Lemma 4.5:

Lemma 4.8. *For any $c > 0$ and $\Delta \geq 1$, there exists $\widehat{c} > 0$ with the following property. Let A, B be subsets of a metric space (Z, d) with $A \cap B = X$ and $\angle_d(A, B) \geq c$. Let $w \in Z$ and let us consider the metric space $(Z \setminus \{w\}, \widehat{d})$ given by Lemma 4.5. Then $\angle_{\widehat{d}}(A \setminus \{w\}, B \setminus \{w\}) \geq \widehat{c}$.*

Proof. Fix $a \in A \setminus \{w\}$, $b \in B \setminus \{w\}$ and $x \in X$ such that $d(a, b) \geq c(d(a, x) + d(x, b))$.

We first observe that it is enough to find a constant $C = C(c, \Delta)$ and a point $x' \in X \setminus \{w\}$ such that

$$\min\{\widehat{d}(a, x'), \widehat{d}(b, x')\} \leq C\widehat{d}(a, b)\,.$$

In this case, it follows from the triangle inequality that $\max\{\widehat{d}(a, x'), \widehat{d}(b, x')\} \leq (C+1)\widehat{d}(a, b)$ so that

$$\widehat{d}(a, b) \geq \frac{1}{2(C+1)}(\widehat{d}(a, x') + \widehat{d}(x', b))\,.$$

We rely on the notations of Lemma 4.5 and shall use q instead of $\widehat{d}$. Let $\alpha \in (0,1)$ be a small constant which will only depend on c and Δ. If $\delta(x) \geq \alpha\delta(a)$ then

$$q(a, b) \geq c\frac{d(b, x)}{\delta(a)\delta(b)} \geq c\alpha q(b, x)\,;$$

similarly if $\delta(x) \geq \alpha\delta(b)$.

Let us now assume that $\delta(x) \leq \alpha \min\{\delta(a), \delta(b)\}$. On the hand, the triangle inequality implies that

$$d(a, b) \geq cd(b, x) \geq c(\delta(b) - \delta(x)) \geq c(1-\alpha)\delta(b)$$

holds so that $q(a, b) \geq c(1-\alpha)/\delta(a)$. On the other hand, there is a point $x' \in X$ such that

$$d(x, x') \geq \frac{1}{2\Delta} d(x, a) \geq \frac{1}{2\Delta}(1-\alpha)\delta(a)$$

so that

$$\delta(x') \geq d(x, x') - \delta(x) \geq \left(\frac{1-\alpha}{2\Delta} - \alpha\right)\delta(a)\,.$$

Choosing α small enough with respect to Δ, we get $\delta(x') \geq (1/4\Delta)\delta(a)$, so that

$$q(a, x') \leq \frac{\delta(a) + \delta(x')}{\delta(a)\delta(x')} \leq \frac{1+4\Delta}{\delta(a)} \leq \frac{1+4\Delta}{c(1-\alpha)} q(a, b)\,.$$

■

We may now turn to the proof of the main result of the section.

Proof. (Theorem 4.4) We first notice that Theorem 2.3 and Proposition 4.2 imply that there is a distortion function θ_1 such that the maps $(X, d_\Sigma) \xrightarrow{h_X} (X, d_\Sigma)$ and $(L_K, d_\Sigma) \xrightarrow{h_K} (L_{h_X(K)}, d_\Sigma)$, $K \in \mathcal{P}$, are θ_1-quasi-Möbius.

Let us pick $w \in X$ $(\subset \Sigma)$ and apply Lemma 4.5 to (Σ, w) and denote by (Z, d) the resulting metric space. Let (Z', d') be the metric space obtained from $(\Sigma, h(w))$. The theorem follows if we prove that $h : Z \to Z'$ is quasisymmetric with a distortion function which only depends on θ, η and Δ.

Proposition 4.2 implies that $h_X : (X \cap Z, d) \to (X \cap Z', d')$ is η_1-quasisymmetric, where η_1 only depends on θ. Similarly, there is some θ_2 which only depends on θ such that $(L_K \cap Z, d) \xrightarrow{h_K} (L_{h_X(K)} \cap Z', d')$ is θ_2-quasi-Möbius for all $K \in \mathcal{P}$.

Now, from Theorem 2.3, we know the existence of Δ_1 such that $\mathrm{diam}_\Sigma L_K \le \Delta_1 \mathrm{diam}_\Sigma K$ for all $K \in \mathcal{P}$. Lemma 4.6 implies the same property on Z and Z' with another constant Δ_2. So by Corollary 4.3, there is a distortion function η_2 such that $(L_K \cap Z, d) \xrightarrow{h_K} (L_{h_X(K)} \cap Z', d')$ is η_2-quasisymmetric for all $K \in \mathcal{P}$.

For all $K \in \mathcal{P}$, we have $(L_K \cap X) = K$, and the separating property (2.4) implies that $\angle(L_K, X) \ge c_0$ for some constant which only depends on η and Δ_0. Lemma 4.8 provides us with uniform positive angles in Z and Z'. Therefore, Proposition 4.7 implies that $h : (L_K \cup X) \cap Z \to (h(L_K) \cup X) \cap Z'$ is η_3-quasisymmetric for some η_3 which only depends on c_0, η_1 and η_2, so on η and Δ_0. Similarly, given $K_1 \ne K_2 \in \mathcal{P}$, $(L_{K_1} \cup X) \cap (L_{K_2} \cup X) = X$ and $\angle((L_{K_1} \cup X), (L_{K_2} \cup X)) \ge c_0$ so that $h|_{(L_{K_1} \cup L_{K_2} \cup X) \cap Z}$ is η_4-quasisymmetric onto its image, with η_4 which only depends on η and Δ_0. If we pick a third $K_3 \in \mathcal{P}$, then we obtain that $h|_{(L_{K_1} \cup L_{K_2} \cup L_{K_3} \cup X) \cap Z}$ is η_5-quasisymmetric onto its image, with η_5 which only depends on η and Δ_0.

This is enough to conclude that $h : Z \to Z'$ is η_5-quasisymmetric. ■

5. Quasiconvex subgroups of one-ended hyperbolic groups

Background on hyperbolic groups include [Gro, GdlH, KB].

Let G be a word hyperbolic group. The action of G on its boundary ∂G is that of a *uniform convergence group* i.e., the diagonal action on the set of distinct triples is properly discontinuous and cocompact. When G is one-ended, then its boundary is connected, locally connected and without (global) cut points.

Given a metric in its gauge $\mathcal{G}(G)$, the group G acts as a uniform quasi-Möbius group. Moreover, there exists a constant $m > 0$ such that, for any distinct points $x_1, x_2, x_3 \in \partial G$, there is some $g \in G$ such that $\{g(x_1), g(x_2), g(x_3)\}$ is m-separated. It follows that ∂G is doubling. and LLC when one-ended, see [Haï1] and the references therein.

5.1. Invariant null-sequences.

5.1. Invariant null-sequences. We establish some general properties in the spirit of [KK, Thm 5] and [Bon2, Prop. 1.4] where the authors dealt with carpets.

Proposition 5.1. *Let G be a non-elementary hyperbolic group and $\mathcal{P}$ a G-invariant null-sequence in ∂G, each element containing at least two points. Then*

(a) *the set $\mathcal{P}/G$ is finite;*
(b) *there exists a distortion function η such that, for any $K \in \mathcal{P}$ and $K' \in G(K)$, there exists $g \in G$ such that $g|_K : K \to K'$ is η-quasisymmetric;*
(c) *for any $K \in \mathcal{P}$, the stabilizer G_K of K is infinite and acts on K as a uniform convergence group;*
(d) *the boundary ∂G is doubling and porous relatively to $\mathcal{P}$ if the elements of $\mathcal{P}$ are connected;*

(e) *if the elements of* $\mathcal{P}$ *are pairwise disjoint and connected, then they are uniformly relatively separated i.e., there is a constant* $s > 0$ *such that* $\mathrm{dist}(K_1, K_2) \geq s \min\{\mathrm{diam}\, K_1, \mathrm{diam}\, K_2\}$ *for every distinct pairs* $K_1, K_2 \in \mathcal{P}$.

In other words, (c) means that G_K is a *quasiconvex subgroup* of G. These properties may be essentially established with the conformal elevator principle [Haï1, Prop. 4.6]:

Proposition 5.2. (Conformal elevator principle) *Let* G *be a non-elementary hyperbolic group and consider its boundary* ∂G *endowed with a metric from its gauge. Then there exist definite sizes* $r_0 \geq \delta_0 > 0$ *and a distortion function* η *such that, for any* $x \in X$*, and any* $r \in (0, \mathrm{diam}\,\partial G/2]$*, there exists* $g \in G$ *such that* $g(B(x,r)) \supset B(g(x), r_0)$*,* $\mathrm{diam}\, B(g(x), r_0) \geq 2\delta_0$ *and* $g|_{B(x,r)}$ *is* η*-quasisymmetric.*

We draw the following consequence of the conformal elevator principle. Let $\varepsilon \in (0,1)$, and let us consider $y \in B(x,r)$ such that $d(g(x), g(y)) \geq \delta_0$, and let $z \notin B(x, \varepsilon r)$. Then either $z \notin B(x,r)$ so that $d(g(x), g(z)) > r_0$, or

$$d(g(x), g(z)) \geq \frac{1}{\eta(1/\varepsilon)} d(g(x), g(y))$$

so that we have

$$g(B(x, \varepsilon r)) \supset B(g(x), \delta_0/\eta(1/\varepsilon)) \tag{5.1}$$

Proof. (Prop. 5.1) Let us fix a metric in $\mathcal{G}(G)$ and let $m > 0$ be such that any distinct triple of ∂G can be m-separated by an element of G. Given $\delta > 0$, we let $\mathcal{P}_\delta$ denote the subset of elements K of $\mathcal{P}$ such that $\mathrm{diam}\, K \geq \delta$; this set is finite since $\mathcal{P}$ is a null-sequence and non-empty for small enough δ.

For all $K \in \mathcal{P}$, we can find two points $x_1, x_2 \in K$ and a group element $g \in G$ such that $\{g(x_1), g(x_2)\}$ is m-separated: this implies that $g(K) \in \mathcal{P}_m$, so that $\mathcal{P}$ is composed of finitely many orbits and (a) holds.

Let r_0 and δ_0 be the constants arising from the conformal elevator principle. To prove (b), we notice that since $\mathcal{P}_{\delta_0}$ is finite and that quasi-Möbius mappings between compact sets are quasisymmetric, it is enough to prove that any $K \in \mathcal{P}$ can be mapped by a uniform quasisymmetric map to an element of $\mathcal{P}_{\delta_0}$. But this is exactly what the conformal elevator principle does.

Let us fix $K \in \mathcal{P}$ and let us first assume that K contains at least three points. Let us enumerate $\mathcal{P}_m \cap G(K) = \{K_1, \ldots, K_N\}$. For each $j \in \{1, \ldots, N\}$, one can find $g_j \in G$ such that $g_j(K_j) = K$. Since $\mathcal{P}_m \cap G(K)$ is finite and $\{g_j^{-1},\, j = 1, \ldots, N\}$ are uniformly continuous, there is some $m' \in (0, m]$ such that, for all $j \in \{1, \ldots, N\}$ and all $x, y \in K_j$ with $d(x,y) \geq m$, one has $d(g_j(x), g_j(y)) \geq m'$. Since G_K is a subgroup of G, its action on the set of distinct triples of K is automatically properly discontinuous. Let us prove that it is also cocompact. Let $x_1, x_2, x_3 \in K$ and consider $g \in G$ such that $\{g(x_1), g(x_2), g(x_3)\}$ is m-separated. It follows that $g(K) = K_j$ for some j, hence $(g_j \circ g) \in G_K$ and $\{(g_j \circ g)(x_1), (g_j \circ g)(x_2), (g_j \circ g)(x_3)\}$ is m'-separated.

If K has only two points x, y, then it suffices to prove that its stabilizer is infinite: this will imply it is two-ended, hence is quasiconvex in G. Pick a sequence (x_n) accumulating x. As above, we may find infinitely many $g_n \in G$ such that $g_n\{x, x_n, y\}$ are m-separated; the same argument as above proves that G_K is infinite. This proves (c).

Let us prove that ∂G is relatively doubling. Pick $x \in \partial G$ and $r \in (0, \operatorname{diam} G/2]$ and let us apply Proposition 5.2: there exists $g \in G$ such that $g(B(x,r)) \supset B(g(x), r_0)$ and $g|_{B(x,r)}$ is η-quasisymmetric.

Since the restriction of g is η-quasisymmetric, it follows from Lemma 2.2 that, for each boundary component K which intersects $B(x,r)$,

$$2\eta\left(\frac{\operatorname{diam} B(x,r)}{\operatorname{diam}(B(x,r)\cap K)}\right) \geq \frac{\operatorname{diam} g(B(x,r))}{\operatorname{diam} g(B(x,r)\cap K)}\,.$$

So if we assume that $\operatorname{diam}(B(x,r)\cap K) \geq \varepsilon r$, then $\operatorname{diam} g(B(x,r)\cap K) \geq \delta_0/\eta(2/\varepsilon)$. But since $\mathcal{P}$ is a null-sequence, there are only finitely many such boundary components.

We now assume that the elements of $\mathcal{P}$ are connected.

The proof that ∂G is relatively porous is similar: we apply the conformal elevator principle as above. It follows from (5.1) that $g(B(x,r/2)) \supset B(g(x), r_0/\eta(2))$. But since the action of G is minimal on ∂G which is compact, there exists a constant $\delta_1 > 0$ such that, for $y \in \partial G$, there exists $K_y \in \mathcal{P}$ with the property that $K_y \cap B(y, r_0/\eta(2)) \neq \emptyset$ and $\operatorname{diam} K_y \geq \delta_1$. Let $K'_{g(x)}$ be a connected component of $K_{g(x)} \cap g(B(x,r))$ which intersects $B(g(x), r_0/\eta(2))$; it follows from above that $\operatorname{diam} K'_{g(x)} \geq c r_0$ for some universal constant $c > 0$. By construction, $K' = g^{-1}(K'_{g(x)})$ is a subset of an element of $\mathcal{P}$ which intersects $B(x, r/2)$; moreover, Lemma 2.2 implies that

$$\operatorname{diam} K' \geq (1/2\eta(2/c)) r\,.$$

We now turn to the proof of (e). Let K_1 and K_2 be two distinct boundary components such that $\operatorname{diam} K_1 \leq \operatorname{diam} K_2$. Since $\mathcal{P}$ is a null-sequence, we may assume that $\operatorname{diam} K_1 \leq r_0$, and that $\operatorname{dist}(K_1, K_2) \leq \operatorname{diam} K_1/2$. Let $x \in K_1$ and $y \in K_2$ be such that $d(x,y) = \operatorname{dist}(K_1, K_2)$ and set $r = \operatorname{diam} K_1$. Let K'_2 be the component of $K_2 \cap B(x,r)$ which contains y so that $\operatorname{diam} K'_2 \geq r/2$ and $\operatorname{dist}(K_1, K'_2) = \operatorname{dist}(K_1, K_2) = d(x,y)$. Apply the conformal elevator principle to $B(x,r)$. It follows that $\operatorname{diam} g(K_1) \geq r_0$ and, from Lemma 2.2, we may deduce that

$$\operatorname{diam} g(K_2) \geq \operatorname{diam} g(K'_2) \geq \frac{r_0}{2\eta(2)}\,.$$

Since $\mathcal{P}$ is a null-sequence and the components are pairwise disjoint, there is a constant $s_0 > 0$ independent from K_1 and K_2 such that

$$d(g(x), g(y)) \geq \operatorname{dist}(g(K_1), g(K'_2)) \geq \operatorname{dist}(g(K_1), g(K_2)) \geq s_0 r_0\,.$$

Hence, applying a last time Lemma 2.2 yields

$$d(x,y) \geq \frac{r}{2\eta^{-1}(s_0)} \geq \frac{\operatorname{diam} K_1}{2\eta^{-1}(s_0)}$$

and so

$$\operatorname{dist}(K_1, K_2) \geq \frac{1}{2\eta^{-1}(s_0)} \min\{\operatorname{diam} K_1, \operatorname{diam} K_2\}\,.$$

■

5.2. **The extension property.** We give a criterion which enables to extend an action to a larger space. A *convergence group action* is an action of a group on a metrizable compactum X such that its diagonal action on the set of distinct triples of X is properly discontinuous. The *limit set* is by definition the unique minimal closed invariant subset of X.

Definition 5.3 (Extension property). *Let X be a metrizable continuum and $Y \subset X$ be a compact subset. We say that the pair (Y, X) has the extension property if any convergence action of a group on Y is the restriction of a convergence action on X by the same group. If X is supplied with a metric, we say that the pair (Y, X) has the conformal extension property if any action of a group which acts by uniform quasi-Möbius homeomorphisms on Y can be extended to an action on X by uniform quasi-Möbius homeomorphisms.*

Remark 5.4. A discrete group G of uniformly quasi-Möbius self-homeomorphisms of a compact metric space Z has the convergence property.

These extension properties are motivated by the following result due to Casson and Jungreis [CJ], Gabai [Gab], Hinkkanen [Hin1, Hin2] and Markovic [Mak1].

Theorem 5.5. *(a) Any faithful convergence action of a group on the unit circle is conjugate to an action of a Fuchsian group.*

(b) Any uniformly quasi-Möbius group of homeomorphisms on the unit circle is quasisymmetrically conjugate to a group of Möbius transformations.

So we may conclude that the extension property is not void.

Corollary 5.6. *The pair $(\mathbb{S}^1, \mathbb{D})$ has both the extension and conformal extension properties.*

Proof. The extension property is a direct consequence of Theorem 5.5 (a): if G is a convergence group of $\mathbb{S}^1$, there is a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $G' = hGh^{-1}$ is a group of Möbius transformations. Therefore, this group G' acts canonically on $\mathbb{D}$ by Möbius transformations as well. Let $H : \mathbb{D} \to \mathbb{D}$ be a homeomorphism which extends h. The group $H^{-1}G'H$ extends the action of G faithfully.

If G is a group of uniform quasi-Möbius homeomorphisms. By Theorem 5.5 (b), there exists a quasisymmetric homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $G' = hGh^{-1}$ is a group of Möbius transformations. Let $H : \mathbb{D} \to \mathbb{D}$ be a quasiconformal map which extends h [Ahl]. The group $H^{-1}G'H$ is a uniform group of quasi-Möbius maps which extends the action of G faithfully.■

Theorem 5.7. *Let G be a group acting on a metrizable continuum X as a convergence group action. Let $K_0 \subset X$ be a subcontinuum such that $G(K_0)$ defines an admissible sequence. We also assume that there exists an embedding $\psi : K_0 \to L$, where L is a metrizable continuum such that $(\psi(K_0), L)$ has the extension property. For each $K \in G(K_0)$, let $g_K \in G$ map K_0 to K, and let us consider the space Σ obtained by Proposition 2.1 with gluing maps $(\psi \circ g_K^{-1})$. Then we may extend the action of G on X to Σ as a convergence group action with same limit set.*

Proof. Set $K' = \psi(K_0)$, let $\varphi_0 : K' \to K_0$ denote its inverse and let $\varphi_K = g_K \circ \varphi_0 : K' \to K$ and $\psi_K = \varphi_K^{-1}$. For each K, we set $L_K = L$ and we let Σ be the metric space obtained from Proposition 2.1.

Denote by H the set of self-homeomorphisms of K' of the form $\varphi_{g(K)}^{-1} \circ g \circ \varphi_K$, among all $K \in G(K_0)$ and $g \in G$. Given h_1 and h_2 in H, we let $g_1, g_2 \in G$ and $K_1, K_2 \in G(K_0)$ be such that $h_j = \varphi_{g_j(K_j)}^{-1} \circ g_j \circ \varphi_{K_j}$. Then

$$\begin{aligned} h_1 \circ h_2 &= \varphi_{g_1(K_1)}^{-1} \circ g_1 \circ \varphi_{K_1} \circ \varphi_{g_2(K_2)}^{-1} \circ g_2 \circ \varphi_{K_2} \\ &= \varphi_{g_1(K_1)}^{-1} \circ g_1 \circ (g_{K_1} \circ \varphi_0) \circ (\varphi_0^{-1} \circ g_{g_2(K_2)}^{-1}) \circ g_2 \circ \varphi_{K_2} \\ &= \varphi_{g_1(K_1)}^{-1} \circ (g_1 \circ g_{K_1} \circ g_{g_2(K_2)}^{-1} \circ g_2) \circ \varphi_{K_2} . \end{aligned}$$

We may check that $(g_1 \circ g_{K_1} \circ g_{g_2(K_2)}^{-1} \circ g_2)(K_2) = g_1(K_1)$ so that $h_1 \circ h_2 \in H$ and H is a group of homeomorphisms. Note that H is isomorphic to the stabilizer of K_0: if $h = \varphi_{g(K)}^{-1} \circ g \circ \varphi_K$, then we may write

$$h = \varphi_0^{-1} \circ (g_{g(K)}^{-1} \circ g \circ g_K) \circ \varphi_0$$

and

$$(g_{g(K)}^{-1} \circ g \circ g_K)(K_0) = (g_{g(K)}^{-1} \circ g)(K) = g_{g(K)}^{-1}(g(K)) = K_0 \,.$$

Therefore, H is a convergence group on K'.

By the extension property, H acts on L as a convergence group as well.

We now define an action of G on Σ. Let $g \in G$. Fix $K \in G(K_0)$, and let $h = \varphi_{g(K)}^{-1} \circ g \circ \varphi_K : K' \to K'$ and $\widehat{h} : L_K \to L_{g(K)}$ be its extension. By construction, $g|_K = (\psi_{g(K)}^{-1} \circ \widehat{h} \circ \psi_K)|_K$ so that Lemma 4.1 implies that these maps patch up into a homeomorphism $\widehat{g} : \Sigma \to \Sigma$.

Let $\widehat{g}_j \in G$, $j = 1, 2$. On X, we find g_1, g_2 such that $\widehat{g}_j = g_j$ so that $(\widehat{g}_1 \circ \widehat{g}_2)|_X = (g_1 \circ g_2)|_X$. Set $g = g_1 \circ g_2$ and $\widehat{g}$ its extension to Σ. We have to prove that $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g}$.

Fix $K \in G(K_0)$, and let us prove that $(\widehat{g}_1 \circ \widehat{g}_2)|_{L_K} = \widehat{g}|_{L_K}$.

We let $(h_1, \widehat{h}_1)$, $(h_2, \widehat{h}_2)$ and $(h, \widehat{h})$ be associated to (g_2, K), $(g_1, g_2(K))$, and (g, K). On K', we have

$$\begin{aligned} h_1 \circ h_2 &= (\varphi_{g_1(g_2(K))}^{-1} \circ g_1 \circ \varphi_{g_2(K)}) \circ (\varphi_{g_2(K)}^{-1} \circ g_2 \circ \varphi_K) \\ &= \varphi_{(g_1 \circ g_2)(K)}^{-1} \circ (g_1 \circ g_2) \circ \varphi_K \\ &= \varphi_{g(K)}^{-1} \circ g \circ \varphi_K = h \,. \end{aligned}$$

Hence, the extension property implies that $\widehat{h}_1 \circ \widehat{h}_2 = \widehat{h}$.

It follows that the extended maps define an action of G on Σ. Since $(G(K_0))$ is admissible, the limit set Λ of this action coincides with the embedded copy of X. This embedding is equivariant by construction. ■

We now apply this construction to hyperbolic groups.

Let G be a one-ended hyperbolic group, and let us supply ∂G with a metric from its gauge. We assume that there exists a continuum $K_0 \subset \partial G$ such that $G(K_0)$ forms an admissible sequence of subcontinua of ∂G. We also assume that there exists a quasisymmetric embedding $\psi : K_0 \to L$, where L is a metric continuum such that $(\psi(K_0), L)$ has the conformal extension property.

Theorem 5.8. *There exist gluing functions of L along $G(K_0)$ so that the compact metric space Σ constructed by Theorem 2.3 can be endowed with an action of G by uniformly quasi-Möbius maps such that there is an equivariant bi-Lipschitz homeomorphism $\varphi : \partial G \to \Lambda$ where $\Lambda \subset \Sigma$ is the limit set of the action of G.*

Proof. According to Proposition 5.1 (b), there exists η such that, for any $K \in G(K_0)$, there is some $g_K : K_0 \to K$ which is η-quasisymmetric. We use the same notation as above and note that $\operatorname{diam} L_K \le \Delta_0 \operatorname{diam} \psi_K(K)$ with $\Delta_0 = \operatorname{diam} L / \operatorname{diam} K'$, and that φ_K and ψ_K are uniformly quasisymmetric. We let Σ be the metric space obtained from Proposition 2.1 and Theorem 2.3. The embedding of ∂G in Σ is bi-Lipschitz by Theorem 2.3.

Since G is uniformly quasi-Möbius and the φ_K's are uniformly quasisymmetric, the group H defined as in Theorem 5.7 is also a uniform quasi-Möbius group. According to Theorem

5.7, the group G acts as a convergence group on Σ with limit set ∂G. Theorem 4.4 implies that this action is also uniformly quasi-Möbius.

This embedding is bi-Lipschitz by Theorem 2.3 and equivariant by construction. ■

6. Hyperbolic groups with planar boundaries

6.1. **Hyperbolizable manifolds.** We quickly review definitions and properties of Haken 3-manifolds and of convex-cocompact Kleinian groups. Basic references include [Thu, Mor, Mad]. The following exposition is much inspired by [And].

A *Kleinian group* is a discrete subgroup of $\mathbb{P}SL_2(\mathbb{C})$ which we view as acting both on hyperbolic 3-space $\mathbb{H}^3$ via orientation-preserving isometries and on the Riemann sphere $\widehat{\mathbb{C}}$ via Möbius transformations.

The action of a Kleinian group G partitions $\widehat{\mathbb{C}}$ into the *domain of discontinuity* Ω_G, which is the largest set of $\widehat{\mathbb{C}}$ on which G acts discontinuously, and the *limit set* Λ_G, which is the minimal G-invariant compact subset of $\widehat{\mathbb{C}}$.

When G is torsion-free, we may associate a 3-manifold $M_G = (\mathbb{H}^3 \cup \Omega_G)/G$, canonically endowed with a complete hyperbolic structure in its interior, which is called the *Kleinian manifold*. We say that G is *convex-cocompact* if M_G is compact. Conversely, a compact 3-manifold M is *hyperbolizable* if there exists a discrete subgroup of isometries G such that M is homeomorphic to M_G (this whole presentation rules out tori in ∂M since they are not relevant to the present work). We say that M is *uniformized* by G. Note that G is isomorphic to the fundamental group of M, and that it is necessarily word-hyperbolic. Moreover, the boundary ∂M is a union of finitely many hyperbolic compact surfaces. When M is orientable, then G is a convex-cocompact Kleinian group.

Let M be a compact hyperbolizable 3-manifold with boundary. A surface S is *properly embedded* in M if S is compact and orientable and if either $S \cap \partial M = \partial S$ or S is contained in ∂M. A properly embedded surface S is *incompressible* if S is not homeomorphic to the 2-sphere and if the inclusion $i : S \to M$ gives rise to an injective morphism $i_* : \pi_1(S, x) \to \pi_1(M, x)$. A *Haken manifold* is a manifold which contains an incompressible surface. In our situation, as $\partial M \neq \emptyset$, M is always Haken.

We say that M has *incompressible boundary* if each component of ∂M is incompressible. This is equivalent to the connectedness of the limit set of the group uniformizing M.

A surface S in M is *non-peripheral* if it is properly embedded and if the inclusion $i : S \to M$ is not homotopic to a map $f : S \to M$ such that $f(S) \subset \partial M$. A surface S is *essential* if it is properly embedded, incompressible and non-peripheral. An *acylindrical compact manifold* has incompressible boundary and no essential annuli: the limit set of any uniformizing group is homeomorphic to the Sierpiński carpet.

A compact *pared manifold* (M, P) is given by a 3-manifold M as above together with a finite collection of pairwise disjoint incompressible annuli $P \subset \partial M$ such that any cylinder in $C \subset M$ with boundaries in P can be homotoped relatively to its boundary into P. We say the paring is *acylindrical* if $\partial M \setminus P$ is incompressible and every incompressible cylinder disjoint from P and with boundary curves in ∂M can be homotoped into $\partial M \setminus P$ relatively to ∂M.

If M has compressible boundary, it can be cut along *compression disks* into finitely many pieces each of which has incompressible boundary. Given a compact hyperbolizable manifold

M with incompressible boundary, we may cut it into finitely many pieces along essential annuli so that the remaining pieces are acylindrical pared manifolds.

We will use the following form of Thurston's uniformization theorem for Haken manifolds:

Theorem 6.1. *Let M be a compact irreducible Haken 3-manifold with word hyperbolic fundamental group. Then M is hyperbolizable.*

If the orientable case is usually stated, see for instance [Thu, Thm 2.3] and [Mor, Thm A'], this is not the case for non-orientable manifolds. It can be deduced from the uniformization of orientable manifolds: taking the orientable double cover, we obtain a representation of its fundamental group as a group generated by a convex-cocompact Kleinian group and an orientation-reversing quasiconformal involution: this group is thus uniformly quasiconformal, hence conjugate to a group of Möbius transformations according to Sullivan's straightening theorem [Sul2].

In particular, one obtains the following [McM2, Cor. 4.9]

Corollary 6.2. *A compact 3-manifold M is homeomorphic to the Kleinian manifold of a convex-cocompact Kleinian group G with $\Lambda_G \neq \widehat{\mathbb{C}}$ if and only if M is irreducible, orientable with non-empty boundary and its fundamental group contains no subgroup isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$.*

6.2. **Planar action of word hyperbolic groups.** For the proof of Theorem 1.7, we need some properties on the boundaries of hyperbolic groups. Some may be established just using the planarity. Others seem more specific to planar actions.

We start with a lemma which describes the topology of the boundary of a one-ended hyperbolic group provided it is planar.

Proposition 6.3. *Let G be a one-ended hyperbolic group with a planar boundary. Then either ∂G is homeomorphic to the sphere or ∂G is a degenerate carpet. In the latter case, any embedding defines an admissible collection of boundary components.*

Proof. If ∂G is not a sphere, then it is one-dimensional. But the boundary of a one-ended group has no global cut point and is locally connected. Therefore, Lemma 3.4 implies that it is a degenerate carpet.

Let us fix an embedding $\varphi : \partial G \to S^2$ and define $\mathcal{P}$ to be the collection of boundaries of the components of $S^2 \setminus \varphi(\partial G)$. Since ∂G is locally connected, it follows from [Why2, Thm. VI.4.4] that $\mathcal{P}$ is an admissible null-sequence. ∎

We may now establish some stronger assumptions when G admits a planar action.

Proposition 6.4. *Let G be a one-ended hyperbolic group with a planar action. Then*

(1) *the boundary components form an invariant null-sequence;*
(2) *every boundary component K is a uniform quasicircle and G_K is virtually a cocompact Fuchsian group.*

Proof. (Prop. 6.4) Let $\varphi : \partial G \to S^2$ be the topological embedding such that G leaves invariant the set $\mathcal{P}$ of boundary components of $S^2 \setminus \varphi(\partial G)$. We know from Proposition 6.3 that $\mathcal{P}$ is a null-sequence; it is invariant by definition.

Proposition 6.3 also implies that each boundary component is a Jordan curve. By Proposition 5.1, for any $K \in \mathcal{P}$, G_K is a uniform convergence group on K. We may then conclude from [Gab, CJ] that G_K is virtually a cocompact Fuchsian group and K is a quasicircle. Since G acts by uniform quasi-Möbius maps and there are only finitely many orbits of boundary components, we conclude that $\mathcal{P}$ is formed of uniform quasicircles. ■

We may now prove Theorem 1.7.

PROOF. (Theorem 1.7) Let $\varphi : \partial G \to S^2$ be the topological embedding such that G leaves invariant the set $\mathcal{P}$ of boundary components of $S^2 \setminus \varphi(\partial G)$. By Proposition 6.4, each peripheral circle is uniformly quasisymmetric equivalent to the unit circle.

Note that ∂G is LLC, relatively doubling and porous with respect to its boundary components. Therefore, Corollary 3.5 applies, so ∂G can be quasisymmetrically embedded in the sphere.

Since $(\mathbb{S}^1, \mathbb{D})$ has the extension property according to Corollary 5.6, Theorem 5.8 enables us to extend the action of G to the whole sphere as a group of uniform quasi-Möbius transformations. By Sullivan's straightening theorem [Sul2], this action is conjugate to a group of Möbius transformations. The action being properly discontinuous on the complement of ∂G, the group is discrete.

If G does not act faithfully on ∂G, then there is a normal finite subgroup F such that G/F is isomorphic to a discrete subgroup of isometries of $\mathbb{H}^3$. In order to conclude, we use material developed in §§ 6.4.2 and 6.5. According to Lemma 6.16 below and [Ago, Lemma 2.8], the group G is virtually special, hence virtually torsion-free by Corollary 6.20.

Therefore, G has a torsion-free finite index subgroup which is isomorphic to a convex-cocompact Kleinian group. ■

Remark 6.5. In the case of carpets, one could construct the Kleinian group differently. From [Bon2, Thm 1.1], we know that ∂G is quasisymmetric equivalent to a round carpet Λ (with all boundary components a round circle) of measure 0 (since the carpet is porous in $\widehat{\mathbb{C}}$), and G acts by quasisymmetric homeomorphisms. Therefore, [BKM, Thm 1.1] implies that each element of G acting on Λ is actually the restriction of a Möbius transformation.

PROOF. (Remark 1.11) There are several approaches to see whether G is quasi-isometric to a convex subset of $\mathbb{H}^3$ with geodesic boundary or not. The first is based on hyperbolic geometry. If G is quasi-isometric to a convex subset with geodesic boundary, then it is virtually the fundamental group of a compact hyperbolic manifold with geodesic boundary. It follows that this manifold is acylindrical so the limit set is homeomorphic to the Sierpiński carpet, see e.g. [McM2]. The converse is one of the main steps of Thurston's hyperbolization theorem, see [McM1].

Another route goes as follows. When ∂G is a carpet, peripheral circles are disjoint uniform quasicircles so they are also uniformly separated by Proposition 5.1. Hence, [Bon2, Thm 1.1] implies that ∂G is quasisymmetric equivalent to a round carpet Λ (with all boundary component a round circle). It now follows from [BS] that G is quasi-isometric to a convex subset of $\mathbb{H}^3$ with geodesic boundary.

When ∂G is not a carpet, then ∂G admits local cut points [KK, Thm 4]. Either it is homeomorphic to a circle so that G is virtually a cocompact Fuchsian group. Or the local cut points are structured in equivalence classes as described by Bowditch [Bow1], see also below. Note that local cut points have to belong to boundary components: if $x \in \partial G$ does not belong to any boundary component, [Why2, VI.4.5] asserts that we can find a nested sequence of

Jordan curves contained in ∂G enclosing x in any neighborhood of x. This prevents x to be a local cut point. These cut points are associated to two-ended hyperbolic groups: every component of the complement which contains one of the fixed points has to contain the other one: therefore, there are components which intersect in at least two different points, preventing them to be mapped simultaneaously to round disks. ■

We may now prove Corollary 1.12.

Proof. (Corollary 1.12) If Λ_G is connected, then Theorem 1.7 and its proof imply that G is isomorphic to a convex-cocompact Kleinian group with conjugate actions by a homeomorphism which extend to the whole sphere. Moreover, according to [MT, Cor. 4.5], the action of each stabilizer of each component is essentially unique, so we may build a conjugacy using similar considerations as in the proof of Theorem 5.7.

Let us assume that Λ_G is not connected, and let us write $\Omega_G = S^2 \setminus \Lambda_G$. We apply the decomposition techniques from [AM], see also [MS]. Let $p : \Omega_G \to \Omega_G/G$ be the covering and N be the normal subgroup defining the covering; we consider a finite set of simple, disjoint loops $M = \{u_1, \ldots, u_n\}$ on Ω_G/G such that there exist (minimal) positive integers $a_1, \ldots, a_n$ so that the normal subgroup N_M generated by $u_1^{a_1}, \ldots, u_n^{a_n}$ is a subgroup of N. Let $\Gamma = p^{-1}(\{u_1, \ldots, u_n\})$: this is a countable set of pairwise disjoint homotopically non-trivial simple loops in Ω_G.

We construct a tree T as follows. Let the set of vertices be the connected components of $S^2 \setminus \cup_{\gamma\in\Gamma}\gamma$ and put an edge between two such components if they share a curve of Γ on their boundary. Let us observe that Γ is a null-sequence since Γ/G is finite and the curves live in the set of discontinuity of G. Since each curve is a Jordan curve which separates S^2, it follows that T is simply connected and that the ends of T correspond to a nested sequence of disks so that T is connected. This implies that T is a tree.

By construction, G acts simplicially on T and T/G is finite since Γ/G is finite. Moreover, an edge stabilizer corresponds to the stabilizer of a curve $\gamma \in \Gamma$; since $\gamma \subset \Omega_G$ and G is torsion-free, each edge stabilizer is trivial and there are no edge inversions. It follows that this action yields a decomposition of G as a free product. SInce G is hyperbolic, it is accessible so the number n of edges in T/G is bounded. But, according to [Mas, Lemma 5], if $N_M \neq N$, then we can find u_{n+1} disjoint from M and a_{n+1} with $u_{n+1}^{a_{n+1}} \in N$, so may repeat the above construction with $M \cup \{u_{n+1}\}$. The accessibility of G implies that this process has to stop, meaning that we end up with a multicurve M such that $N_M = N$.

This implies in particular that Ω_G/G is a finite union of compact surfaces. Let $T_0' \subset T/\Gamma$ be a maximal tree and let us consider a connected lift $T_0 \subset T$ so that each vertex is represented in T exactly once; the stabilizers of vertices of T_0 are quasiconvex subgroups according to [Bow1, Prop. 1.2]. Moreover, they are either trivial or one-ended for otherwise Γ/G would not generate N.

Let $v \in T_0$; if its stabilizer G_v is trivial, we may associate a 3-ball M_v; otherwise G_v is a one-ended torsion-free planar group hence it is conjugate to a discrete group of Möbius transformations by Theorem 1.7, and so G_v is isomorphic to the fundamental group of a hyperbolic manifold M_v. Each edge orbit attached to v corresponds to a simple closed curve on ∂M_v which bounds a disk. The graph T/G tells us how to build a manifold M by gluing the different M_v's along the disks bounded by the curves above [SW]. We obtain in this way a Haken manifold which satisfies the assumptions of Theorem 6.1. Therefore, G is isomorphic to a discrete subgroup H of isometries of $\mathbb{H}^3$.

By the construction of M, the isomorphism between G and H yields an equivariant homeomorphism between Ω_G and Ω_H, so we may find a function $f : S^2 \to \widehat{\mathbb{C}}$ such that (a) $f \circ G = H \circ f$, (b) $f|_{\Lambda_G} : \Lambda_G \to \Lambda_H$ and $f|_{\Omega_G} : \Omega_G \to \Omega_H$ are homeomorphisms. Therefore, [Bow2, Prop. 5.5] implies that this conjugacy is a global homeomorphism. ■

6.3. **The JSJ decomposition.** We first summarize briefly the JSJ-decomposition of a non-Fuchsian one–ended hyperbolic group G following Bowditch [Bow1]. Then we will focus on specific decompositions for groups with planar boundary.

6.3.1. *General properties.* There exists a canonical simplicial minimal action of G on a simplicial tree $T = (V, E)$ without edge inversions such that T/G is a finite graph and which enjoys the following properties, cf. [Bow1, Thm 5.28, Prop. 5.29]. If v is a vertex (resp. e an edge), we will denote by G_v (resp. G_e) its stablizer, and by Λ_v (resp. Λ_e) the limit set of G_v (resp. G_e). Let E_v denote the set of edges incident to $v \in T$. Every vertex and edge group is quasiconvex in G. Each edge group G_e is two-ended and $\partial G \setminus \Lambda_e$ is not connected. A vertex v of T belongs to exactly one of the following three exclusive types.

> **Type I (elementary).**— The vertex has bounded valence in T. Its stabilizer G_v is two-ended, and the connected components of $\partial G \setminus \Lambda_v$ are in bijection with the edges incident to v.
> **Type II (surface).**— The limit set Λ_v is cyclically separating and the stabilizer G_v of such a vertex v is a non-elementary virtually free group canonically isomorphic to a convex-cocompact Fuchsian group. The incident edges are in bijection with the peripheral subgroups of that Fuchsian group.
> **Type III (rigid).**— Such a vertex v does not belong to a class above. Its stabilizer G_v is a non-elementary quasiconvex subgroup. Every local cut point of ∂G in Λ_v is in the limit set of an edge stabilizer incident to v; see Lemma 6.6 for more properties of rigid type vertices.

No two vertices of the same type are adjacent, nor surface type and rigid can be adjacent either. The action of G preserves the types. Therefore, the edges incident to a vertex v of surface type or rigid type are split into finitely many G_v-orbits.

Let v be a vertex of T of rigid type; if $e \in E_v$ is incident to v, let $C_v(e)$ denote the connected component of $\partial G \setminus \Lambda_e$ which contains Λ_v and set $Z_e = \partial G \setminus C_v(e)$: this is a connected, locally connected, compact set by construction. Let us define the following equivalence relation $\sim_v$ on ∂G. Say $x \sim_v y$ if $x = y$ or if there exists an edge $e \in E_v$ incident to v such that $\{x, y\}$ is a subset of Z_e. Let $Q_v = \partial G / \sim_v$ be endowed with the quotient topology and let $p_v : \partial G \to Q_v$ be the canonical projection. Note that preimages of points are either points or one of the Z_e's, so they are always connected and the map p_v is *monotone.*

Lemma 6.6. *If v is of rigid type, then the space Q_v is a Hausdorff, compact, connected and locally connected space. Moreover, no pair of points can disconnect Q_v and the local cut points of Q_v correspond to the non-trivial classes of $\sim_v$ which disconnect locally Q_v in exactly two components. The group G_v acts on Q_v as a geometrically finite group and there are finitely many orbits of parabolic points.*

Following Bowditch [Bow3], we say that a subgroup $H < G_v$ is *parabolic* if it is infinite, fixes some point of Q_v, and contains no loxodromics. In this case, the fixed point of H is unique. We refer to it as a *parabolic point.* The stabilizer of a parabolic point is necessarily a parabolic group. There is thus a natural bijective correspondence between parabolic points in Q_v and maximal parabolic subgroups of G_v. We say that a parabolic group, H, with fixed point p, is *bounded* if the quotient $(Q_v \setminus \{p\})/H$ is compact. (It is necessarily Hausdorff.) We say that p is a *bounded parabolic point* if its stabilizer is bounded. A *conical limit point* is a point $y \in Q_v$ such that there exists a sequence $(g_j)_{j\geq 0}$ in G_v, and distinct points $a, b \in Q_v$, such that $g_j(y)$ tends to a and $g_j(x)$ tends to b for all $x \in Q_v \setminus \{y\}$. We finally say that the action of G_v on Q_v is *geometrically finite* if every point is either conical or bounded parabolic (they cannot be both simultaneously).

Proof. Since ∂G is locally connected, it follows that $\{Z_e\}_{e\in E_v}$ is a null-sequence, so $\sim_v$ defines an upper semicontinuous decomposition of ∂G and Q_v is Hausdorff (see also [Car, Cor. 6.16]). It follows that Q_v is connected, locally connected and compact as the image under a continuous map of a Hausdorff, connected, locally connected and compact set into a Hausdorff space. See [Why2, Chapter VII, § 2,3] for details.

Cut points and local cut points of Q_v yield (local) cut points of ∂G by pull-backs under p_v. Let $x \in Q_v$; if $x = p_v(Z_e)$, then it does not disconnect Q_v since Z_e does not ($\partial G \setminus Z_e = C_v(e)$); otherwise, since v is of rigid type, it follows that $p_v^{-1}(x)$ is not a local point, so neither is x. We also conclude that the only possible local points correspond to the non-trivial classes.

Fix an incident edge e. By construction, $\partial G \setminus Z_e$ has exactly two ends, each accumulating one single point of Λ_e. Hence, we may consider two disjoint connected neiborhoods N_1 and N_2 of Λ_e in $(\partial G \setminus Z_e) \cup \Lambda_e$; if there are small enough then $p_v(N_1)$ and $p_v(N_2)$ cover a neigborhood of $p_v(\Lambda_e)$ and they intersect exactly at that point. This implies that it is a local cut point with two ends.

The action of G_v permutes the fibers of p_v, hence G_v acts on Q_v. Since the action of G_v is a convergence action and is minimal on Λ_v it is also the case on Q_v.

If $e \in E_v$ is an incident edge, then $(\Lambda_v \setminus \Lambda_e)/G_e$ is compact by [Bow2], so $(Q_v \setminus p_v(\Lambda_e))/G_e = p_v(\Lambda_v \setminus p_v(\Lambda_e))/G_e$ is compact as well. Thus, $p_v(\Lambda_e)$ is a bounded parabolic point.

Note that the action on Λ_v is uniform so every point is conical [Bow2]. Thus, if $y \in \Lambda_v$, we may find a sequence $(g_j)_{j\geq 0}$ in G_v, and distinct points $a, b \in \Lambda_v$, such that $g_j(y)$ tends to a and $g_j(x)$ tends to b for all $x \in \Lambda_v \setminus \{y\}$. To conclude that $p_v(y)$ is also conical, it suffices to make sure that we may choose a and b not simultaneously in some Λ_e. Let us assume that y is in no limit set of an incident edge and that there is indeed some incident edge e such that $\Lambda_e = \{a, b\}$. Pick a compact fundamental domain K of $(\Lambda_v \setminus \Lambda_e)/G_e$. Then for any j, we may find $h_j \in G_e$ such that $h_j g_j(y) \in K$. Then it follows that, extracting a subsequence if necessary, $(h_j g_j(x))_j$ tends to b for all $x \in \Lambda_v$ but $(h_j g_j(y))_n$ remains far from Λ_e. Therefore $p_v(y)$ is conical as well.

This proves that the action of G_v on Q_v is geometrically finite. Since there are only finitely many G_v-orbit of incident edges to v in T, there are only finitely many parabolic orbits in Q_v.

Let us prove that no pair of points separates Q_v by contradiction. Following Bowditch [Bow1, § 3], say two points x and y in Q_v are equivalent, $x \sim y$, if $x = y$ or $Q_v \setminus \{x, y\}$ is disconnected. Since every local cut point disconnects Q_v locally into two components, this defines an equivalence relation on Q_v according to [Bow1, Lma 3.1]. Moreover, [Bow1, Lma 3.7] implies that each class is closed. So, let us assume that $x \in Q_v$ belongs to a non-trivial class and let $y \sim x$, $y \neq x$. Since x is necessarily a local cut, it is also a parabolic point. Therefore, one can find a sequence (g_n) stabilizing x such that $g_n(y)$ tends to x. It follows that no point is isolated in a non-trivial class. Therefore, each non-trivial class is a perfect compact subset of local cut points of Q_v. But such a set is always uncountable, contradicting that there are at most countably many parabolic points. Hence each class is trivial and no pair of points can separate Q_v. ■

6.3.2. *Planar action of stabilizers of vertices of rigid type.* In this paragraph, we assume that G is a one-ended hyperbolic group with planar boundary. The goal is to analyze vertices of rigid type and interpret the incident edges as an acylindrical paring for the Kleinian manifold when this vertex stabilizer is isomorphic to the fundamental group of a compact 3-manifold with boundary.

We assume that we are given an embedding $\varphi : \partial G \to S^2$, and we write $\Lambda_G = \varphi(\partial G)$. We will identify in the sequel subsets of ∂G with subsets of Λ_G via the map φ.

Proposition 6.7. *Let G be a one-ended hyperbolic group with planar boundary. Let v be a vertex of rigid type in its JSJ decomposition. Then the action of G_v on Λ_v extends to a convergence action of S^2 with limit set Λ_v.*

The proof of this proposition will require several steps, which we outline right now. We will first prove that we may find a degree 1 map of S^2 transforming Λ_v onto a homeomorphic copy of Q_v; this will enable us to prove that Q_v is a degenerate carpet and that G_v acts as a geometrically finite convergence group. This action is planar and can be extended to a convergence action on S^2 with limit set the copy of Q_v. We may then lift this action to an action of S^2 with limit set Λ_v.

Lemma 6.8. *There exists a pseudoisotopy $(\psi_t)_{t\in[0,1]}$ of the sphere such that, writing $\psi = \psi_1$, $\psi(\Lambda_v)$ is homeomorphic to Q_v and such that fibers are points except at non-trivial classes of $\sim_v$ where fibers are closed arcs. Moreover, $\widehat{\Lambda}_v$ is connected and locally connected, where $\widehat{\Lambda}_v$ denotes the inverse image of $\psi(\Lambda_v)$ under ψ.*

A pseudoisotopy is a continuous map $\psi : [0,1] \times S^2 \to S^2$ where, for all $t \in [0,1)$, $\psi_t : x \mapsto \psi(t,x)$ is a homeomorphism of S^2 and ψ_0 is the identity.

Proof. Let $e \in E_v$ be an edge incident to v. Since Λ_e disconnects Z_e from Λ_v, we may find an arc $c_e \subset Z_e$ joining Λ_e.

Proceeding as above for all edges incident to v, we obtain a family of arcs $\{c_e\}_{e\in E_v}$. Since the sets Z_e are disjoint, these curves are pairwise disjoint. We wish to prove that the partition $\mathcal{G}$ of the sphere into these arcs and single points is upper semicontinuous, cf. [Why2, Chap. VII]. Since the non-trivial elements of this collection form a countable set, it is enough to prove that they form a null-sequence. This follows from the local connectivity of ∂G since it implies that $(Z_e)_{e\in E_v}$ forms a null-sequence.

Since the elements of $\mathcal{G}$ are connected compact non-separating subsets of S^2, Moore's Theorem [Dav, Theorem 25.1] implies that the quotient $S^2/\mathcal{G}$ is homeomorphic to the sphere. Note that $\mathcal{G}$ and $\sim_v$ agree on Λ_v so that both quotients are homeomorphic (to Q_v). By [Dav, Theorems 13.4, 25.1], the decomposition $\mathcal{G}$ of S^2 has the property of being *strongly shrinkable*: there is a pseudoisotopy $\psi_t : S^2 \to S^2, t \in [0,1]$ such that the fibers of ψ agree with $\mathcal{G}$.

The set $\widehat{\Lambda}_v$ is clearly connected since ψ is monotone. It remains to prove it is also locally connected. Let $x \in \widehat{\Lambda}_v$ and let us consider a nested family of connected neighborhoods (V_n) of $\psi(x)$ with $\cap V_n = \{\psi(x)\}$. It follows that $\psi^{-1}(V_n)$ is also a sequence of nested connected neighborhoods of x and that $\cap\psi^{-1}(V_n) = \psi^{-1}(\{\psi(x)\})$. Therefore, we already know that $\widehat{\Lambda}_v$ is locally connected at points which are fibers of ψ. If x belongs to the interior of some arc c_e, then we may also construct a basis of connected neighborhoods as the arc is isolated in $\widehat{\Lambda}_v$. If

$x \in \Lambda_e$, then $\psi^{-1}(V_n) \setminus c_e$ has two connected components, one of them —W_n— containing x in its closure. We may then add to W_n a small subarc of c_e to obtain a connected neighborhood W_n' in $\widehat{\Lambda}_v$ of x so that $\cap W_n' = \{x\}$. ■

From now on, we let $Q_v = \psi(\Lambda_v)$.

Lemma 6.9. *The set Q_v is a degenerate carpet such that, for any connected component Ω of $S^2 \setminus Q_v$, $Q_v \setminus \partial\Omega$ is connected.*

Proof. Note that Λ_v is one-dimensional (otherwise we would have $G_v = G$), hence Q_v as well. By Lemma 6.6 and Lemma 3.4, Q_v is a degenerate carpet.

Let Ω be a component of $S^2 \setminus Q_v$, and let us prove that $Q_v \setminus \overline{\Omega}$ is connected: let Q_1 and Q_2 be a partition of $Q_v \setminus \overline{\Omega}$ into two open disjoint sets. Let V be a connected component of $S^2 \setminus (Q_v \cup \Omega)$.

Note that ∂V can intersect $\partial\Omega$ at most at a single point, otherwise we would be able to disconnect Q_v by removing two points, contradicting Lemma 6.6. Therefore $\partial V \cap (Q_v \setminus \partial\Omega)$ is connected: it is either contained in Q_1 or in Q_2.

We may now define, for $j = 1, 2$, U_j to be the union of Q_j with all the components V the boundaries of which are contained in Q_j: we obtain in this way two disjoint open sets covering $S^2 \setminus \overline{\Omega}$. Since the latter is connected, U_1 or U_2 is empty, hence $Q_v \setminus \partial\Omega$ is connected as well. ■

By definition of ψ, we may define an action of G_v on Q_v such that $G_v \circ \psi = \psi \circ G_v$.

Lemma 6.10. *The projection ψ yields a minimal planar action of G_v on Q_v as a geometrically finite convergence action with finitely many orbits of parabolic points. Moreover, there are only finitely many orbits of boundary components of Q_v and, for any connected component Ω of $S^2 \setminus Q_v$, the stabilizer of $\partial\Omega$ is also geometrically finite.*

Proof. The first statement of the lemma is contained in Lemma 6.6.

We note that, according to Lemma 6.9, boundaries of complementary components do not separate Q_v, so they are preserved by G_v, and the action is planar.

Let $\mathcal{P} = \{\psi^{-1}(\partial\Omega) \cap \Lambda_v,\ \Omega \in \pi_0(S^2 \setminus Q_v)\}$. Since the action on Q_v is planar, the collection $\mathcal{P}$ is G_v-invariant. Note that each element is contained in the boundary of a connected component of $S^2 \setminus \widehat{\Lambda}_v$, cf. Lemma 6.8; since $\widehat{\Lambda}_v$ is a locally connected continuum by Lemma 6.8, we conclude that $\mathcal{P}$ is a G_v-invariant null-sequence. Therefore, Proposition 5.1 applies and proves that each stabilizer of $K \in \mathcal{P}$ is quasiconvex with limit set K and that $\mathcal{P}$ is composed of finitely many orbits. Pushing down by ψ, we obtain finitely many orbits of boundary components, and the stabilizer G_K of each $K \in \mathcal{P}$ provides us with a geometrically finite action of the stabilizer G_Ω of the boundary of each complementary component Ω of Q_v. ■

Corollary 6.11. *The action of G_v on Q_v can be extended to a convergence action on S^2 with limit set Q_v such that $(S^2 \setminus Q_v)/G_v$ is a finite union of surfaces of finite type.*

Proof. It follows from Theorem 5.7, Lemma 6.10 and Proposition 3.2 that the action of G_v to Q_v is the restriction of a convergence action on S^2.

Finally, for each component Ω of $S^2 \setminus Q_v$, the action of its stabilizer G_Ω is geometrically finite, hence isomorphic to a geometrically finite Fuchsian group and we may conclude that Ω/G_v is a surface of finite type. By Lemma 6.10, there are only finitely many of them. ■

PROOF. (Proposition 6.7) We want to lift under ψ the action of G_v given by Corollary 6.11. Each parabolic point $p_e = \psi(\Lambda_e)$ belongs to the boundary of two complementary components according to Lemma 6.6. Let $H_e \subset (S^2 \setminus Q_v)$ be the union of two G_e-invariant horocycles (one in each component) that we choose small enough so that their G_v-orbit are all pairwise disjoint. The map

$$\psi : S^2 \setminus (\cup_{e\in E_v}\psi^{-1}(H_e)) \to S^2 \setminus (\cup_{e\in E_v} H_e)$$

is a homeomorphism so we may lift the action of G_v there.

Let e represent an element of E_v/G_v; the set $S_e = \psi^{-1}(H_e)$ is homeomorphic to a Jordan domain and G_e acts as an elementary convergence group on its boundary. By Theorem 5.7, Lemma 6.10 and Proposition 3.2, the action of G_v extends to a convergence action on S^2. ■

Corollary 6.12. *Let G be a torsion-free one-ended hyperbolic group with a planar boundary and let v be a vertex of rigid type in its JSJ decomposition such that all the stabilizers of its incident edges are isomorphic to $\mathbb{Z}$. We also assume that* $\mathrm{confdim}_{AR} G_v < 2$. *Then there exists a compact hyperbolizable* 3*-manifold M_v with boundary and with fundamental group isomorphic to G_v such that the conjugacy classes of incident edges define a maximal collection of incompressible disjoint simple closed curves on ∂M_v.*

Thickening this multicurve into pairwise disjoint annuli provides us with an acylindrical paring of M_v.

PROOF. According to Proposition 6.7, G_v acts on S^2 with limit set Λ_v. Since its Ahlfors regular conformal dimension is strictly less than two, Corollary 1.12 provides us with a hyperbolizable manifold.

Each generator of a conjugacy class of an incident edge defines a non-trivial curve γ_e in $M_v = \mathbb{H}^3/G_v$. Since the curves $\{c_e\}$ are pairwise disjoint off of Λ_v, it follows that the γ_e are homotopic to simple and pairwise disjoint curves on ∂M_v. If this family was not maximal, we could split M_v along an incompressible annulus disjoint from the γ_e. But this would imply that the JSJ-decomposition of G was not maximal. ■

6.3.3. *Regular decomposition.* We will focus on particular JSJ-decompositions which are suited to manifolds.

Definition 6.13 (Regular JSJ decomposition)**.** *Let G be a one-ended hyperbolic group and let us consider its JSJ decomposition. We say it is regular if the following properties hold:*

- *every two-ended group H which appears as a vertex or an edge group is isomorphic to $\mathbb{Z}$ and stabilizes the components of $\partial G \setminus \Lambda_H$;*
- *the stabilizer of vertices of surface type are free;*
- *the stabilizer of vertices of rigid type is torsion-free.*

The main point comes from the following proposition.

Proposition 6.14. *Let G be a one-ended hyperbolic group with planar boundary and with a regular JSJ decomposition. We assume that the Ahlfors regular conformal dimension of each vertex of rigid type is stricly less than* 2*. Then G is isomorphic to the fundamental group of a compact hyperbolizable* 3*-manifold with boundary.*

PROOF. We first notice that since each elementary group G_v or G_e fixes the components of $\partial G \setminus \Lambda_v$ and $\partial G \setminus \Lambda_e$, they are generated by primitive elements of G.

If v is a vertex of elementary type, we associate a solid torus $M_v = \mathbb{S}^1 \times \mathbb{D}$ on which we consider on its boundary pairwise disjoint incompressible annuli $\mathbb{S}^1 \times \alpha_e$ in bijection with its incident edges, where $\alpha_e \subset \partial\mathbb{D}$ are arcs.

If v is of surface type, then G_v is Fuchsian so it uniformizes a surface S_v; we let $M_v = S_v \times [0,1]$ and, noting that each incident edge in T/G corresponds to a simple closed curve $\gamma_e \subset S_v$ bounding a hole of S_v, we may associate on ∂M_v disjoint annuli $\gamma_e \times [0,1]$.

If v is of rigid type, Corollary 6.12 enables us to associate a pared manifold M_v.

Now, T/G provides us with a manual to build a 3-manifold M with fundamental group isomorphic to G by gluing the M_v's along the annuli [SW]. Thurston's hyperbolization theorem for Haken manifolds shows that M is hyperbolizable. ■

6.4. **Group actions on CAT(0) cube complexes.** A cube complex X is a CW-complex where each n-cell is a standard Euclidean n-cube and such that

(1) each closed cube is embedded into X;
(2) the intersection of two cubes is either empty or a face.

A cube complex is naturally endowed with a length structure such that each n-cell is isometric to a unit Euclidean cube of the same dimension. We will focus on cube complexes which satisfy the CAT(0) condition. We shall say that a group G is *cubulated* if it admits a geometric and cellular action on a CAT(0) cube complex X. We refer for instance to [Sag] and [BH] for details.

We gather some definitions and properties for future reference.

6.4.1. *Hyperplanes.* A fundamental feature of CAT(0) cube complexes comes from hyperplanes. Let us first define a *midcube* of a cube $[(-1/2), 1/2]^n$ to be the intersection of the cube with a linear hyperplane orthogonal to one axis $[(-1/2), 1/2]^n \cap \{x_j = 0\}$, for some $j \in \{1, \ldots, n\}$. A *hyperplane* is a maximal convex subset of a CAT(0) cube complex for which the intersection with any cube is either empty or a midcube.

Hyperplanes have many interesting properties [Sag, Hag]. Among them:

(1) given an edge, there is a unique hyperplane which intersects it orthogonally;
(2) a hyperplane divides a CAT(0) cube complex into exactly two connected components which are both convex.

Say a hyperplane $Y \subset X$ is *essential* if, for any $R > 0$, none of the two components of $X \setminus Y$ is contained in the R-neighborhood of Y. Note that if X is proper and hyperbolic, then any pair of distinct points $x, y \in \partial X$ is separated by an essential hyperplane. If we also assume that X supports a geometric action of a group G, then the action of the stabilizer H of a hyperplane Y is always cocompact.

6.4.2. *Special actions.* Haglund and Wise have defined a particular class of non-positively curved cube complexes named *special*, see [HW] for the precise definition and a proper introduction to the subject. They enjoy two properties which will be of interest for the present work.

Say a hyperbolic group G has a *special action* if it acts cellularly and geometrically on a CAT(0) cube complex X such that X/G is special. In this case,

(1) the group G splits over the stabilizer of any essential hyperplane;
(2) the group G has the QCERF property [HW, Thm 1.3]; see § 6.5 for the definition.

Following Wise [Wis, Ago], define the class $\mathcal{QVH}$ as the smallest class of hyperbolic groups that contains the trivial group $\{1\}$ and is closed under the following operations:

– if $G = A \star_C B$ with $A, B \in \mathcal{QVH}$ and C quasiconvex in G then $G \in \mathcal{QVH}$;
– if $G = A\star_C$ with $A \in \mathcal{QVH}$ and C quasiconvex in G then $G \in \mathcal{QVH}$;
– if $H < G$ with $H \in \mathcal{QVH}$ and $[G : H] < \infty$ then $G \in \mathcal{QVH}$.

A group in $\mathcal{QVH}$ is said to have a *quasiconvex virtual hierarchy.*

Improving on the work of Wise [Wis], Agol, Groves and Manning proved that a hyperbolic group has a quasiconvex virtual hierarchy if and only if it is virtually special [Ago, Thm. A.42]. Moreover, Agol proved that any cubulated hyperbolic group admits a finite index subgroup with a special action [Ago, Thm. 1.1]. In summary, we have

Theorem 6.15. *Let G be a hyperbolic group. The following are equivalent*

– *G is cubulated;*
– *G is virtually special;*
– *G admits a quasiconvex virtual hierarchy.*

We record the following application.

Lemma 6.16. *If G is a convex-cocompact Kleinian group, then G is virtually special.*

Proof. By Brooks' theorem [Bro], we may assume that it is a quasiconvex subroup of a cocompact Kleinian group. By [BW] and Theorem 6.15, the latter is virtually special so [Hag, Thm H] implies that G is as well. ■

6.5. **The QCERF property.** A group G satisfies the *QCERF property* if every quasiconvex subgroup $A < G$ is *separable* i.e., for any $g \in G \setminus A$, there exists a finite index subroup of G which contains A but not g. This property will provide us with regular JSJ decompositions (Theorem 6.18). The QCERF property holds for virtually special hyperbolic groups [HW, Thm 1.3 and Lma 7.5].

6.5.1. *Strong accessibility.* Let G be a non-elementary hyperbolic group. If it is not one-ended then it splits over a finite group [Sta1]. By [Dun], there is a quasiconvex splitting over finite groups such that each vertex group is finite or one-ended; when G is torsion-free, it leads to a free product of a free group with finitely many one-ended groups. We may then consider the JSJ-decomposition of the remaining one-ended vertex subgroups and proceed inductively. If G has no element of order two, then this process stops in finite time [DP]. In the end, we are left with finite groups, virtually Fuchsian groups and/or one-ended hyperbolic groups with no local cut points in their boundaries. If G has planar boundary, then those latter groups are carpet groups [KK, Thm 4].

6.5.2. *A criterion for the QCERF property.* This section is devoted to the proof of the following proposition.

Proposition 6.17. *Let G be a nonelementary hyperbolic group with planar boundary different from the sphere. If* $\mathrm{confdim}_{AR} G < 2$ *or if G has no elements of order two and if* $\mathrm{confdim}_{AR} H < 2$ *for all carpet quasiconvex subgroups H, then G is QCERF.*

Proof. By [HW, Thm 1.3 and Lma 7.5], a group is QCERF provided it is virtually special. According to Theorem 6.15, it suffices to prove that G has quasiconvex virtual hierarchy.

Let us first assume that $\mathrm{confdim}_{AR} G < 2$. Let us split G over finite groups so that every factor is elementary or one-ended. Then G will have a quasiconvex virtual hierarchy provided every rigid group arising in the JSJ-decomposition is also in $\mathcal{QVH}$. But such a group has a planar action by Proposition 6.7 so Theorem 1.7 implies that it is virtually Kleinian and Lemma 6.16 enables us to conclude that G is virtually special.

We now assume that G has no elements of order 2. From the strong accessibility of such groups (§ 6.5.1), we just have to deal with carpet groups. Assuming their conformal dimension is strictly less than two, Corollary 1.9 implies that they are virtually isomorphic to convex-cocompact Kleinian groups so are virtually special according to Lemma 6.16. ■

6.5.3. *The QCERF property and regular JSJ decompositions.* We may now prove the following:

Theorem 6.18. *Let G be a non-elementary hyperbolic group which is QCERF. There is a normal finite index torsion-free subgroup $H < G$ such that all the one-ended groups arising from its strong accessibility have regular JSJ decompositions.*

The key point is the following group-theoretic result.

Proposition 6.19. *Let $A' < A < G$ be groups with $[A : A'] < \infty$ and A' separable in G. Then there exist subgroups A'' and H with the following properties:*

1. *H is a normal subgroup of finite index in G;*
2. *$A'' = H \cap A'$ is a normal subgroup of finite index in A ;*
3. *for all $g \in G$, $(gAg^{-1}) \cap H = gA''g^{-1}$.*

Proof. Since A' has finite index in A, we may pick a set of representatives $\{a_0, \ldots, a_n\}$ of A/A' with a_0 the neutral element. Since A' is separable, there is some finite index subgroup $G_j < G$ which contains A' but not a_j for all $j \in \{1, \ldots, n\}$. Note that this implies that $a_j A' \cap G_j = \emptyset$.

Let $G' = \cap G_j$. Then G' is a finite index subgroup of G with the property that $G' \cap A = A'$. Let H be the largest finite index subgroup of G' normal in G and set $A'' = A \cap H$.

It follows that H is a normal subgroup of finite index in G and that A'' is a normal subgroup of finite index in A. Note that since $A \cap G' = A'$ we also have $A'' = H \cap A'$. Since H is a normal subgroup, it follows that, for all $g \in G$, $(gAg^{-1}) \cap H = g(A \cap H)g^{-1} = gA''g^{-1}$. ■

We include a proof of the following folklore result:

Corollary 6.20. *A hyperbolic group with the QCERF property is virtually torsion-free.*

Proof. A hyperbolic group has finitely many conjugacy classes of torsion elements [GdlH, Prop. 4.13]. Let $\{g_1, \ldots, g_n\}$ be a set of representatives. Set A_j to be the cyclic subgroup generated by g_j and apply Proposition 6.19 to $\{e\} < A_j < G$ to obtain G_j and set $G' = \cap G_j$. We let the reader check that G' is a finite index torsion-free subgroup of G. ■

Proof. (Thm 6.18) Since G is hyperbolic and QCERF, it contains a finite index subgroup H' which is torsion-free (and QCERF) by Corollary 6.20. It follows that its action on ∂G is faithful. We now use the strong accessibility to decompose H' until we obtain free groups, Fuchsian groups and one-ended subgroups with no local cut points: let us first write H'

as a free product of one-ended groups with a free group and let us denote by $\mathcal{H}'_1$ the set of vertices of this decomposition. To each non-Fuchsian one-ended subgroup, we consider its JSJ-decomposition; let $\mathcal{H}'_2$ be the collection of groups obtained as vertices and edges of the graph of groups describing those elements of $\mathcal{H}'_1$. We proceed inductively so that $\mathcal{H}'_{2n+1}$ denotes the factors of the decompositon as a free product by a free group with one-ended groups of the elements of $\mathcal{H}'_{2n}$ and $\mathcal{H}'_{2n+2}$ consists of the vertices and edges obtained by the JSJ-decomposition of the non-Fuchsian one-ended elements of $\mathcal{H}'_{2n+1}$.

By [Vav], there is some N such that every element of $\mathcal{H}_N$ is either free, Fuchsian or one-ended without any local cut point on its boundary.

Let $1 \leq 2n+1 \leq N$ and pick $K \in \mathcal{H}_{2n+1}$; each vertex group $K_v \in \mathcal{H}_{2n+2}$ of elementary type coming from its JSJ decomposition contains a cyclic subgroup A_v of finite index which stabilizes all the components of $\partial K \setminus \Lambda_v$. Note that, for any $g \in G$, gA_vg^{-1} satisfy the same conditions at the vertex $g(v)$.

We now apply Proposition 6.19 to each triple (A_v, K_v, H') to obtain finitely many finite index subgroups of H' and we let H denote their intersection.

By construction, H is normal and of finite index in H'. Note that a virtually free torsion-free group is free [Sta2] so that we only need to check the condition on the elementary type vertices and edges. Let $K' \in \mathcal{H}'_{2n+1}$ and let T_K denote the Bass-Serre tree given by its JSJ-decomposition. Then $K = H \cap K'$ acts on T_K and T_K/K provides its JSJ-decomposition. For each vertex v, we have $K_v = K'_v \cap H$. Therefore, Proposition 6.19 ensures that K_v is a subgroup of A_v. Moreover, each edge group K_e is cyclic since it is torsion-free and it fixes the components of $\partial K \setminus \Lambda_e$ since two adjacent vertices are not of the same type and if it is incident to a vertex of elementary type, then it already fixes the complementary components. This garantees that K admits a regular JSJ decomposition. ■

6.6. **Dynamical characterization of groups with planar boundaries.** We first notice that a free group is the fundamental group of a handlebody or a solid torus if its rank is one, so it can always be uniformized by a convex-cocompact Kleinian group.

We start proving Theorem 1.1.

Proof. (Theorem 1.1) Let G be a non-elementary hyperbolic group with planar boundary of Ahlfors regular conformal dimension strictly less than two. According to Proposition 6.17, G is QCERF and so Theorem 6.18 implies that it contains a torsion-free subgroup H of finite index such that the JSJ-decomposition of its one-ended subgroups are regular.

We construct a compact Haken manifold with fundamental group isomorphic to H as follows.

Let us first write $H = H_0 \star H_1 \star \ldots \star H_n$ as a free product of a free group H_0 and of one-ended subgroups. Now, each non-elementary H_j has conformal dimension strictly less than 2. Either H_j is free or with boundary isomorphic to the circle or H_j satisfies the assumptions of Proposition 6.14. In either case, H_j is isomorphic to the fundamental group of a hyperbolizable manifold M_j with boundary. We may then glue these manifolds along disks on their boundaries to obtain a Haken manifold M with fundamental group isomorphic to H. Theorem 6.1 proves that M is hyperbolizable, so that, up to index two, H is isomorphic to a convex-cocompact Kleinian group. ■

Remark 6.21. The proof only uses that the Ahlfors regular onformal dimension of rigid type vertices in the JSJ-decomposition of the subgroups H_j is strictly less than two.

We now turn to Theorem 1.2; we just prove (3) implies (1).

Proof. (Theorem 1.2) Let G be a non-elementary hyperbolic group with planar boundary, no elements of order 2 and with carpet quasiconvex subgroups with conformal dimension strictly less than two. Proposition 6.17 and Theorem 6.18 provide us with a torsion-free finite index subgroup H of G such that each JSJ decomposition appearing in its hierarchy is regular. This group H acts faithfully on ∂G.

We will prove that H is isomorphic to the fundamental group of a hyperbolizable manifold by induction on its hierarchy. The initial cases $n = 1, 2$ follow along the same lines as Theorem 1.1.

Assuming Theorem 1.2 holds up to rank $2n$, we show how to deal with a group H with $2n+2$ generations. Note that each element of $\mathcal{H}_2$ has rank $2n$, so they are isomorphic to fundamental groups of compact Haken manifolds and virtually isomorphic to convex-cocompact Kleinian groups. It follows that their Ahlfors regular conformal dimension is strictly less than 2. Therefore, the same argument as above shows that H is also a virtually convex-cocompact Kleinian group. ■

6.7. **Cubulated groups with planar boundary.** We will deduce Theorem 1.4 from the following:

Theorem 6.22. *Let G be a quasiconvex subgroup of a hyperbolic cubulated group $\widehat{G}$ with boundary homeomorphic to the two-sphere. Then G is virtually a convex-cocompact Kleinian group.*

We start by preparing the ambient group $\widehat{G}$:

Proposition 6.23. *Let G be a cubulated hyperbolic group with boundary homeomorphic to the two-sphere. There exists a finite index torsion-free subgoup which admits a special action on a cube complex such that every hyperplane stabilizer is isomorphic to a cocompact Fuchsian group.*

Proof. Let X be a CAT(0) cube complex on which G acts geometrically.

Let $x, y \in \partial X$ be two distinct points. We first prove that we may separate them with the limit set of a cocompact Fuchsian subgroup of G. We start with a hyperplane of X with stabilizer H which separates x and y. Considering a quasiconvex subgroup, we may assume that H is one-ended. It follows that x and y lie in two different components of $\partial X \setminus \Lambda_H$. Note that the action of H on ∂X is planar. Therefore, by Proposition 6.4, there is a Fuchsian group which separates x and y.

By [BW], G acts on a CAT(0) cube complex Y such that the stabilizer of each hyperplane is virtually Fuchsian. By [Ago, Thm 1.1] and Corollary 6.20, there is a finite index torsion-free subgroup of G which has a special action on X. The stabilizers of the hyperplanes in this subroup are now Fuchsian. ■

Proof. (Thm 6.22) The proof will proceed by induction: we may start with $G < \widehat{G}$ where $\widehat{G}$ is torsion-free and admits a special action on a CAT(0) cube complex $\widehat{X}$, cf. Proposition 6.23. Since G is quasiconvex, its action on X is convex-cocompact according to [Hag, Thm. H]: there is a convex subcomplex $X \subset \widehat{X}$ invariant by G with X/G is compact.

Let $Y \subset X$ be an essential hyperplane of X. Set $\mathcal{Y} = \cup_{g\in G} g(Y)$. We define a graph T as follows: the vertices are the connected components of $X \setminus \mathcal{Y}$ and two vertices form an edge if they are separated by exactly one hyperplane $g(Y)$ for some $g \in G$. Since G has a special action, it follows that T is a tree and that the action of G on T is simplicial, minimal and without edge inversions, cf. [HW]. If we let C be the stabilizer of Y, then we have shown that G is either an amalgamated product $G = A \star_C B$ or an HNN extension $G = A \star_C$, where A and B are stabilizers of components $X \setminus \mathcal{Y}$: their action is also convex-cocompact.

If no vertex group contains a carpet group, then Corollary 1.3 shows that these vertex groups are Kleinian, hence their Ahlfors regular conformal dimension is strictly less than two. We then apply the following proposition, the proof of which is postponed later on:

Proposition 6.24. *With the notation above, if the Ahlfors regular conformal dimension of every vertex group is strictly less than two, then G is conjugate to a convex-cocompact Kleinian group.*

This ends the proof in this case. Otherwise, each vertex group is convex-cocompact in X so we may find a convex subcomplex invariant by the vertex group, and proceed as above until their vertex groups contain no carpet group. This process ends since the action of G is special, so G admits a quasiconvex hierarchy. We obtain a rooted finite tree of quasiconvex subgroups where children of a vertex correspond to a splitting of it. So, we may apply, as above, Corollary 1.3 to the leaves and inductively Proposition 6.24 in order to reconstruct the whole group G. ■

Proof. (Prop. 6.24) We first treat the case $G = A \star_C B$. It follows from Corollary 1.12 that A and B are conjugate to convex-cocompact Kleinian groups.

The hyperplane Y is defined by an orthogonal edge $e \in X^{(1)}$, cf. § 6.4.1. This is also an edge of $\widehat{X}$ so it defines a hyperplane $\widehat{Y} \subset \widehat{X}$. It follows that $Y = \widehat{Y} \cap X$ so $\mathrm{Stab}_G Y = \mathrm{Stab}_{\widehat{G}} \widehat{Y} \cap G$ and C is a quasiconvex subroup of a cocompact Fuchsian group $\widehat{C}$. Moreover, $\widehat{Y}$ is clearly inessential with respect to A and B so we may name the connected components of $\partial\widehat{X} \setminus \Lambda_{\widehat{C}}$ D_A and D_B so that $D_A \cap \Lambda_A = \emptyset$ and $D_B \cap \Lambda_B = \emptyset$. Moreover, it follows from [MT, Cor. 4.6] that the action of C on $\partial\widehat{X}$ is globally conjugate to a convex-cocompact Fuchsian group. Hence there is an equivariant involution $\iota_C : \partial\widehat{X} \to \partial\widehat{X}$ which fixes $\Lambda_{\widehat{C}}$ pointwise and exchanges its complementary components D_A and D_B.

It follows that $(\overline{D_A} \setminus \Lambda_C)/C$ is a subsurface of a boundary component of the Kleinian manifold M_A, $(\overline{D_B} \setminus \Lambda_C)/C$ is also a subsurface of a boundary component of the Kleinian manifold M_B and ι_C induces an orientation reversing homeomorphism between them. We may then define $M_G = M_A \sqcup_{\iota_C} M_B$: this is a Haken manifold with fundamental group isomorphic to G [SW]. Therefore, Theorem 6.1 implies that G is Kleinian and Corollary 1.12 that the actions are conjugate.

If $G = A \star_C$, the proof is similar: there is some element $g_0 \in G \setminus A$ such that the HNN extension is obtained by identifying C with $C' = g_0 C g_0^{-1}$. Since the action is special, C and C' will be contained in cocompact Fuchsian groups $\widehat{C}$ and $\widehat{C}'$ which bound disjoint disks D_C and $D_{C'}$ disjoint from Λ_A. We may then glue the compact surface $(\overline{D_C} \setminus \Lambda_C)/C$ with $(\overline{D_{C'}} \setminus \Lambda_{C'})/C'$, both contained in the boundary of M_A, to obtain a Haken manifold with fundamental group isomorphic to G. ■

We now prove Theorem 1.4 and its corollaries.

Proof. (Thm 1.4) The necessity comes from Lemma 6.16. For the suffciency, Theorem 1.2 tells us that we just need to deal with groups with boundary a carpet or a sphere. For the latter, Theorem 6.22 shows that the group is virtually Kleinian.

For the carpet case, let G be a carpet group and let $H_1, \ldots, H_k$ denote representatives of the peripheral Fuchsian groups, cf. Proposition 6.4. Let us take another copy $(G', H_1', \ldots, H_k')$ of $(G, H_1, \ldots, H_k)$ and consider the graph of groups with vertices G and G' and with k edges identifying each H_j with H_j'. According to [KK, Thm 5], one obtains a hyperbolic group $\widehat{G}$ with boundary the sphere so that G has become a quasiconvex subgroup of $\widehat{G}$.

Since G is cubulated, it follows from Theorem 6.15 that G admits a quasiconvex hierarchy, so $\widehat{G}$ as well and we may conclude that $\widehat{G}$ is cubulated.

We may now apply Theorem 6.22 and conclude that G is a virtually convex-compact Kleinian group. ■

Proof. (Corollaries 1.5 and **??**) The Ahlfors regular conformal gauge of a group is the same for its finite index subgroups. Therefore, if G is cubulated, Theorem 1.4 implies that the group is virtually Kleinian, so either its Ahlfors regular conformal dimension is strictly less than two or it is attained. In the former case, we conclude with Corollary 1.9 and in the latter case, with [BK3]. ■

References

[AM] William Abikoff and Bernard Maskit. Geometric decompositions of Kleinian groups. *Amer. J. Math.* **99**(1977), 687–697.

[Ago] Ian Agol. The virtual Haken conjecture. With an appendix by Ian Agol, Daniel Groves, Jason Manning, 2012.

[Ahl] Lars V. Ahlfors. *Lectures on quasiconformal mappings.* Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10. D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966.

[And] James W. Anderson. Closed essential surfaces in hyperbolizable acylindrical 3-manifolds. *Pacific J. Math.* **183**(1998), 1–19.

[BW] Nicolas Bergeron and Daniel T. Wise. A boundary criterion for cubulation. American Math. J., to appear., 2012.

[Bon1] Mario Bonk. Quasiconformal geometry of fractals. In *International Congress of Mathematicians. Vol. II*, pages 1349–1373. Eur. Math. Soc., Zürich, 2006.

[Bon2] Mario Bonk. Uniformization of Sierpiński carpets in the plane. *Invent. Math.* **186**(2011), 559–665.

[BK1] Mario Bonk and Bruce Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.* **150**(2002), 127–183.

[BK2] Mario Bonk and Bruce Kleiner. Rigidity for quasi-Möbius group actions. *J. Differential Geom.* **61**(2002), 81–106.

[BK3] Mario Bonk and Bruce Kleiner. Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary. *Geom. Topol.* **9**(2005), 219–246 (electronic).

[BKM] Mario Bonk, Bruce Kleiner, and Sergei Merenkov. Rigidity of Schottky sets. *Amer. J. Math.* **131**(2009), 409–443.

[BS] Mario Bonk and Oded Schramm. Embeddings of Gromov hyperbolic spaces. *Geom. Funct. Anal.* **10**(2000), 266–306.

[Bow1] Brian H. Bowditch. Cut points and canonical splittings of hyperbolic groups. *Acta Math.* **180**(1998), 145–186.

[Bow2] Brian H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.

[Bow3] Brian H. Bowditch. Relatively hyperbolic groups. Preprint, 2011.

[BH] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999.

[Bro] Robert Brooks. Circle packings and co-compact extensions of Kleinian groups. *Invent. Math.* **86**(1986), 461–469.

[Can] James W. Cannon. The theory of negatively curved spaces and groups. In *Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989)*, Oxford Sci. Publ., pages 315–369. Oxford Univ. Press, New York, 1991.

[Car] Matías Carrasco. *Jauge conforme des espaces métriques compacts.* PhD thesis, Aix-Marseille University, 2011. Available at `http://tel.archives-ouvertes.fr/tel-00645284`.

[CJ] Andrew Casson and Douglas Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.* **118**(1994), 441–456.

[Dav] Robert J. Daverman. *Decompositions of manifolds*, volume 124 of *Pure and Applied Mathematics.* Academic Press Inc., Orlando, FL, 1986.

[DP] Thomas Delzant and Leonid Potyagailo. Accessibilité hiérarchique des groupes de présentation finie. *Topology* **40**(2001), 617–629.

[Dun] Martin J. Dunwoody. The accessibility of finitely presented groups. *Invent. Math.* **81**(1985), 449–457.

[Gab] David Gabai. Convergence groups are Fuchsian groups. *Ann. of Math. (2)* **136**(1992), 447–510.

[GdlH] Étienne Ghys and Pierre de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*, volume 83 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[Gro] Mikhael Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Hag] Frédéric Haglund. Finite index subgroups of graph products. *Geom. Dedicata* **135**(2008), 167–209.

[HW] Frédéric Haglund and Daniel T. Wise. Special cube complexes. *Geom. Funct. Anal.* **17**(2008), 1551–1620.

[Haï1] Peter Haïssinsky. Géométrie quasiconforme, analyse au bord des espaces métriques hyperboliques et rigidités [d'après Mostow, Pansu, Bourdon, Pajot, Bonk, Kleiner. . .]. *Astérisque* **326**(2009), Exp. No. 993, ix, 321–362 (2010). Séminaire Bourbaki. Vol. 2007/2008.

[Haï2] Peter Haïssinsky. A sewing problem in metric spaces. *Ann. Acad. Sci. Fenn. Math.* **34**(2009), 319–345.

[HP] Peter Haïssinsky and Kevin M. Pilgrim. Coarse expanding conformal dynamics. *Astérisque* **325**(2009), viii+139 pp. (2010).

[Hei] Juha Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[Hin1] Aimo Hinkkanen. Uniformly quasisymmetric groups. *Proc. London Math. Soc. (3)* **51**(1985), 318–338.

[Hin2] Aimo Hinkkanen. The structure of certain quasisymmetric groups. *Mem. Amer. Math. Soc.* **83**(1990), iv+87.

[KB] Ilya Kapovich and Nadia Benakli. Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 39–93. Amer. Math. Soc., Providence, RI, 2002.

[KK] Michael Kapovich and Bruce Kleiner. Hyperbolic groups with low-dimensional boundary. *Ann. Sci. École Norm. Sup. (4)* **33**(2000), 647–669.

[MT] John M. Mackay and Jeremy T. Tyson. *Conformal dimension*, volume 54 of *University Lecture Series.* American Mathematical Society, Providence, RI, 2010. Theory and application.

[Mad] Albert Marden. *Outer circles.* Cambridge University Press, Cambridge, 2007. An introduction to hyperbolic 3-manifolds.

[Mak1] Vladimir Markovic. Quasisymmetric groups. *J. Amer. Math. Soc.* **19**(2006), 673–715 (electronic).

[Mak2] Vladimir Markovic. Criterion for Cannon's conjecture. `arXiv:1205.5747`, 2012.

[MS] Gaven J. Martin and Richard K. Skora. Group actions of the 2-sphere. *Amer. J. Math.* **111**(1989), 387–402.

[MT] Gaven J. Martin and Pekka Tukia. Convergence and Möbius groups. In *Holomorphic functions and moduli, Vol. II (Berkeley, CA, 1986)*, volume 11 of *Math. Sci. Res. Inst. Publ.*, pages 113–140. Springer, New York, 1988.

[Mas] Bernard Maskit. A theorem on planar covering surfaces with applications to 3-manifolds. *Ann. of Math. (2)* **81**(1965), 341–355.

[Mat] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[McM1] Curt McMullen. Iteration on Teichmüller space. *Invent. math.* **99**(1990), 425–454.

[McM2] Curtis T. McMullen. The classification of conformal dynamical systems. In *Current developments in mathematics, 1995 (Cambridge, MA)*, pages 323–360. Internat. Press, Cambridge, MA, 1994.

[Mor] John Morgan. On Thurston's uniformization theorem for three dimensional manifolds. In *The Smith Conjecture*, pages 37–125. Academic Press, 1984.

[Sag] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)* **71**(1995), 585–617.

[SW] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.

[Sta1] John Stallings. *Group theory and three-dimensional manifolds.* Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1969, Yale Mathematical Monographs, 4.

[Sta2] John R. Stallings. On torsion-free groups with infinitely many ends. *Ann. of Math. (2)* **88**(1968), 312–334.

[Sul1] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.* **50**(1979), 171–202.

[Sul2] Dennis Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 465–496, Princeton, N.J., 1981. Princeton Univ. Press.

[Thu] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)* **6**(1982), 357–381.

[Väi] Jussi Väisälä. Quasi-Möbius maps. *J. Analyse Math.* **44**(1984/85), 218–234.

[Vav] Diane M. Vavrichek. Strong accessibility for hyperbolic groups. *Algebr. Geom. Topol.* **8**(2008), 1459–1479.

[Why1] Gordon Thomas Whyburn. Topological characterization of the Sierpiński curve. *Fund. Math.* **45**(1958), 320–324.

[Why2] Gordon Thomas Whyburn. *Analytic topology.* American Mathematical Society Colloquium Publications, Vol. XXVIII. American Mathematical Society, Providence, R.I., 1963.

[Wis] Daniel T. Wise. The structure of groups with a quasiconvex hierarchy. preprint, 2011.

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