



HAL
open science

Classical plate buckling theory as the small-thickness limit of three-dimensional incremental elasticity

David Steigmann, Raymond W. Ogden

► **To cite this version:**

David Steigmann, Raymond W. Ogden. Classical plate buckling theory as the small-thickness limit of three-dimensional incremental elasticity. *Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 2012, - (-), pp.1-14. hal-00773268

HAL Id: hal-00773268

<https://hal.science/hal-00773268>

Submitted on 12 Jan 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Classical plate buckling theory as the small-thickness limit of three-dimensional incremental elasticity

David J. Steigmann¹ and Ray W. Ogden^{2,3}

¹Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

²School of Engineering, University of Aberdeen, Aberdeen AB24 3UE, UK

³School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QW, UK

Abstract

Classical plate buckling theory is obtained systematically as the small-thickness limit of the three-dimensional linear theory of incremental elasticity with null incremental data. Various *a priori* assumptions associated with classical treatments of plate buckling, including the Kirchhoff–Love hypothesis, are here *derived* rather than imposed, and the conditions under which they emerge are stated precisely.

Keywords: Plate buckling; Incremental elasticity; Elastic stability

1 Introduction

In this work we study the relationship between classical plate-buckling theory and the linear three-dimensional theory of incremental elasticity. We show that the former model, despite the seemingly *ad hoc* assumptions underpinning its foundations [1, 2], emerges naturally as the limit of the latter if the thickness is sufficiently small. Specifically, the classical theory furnishes the leading order model in the small-thickness limit. The methods we employ, while entirely elementary, deliver precisely the same results as would be obtained by using the method of gamma convergence, which has become a popular framework for effecting the so-called dimension reduction procedure in static elasticity theory.

In the engineering literature classical plate-buckling theory is derived on the basis of the following *ad hoc* assumptions, adapted from pages 137 and 138 of [1]:

- The state of stress is approximately plane and parallel to the middle surface.
- Normals to the undeformed middle surface remain normal to the deformed middle surface.
- The fundamental state is linear, i.e. the underlying deformation of the plate prior to buckling may be described using linear elasticity theory.

Our development parallels that of [2], which is based on the idea of minimizing the second variation of the energy at equilibrium, yielding linear incremental elasticity with null incremental data. The contribution of the present work is the relaxation of the *a priori* assumptions and hence the strengthening of the theoretical underpinnings of the theory. Here the classical assumptions are derived as *consequences* of the three-dimensional theory in the small-thickness limit. The only important hypotheses we make concern the regularity of solutions of the three-dimensional incremental elasticity problem in the presence of strong ellipticity, together with the stipulation that the pre-stress vanishes with plate thickness. Some discussion of the former is provided in Section 4; the latter is introduced there to ensure that the considered model furnishes a meaningful minimization problem.

Prerequisite material on nonlinear elasticity theory is summarized in Section 2. This includes a review of the connection between stability and uniqueness of equilibria in the dead-load problem. The linear incremental theory, which provides the basis for the study of bifurcation of equilibria, is outlined in Section 3, and the small-thickness limit of the theory is developed in detail in Section 4, which constitutes the main part of the paper.

2 Background theory

The standard equilibrium boundary-value problem in the absence of body forces is given by

$$\text{Div} \mathbf{P} = \mathbf{0} \quad \text{in } \kappa \quad \text{with} \quad \mathbf{P} \mathbf{n} = \mathbf{t} \quad \text{on} \quad \partial \kappa_t \quad \text{and} \quad \boldsymbol{\chi} = \boldsymbol{\phi} \quad \text{on} \quad \partial \kappa_\phi, \quad (1)$$

where κ is a fixed reference configuration with piecewise smooth boundary $\partial \kappa = \partial \kappa_t \cup \partial \kappa_\phi$, \mathbf{n} is the unit outward normal to the boundary, Div is the divergence operation based on position $\mathbf{x} \in \kappa$, $\boldsymbol{\chi}(\mathbf{x})$ is the deformation function, yielding the position $\mathbf{y} = \boldsymbol{\chi}(\mathbf{x})$ of the material point \mathbf{x} after deformation, and \mathbf{t} and $\boldsymbol{\phi}$ are specified functions. Here we use the convention that the divergence operates on the second index when applied to a second-order tensor; for example, in Cartesian components $(\text{Div} \mathbf{P})_i = \partial P_{i\alpha} / \partial x_\alpha$. Also, \mathbf{P} is the first Piola–Kirchhoff stress tensor, which is given by

$$\mathbf{P} = \mathcal{W}_{\mathbf{F}}, \quad (2)$$

where $\mathcal{W}(\mathbf{F})$ is the strain-energy function (per unit volume), \mathbf{F} is the deformation gradient tensor given by $\mathbf{F} = \text{D} \boldsymbol{\chi}$ and D is the gradient operation based on \mathbf{x} . We suppose the reference configuration to be one that could in principle be occupied by the body and thus impose $\det \mathbf{F} > 0$. Bold subscripts are used to denote the gradients of the strain energy with respect to the indicated tensor, as in (2). Another example is the tensor of elastic moduli, denoted \mathcal{M} , defined as the second gradient of \mathcal{W} , i.e.

$$\mathcal{M} = \mathcal{W}_{\mathbf{F}\mathbf{F}}, \quad (3)$$

which figures prominently in this work.

The second Piola–Kirchhoff stress, denoted \mathbf{S} , is defined in terms of the first Piola–Kirchhoff stress by

$$\mathbf{P} = \mathbf{F} \mathbf{S}. \quad (4)$$

We satisfy the moment-of-momentum balance identically by taking the strain-energy function to be invariant under superposed rigid motions. This ensures the symmetry of \mathbf{S} , which is then given by

$$\mathbf{S} = \mathcal{U}_{\mathbf{E}}, \quad (5)$$

where

$$\mathcal{U}(\mathbf{E}) = \mathcal{W}(\mathbf{F}) \quad (6)$$

and $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$ is the Green–Lagrange strain tensor.

The considerations of this paper involve smooth deformations. In particular, we exclude non-smooth deformations of shear-band type that entail discontinuities in the gradient of \mathbf{F} associated with loss of ellipticity [4]. Accordingly, we limit attention to deformations which are such as to satisfy the strong-ellipticity condition

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathcal{M}(\mathbf{F})[\mathbf{a} \otimes \mathbf{b}] > 0 \quad \text{for all } \mathbf{a} \otimes \mathbf{b} \neq \mathbf{0}. \quad (7)$$

Here and elsewhere we use the notation $\mathcal{L}[\mathbf{A}]$, where \mathcal{L} and \mathbf{A} are fourth- and second-order tensors respectively, to represent the second-order tensor with Cartesian components $\mathcal{L}_{ijkl}A_{kl}$. The tensor \mathcal{L} is said to possess major symmetry if $\mathcal{L}^T = \mathcal{L}$, where the transpose \mathcal{L}^T is defined by $\mathbf{B} \cdot \mathcal{L}^T[\mathbf{A}] = \mathbf{A} \cdot \mathcal{L}[\mathbf{B}]$, and minor symmetry if $\mathbf{A} \cdot \mathcal{L}[\mathbf{B}] = \mathbf{A}^T \cdot \mathcal{L}[\mathbf{B}]$ and $\mathbf{A} \cdot \mathcal{L}[\mathbf{B}] = \mathbf{A} \cdot \mathcal{L}[\mathbf{B}^T]$. For example, \mathcal{M} possesses major symmetry but not minor symmetry. We also use the Euclidean inner product of second-order tensors; for example, for second-order tensors \mathbf{A} and \mathbf{B} , this is defined by $\mathbf{A} \cdot \mathbf{B} = \text{tr}(\mathbf{A}\mathbf{B}^T)$. The norm of a second-order tensor \mathbf{A} , denoted $|\mathbf{A}|$, is $\sqrt{\mathbf{A} \cdot \mathbf{A}}$. The dot notation is also used for the conventional scalar product of two vectors.

The present boundary-value problem is of mixed type, involving position and traction data on complementary parts of the boundary. The simplest among these, to which attention is here confined, entails the assignment of *dead*, i.e. configuration-independent, tractions. Thus, in (1) \mathbf{t} and ϕ are to be regarded as assigned functions of \mathbf{x} .

In these circumstances the potential energy of the body, consisting of the strain energy and the load potential, is the functional of $\chi(\mathbf{x})$ given by

$$\mathcal{E}[\chi] = \int_{\kappa} \mathcal{W}(\mathbf{F})dv - \int_{\partial\kappa_t} \mathbf{t} \cdot \chi da. \quad (8)$$

It is well known (see, for example, [4]) that deformations satisfying (1) render \mathcal{E} stationary, and conversely. This of course is the virtual-work principle, specialized to the present position/dead-load problem. According to the energy criterion of elastic stability [5], the deformation χ is stable if and only if it minimizes the energy relative to kinematically admissible alternatives, that is if and only if

$$\mathcal{E}[\chi] < \mathcal{E}[\chi + \Delta\chi] \quad (9)$$

for all $\Delta\chi(\mathbf{x})$ vanishing on $\partial\kappa_\phi$ but not vanishing identically in κ . This implies that $\mathcal{E}[\cdot]$ is stationary at χ and hence that the latter is equilibrated, i.e. it solves the boundary-value problem (1).

As is well known, (9) also implies that solutions of (1) are unique. To prove this claim we adapt an argument of Hill [6] and suppose that $\boldsymbol{\chi}_1(\mathbf{x})$ and $\boldsymbol{\chi}_2(\mathbf{x})$ are two solutions of the boundary-value problem. If both are strict minimizers of the energy, then, by selecting $\boldsymbol{\chi}$ and $\Delta\boldsymbol{\chi}$ appropriately it follows that

$$\int_{\kappa} [\mathcal{W}(\mathbf{F}_2) - \mathcal{W}(\mathbf{F}_1) - \mathbf{P}_1 \cdot (\mathbf{F}_2 - \mathbf{F}_1)] dv > 0 \quad (10)$$

and

$$\int_{\kappa} [\mathcal{W}(\mathbf{F}_1) - \mathcal{W}(\mathbf{F}_2) - \mathbf{P}_2 \cdot (\mathbf{F}_1 - \mathbf{F}_2)] dv > 0 \quad (11)$$

provided that $\mathbf{F}_2 \neq \mathbf{F}_1$, where $\mathbf{F}_i = D\boldsymbol{\chi}_i$, $i = 1, 2$, and $\mathbf{P}_i = \mathcal{W}_{\mathbf{F}}(\mathbf{F}_i)$ are the associated stresses. These inequalities follow from the fact that $\boldsymbol{\chi}_i$, $i = 1, 2$, both satisfy (1). Adding the two inequalities above furnishes

$$\int_{\kappa} (\mathbf{P}_2 - \mathbf{P}_1) \cdot (\mathbf{F}_2 - \mathbf{F}_1) dv > 0, \quad \mathbf{F}_2 \neq \mathbf{F}_1. \quad (12)$$

However, $\text{Div}(\mathbf{P}_2 - \mathbf{P}_1) = \mathbf{0}$. Forming the inner product of this with $\boldsymbol{\chi}_2 - \boldsymbol{\chi}_1$, integrating the resulting equation over κ and invoking the boundary conditions, we arrive at

$$\int_{\kappa} (\mathbf{P}_2 - \mathbf{P}_1) \cdot (\mathbf{F}_2 - \mathbf{F}_1) dv = 0, \quad (13)$$

which is reconciled with (12) if and only if $\mathbf{F}_2 = \mathbf{F}_1$; integration then yields $\boldsymbol{\chi}_2 = \boldsymbol{\chi}_1$ apart from a rigid-body translation, which vanishes by virtue of the position data. This proves the claim that stability, in the sense of the energy criterion, implies uniqueness of solution of the mixed position/dead-load problem. Conversely, non-uniqueness implies that equilibria are not stable in the sense of strict inequality in (9).

The deformation $\boldsymbol{\chi}$ is said to be neutrally stable if the inequality in (9) is semi-strict, with equality holding for some admissible $\Delta\boldsymbol{\chi}$. This again implies that $\boldsymbol{\chi}$ is equilibrated, but in this case non-uniqueness is possible because strict inequality in (9) is sufficient for uniqueness of solution of the present equilibrium boundary-value problem. Non-uniqueness of solutions then implies that the strict inequality does not obtain. Accordingly, non-uniqueness signals a failure of stability and thus a potential instability. This is Euler's well-known adjacent-equilibrium criterion of elastic stability, adapted to nonlinear elasticity.

We remark that the energy criterion of elastic stability as stated is heuristic in the sense that no rigorous connection to stability in the dynamical sense is known in the case of continuous systems [5]. Nevertheless, Koiter [7] has given convincing arguments in support of this criterion, the absence of a rigorous foundation for it notwithstanding.

3 Three-dimensional incremental elasticity and the Trefftz–Hill bifurcation criterion

In this section, for the sake of completeness, we summarize the basic theory of incremental elasticity as it relates to bifurcation of equilibria. This theory, together with extensions to

configuration-dependent loading, is discussed in detail in [4].

Let $\boldsymbol{\chi}(\mathbf{x}; \epsilon)$ be a one-parameter family of kinematically possible deformations satisfying fixed position data:

$$\boldsymbol{\chi}(\mathbf{x}; \epsilon) = \boldsymbol{\phi}(\mathbf{x}) \quad \text{on} \quad \partial\kappa_\phi. \quad (14)$$

The potential energy, given by (8), may then be regarded as a function of ϵ , which we write as

$$F(\epsilon) = \mathcal{E}[\boldsymbol{\chi}(\mathbf{x}; \epsilon)]. \quad (15)$$

We assume the deformation corresponding to $\epsilon = 0$ to be equilibrated, and expand F for small ϵ , that is for small displacements from the equilibrium configuration, obtaining

$$F(\epsilon) = F(0) + \epsilon^2 \mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}] + o(\epsilon^2), \quad (16)$$

where

$$\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}] = \frac{1}{2} \ddot{F}, \quad (17)$$

in which the superposed dots stand for derivatives with respect to ϵ , evaluated at $\epsilon = 0$, and $\boldsymbol{\chi}(\mathbf{x}) = \boldsymbol{\chi}(\mathbf{x}; 0)$ is the underlying finite equilibrium deformation. Thus \mathcal{G} is the second variation of the potential energy at the considered equilibrium state. The first variation vanishes by the virtual-work principle, and this fact is reflected in (16).

To compute these variations we proceed from (8) and (15), obtaining

$$F'(\epsilon) = \int_{\kappa} \mathcal{W}_{\mathbf{F}} \cdot \mathbf{D}\boldsymbol{\chi}' dv - \int_{\partial\kappa} \mathbf{t} \cdot \boldsymbol{\chi}' da, \quad (18)$$

where, in this section, primes are used to denote derivatives at any value of ϵ and use has been made of the fact that $\boldsymbol{\chi}'$ vanishes on $\partial\kappa_\phi$ to justify extension of the domain of the integral from $\partial\kappa_t$ to all of $\partial\kappa$. Using (2) we easily reduce this to

$$F'(\epsilon) = \int_{\partial\kappa_t} (\mathbf{P}\mathbf{n} - \mathbf{t}) \cdot \boldsymbol{\chi}' da - \int_{\kappa} \boldsymbol{\chi}' \cdot \text{Div}\mathbf{P} dv, \quad (19)$$

which vanishes at $\epsilon = 0$, as claimed, by virtue of (1). Proceeding to the second derivative we use (18) to derive

$$F''(\epsilon) = \int_{\kappa} \mathcal{W}_{\mathbf{F}} \cdot \mathbf{D}\boldsymbol{\chi}'' dv - \int_{\partial\kappa} \mathbf{t} \cdot \boldsymbol{\chi}'' da + \int_{\kappa} \mathbf{D}\boldsymbol{\chi}' \cdot \mathcal{M}(\mathbf{F})[\mathbf{D}\boldsymbol{\chi}'] dv, \quad (20)$$

in which the first two terms cancel at $\epsilon = 0$ by virtue of the argument leading from (18) to (19).

Evaluating at $\epsilon = 0$ and comparing with (17) yields

$$\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}] = \frac{1}{2} \int_{\kappa} \dot{\mathbf{F}} \cdot \mathcal{M}(\mathbf{F})[\dot{\mathbf{F}}] dv, \quad (21)$$

in which $\dot{\mathbf{F}} = D\dot{\boldsymbol{\chi}}$ is the incremental deformation gradient and $\mathbf{F} = D\boldsymbol{\chi}$ is now the gradient of the underlying finite equilibrium deformation. If the latter is stable, then from (9) it is necessary that

$$0 < [F(\epsilon) - F(0)]/\epsilon^2 = \mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}] + o(\epsilon^2)/\epsilon^2, \quad (22)$$

and passing to the limit yields

$$\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}] > 0. \quad (23)$$

Later, we shall require an expression for the second variation based on the strain-energy function $\mathcal{U}(\mathbf{E})$. This is most easily obtained from the incremental form of (4), namely

$$\dot{\mathbf{P}} = \dot{\mathbf{F}}\mathbf{S} + \mathbf{F}\dot{\mathbf{S}}, \quad (24)$$

where

$$\dot{\mathbf{S}} = \mathbf{C}(\mathbf{E})[\dot{\mathbf{E}}], \quad \mathbf{C} = \mathcal{U}_{\mathbf{E}\mathbf{E}}, \quad (25)$$

and

$$\dot{\mathbf{E}} = \frac{1}{2}(\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}}), \quad (26)$$

the fourth-order tensor \mathbf{C} possessing both major and minor symmetry. When combined with (21), this furnishes an expression for the second variation that renders explicit the role played by the pre-stress $\mathbf{S}(\mathbf{x})$ induced by $\boldsymbol{\chi}(\mathbf{x})$. In particular, we find, using

$$\dot{\mathbf{P}} = \mathcal{M}(\mathbf{F})[\dot{\mathbf{F}}], \quad (27)$$

that

$$\mathcal{M}(\mathbf{F})[\mathbf{A}] = \mathbf{A}\mathbf{S} + \frac{1}{2}\mathbf{F}\mathbf{C}(\mathbf{E})[\mathbf{A}^T\mathbf{F} + \mathbf{F}^T\mathbf{A}] \quad (28)$$

for any (second-order) tensor \mathbf{A} .

In view of the argument following (13), stability is lost in the strict sense if $\boldsymbol{\chi}$ is such that

$$\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}] = 0 \quad (29)$$

for some non-trivial admissible $\dot{\boldsymbol{\chi}}$. To see how this relates to bifurcation of equilibrium, suppose $\boldsymbol{\chi}(\mathbf{x}; \mu)$ is a one-parameter family of *equilibrium* deformations, satisfying the boundary-value problem (1) at all values of μ in some open interval containing zero, say. The equations in (1) can then be differentiated with respect to μ . Evaluating the resulting system at $\mu = 0$ and identifying the associated value of the μ -derivative of $\boldsymbol{\chi}(\mathbf{x}; \mu)$ with $\dot{\boldsymbol{\chi}}$ above, we derive

$$\text{Div } \dot{\mathbf{P}} = \mathbf{0} \quad \text{in } \kappa, \quad \dot{\mathbf{P}}\mathbf{n} = \mathbf{0} \quad \text{on } \partial\kappa_t \quad \text{and} \quad \dot{\boldsymbol{\chi}} = \mathbf{0} \quad \text{on } \partial\kappa_\phi. \quad (30)$$

This is a homogeneous linear boundary-value problem for the incremental bifurcation $\dot{\boldsymbol{\chi}}$. It is also the linearization of the equilibrium boundary-value problem for the deformation $\boldsymbol{\chi} + \dot{\boldsymbol{\chi}}$ in which $\boldsymbol{\chi}$ is equilibrated.

Consider now the variation of the functional \mathcal{G} with respect to $\dot{\boldsymbol{\chi}}$, i.e.

$$\delta\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}, \delta\dot{\boldsymbol{\chi}}] = \frac{1}{2} \int_{\kappa} \{D(\delta\dot{\boldsymbol{\chi}}) \cdot \dot{\mathbf{P}} + \dot{\mathbf{F}} \cdot \boldsymbol{\mathcal{M}}(\mathbf{F})[D(\delta\dot{\boldsymbol{\chi}})]\} dv = \int_{\kappa} D(\delta\dot{\boldsymbol{\chi}}) \cdot \dot{\mathbf{P}} dv, \quad (31)$$

where use has been made of (21), (27) and the major symmetry of $\boldsymbol{\mathcal{M}}$. This may be re-written as

$$\delta\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}, \delta\dot{\boldsymbol{\chi}}] = \int_{\partial\kappa_t} \delta\dot{\boldsymbol{\chi}} \cdot \dot{\mathbf{P}} n da - \int_{\kappa} \delta\dot{\boldsymbol{\chi}} \cdot \text{Div} \dot{\mathbf{P}} dv, \quad (32)$$

which is seen to vanish for any admissible $\delta\dot{\boldsymbol{\chi}}$ if and only if the equations in (30) are satisfied. Accordingly, a deformation $\boldsymbol{\chi}(\mathbf{x})$ is stable only if the inequality

$$\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}] \geq 0 \quad (33)$$

is satisfied in the *strict* sense for all kinematically admissible $\dot{\boldsymbol{\chi}}$, and *potentially* unstable if the equality is satisfied for non-zero $\dot{\boldsymbol{\chi}}$, that is if the linearized bifurcation problem (30) has a non-trivial solution. The latter claim follows from the fact that such $\dot{\boldsymbol{\chi}}$ annul \mathcal{G} ; to see this we use (30) and (32) to obtain $\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}] = \frac{1}{2} \delta\mathcal{G}[\boldsymbol{\chi}, \dot{\boldsymbol{\chi}}, \dot{\boldsymbol{\chi}}]$, which vanishes for all solutions $\dot{\boldsymbol{\chi}}$ to the bifurcation problem. In the present setting this criterion is due to Hill [6]. When this criterion is satisfied, the energy comparison is dominated by the higher-order variations; for example, instability of the underlying finite equilibrium deformation is assured if the third variation can be made non-zero. It is in this sense that a non-trivial bifurcation signals a potential instability.

Following standard practice, we refer to such solutions as buckling modes or eigenmodes [4, 6]. It is not possible to distinguish between neutral stability and instability of the equilibrium deformation $\boldsymbol{\chi}$ on the basis of linearized bifurcation theory; for this it is necessary to consider nonlinear terms. These comprise the subject of post-bifurcation theory [2, 8], which, however, falls outside the scope of the present work.

Henceforth we use the notation

$$\tilde{\mathbf{u}}(\mathbf{x}) = \dot{\boldsymbol{\chi}}(\mathbf{x}), \quad \tilde{\mathbf{H}}(\mathbf{x}) = \dot{\mathbf{F}}(\mathbf{x}) \quad (34)$$

for the incremental displacement and displacement gradient fields, respectively; thus, $\tilde{\mathbf{H}} = D\tilde{\mathbf{u}}$.

4 Two-dimensional model for thin plates

4.1 Plates as thin prismatic bodies

It proves advantageous to parametrize the reference configuration κ in the form

$$\mathbf{x} = \mathbf{r} + \varsigma \mathbf{k}, \quad (35)$$

where $\mathbf{r} \in \omega$ (the plate midsurface), \mathbf{k} is the fixed unit normal to the plate and $\varsigma \in [-h/2, h/2]$; h is the plate thickness, assumed here to be uniform and small in the sense

that $h/l \ll 1$, where l is any other length scale in the problem such as a typical spanwise dimension of ω . The latter is presumed to be simply connected, so that Green's integral theorem may be applied. It simplifies matters to adopt l as the unit of length, i.e. $l = 1$, $h \ll 1$. In other words, we assume all length scales to be non-dimensionlized by l *a priori*, and henceforth regard h as a small dimensionless parameter.

Let

$$\hat{\mathbf{u}}(\mathbf{r}, \varsigma) = \tilde{\mathbf{u}}(\mathbf{r} + \varsigma \mathbf{k}), \quad \hat{\mathbf{H}}(\mathbf{r}, \varsigma) = \tilde{\mathbf{H}}(\mathbf{r} + \varsigma \mathbf{k}), \quad (36)$$

and let $\nabla(\cdot)$ and $(\cdot)'$, respectively, stand for the (two-dimensional) gradient with respect to \mathbf{r} at fixed ς and the derivative $\partial(\cdot)/\partial\varsigma$ at fixed \mathbf{r} . Further, let

$$\mathbf{1} = \mathbf{I} - \mathbf{k} \otimes \mathbf{k}, \quad (37)$$

where \mathbf{I} is the identity for three-space; this is the projection onto the translation (vector) space ω' of ω . Using it we derive (see [9])

$$\tilde{\mathbf{H}}\mathbf{1} = \nabla\hat{\mathbf{u}}, \quad \tilde{\mathbf{H}}\mathbf{k} = \hat{\mathbf{u}}', \quad (38)$$

and the consequent orthogonal decomposition

$$\hat{\mathbf{H}} = \nabla\hat{\mathbf{u}} + \hat{\mathbf{u}}' \otimes \mathbf{k}. \quad (39)$$

The subsequent development requires

$$\mathbf{H}_0 = \nabla\mathbf{u} + \mathbf{a} \otimes \mathbf{k}, \quad \mathbf{H}'_0 = \nabla\mathbf{a} + \mathbf{b} \otimes \mathbf{k}, \quad \mathbf{H}''_0 = \nabla\mathbf{b} + \mathbf{c} \otimes \mathbf{k}, \quad (40)$$

where the latter two are obtained by differentiating (39) with respect to ς , the zero subscript indicates evaluation at $\varsigma = 0$, and

$$\mathbf{u} = \hat{\mathbf{u}}_0, \quad \mathbf{a} = \hat{\mathbf{u}}'_0, \quad \mathbf{b} = \hat{\mathbf{u}}''_0, \quad \mathbf{c} = \hat{\mathbf{u}}'''_0 \quad (41)$$

are the coefficient vectors in the thickness-wise power expansion

$$\hat{\mathbf{u}} = \mathbf{u} + \varsigma\mathbf{a} + \frac{1}{2}\varsigma^2\mathbf{b} + \frac{1}{6}\varsigma^3\mathbf{c} + \dots \quad (42)$$

of the three-dimensional incremental displacement field.

Remark: We have assumed more regularity than has been proved, particularly for the underlying finite-deformation equilibrium problem. It is in this sense that our analysis is formal. Thus we pause to discuss the degree of regularity required here and henceforth. It is well known that in the presence of strong ellipticity, any piecewise C^2 equilibrium deformation, suffering a potential jump in the gradient of \mathbf{F} across a regular surface in κ , is in fact C^2 everywhere in the interior of κ . By induction, it follows easily that it is then C^n for arbitrary positive integral n . The analysis presented in this work requires only $n = 3$. Although the premise of piecewise C^2 regularity has not been proved for problems of the kind considered, it is nevertheless an assumption that is consistent with the adopted

condition of strong ellipticity. Regarding regularity, much sharper results, again relying on strong ellipticity, are available in the classical linear theory [3], and it seems likely that much of this can be extended to the incremental theory. However, these matters are better left to experts in the theory of partial differential equations, and so we do not dwell on them here. Because we are concerned with equilibria, we construct an expression for the plate energy that presumes C^3 continuity at the outset, this then forming the admissible class of functions, subject to additional boundary conditions to be discussed.

Let c^* be the line orthogonal to ω and intersecting $\partial\omega$ at a point with position \mathbf{r} , and let $\partial\kappa_c = \partial\omega \times c$, where c is the collection of such lines, be the cylindrical generating surface of the plate-like region κ obtained by translating the points of $\partial\omega$ along their associated lines c^* . Let s measure arclength on the curve $\partial\omega$ with unit tangent $\boldsymbol{\tau}$ and rightward unit normal $\boldsymbol{\nu} = \boldsymbol{\tau} \times \mathbf{k}$.

We suppose that $\partial\kappa_t$ consists of the major surfaces of the plate together with a part $\partial\omega_t \times c$ of $\partial\kappa_c$, where $\partial\omega_t \subset \partial\omega$. The traction data (30)₂ then furnish $\dot{\mathbf{P}}\mathbf{1}\boldsymbol{\nu} = \mathbf{0}$ on $\partial\omega_t \times c$. In particular,

$$\dot{\mathbf{P}}_0\mathbf{1}\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \partial\omega_t, \quad (43)$$

where $\dot{\mathbf{P}}_0$ is the restriction of $\dot{\mathbf{P}}$ to ω . Position is then assigned, and the incremental displacement vanishes, on $\partial\omega_\phi \times c$, where $\partial\omega_\phi = \partial\omega \setminus \partial\omega_t$. Thus, $\hat{\mathbf{u}}$ vanishes identically on $\partial\omega_\phi \times c$, implying that

$$\mathbf{u} = \mathbf{a} = \mathbf{b} = \cdots = \mathbf{0} \quad \text{on} \quad \partial\omega_\phi. \quad (44)$$

4.2 Small-thickness expansion of the second variation

We seek the optimal expression for the functional G in the expansion

$$\mathcal{G} = G + o(h^3) \quad (45)$$

of the second variation. To this end we write \mathcal{G} as the iterated integral

$$\mathcal{G} = \frac{1}{2} \int_\omega \int_{-h/2}^{h/2} \dot{\mathbf{P}} \cdot \tilde{\mathbf{H}} d\zeta da \quad (46)$$

and use the Leibniz rule to generate the Taylor expansion

$$\int_{-h/2}^{h/2} \dot{\mathbf{P}} \cdot \tilde{\mathbf{H}} d\zeta = h(\dot{\mathbf{P}} \cdot \hat{\mathbf{H}})_0 + \frac{1}{24}h^3(\dot{\mathbf{P}} \cdot \hat{\mathbf{H}})''_0 + o(h^3), \quad (47)$$

where again a prime is used to identify a derivative with respect to ζ and the subscript zero signifies evaluation on the plane ω defined by $\zeta = 0$. We obtain

$$(\dot{\mathbf{P}} \cdot \hat{\mathbf{H}})_0 = \dot{\mathbf{P}}_0 \cdot \mathbf{H}_0, \quad (\dot{\mathbf{P}} \cdot \hat{\mathbf{H}})''_0 = 4\dot{\mathbf{P}}'_0 \cdot \mathbf{H}'_0 + 2\dot{\mathbf{P}}_0 \cdot \mathbf{H}''_0 + \mathbf{H}_0 \cdot \mathcal{M}''_0[\mathbf{H}_0] - 2\mathbf{H}'_0 \cdot \mathcal{M}_0[\mathbf{H}'_0], \quad (48)$$

having used the intermediate result

$$\dot{\mathbf{P}}'_0 = \mathcal{M}'_0[\mathbf{H}_0] + \mathcal{M}_0[\mathbf{H}'_0], \quad (49)$$

which follows from (27). For uniform materials, the thickness-wise derivatives \mathcal{M}' and \mathcal{M}'' are induced by the through-thickness variation of the gradient \mathbf{F} of the underlying finite deformation $\boldsymbol{\chi}$. Their computation requires higher-order elastic moduli. For example, $\mathcal{M}'_{ijkl} = \mathcal{A}_{ijklmn} F'_{mn}$, where \mathcal{A} is the tensor of second-order moduli [4]. In this way the finite deformation may be used to generate a through-thickness *functional gradient* in the incremental response.

The reason why the expansion is terminated at order h^3 is discussed in Subsection 4.5 below. Briefly, this is the lowest order at which the derived two-dimensional model remains well posed while accommodating buckling in the presence of compressive pre-stress.

To keep the treatment as simple as possible, and to make more direct contact with classical plate buckling theory, we suppose the initial finite deformation to be such that \mathbf{F} does not vary through the thickness of the plate. In this case the higher-order moduli are not needed. This yields a dramatic simplification of the model while making allowance for in-plane functional gradients. Using (40) we then have

$$(\dot{\mathbf{P}} \cdot \hat{\mathbf{H}})_0 = \dot{\mathbf{P}}_0 \mathbf{1} \cdot \nabla \mathbf{u} + \dot{\mathbf{P}}_0 \mathbf{k} \cdot \mathbf{a}, \quad (\dot{\mathbf{P}} \cdot \hat{\mathbf{H}})''_0 = 2(\dot{\mathbf{P}}'_0 \mathbf{1} \cdot \nabla \mathbf{a} + \dot{\mathbf{P}}'_0 \mathbf{k} \cdot \mathbf{b} + \dot{\mathbf{P}}_0 \mathbf{1} \cdot \nabla \mathbf{b} + \dot{\mathbf{P}}_0 \mathbf{k} \cdot \mathbf{c}). \quad (50)$$

4.3 Refinement of the model

If (49) and (50) are substituted into (47), the functional G in (45) is found to depend on the vector fields \mathbf{u} , \mathbf{a} , \mathbf{b} and \mathbf{c} . At this level of the development these fields are independent and each would generate an associated Euler equation and natural boundary condition. The resulting system would then constitute the linearized bifurcation problem for the thin plate. However, the functional G may be optimized *vis à vis* the three-dimensional theory by imposing certain *a priori* constraints among the vector fields. The resulting expression, derived below, involves the single field \mathbf{u} and furnishes the optimal order- h^3 approximation to the second variation for a given mid-surface incremental displacement field, in the sense that it automatically encodes restrictions arising in the three-dimensional parent theory.

Thus, for example, if the lateral surfaces of the plate are traction-free, then (30)₂ requires that $\dot{\mathbf{P}}^\pm \mathbf{k} = \mathbf{0}$, where $\dot{\mathbf{P}}^\pm = \dot{\mathbf{P}}|_{\varsigma=\pm h/2}$. Adding and subtracting the two Taylor expansions ($\mathbf{0} =$) $\dot{\mathbf{P}}^\pm \mathbf{k} = \dot{\mathbf{P}}_0 \mathbf{k} \pm (h/2) \dot{\mathbf{P}}'_0 \mathbf{k} + (h^2/8) \dot{\mathbf{P}}''_0 \mathbf{k} + O(h^3)$, we then derive the estimates

$$\dot{\mathbf{P}}_0 \mathbf{k} = O(h^2), \quad \dot{\mathbf{P}}'_0 \mathbf{k} = O(h^2), \quad (51)$$

so that $\dot{\mathbf{P}}_0 \mathbf{k}$ and $\dot{\mathbf{P}}'_0 \mathbf{k}$ may be suppressed *in the coefficient of h^3* in the order- h^3 expansion of the second variation with no adverse effect on accuracy. Accordingly, we impose

$$\dot{\mathbf{P}}_0 \mathbf{k} = \mathbf{0} = \dot{\mathbf{P}}'_0 \mathbf{k} \quad (52)$$

in (50)₂.

In fact, (52) may be solved explicitly for \mathbf{a} and \mathbf{b} . To prove this we use (40)₁ to write

$$\dot{\mathbf{P}}_0 \mathbf{k} = (\mathcal{M}[\nabla \mathbf{u}]) \mathbf{k} + \mathbf{A}_{(\mathbf{k})} \mathbf{a}, \quad (53)$$

where the tensor $\mathbf{A}_{(\mathbf{k})}$ (in other contexts referred to as the *acoustic tensor*) is defined by

$$\mathbf{A}_{(\mathbf{k})}\mathbf{v} = (\mathcal{M}[\mathbf{v} \otimes \mathbf{k}])\mathbf{k} \quad (54)$$

for all 3-vectors \mathbf{v} . That this is positive definite and hence invertible follows from the strong-ellipticity inequality (7). Accordingly, (52)₁ determines \mathbf{a} uniquely in terms of $\nabla\mathbf{u}$, i.e.

$$\mathbf{a} = \mathbf{g}(\nabla\mathbf{u}), \quad (55)$$

where \mathbf{g} is a function determined by material properties and the underlying finite deformation. Explicitly,

$$\mathbf{g}(\nabla\mathbf{u}) = -\mathbf{A}_{(\mathbf{k})}^{-1}(\mathcal{M}[\nabla\mathbf{u}])\mathbf{k}. \quad (56)$$

In the same way, (49) and (52)₂ determine \mathbf{b} . In the special case of no through-thickness functional gradient, the relationship is

$$\mathbf{b} = \mathbf{g}(\nabla\mathbf{a}). \quad (57)$$

The latter is altered in the presence of a through-thickness functional gradient, but the procedure delivers \mathbf{b} explicitly in any case. Thus we replace (50)₂ by

$$(\dot{\mathbf{P}} \cdot \hat{\mathbf{H}})_0'' = 2(\dot{\mathbf{P}}_0' \mathbf{1} \cdot \nabla\mathbf{a} + \dot{\mathbf{P}}_0 \mathbf{1} \cdot \nabla\mathbf{b}) + O(h^2), \quad (58)$$

in which \mathbf{a} and \mathbf{b} are subject to (55) and (57), respectively.

We are *not* justified in suppressing $\dot{\mathbf{P}}_0\mathbf{k}$ in the order- h term, however, as (51)₁ implies that the former makes a net contribution at order h^3 and is thus comparable to other terms that have been retained. We return to this point below.

The model is further simplified by using

$$\dot{\mathbf{P}}_0 \mathbf{1} \cdot \nabla\mathbf{b} = \operatorname{div}[(\dot{\mathbf{P}}_0 \mathbf{1})^T \mathbf{b}] - \mathbf{b} \cdot \operatorname{div}(\dot{\mathbf{P}}_0 \mathbf{1}), \quad (59)$$

where div is the (two-dimensional) divergence with respect to position \mathbf{r} on the midplane ω . To assess this we use the decomposition $\dot{\mathbf{P}} = \dot{\mathbf{P}} \mathbf{1} + \dot{\mathbf{P}} \mathbf{k} \otimes \mathbf{k}$ of the incremental stress and evaluate (30)₁, expressed in the form $\operatorname{div}(\dot{\mathbf{P}} \mathbf{1}) + \dot{\mathbf{P}} \mathbf{k} = \mathbf{0}$, on ω , obtaining

$$\operatorname{div}(\dot{\mathbf{P}}_0 \mathbf{1}) + \dot{\mathbf{P}}_0' \mathbf{k} = \mathbf{0}. \quad (60)$$

The estimate (51)₂ then furnishes

$$\dot{\mathbf{P}}_0 \mathbf{1} \cdot \nabla\mathbf{b} = \operatorname{div}[(\dot{\mathbf{P}}_0 \mathbf{1})^T \mathbf{b}] + O(h^2). \quad (61)$$

Using Green's theorem together with the data (43) and (44), we conclude that the term $\dot{\mathbf{P}}_0 \mathbf{1} \cdot \nabla\mathbf{b}$ may be suppressed without adversely affecting the accuracy of the order- h^3 expansion of the second variation. We are thus left with

$$G = \int_{\omega} \bar{W} da, \quad (62)$$

where

$$\bar{W} = \frac{1}{2}h(\dot{\mathbf{P}}_0\mathbf{1} \cdot \nabla\mathbf{u} + \dot{\mathbf{P}}_0\mathbf{k} \cdot \mathbf{a}) + \frac{1}{24}h^3\dot{\mathbf{P}}'_0\mathbf{1} \cdot \nabla\mathbf{a} \quad (63)$$

in which (55) and (57) are imposed in the coefficient of h^3 .

If the present model is to apply on the closure of ω , then the values of \mathbf{a} and \mathbf{b} obtained from (55) and (57) must be such that (44) is satisfied. This imposes a restriction on admissible data. In particular, the decomposition [9]

$$\nabla\mathbf{u} = \mathbf{u}_s \otimes \boldsymbol{\tau} + \mathbf{u}_\nu \otimes \boldsymbol{\nu}, \quad (64)$$

where \mathbf{u}_s and \mathbf{u}_ν are the tangential (arclength) and normal derivatives of \mathbf{u} on $\partial\omega$, together with (55), implies that the condition $\mathbf{a} = \mathbf{0}$ on $\partial\omega_\phi$ is satisfied provided that \mathbf{u}_s and \mathbf{u}_ν vanish there. The data for \mathbf{u} and \mathbf{a} are thus tantamount to the clamping conditions

$$\mathbf{u} = \mathbf{u}_\nu = \mathbf{0} \quad \text{on} \quad \partial\omega_\phi. \quad (65)$$

However, the model to be derived is overspecified if (65) is imposed together with $\mathbf{b} = \mathbf{0}$ on $\partial\omega_\phi$. Accordingly, the condition on \mathbf{b} must either be relaxed or the equations must be used only in the interior of the plate, away from $\partial\omega$, and their predictions then matched to those of the three-dimensional theory in a region adjoining it. We pursue the first alternative in this work.

The energy \bar{W} involves \mathbf{a} only in the coefficient of h . It occurs algebraically, in the combination

$$H(\nabla\mathbf{u}, \mathbf{a}) = \frac{1}{2}\mathbf{H}_0 \cdot \mathcal{M}[\mathbf{H}_0]. \quad (66)$$

The variation of this term with respect to \mathbf{a} is

$$\delta H = \delta\mathbf{a} \cdot (\mathcal{M}[\mathbf{H}_0])\mathbf{k}. \quad (67)$$

The Euler equation for the functional G associated with the variable \mathbf{a} is thus given by (52)₁, which is solved by (55). Strong ellipticity implies that the latter furnishes the minimum of H with respect to \mathbf{a} and hence also the pointwise minimum of \bar{W} with respect to \mathbf{a} . This claim is proved by adapting, essentially *verbatim*, an argument discussed in [10] (eqs. (22)–(28) therein). Because we can do no better than minimize the energy function G pointwise, the adoption of (55) in the coefficient of h yields the optimal criterion for buckling. This follows since a field $\{\mathbf{u}, \mathbf{a}\}$ that satisfies the bifurcation criterion in which (55) is imposed will, in the alternative case, satisfy the semi-strict inequality (33). Said differently, the adoption of (55) promotes bifurcation, whereas its onset is delayed in the alternative case. This furnishes justification for the assumption of plane stress listed in the Introduction on which classical treatments are based.

Altogether, the energy function is then given by (62) in which \bar{W} is replaced by

$$W(\nabla\mathbf{u}, \nabla\nabla\mathbf{u}) = \frac{1}{2}h\dot{\mathbf{P}}_0\mathbf{1} \cdot \nabla\mathbf{u} + \frac{1}{24}h^3\dot{\mathbf{P}}'_0\mathbf{1} \cdot \nabla\mathbf{a}, \quad (68)$$

with (55) and (57) incorporated in all terms.

The buckling equations are simply the Euler equations and natural boundary conditions associated with the functional G . In Cartesian tensor notation, these are [9]

$$T_{i\alpha,\alpha} = 0 \quad \text{in } \omega \quad (69)$$

and

$$T_{i\alpha}\nu_\alpha - (M_{i\alpha\beta}\nu_\alpha\tau_\beta)_s = 0, \quad M_{i\alpha\beta}\nu_\alpha\nu_\beta = 0 \quad \text{on } \partial\omega_t, \quad (70)$$

respectively, where

$$T_{i\alpha} = N_{i\alpha} - M_{i\beta\alpha,\beta}, \quad \text{with } N_{i\alpha} = \frac{\partial W}{\partial u_{i,\alpha}} \quad \text{and } M_{i\alpha\beta} = \frac{\partial W}{\partial u_{i,\alpha\beta}}. \quad (71)$$

Here $u_i = \mathbf{u} \cdot \mathbf{e}_i$, with $\mathbf{e}_3 = \mathbf{k}$, are the orthogonal components of \mathbf{u} , the subscript s again represents the arclength derivative along $\partial\omega$ (traversed counterclockwise), with unit tangent $\boldsymbol{\tau} = \mathbf{k} \times \boldsymbol{\nu}$ to $\partial\omega$ and rightward unit normal $\boldsymbol{\nu}$; Greek subscripts preceded by commas are used to denote partial derivatives with respect to the in-plane Cartesian coordinates $r_\alpha = \mathbf{r} \cdot \mathbf{e}_\alpha$. The boundary conditions (70)_{1,2}, respectively, are to be interpreted as the vanishing of the incremental force and moment (per unit length) on $\partial\omega_t$.

4.4 Reflection symmetry and a restriction on the pre-stress

In view of the restrictions imposed in the classical theory [2], we limit attention to strain-energy functions that exhibit reflection symmetry with respect to the midplane ω , i.e. $\mathcal{U}(\mathbf{E}) = \mathcal{U}(\mathbf{Q}^T \mathbf{E} \mathbf{Q})$ with $\mathbf{Q} = \mathbf{I} - 2\mathbf{k} \otimes \mathbf{k}$. This implies that the function $\mathcal{U}'(E_{ij}) = \mathcal{U}'(E_{kl}\mathbf{e}_k \otimes \mathbf{e}_l)$ depends on $E_{3\alpha}$ ($= E_{\alpha 3}$) through their squares and the product $E_{31}E_{32}$ (see [11], section 5.4(a)). An example is furnished by any material that is isotropic relative to κ . Following the classical theory, we further suppose the underlying finite deformation to be such that the associated second Piola–Kirchhoff stress is a function only of the in-plane coordinates and subject to null-traction conditions on the major surfaces; the latter are equivalent to $\mathbf{S}^\pm \mathbf{k} = \mathbf{0}$, where $\mathbf{S}^\pm = \mathbf{S}|_{\varsigma=\pm h/2}$. These conditions yield the pointwise restriction

$$\mathbf{S}\mathbf{k} = \mathbf{0} \quad (72)$$

on the pre-stress throughout the plate, implying that the underlying finite deformation is associated with a state of *plane stress*.

Let $\gamma_\alpha = E_{\alpha 3} = E_{3\alpha}$ be the transverse shear strains associated with the finite pre-strain, and let $\Gamma(\gamma_\alpha)$ be the function obtained by holding fixed all components of \mathbf{E} other than the γ_α in the strain-energy function. Then,

$$\frac{\partial \Gamma}{\partial \gamma_\alpha} = \mathbf{e}_\alpha \cdot (\mathcal{U}_{\mathbf{E}})\mathbf{k}, \quad (73)$$

and these vanish by virtue of (5) and (72). In a material that exhibits reflection symmetry, these restrictions are automatically satisfied at $\gamma_\alpha = 0$ because the strain energy is then an even function of the transverse shears. The corresponding strain tensor is of the form

$$\mathbf{E} = \boldsymbol{\epsilon} + \frac{1}{2}(\lambda^2 - 1)\mathbf{k} \otimes \mathbf{k}, \quad (74)$$

where $\boldsymbol{\epsilon} = E_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ and λ is the transverse stretch. At any point (x_α, ς) , the corresponding deformation gradient has the form

$$\mathbf{F} = \mathbf{f} + \lambda \mathbf{n} \otimes \mathbf{k}, \quad (75)$$

where \mathbf{n} is a local unit normal to the material surface $\varsigma = \text{constant}$ after deformation; \mathbf{f} maps ω' to the local tangent plane of this surface. Moreover, in the presence of strong ellipticity (cf. (7)), this is the only mode of deformation that is consistent with (72) (for proof, see page 288 of [12]). Therefore reflection symmetry and strong ellipticity, combined with (72), yield deformations in which the transverse shear strain necessarily vanishes. We also suppose the orientations of the planes $\varsigma = \text{constant}$ to remain unaltered by the finite deformation, this being consistent with reflection symmetry and the restriction (72). This amounts to putting $\mathbf{n} = \mathbf{k}$, in which case \mathbf{f} is an invertible map from ω' to itself.

The condition $\dot{\mathbf{P}}_0 \mathbf{k} = \mathbf{0}$ implies, via (24) and (72), that $\dot{\mathbf{S}}_0 \mathbf{k} = \mathbf{0}$, i.e. that

$$\{\mathcal{C}(\mathbf{E})[\dot{\mathbf{E}}_0]\} \mathbf{k} = \mathbf{0}, \quad (76)$$

wherein all orthogonal components of the tensor \mathcal{C} of elastic moduli with an odd number of subscripts equal to 3 vanish, by virtue of reflection symmetry [11], and

$$2\dot{\mathbf{E}}_0 = \mathbf{f}^T(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T \mathbf{f} + \mathbf{k} \otimes (\mathbf{f}^T \boldsymbol{\alpha} + \lambda \nabla w) + (\mathbf{f}^T \boldsymbol{\alpha} + \lambda \nabla w) \otimes \mathbf{k} + 2\lambda a \mathbf{k} \otimes \mathbf{k}, \quad (77)$$

where we have used the orthogonal decompositions

$$\mathbf{u} = \mathbf{v} + w \mathbf{k}, \quad \mathbf{a} = \boldsymbol{\alpha} + a \mathbf{k}, \quad \text{with } \mathbf{v} = \mathbf{1} \mathbf{u} \quad \text{and} \quad \boldsymbol{\alpha} = \mathbf{1} \mathbf{a}. \quad (78)$$

In particular, the in-plane and transverse displacements of points on ω are \mathbf{v} and w , respectively, the latter being the variable of principal interest in plate-buckling theory. Equation (76) is equivalent to (52)₁ when (72) is satisfied. Its components are

$$\mathcal{C}_{\alpha 3 \beta 3} \dot{E}_{0 \beta 3} = 0, \quad \mathcal{C}_{33 \alpha \beta} \dot{E}_{0 \alpha \beta} + C \dot{E}_{033} = 0, \quad (79)$$

where $C = \mathcal{C}_{3333}$.

Under the stated restrictions on the material and the underlying finite deformation it follows from (28) that the acoustic tensor defined by (54) reduces to

$$\mathbf{A}_{(\mathbf{k})} = A_{\alpha\beta} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta + A \mathbf{k} \otimes \mathbf{k}, \quad \text{where } A_{\alpha\beta} = f_{\alpha\lambda} f_{\beta\mu} \mathcal{C}_{\lambda 3 \mu 3} \quad \text{and} \quad A = \lambda^2 C. \quad (80)$$

The hypothesis of strong ellipticity implies that $A > 0$ and that $(A_{\alpha\beta})$ is positive definite. Because $\det \mathbf{f} > 0$ (cf. (75)), the matrix $(\mathcal{C}_{\alpha 3 \beta 3})$ is also positive definite and (79)₁ then furnishes $\dot{E}_{0\alpha 3} = 0$, while (79)₂ delivers \dot{E}_{033} in terms of $\dot{E}_{0\alpha\beta}$. Thus,

$$\mathbf{f}^T \boldsymbol{\alpha} = -\lambda \nabla w \quad \text{and} \quad \lambda a = -C^{-1} \mathcal{C}_{33 \alpha \beta} \dot{E}_{0 \alpha \beta}, \quad (81)$$

in which

$$2\dot{E}_{0\alpha\beta} = f_{\lambda\alpha} v_{\lambda,\beta} + f_{\lambda\beta} v_{\lambda,\alpha}. \quad (82)$$

These combine with (78)₂ to furnish the function \mathbf{g} in (56).

Equations (81)_{1,2} comprise the classical Kirchhoff–Love hypothesis with thickness distension. These correspond to the kinematic assumption listed in the Introduction pertaining to the preservation of the surface normal and adopted *a priori* in classical treatments. Here, of course, these conditions are derived rather than postulated.

Further, from (57),

$$\mathbf{f}^T \boldsymbol{\beta} = -\lambda \nabla a \quad \text{and} \quad \lambda b = -C^{-1} \mathcal{C}_{33\alpha\beta} \dot{E}'_{0\alpha\beta}, \quad (83)$$

where

$$2\dot{E}'_{0\alpha\beta} = f_{\lambda\alpha} \alpha_{\lambda,\beta} + f_{\lambda\beta} \alpha_{\lambda,\alpha} \quad (84)$$

and

$$\mathbf{b} = \boldsymbol{\beta} + b\mathbf{k} \quad \text{with} \quad \boldsymbol{\beta} = \mathbf{1}b. \quad (85)$$

The latter also follow directly by imposing $\dot{\mathbf{S}}'_0 \mathbf{k} = \mathbf{0}$, which is equivalent, granted (72), to $\dot{\mathbf{P}}'_0 \mathbf{k} = \mathbf{0}$ (cf. (24)).

Using these results together with the symmetries of the inner product, we find that

$$\dot{\mathbf{P}}'_0 \mathbf{1} \cdot \nabla \mathbf{a} = \nabla a \cdot \mathbf{S}(\nabla a) + (\nabla \boldsymbol{\alpha}) \mathbf{S} \cdot \nabla \boldsymbol{\alpha} + \dot{\mathbf{S}}'_0 \cdot \mathbf{f}^T \nabla \boldsymbol{\alpha}, \quad (86)$$

where

$$\dot{S}'_{0\alpha\beta} = \mathcal{D}_{\alpha\beta\gamma\delta} \dot{E}'_{0\gamma\delta}, \quad (87)$$

and

$$\mathcal{D}_{\alpha\beta\gamma\delta} = \mathcal{C}_{\alpha\beta\gamma\delta} - C^{-1} \mathcal{C}_{\alpha\beta 33} \mathcal{C}_{\gamma\delta 33} \quad (88)$$

are the plane-stress elastic moduli.

We have in mind the idea that a deformation satisfying the Euler equations (69)–(71) should furnish a minimum of the quadratic functional G . However, for a minimum to exist it is necessary that the operative Legendre–Hadamard condition be satisfied. In the present context this is the requirement that the term in the integrand W in (68) involving the highest-order derivative $u_{i,\alpha\beta}$ (the components of $\nabla \nabla \mathbf{u}$) be positive definite when $u_{i,\alpha\beta}$ is replaced by $y_i z_\alpha z_\beta$ for any three-vector \mathbf{y} and any two-vector \mathbf{z} [13]. The choice $y_3 = 0$ is seen to reduce this requirement to the restriction

$$\nabla a \cdot \mathbf{S}(\nabla a) > 0 \quad (89)$$

in which ∇a can be assigned arbitrary values by choice of $y_\mu z_\alpha z_\beta$. The requirement thus limits the present model to a positive-definite pre-stress. Ironically, this in turn precludes its application to precisely the kinds of problems it is meant to address.

We note that the restriction arises from the coefficient of h^3 in the energy function for the plate. To remove it, and thus to restore the applicability of the theory to plate buckling, we assume that

$$|\mathbf{S}| = o(1) \quad (90)$$

for small h , that is \mathbf{S} tends to zero as $h \rightarrow 0$. This is a restriction on solutions of the underlying finite-deformation problem. If the restriction is satisfied, all terms involving \mathbf{S}

may be suppressed in the coefficient of h^3 while preserving the accuracy of the order- h^3 expansion of the energy. This furnishes the simplification

$$W(\nabla \mathbf{u}, \nabla \nabla \mathbf{u}) = \frac{1}{2} h \dot{\mathbf{P}}_0 \mathbf{1} \cdot \nabla \mathbf{u} + \frac{1}{24} h^3 \dot{\mathbf{S}}'_0 \cdot \mathbf{f}^T \nabla \boldsymbol{\alpha}, \quad (91)$$

of the order- h^3 plate energy, where

$$\dot{\mathbf{P}}_0 \mathbf{1} \cdot \nabla \mathbf{u} = \nabla w \cdot \mathbf{S}(\nabla w) + (\nabla \mathbf{v}) \mathbf{S} \cdot \nabla \mathbf{v} + \dot{\mathbf{S}}_0 \cdot \mathbf{f}^T \nabla \mathbf{v}, \quad (92)$$

and

$$\dot{S}_{0\alpha\beta} = \mathcal{D}_{\alpha\beta\gamma\delta} \dot{E}_{0\gamma\delta}. \quad (93)$$

Remark: The necessary condition (89), which does not apply in the three-dimensional theory, arises from the fact that the order- h^3 truncation of the energy does not account fully for the energy of the Ansatz (42). This requires that the pre-stress be limited in accordance with (90) if the order- h^3 truncation is to furnish a well-posed minimization problem. Alternatively, ill-posedness may be eliminated by retaining all terms in the energy associated with a finite truncation of the power expansion (42). Such a procedure would lead to a more complicated model, which in any case would be limited by the approximations inherent in any such truncation. Later, we show that (90) is in fact not restrictive as it is satisfied by the pre-stress associated with a non-trivial bifurcation mode.

From the structure of the foregoing it is clear that the energy decouples into the sum of pure stretching and pure bending energies; thus, from (87) and (93),

$$W = W_s + W_b, \quad (94)$$

where

$$W_s = \frac{1}{2} h \{ (\nabla \mathbf{v}) \mathbf{S} \cdot \nabla \mathbf{v} + \mathbf{f}^{tT} \nabla \mathbf{v} \cdot \mathcal{D}[\mathbf{f}^T \nabla \mathbf{v}] \}, \quad W_b = \frac{1}{2} h \nabla w \cdot \mathbf{S}(\nabla w) + \frac{1}{24} h^3 \mathbf{f}^T \nabla \boldsymbol{\alpha} \cdot \mathcal{D}[\mathbf{f}^T \nabla \boldsymbol{\alpha}], \quad (95)$$

with $\boldsymbol{\alpha}$ given by (81), and \mathcal{D} is the fourth-order tensor defined by (88). In view of (90), the *leading-order* (order- h) stretching energy is obtained by suppressing the term in (95)₁ involving \mathbf{S} explicitly, yielding

$$W_s = \frac{1}{2} h \mathbf{f}^T \nabla \mathbf{v} \cdot \mathcal{D}[\mathbf{f}^T \nabla \mathbf{v}]. \quad (96)$$

4.5 Recovery of the classical model

The problem may be further simplified by exploiting the full implications of the restriction (90) on the pre-stress. If the reference configuration κ is stress free, if h is sufficiently small and if the strain energy $\mathcal{U}(\mathbf{E})$ is convex in a neighborhood of the origin in strain space, as is typically assumed, then the initial strain is likewise small, vanishing in the zero-thickness limit; thus $|\mathbf{E}| = o(1)$. The pre-strain may then be suppressed in the coefficient of h^3 with

no effect on the order- h^3 accuracy of the strain-energy function. This permits \mathcal{D} , which in principle is evaluated at the underlying finite strain, to be replaced by

$$\mathcal{D}^{(\kappa)} = \mathcal{D}|_{\mathbf{E}=\mathbf{0}}, \quad (97)$$

the tensor of linear-elastic moduli relative to κ . Our restriction to uniform materials means that $\mathcal{D}^{(\kappa)}$ is *spatially uniform*, yielding a substantial simplification of the model which is fully consistent with order- h^3 accuracy.

Beyond this, the same degree of accuracy is preserved by imposing $\mathbf{F} \in \text{Orth}^+$ in the coefficient of h^3 , where Orth^+ is the group of proper-orthogonal tensors, or rotations. To see this we simply polar-decompose \mathbf{F} as the product of a rotation \mathbf{R} and the right stretch tensor in which the latter is of order unity by virtue of the restriction on $|\mathbf{E}|$. Thus, $\mathbf{F} = \mathbf{R} + o(1)$ with $\mathbf{R} \in \text{Orth}^+$. This in turn implies that order- h^3 accuracy is preserved by substituting $\mathbf{R}^T \nabla \boldsymbol{\alpha}$ in place of $\mathbf{f}^T \nabla \boldsymbol{\alpha}$ ($= \mathbf{F}^T \nabla \boldsymbol{\alpha}$) in the coefficient of h^3 , in which $\mathbf{f}^T \boldsymbol{\alpha} = -\lambda \nabla w$ (cf. (81))₁ is replaced by $\mathbf{R}^T \boldsymbol{\alpha} = -\nabla w$.

An important further simplification follows from eqs. (8) and (9) of [14], which imply that the *gradient* of \mathbf{R} is small if the gradient of the strain is small. Thus, *we confine our further attention to the practically important case in which the gradient of \mathbf{E} is of order $o(1)$* . This in turn yields the conclusion that, to within an error of order $o(1)$, \mathbf{R} is *spatially uniform*. The error contributes at order $o(h^3)$ and may be suppressed in the present model with no adverse effect on accuracy. This yields the consistent-order estimate $\mathbf{f}^T \nabla \boldsymbol{\alpha} = -\nabla \nabla w$, allowing the bending energy to be replaced by

$$W_b = \frac{1}{2} h \nabla w \cdot \mathbf{S}(\nabla w) + \frac{1}{24} h^3 \nabla \nabla w \cdot \mathcal{D}^{(\kappa)}[\nabla \nabla w], \quad (98)$$

with *no effect* on the accuracy of the truncation (95)₂.

Remark: This expression furnishes the rationale for terminating the thickness-wise expansion of the energy at order h^3 . In particular, this is the lowest order at which the bending energy remains well posed in the presence of a compressive pre-stress, and hence the lowest order at which a two-dimensional model, derived from the three-dimensional theory, can furnish a meaningful basis on which classical plate buckling may be analyzed.

The stress \mathbf{S} associated with the underlying pre-buckling deformation satisfies (1)₁ with (4). As before, the assumption (90), taken together with the stated restriction on the gradient of the pre-strain and our constitutive hypotheses, imply that the *leading order* restriction on this stress is obtained on replacing the factor \mathbf{F} in (4) by a *uniform* rotation \mathbf{R} . This furnishes

$$\text{Div} \mathbf{S} = \mathbf{0} \quad \text{in } \kappa, \quad (99)$$

to *leading order* in thickness, where (cf. (74))

$$\mathbf{S} = \mathcal{D}^{(\kappa)}[\boldsymbol{\epsilon}] \quad (100)$$

in which $|\boldsymbol{\epsilon}| = o(1)$.

Similarly, the leading order stretching energy may be replaced by

$$W_s = \frac{1}{2}h\nabla\bar{\mathbf{v}} \cdot \mathcal{D}^{(\kappa)}[\nabla\bar{\mathbf{v}}] \quad (101)$$

with no effect on (order- h) accuracy, where $\bar{\mathbf{v}} = \mathbf{R}^T\mathbf{v}$ is a rigidly-rotated displacement field. Our assumptions imply that $\mathcal{D}^{(\kappa)}$ is positive definite and hence, as in the classical theory of generalized plane stress, that $\mathbf{v} = \mathbf{0}$ when $\partial\omega_\phi$ is non-empty.

We have thus justified the third main assumption of the classical treatments; namely, that the underlying pre-buckling deformation associated with the order- h^3 model may be described using classical linear elasticity theory. This follows from the fact that the model calls for the ground-state moduli $\mathcal{D}^{(\kappa)}$ while the pre-stress \mathbf{S} satisfies the equilibrium and constitutive equations of the linear theory.

Remark: In plate theory based *a priori* on conventional three-dimensional linear elasticity theory with *initial* stress [15, 16], as distinct from the incremental theory in which *pre-stress* is induced by the pre-buckling deformation, both the initial stress and the elastic moduli are constrained by the nature of the material symmetry in the configuration κ . Thus, for example, if the material is isotropic relative to κ , the *initial stress* is a uniform pure pressure that vanishes by virtue of the traction data on the lateral surfaces of the plate. Consequently, that theory does not yield a plate buckling model in the case of isotropy. In contrast, here the *pre-stress* is induced by the underlying $o(1)$ strain; the symmetry of the material in κ manifests itself only through the moduli via (100), whereas the pre-stress is delivered by the linearly elastic boundary-value problem. The importance of the distinction between initial stress and pre-stress is discussed further in [17].

Classical plate-buckling theory [1, 2] is associated with the special case of (90) in which

$$\mathbf{S} = h^2\bar{\mathbf{S}} + o(h^2), \quad \text{with} \quad |\bar{\mathbf{S}}| = O(1), \quad (102)$$

yielding

$$W_b = \frac{1}{2}h^3 \left\{ \nabla w \cdot \bar{\mathbf{S}}(\nabla w) + \frac{1}{12}\nabla\nabla w \cdot \mathcal{D}^{(\kappa)}[\nabla\nabla w] \right\}. \quad (103)$$

The associated boundary-value problem, given by (69)–(71) with $i = 3$, consists of the equations

$$\frac{1}{12}\mathcal{D}_{\alpha\beta\lambda\mu}^{(\kappa)}w_{,\alpha\beta\lambda\mu} = \bar{S}_{\alpha\beta}w_{,\alpha\beta} \quad \text{in} \quad \omega \quad (104)$$

and boundary conditions

$$w = 0, \quad \boldsymbol{\nu} \cdot \nabla w = 0 \quad \text{on} \quad \partial\omega_\phi, \quad (105)$$

with

$$\left(\bar{S}_{\alpha\beta}w_{,\beta} - \frac{1}{12}\mathcal{D}_{\alpha\beta\lambda\mu}^{(\kappa)}w_{,\beta\lambda\mu} \right) \nu_\alpha - \frac{1}{12} \left(\mathcal{D}_{\alpha\beta\lambda\mu}^{(\kappa)}w_{,\lambda\mu} \nu_\alpha \tau_\beta \right)_s = 0, \quad \mathcal{D}_{\alpha\beta\lambda\mu}^{(\kappa)}w_{,\lambda\mu} \nu_\alpha \nu_\beta = 0 \quad \text{on} \quad \partial\omega_t. \quad (106)$$

This is the classical plate-buckling problem for anisotropic materials, incorporating that for isotropic materials found in numerous texts and monographs.

This problem contains no small parameters and thus does not exhibit localized boundary-layer effects. Rather, the bifurcation modes are global, as is well known in the technical literature [1, 2]. Further, in this case W_b furnishes the *rigorous leading-order energy* of the thin plate. This follows on dividing the exact energy \mathcal{G} (cf. (45)) by h^3 and passing to the limit.

The scaling (102) represents the smallest initial stress for which a non-trivial bifurcation mode can exist in the order- h^3 model because smaller initial stresses contribute at order $o(h^3)$ and therefore play no role in the model, whereas the positivity of $\mathcal{D}^{(\kappa)}$ then allows only the trivial solution $w = 0$ to the boundary-value problem. The relevance of the model (104)–(106) is reflected by the large number of plate-buckling problems that have been solved on the basis of the classical theory, all of which exhibit eigenvalues $\bar{\mathbf{S}}$ that satisfy (102)₂. These furnish *post-facto* justification for the imposition of the restriction (90) on the pre-stress, which we earlier motivated merely by the desire to explore the applicability of the order- h^3 model to plate buckling.

Acknowledgment

DJS thanks the University of Aberdeen for support of his sabbatical visit.

References

- [1] H.L. Cox, 1963, *The Buckling of Plates and Shells*. Pergamon-McMillan, New York.
- [2] A.M.A. van der Heijden, 2009, *W.T. Koiter's Elastic Stability of Solids and Structures*. Cambridge University Press.
- [3] G. Fichera, 1972, Existence Theorems in Elasticity. In: *Handbuch der Physik*, vol. VIa/2 (C. Truesdell, ed.), pp. 347–389. Springer, Berlin.
- [4] R.W. Ogden, 1997, *Non-linear Elastic Deformations*. Dover, New York.
- [5] R.J. Knops and E.W. Wilkes, 1973, Theory of Elastic Stability. In: *Handbuch der Physik*, vol. VIa/3 (C. Truesdell, ed.), pp. 125–302. Springer, Berlin.
- [6] R. Hill, 1957, On uniqueness and stability in the theory of finite elastic strain. *J. Mech. Phys. Solids* 5, 229–241.
- [7] W.T. Koiter, 1975, A basic open problem in the theory of elastic stability. Joint IU-TAM/IMU Symposium on Applications of Methods of Functional Analysis to Problems in Mechanics, Sept. 1-6, Part XXIX. Marseille.

- [8] Y.B. Fu, 2001, Perturbation Methods and Nonlinear Stability Analysis. In: Nonlinear Elasticity, Theory and Applications (Y.B. Fu and R.W. Ogden, eds.). London Mathematical Society Lecture Note Series, 283. Cambridge University Press.
- [9] D.J. Steigmann, 2008, Two-dimensional models for the combined bending and stretching of plates and shells based on three-dimensional linear elasticity. *Int. J. Eng. Sci.* 46, 654–76.
- [10] D.J. Steigmann, 2012, Refined theory for linearly elastic plates: laminae and laminates. *Math. Mech. Solids* (doi: 10.1177/1081286511419971).
- [11] A.E. Green and W. Zerna, 1968, *Theoretical Elasticity*, 2nd edn. Oxford University Press.
- [12] D.J. Steigmann, 2010, Applications of Polyconvexity and Strong Ellipticity to Nonlinear Elasticity and Elastic Plate Theory, in: *CISM Course on Applications of Poly-, Quasi-, and Rank-One Convexity in Applied Mechanics*, vol. 516 (J. Schröder and P. Neff, eds.). Springer, Wien and New York, pp. 265–299.
- [13] M.G. Hilgers and A.C. Pipkin, 1992, The Graves condition for variational problems of arbitrary order. *IMA. J. Appl. Math.* 48, 265–269.
- [14] R.T. Shield, 1973, The rotation associated with large strain. *SIAM J. Appl. Math.* 25, 483–491.
- [15] R. Paroni, 2006, Theory of linearly elastic residually stressed plates. *Math. Mech. Solids* 11, 137–159.
- [16] D.J. Steigmann, 2009, Linear theory for the bending and extension of a thin, residually stressed, fiber-reinforced lamina. *Int. J. Eng. Sci.* 47, 1367–1378.
- [17] R.W. Ogden and B. Singh, 2011, Propagation of waves in an incompressible transversely isotropic solid with initial stress: Biot revisited. *J. Mech. Mat. Struct.* 6, 453–477.