THE HOMOTOPY THEORY OF BIALGEBRAS OVER PAIRS OF OPERADS

SINAN YALIN

Abstract. We endow the category of coalgebras over an operad in connective non-negatively graded chain complexes with a cofibrantly generated model category structure. Then, we prove our main result : the category of bialgebras over a pair of operads (P, Q) in distribution admits a cofibrantly generated model category structure inherited from those of the P-algebras and the Q^*-coalgebras. This is the first result of this type for a bialgebras category. It allows to do classical homotopical algebra in various categories such as associative bialgebras, Lie bialgebras or Poisson bialgebras in connective non-negatively graded chain complexes.

Keywords : operads, bialgebras category, homotopical algebra.
AMS : 18G55 ; 18D50.

Contents

Introduction

Operads and props appeared during the 60's as universal tools parametrizing operations of the form $X^{\otimes m} \to X^{\otimes n}$ on an object X of a symmetric monoidal category. The notion of a prop has been introduced by MacLane in algebra [22]. A special kind of prop named category of operators in standard form by Boardman and Vogt arised in the study of iterated loop spaces (see [1] and [2]). Peter May defined the axioms of operads to deal with such structures [26]. The first striking result involving operads was the famous recognition principle, identifying up to homotopy iterated loop spaces and algebras over the little disks operads. More than two decades after their appearance in topology, a rebirth of interest for operads led to fundamental outcomes in many fields of Mathematics. Let us mention for instance the study of moduli spaces of curves, the Deligne conjecture and the deformation quantization of Poisson manifolds.

At the end of the 60's, another major theory emerged in the work of Quillen under the name homotopical algebra [27]. Quillen developped the framework of model categories by analogy with the properties of continuous maps in topological spaces: fibrations, cofibrations and homotopy equivalences. Model categories are aimed to transpose such properties in a general categorical setting, providing an effective and explicit localization functor with respect to these homotopies. Model categories capture the homotopical information of various categories such as differential graded algebras and spectra for instance. Homotopical algebra insights in the operadic setting initiated with the work of Ginzburg and Kapranov on Koszul duality of operads [15], and pursued later in Hinich's work [16]. Operads form themselves a model category. Algebras over a suitable operad can also be endowed with a model category structure. Moreover, the Koszul duality of operads allows to build a special kind of cofibrant resolution of a given operad, called its minimal model. It turns out that these minimal models, inspired by techniques from the rational homotopy theory, are the right objects to encode the notion of (strongly) homotopy algebras over an operad [24].These devices provided the perfect framework to study the cohomology theory and the deformation theory of various sorts of algebras.

However, if one wants to deal with bialgebras it becomes necessary to use general props instead of operads. Structures with products and coproducts started to be popularized with the work of Drinfeld on quantum groups (see [6] and [7]), in particular the examples of bialgebras, Hopf algebras and Lie bialgebras. Hopf algebras play also a crucial role in the construction of quantum knot invariants. Later, such bialgebraic structures appeared with the birth of string topology (see [3],[4] and [5]). One may then wonder how to transpose homotopical algebra methods in the prop setting, as it has been done before for operads. This is the motivating question of this paper. Props themselves form a model category [13]. However, algebras over a prop do not have any known model category structure in general. The only result of this type holds in cartesian categories [18]. It excludes the differential graded context we are interested in. Another approach to capture and understand the homotopical information of a bialgebras category has been settled by the author in [28] via the notion of classifying space of a category.

The main goal of this memoir is to use the framework of model categories, convenient to develop later deformation theory of various bialgebraic structures. For this aim, we use the notion of mixed distributive law introduced by Fox and Markl in [9]. The general idea is that many of the bialgebras encountered in nature are of the following form. There is an operad encoding the operations (several inputs and one single output) and another operad encoding the cooperations (one single input and several outputs). The distributive law then formalizes the interplay between these operads, i.e the compatibilities between operations and cooperations. This formalism includes the aforementioned examples.

The existence of a cofibrantly generated model category structure on algebras over a suitable operad is a classical result, see [16]. When working over a field of characteristic zero, such a structure exists for any operad. We start in section 2 with a detailed construction of this model structure for an operad in the category $Ch_{\mathbb{K}}$ of chain complexes over a field of characteristic zero. The plan and methods followed in this section will serve as guidelines for the remaining part of the paper. Let $Ch_{\mathbb{K}}^{+}$ be the full subcategory of $Ch_{\mathbb{K}}$ of connective chain complexes. We denote by ${}^{P}Ch_{\mathbb{K}}^{+}$ the category of P-coalgebras in $Ch_{\mathbb{K}}^{+}$. The first main result of this paper is the existence of a model category structure for coalgebras over an operad:

Theorem 0.1. *The category of P-coalgebras ${}^{P}Ch_{\mathbb{K}}^{+}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^{P}Ch_{\mathbb{K}}^{+}$ is*
(i) a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}^{+}$;
(ii) a cofibration if $U(f)$ is a cofibration in $Ch_{\mathbb{K}}^{+}$;
(iii) a fibration if f has the right lifting property with respect to acyclic cofibrations.

We prove this theorem via the following steps. First, we prove two crucial results. The first is the structure of the cofree coalgebra over an operad. The second one is based on the construction, for any P-coalgebra A, of its enveloping cooperad. It expresses the coproduct of a A with a cofree coalgebra in terms of the evaluation of the associated enveloping cooperad functor. Axioms MC2 and MC3 are obvious. Axioms MC1 is proved in an analogue way than in the case of algebras. The main difficulty lies in the proofs of MC4 and MC5. For this aim, we use proofs inspired from that of [14] and adapted to our operadic setting. In order to produce the desired factorization axioms, our trick here is to use a slightly modified version of the usual small object argument. We use smallness with respect to injections systems.

Our main result is obtained by transfering the previous model category structure. We denote by ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ the category of $(P,Q$-bialgebras in $Ch_{\mathbb{K}}^{+}$, where P encodes the operations and Q the cooperations. We use an adjunction

$$U :_{P}^{Q} Ch_{\mathbb{K}}^{+} \rightleftarrows_{P} Ch_{\mathbb{K}}^{+} : Q^{*}.$$

The model category structure on (P,Q)-bialgebras is then given by the following theorem:

Theorem 0.2. *The category of (P,Q)-bialgebras ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ is*
(i) a weak equivalence if $U(f)$ is a weak equivalence in ${}^{Q}Ch_{\mathbb{K}}^{+}$ (i.e a weak equivalence in $Ch_{\mathbb{K}}^{+}$ by definition of the model structure on ${}^{Q}Ch_{\mathbb{K}}^{+}$);
(ii) a fibration if $U(f)$ is a fibration in ${}^{Q}Ch_{\mathbb{K}}^{+}$;
(iii) a cofibration if f has the left lifting property with respect to acyclic fibrations.

The main difficulty is the proof of MC5. We use mainly our refined version of the small object argument combined with a result about cofibrations in algebras over an operad due to Hinich [16].

1. Preliminary notions

In this section, we first recall some notions and facts about Σ-modules, operads and algebras over operads. Then we review the interplay between monads and comonads by means of distributive laws and make the link with operads. It leads us to the crucial definition of bialgebras over pairs of operads in distribution. Finally, we recall a classical tool of homotopical algebra, namely the small object argument, aimed to produce factorizations in model categories. The material of this section is taken from [20], [9] and [17].

1.1. **Operads and their algebras.** For simplicity, definitions of this subsection are given in the category of vector spaces $Vect_{\mathbb{K}}$, where $\mathbb{K}$ is a field of characteristic zero. They extend readily to the category of non-negatively graded chain complexes $Ch_{\mathbb{K}}$, which will be our base category for the next sections.

1.1.1. *Σ-modules, Schur functors and operads.* Let us start with Σ-modules and their associated Schur functors:

Definition 1.1. (1) A Σ-module is a family $M = \{M(n)\}_{n\in\mathbb{N}}$ of right $\mathbb{K}[\Sigma_n]$-modules $M(n)$, where Σ_n is the symmetric group of permutations of $\{1,...,n\}$. It is connected if $M(0)=0$, simply connected if moreover $M(1)=0$. It is finite dimensional if for every $n\in\mathbb{N}$, the vector space $M(n)$ is of finite dimension. For any element $x\in M(n)$, the integer n is the arity of x.

(2) A morphism of Σ-modules $f: M\rightarrow N$ is a family of Σ_n-equivariant maps $f_n: M(n)\rightarrow N(n)$. When all the f_n are injective, M is a sub-Σ-module of N.

To each Σ-module corresponds a Schur functor:

Definition 1.2. Let M be a Σ-module, its Schur functor $M: Vect_{\mathbb{K}}\rightarrow Vect_{\mathbb{K}}$ is defined by

$$M(V)=\bigoplus_{n\in\mathbb{N}} M(n)\otimes_{\Sigma_n} V^{\otimes n}$$

where Σ_n acts on $V^{\otimes n}$ by permuting variables: for any $\sigma\in\Sigma_n$ and $(v_1,...,v_n)\in V^{\otimes n}$, $\sigma.(v_1,...,v_n)=(v_{\sigma^{-1}(1)},...,v_{\sigma^{-1}(n)})$. Then $M(n)\otimes_{\Sigma_n} V^{\otimes n}$ is the space of coinvariants under the diagonal action of Σ_n.

Example 1.3. The Schur functor of $I=(0,\mathbb{K},0,...,0,...)$ is the identity functor.

Now let us give two definitions of an operad. The first is a concise "monoidal" definition in terms of Schur functor:

Definition 1.4. (1) Consider the category of endofunctors $Vect_{\mathbb{K}}\rightarrow Vect_{\mathbb{K}}$ endowed with the functor composition product. An operad is a Σ-module $P=\{P(n)\}_{n\in\mathbb{N}}$ whose Schur functor forms a monoid for the composition product. It means that there are two natural transformations, the composition $\gamma: P\circ P\rightarrow P$ and the unit $\iota: I\rightarrow P$ satisfying the usual monoid axioms:

-*associativity*:

$$\begin{tikzcd} P\circ P\circ P \arrow[r, "P\gamma"] \arrow[d, "\gamma P"'] & P\circ P \arrow[d, "\gamma"] \\ P\circ P \arrow[r, "\gamma"'] & P \end{tikzcd}$$

-*unitarity*:

$$\begin{tikzcd} I\circ P \arrow[r, "\iota P"] \arrow[dr, "="'] & P\circ P \arrow[d, "\gamma"] & P\circ I \arrow[l, "P\gamma"'] \arrow[dl, "="] \\ & P & \end{tikzcd}$$

Such a structure is also called a monad on $Vect_{\mathbb{K}}$.

(2) A morphism of operads $f : P \to Q$ is a natural transformation commuting with the monoid structures.

The second one is the classical definition due to Peter May. One can show that these two definitions coincide (see [20] for a proof):

Definition 1.5. (1) An operad is a Σ-module $P = \{P(n)\}_{n\in\mathbb{N}}$ endowed with $\mathbb{K}$-linear applications called the operadic compositions

$$\gamma(k_1,...,k_n) : P(n)\otimes P(k_1)\otimes ...\otimes P(k_n) \to P(k_1+...+k_n)$$

for $n \geq 1$ and $k_1,...,k_n \geq 0$, and a unit application $\eta : \mathbb{K} \to P(1)$. These applications satisfy the following axioms:

associativity. Let $n \geq 1$ and $m_1,...,m_n,k_1,...,k_m$ natural integers, where $m = m_1+...+m_n$. Let us denote $g_s = m_1+...+m_{s-1}$ and $h_s = k_{g_s+1}+...+k_{g_s+m_s}$ for $1 \leq s \leq n$. Then the following diagram commutes:

$$\begin{tikzcd}
(P(n)\otimes\bigotimes_{s=1}^n P(m_s))\otimes\bigotimes_{r=1}^m P(k_r) \arrow[r, "\gamma(m_1,...,m_n)\otimes id"] \arrow[d, "permutations"'] & P(m)\bigotimes_{r=1}^m P(k_r) \arrow[d, "\gamma(k_1,...,k_m)"] \\
P(n)\otimes\bigotimes_{s=1}^n (P(m_s)\otimes\bigotimes_{q=1}^{m_s} P(k_{g_s+q})) \arrow[r, "\gamma(h_1,...,h_n)\circ id\otimes(\bigotimes_{s=1}^n \gamma(k_{g_s+1},...,k_{g_s+m_s}))"'] & P(k_1+...+k_m)
\end{tikzcd}$$

equivariance. Let $n \geq 1$, $k_1,...,k_n$ be natural integers and $\sigma \in \Sigma_n$, $\tau_1 \in \Sigma_{k_1},...,\tau_n \in \Sigma_{k_n}$ be permutations. Let us note $\sigma(k_1,...,k_n) \in \Sigma_{k_1+...+k_n}$ the permutation permuting the blocks $(1,...,k_1),...,(k_{n-1}+1,...,k_n)$ as σ permutes the elements of $\{1,...,n\}$. Denote also by $\tau_1\oplus...\oplus\tau_n \in \Sigma_{k_1+...+k_n}$ the blockwise sum of the permutations τ_i, that is if $x \in \{k_{i-1}+1,...,k_i\}$ then $(\tau_1\oplus...\oplus\tau_n) = \tau_i(x)$. Then the following diagrams commute:

$$\begin{tikzcd}
P(n)\otimes P(i_1)\otimes ...\otimes P(i_n) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\gamma(i_1,...,i_n)"'] & P(n)\otimes P(i_{\sigma(1)})\otimes ...\otimes P(i_{\sigma(n)}) \arrow[d, "\gamma(i_{\sigma(1)},...,i_{\sigma(n)})"] \\
P(i_1+...+i_n) \arrow[r, "\sigma(i_{\sigma(1)},...,i_{\sigma(n)})"'] & P(i_{\sigma(1)}+...+i_{\sigma(n)})
\end{tikzcd}$$

$$\begin{tikzcd}
P(n)\otimes P(i_1)\otimes ...\otimes P(i_n) \arrow[r, "id\otimes\tau_1\otimes...\otimes\tau_n"] \arrow[d, "\gamma(i_1,...,i_n)"'] & P(n)\otimes P(i_{\sigma(1)})\otimes ...\otimes P(i_{\sigma(n)}) \arrow[d, "\gamma(i_{\sigma(1)},...,i_{\sigma(n)})"] \\
P(i_1+...+i_n) \arrow[r, "\tau_1\oplus...\oplus\tau_n"'] & P(i_1+...+i_n)
\end{tikzcd}$$

unitarity. For every $n \geq 1$ the following diagrams commute :

$$\begin{tikzcd}
\mathbb{K}\otimes P(n) \arrow[r, "\cong"] \arrow[d, "\eta\otimes id"'] & P(n) \\
P(1)\otimes P(n) \arrow[ur, "\gamma(n)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
P(n)\otimes\mathbb{K}^{\otimes n} \arrow[r, "\cong"] \arrow[d, "id\otimes\eta^{\otimes n}"'] & P(n) \\
P(n)\otimes P(1)^{\otimes n} \arrow[ur, "\gamma(1,...,1)"'] &
\end{tikzcd}$$

(2) A morphism of operads $f : P \to Q$ is a family $\{f(n) : P(n) \to Q(n)\}_{n\in\mathbb{N}}$ of Σ_n-equivariant $\mathbb{K}$-linear maps commuting with operadic compositions and preserving units.

Remark 1.6. When $\mathbb{K}$ is an infinite field, we have a fully faithful embedding of the category of Σ-modules in the category of endofunctors of $Vect_{\mathbb{K}}$. However, this is no more true if we suppose $\mathbb{K}$ to be finite. We refer the reader to [21] about this fact. This is a key point to obtain the equivalence between our two definitions of operads.

We can also define suboperads and operadic ideals:

Definition 1.7. Let $P = \{P(n)\}_{n\geq 0}$ and $R = \{R(n)\}_{n\geq 0}$ be two operads. The operad R is a suboperad of P if for every $n \geq 0$, the space $R(n)$ is a sub-$\mathbb{K}[\Sigma_n]$-module of $P(n)$ and if all operadic compositions of R are restrictions of that of P.

Definition 1.8. An ideal in the operad P is a collection $\mathcal{I} = \{\mathcal{I}(n)\}_{n\geq 0}$ of Σ_n-invariant subspaces $\mathcal{I}(n) \subset P(n)$ such that $\gamma(f, g_1, ..., g_n) \in \mathcal{I}(k_1 + ... + k_n)$ if $f \in \mathcal{I}(n)$ or $g_i \in \mathcal{I}(k_i)$ for some $1 \geq i \geq n$.

1.1.2. *Algebras over operads.* Operads are aimed to parametrize various kind of algebraic structures: associative, commutative, Poisson or Lie algebras for instance. This leads us to the general notion of algebra over an operad. The operads we carry about in this paper are algebraic operads, but the reader should note that the first examples of operads where topological operads, namely the little disks operads, introduced in homotopy theory in the 60s in order to understand the structure of iterated loop spaces. We can formulate two alternative definitions of an algebra over an operad:

Definition 1.9. (1) Let P be an operad. A P-algebra is a vector space A endowed with a linear application $\gamma_A : P(A) \to A$ such that the following diagrams commute

$$\begin{tikzcd}
(P\circ P)(A) \arrow[r, "P(\gamma_A)"] \arrow[d, "\gamma(A)"'] & P(A) \arrow[d, "\gamma_A"] \\
P(A) \arrow[r, "\gamma_A"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\iota(A)"] \arrow[dr, "="'] & P(A) \arrow[d, "\gamma_A"] \\
& A
\end{tikzcd} .$$

(2) A morphism of P-algebras $f : A \to B$ is a linear application such that the following diagram commutes :

$$\begin{tikzcd} P(B) \arrow[r, "\gamma_A"] \arrow[d, "P(f)"'] & A \arrow[d, "f"] \\ P(B) \arrow[r, "\gamma_B"'] & B \end{tikzcd}$$

Definition 1.10. Let P be an operad. A P-algebra is a vector space A equiped with linear applications $\alpha_n : P(n) \otimes A^{\otimes n} \to A$ for $n \in \mathbb{N}$ satisfying the following axioms:

associativity. Let $n \geq 1$ and $k_1, ..., k_n$ be natural integers, then the following diagram commutes:

$$\begin{tikzcd} (P(n) \otimes \bigotimes_{s=1}^n P(k_s)) \otimes \bigotimes_{s=1}^n A^{\otimes k_s} \arrow[r, "\gamma(k_1,...,k_n)\otimes id"] \arrow[d, "permutation"'] & P(A) \arrow[d, "\alpha_{k_1+...+k_n}"] \\ P(n) \otimes \bigotimes_{s=1}^n (P(k_s) \otimes A^{\otimes k_s}) \arrow[r, "\alpha_n \circ (id \otimes \bigotimes_{s=1}^n \alpha_{k_s})"'] & A \end{tikzcd}$$

equivariance. For every $n \geq 1$ and $\sigma \in \Sigma_n$ the following diagram commutes:

$$\begin{tikzcd} P(n) \otimes A^{\otimes n} \arrow[rr, "\sigma \otimes \sigma^{-1}"] \arrow[dr, "\alpha_n"'] & & P(n) \otimes A^{\otimes n} \arrow[dl, "\alpha_n"] \\ & A & \end{tikzcd}$$

unitarity. For every $n \geq 1$ the following diagram commutes:

$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "\cong"] \arrow[d, "\eta \otimes id"'] & A . \\ P(1) \otimes A \arrow[ur, "\alpha_1"'] & \end{tikzcd}$$

We will denote ${}_PVect_{\mathbb{K}}$ the category of P-algebras in vector spaces and ${}_PCh_{\mathbb{K}}$ the category of P-algebras in non-negatively graded chain complexes. Let us give some fundamental examples of operads.

Example 1.11. Let $As : Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$ the functor defined by

$$As(V) = \bigoplus_{n \geq 1} V^{\otimes n}.$$

As a Σ-module we have $As = \mathbb{K}[\Sigma_n]$ for $n \geq 1$ and $As(0) = 0$. The composition product γ of As is given by the composition of non commutative polynomials. Every non-unitary associative algebra is an algebra over the operad As. If A is an associative algebra and γ_A the application of definition 1.9, then the component $\gamma_A(2)$ of γ_A in arity 2 determines the associative product on A. We can parametrize unitary associative algebras by slightly modifying As and defining uAs by $uAs(0) = \mathbb{K}$ and $uAs(n) = \mathbb{K}[\Sigma_n]$ for $n \geq 1$. The unit of A is then given by the component $\mathbb{K} \to A$ of γ_A in arity 0.

Example 1.12. Let $Com : Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$ be the functor defined by

$$Com(V) = \bigoplus_{n\geq 1} (V^{\otimes n})_{\Sigma_n}$$

where $(V^{\otimes n})_{\Sigma_n}$ is the quotient of $V^{\otimes n}$ by the left action of Σ_n. As a Σ-module, we have $Com(0) = 0$ and $Com(n) = \mathbb{K}$ for $n \geq 1$. The composition product γ of Com is given by the composition of polynomials. The operad Com parametrizes non-unitary associative and commutative algebras in the way we explained before for As.

Example 1.13. The operad Lie is a more involved example. There is a Schur functor $Lie : Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$ sending a vector space V to the free Lie algebra $Lie(V) \subset T(V)$, where $T(V) = As(V)$ is the tensor algebra of V. This is the subspace generated by V under the commutator $[v, w] = vw - wv$. One can show that there is an operad structure on Lie induced by that of As. One can also show that $Lie(V)$ is the space of primitive elements in the bialgebra structure of $T(V)$. The explicit description of $Lie(n)$ for every n is more complicated than the previous examples, and we refer to [20] for more details.

Example 1.14. To any vector space V we can associate its endomorphism operad End_V defined by $End_V(n) = Hom_{\mathbb{K}}(V^{\oplus n}, V)$. The right action of Σ_n on $End_V(n)$ is induced by its left action on $V^{\oplus n}$. The operadic compositions $\gamma(k_1, ..., k_n)$ are given by partial compositions of morphisms:

$$\gamma(f; f_1, ..., f_n)(v_1, ...v_{k_1+...+k_n}) = f(f_1(v_1, ..., v_{k_1}), ..., f_n(v_{k_1+...+k_{n-1}+1}, ...v_{k_1+...+k_n}))$$

where $f \in Hom_{\mathbb{K}}(V^{\otimes n}, V)$ and $f_i \in Hom_{\mathbb{K}}(V^{\otimes k_i}, V)$.

The endomorphism operad allows us to give a third definition of algebras over operads equivalent to definitions 1.9 and 1.10:

Definition 1.15. Let P be an operad. A P-algebra is the data of a vector space A and an operad morphism $P \to End_A$.

It is well known that the tensor algebra and the symmetric algebra constructions are respectively the free associative algebra functor and the free commutative algebra functor. There exists a more general notion of free algebra functor in the operadic setting:

Definition 1.16. Let V be a vector space. In the category of P-algebras, a P-algebra $\mathcal{F}(V)$ endowed with a linear map $i : V \to \mathcal{F}(V)$ is the free P-algebra on V if it satisfies the following universal property: for every P-algebra A and every linear application $f : V \to A$, there exists a unique factorization

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{F}(V) \arrow[d, "\overline{f}"] \\ & A \end{tikzcd}$$

where $\overline{f}$ is a morphism of P-algebras. A free algebra is unique up to isomorphism. The functor $\mathcal{F} : Vect_{\mathbb{K}} \to_P Vect_{\mathbb{K}}$ is called the free P-algebra functor and is by definition the left adjoint of the forgetful functor $U :_P Vect_{\mathbb{K}} \to Vect_{\mathbb{K}}$.

For every vector space V, we can equip $P(V)$ with a P-algebra structure by setting $\gamma_{P(V)} = \gamma(V) : P(P(V)) \to P(V)$.

Proposition 1.17. *(see [20], proposition 5.2.6) The P-algebra $(P(V), \gamma(V))$ equiped with the map $\iota(V) : I(V) = V \to P(V)$ is the free P-algebra on V.*

1.1.3. *Coalgebras and cooperads.* Let us introduce first the coendomorphism operad :

Definition 1.18. To any vector space V we can associate its coendomorphism operad $coEnd_V$ defined by $coEnd_V(n) = Hom_{\mathbb{K}}(V, V^{\oplus n})$. The right action of Σ_n on $coEnd_V(n)$ is induced by its right action on $V^{\oplus n}$. The operadic compositions $\gamma(k_1, ..., k_n)$ are given by partial compositions of morphisms:

$$\gamma(f; f_1, ..., f_n) = (f_1 \circ f^1) \otimes ... \otimes (f_n \circ f^n) \in Hom_{\mathbb{K}}(V, V^{\otimes k_1 + ... + k_n})$$

where $f \in Hom_{\mathbb{K}}(V, V^{\otimes n})$ and $f_i \in Hom_{\mathbb{K}}(V, V^{\otimes k_i})$. The map f^i is the i^{th} component of f in $V^{\otimes n}$.

Now we can introduce two equivalent definitions of coalgebras over an operad:

Definition 1.19. (1) Let P be an operad. A P-coalgebra is a vector space C equiped with linear applications $\rho_n : P(n) \otimes X \to X^{\otimes n}$ for every $n \geq 0$. These maps are Σ_n-equivariant and associative with respect to the operadic compositions, i.e the following diagram commutes for every $n, k_1, ..., k_n \in \mathbb{N}$:

$$\begin{tikzcd} P(n) \otimes \bigotimes_{s=1}^n P(k_s) \otimes C \arrow[r, "\gamma(k_1,...,k_n)\otimes id"] \arrow[d, "(id\otimes...\otimes id\otimes\rho_n)\circ permutation"'] & P(k_1 + ... + k_n) \otimes C \arrow[d, "\rho_{k_1+...+k_n}"] \\ \bigotimes_{s=1}^n P(k_s) \otimes C^{\otimes n} \arrow[r, "(\bigotimes_{s=1}^n \rho_{k_s})\circ permutation"'] & C^{\otimes(k_1+...+k_n)} \end{tikzcd}$$

If $\mathbb{K}$ is a field of characteristic zero and the $P(n)$ are finite dimensional, then it is equivalent to define applications $\overline{\rho}_n : X \to P(n)^* \otimes_{\Sigma_n} X^{\otimes n}$. (2) A P-coalgebra is a vector space C equiped with an operad morphism $P \to coEnd_C$.

We can also define the dual notion of operads, namely the cooperads:

Definition 1.20. Let $C = \{C(n)\}_{n\in\mathbb{K}}$ a Σ-module such that $C(0) = 0$. It is a cooperad if the associated Schur functor is a comonoid in the endofunctors of $Vect_{\mathbb{K}}$. It means that there exist two natural transformations, the counit $\eta : C \to I$ and the decomposition product $\Delta : C \to C \circ C$ satisfying the following axioms:

coassociativity.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \circ C \arrow[d, "C\Delta"] \\ C \circ C \arrow[r, "\Delta C"'] & C \circ C \circ C \end{tikzcd}$$

counitarity.

$$\begin{tikzcd} & C \arrow[dl, "="'] \arrow[d, "\Delta"] \arrow[dr, "="] & \\ I \circ C & C \circ C \arrow[l, "\eta C"] \arrow[r, "C\eta"'] & C \circ I \end{tikzcd}$$

Such a structure is called a comonad on $Vect_{\mathbb{K}}$. We also suppose that there exists an element $id \in C(1)$ such that $\eta(id) = 1 \in I(1) = \mathbb{K}$, called the identity cooperation.

There is a notion of coalgebra over a cooperad:

Definition 1.21. Let C be a cooperad. A C-coalgebra is a vector space X equiped with a linear application $\rho : X \rightarrow C(X)$ such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\rho"'] & C(X) \arrow[d, "\Delta\rho"] \\ C(X) \arrow[r, "C(\rho)"'] & C(C(X)) \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr, "="'] & C(X) \arrow[d, "\eta\rho"] \\ & X \end{tikzcd} .$$

We can go from operads to cooperads and vice-versa by dualization. Indeed, if C is a cooperad, then the Σ-module P defined by $P(n) = C(n)^* = Hom_{\mathbb{K}}(C(n), \mathbb{K})$ form an operad. Conversely, suppose that $\mathbb{K}$ is of characteristic zero and P is an operad such that each $P(n)$ is finite dimensional. Then the $P(n)^*$ form a cooperad. The additional hypotheses are needed because you have to use, for finite dimensional vector spaces V and W, the isomorphism $(V \otimes W)^* \cong V^* \otimes W^*$ to define properly the decomposition product. Note also that under the same hypotheses, the notion of coalgebra over the operad P is equivalent to the notion of coalgebra over the cooperad P^*.

1.1.4. *Props and bialgebras.* Props encode operations with multiple inputs and outputs and, as operads do for algebras, model the structures of various categories of bialgebras. These bialgebras, for instance associative-coassociative, Lie, Frobenius or Poisson bialgebras appear in a wide range of domains of mathematics. The main problem is that dealing with algebras over props is much more difficult than dealing with algebras over operads. This gap comes from the "combinatorial explosion" of props, which make them quite difficult objects to handle. For instance, contrary to the operadic case, in general there is no free algebra functor. Although props themselves forms a model category, in general algebras over a prop do not form a model category (contrary to algebras over a suitable operad). However, in the most common cases such as associative, Poisson or Lie bialgebras, it turns out that we can construct a model category structure and therefore have access to the usual tools of homotopical algebra. The main goal of this paper is to prove it for a broad class of bialgebras including the aforementioned examples. In the general context, the author has also developed in [28] a study of the classifying space of the category of algebras over a cofibrant prop. There is no model category structure on algebras in this situation. This classifying space is a simplicial set which turns out to be a well defined homotopy invariant of homotopy bialgebras categories.

1.2. **Monads, comonads and distributive laws.** In certain cases, bialgebras can be parametrized by a pair of operads in the following way: one operad encodes the operations, the other encodes the cooperations, such that the concerned bialgebra forms an algebra over the first operad and a coalgebra over the second operad.

The compatibility relations between operations and cooperations are formalized by the notion of distributive law between the two operads. The purpose of this subsection is to explain these notions, starting in the more general context of monads and comonads.

Definition 1.22. Let $\mathcal{C}$ be a category. (a) A monad $\mathcal{T} = (\mathcal{T}, \gamma, \iota)$ in $\mathcal{C}$ is a functor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ equiped with two natural transformations $\gamma : \mathcal{T} \circ \mathcal{T} \to \mathcal{T}$ and $\iota : \mathcal{I} \to \mathcal{T}$ (where $\mathcal{I}$ is the identity functor) satisfying the usual monoid axioms:
-*associativity*:

$$\begin{tikzcd}
\mathcal{T} \circ \mathcal{T} \circ \mathcal{T} \arrow[r, "\mathcal{T}\gamma"] \arrow[d, "\gamma\mathcal{T}"'] & \mathcal{T} \circ \mathcal{T} \arrow[d, "\gamma"] \\
\mathcal{T} \circ \mathcal{T} \arrow[r, "\gamma"'] & \mathcal{T}
\end{tikzcd}$$

-*unitarity*:

$$\begin{tikzcd}
\mathcal{I} \circ \mathcal{T} \arrow[r, "\iota\mathcal{T}"] \arrow[dr, "="'] & \mathcal{T} \circ \mathcal{T} \arrow[d, "\gamma"] & \mathcal{T} \circ \mathcal{I} \arrow[l, "\mathcal{T}\gamma"'] \arrow[dl, "="] \\
 & \mathcal{T} &
\end{tikzcd} .$$

(b) A $\mathcal{T}$-algebra (A, α) is an object A of $\mathcal{C}$ equiped with a morphism $\alpha : \mathcal{T}(A) \to A$ such that the following diagrams commute:

$$\begin{tikzcd}
(\mathcal{T} \circ \mathcal{T})(A) \arrow[r, "\mathcal{T}(\alpha)"] \arrow[d, "\gamma(A)"'] & \mathcal{T}(A) \arrow[d, "\alpha"] \\
\mathcal{T}(A) \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\iota(A)"] \arrow[dr, "="'] & \mathcal{T}(A) \arrow[d, "\alpha"] \\
 & A
\end{tikzcd} .$$

(c) A morphism of $\mathcal{T}$-algebras $(A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ of $\mathcal{C}$ such that the following diagram commute:

$$\begin{tikzcd}
(\mathcal{T}(A) \arrow[r, "\alpha"] \arrow[d, "\mathcal{T}(f)"'] & A \arrow[d, "f"] \\
\mathcal{T}(B) \arrow[r, "\beta"'] & B
\end{tikzcd}$$

We denote $\mathcal{T} - Alg$ the category of $\mathcal{T}$-algebras.

One immediately sees that it corresponds to definitions 1.4 and 1.9, such that operads and their algebras are special cases of monads and their algebras. In a dual way, we can define comonads and coalgebras over comonads:

Definition 1.23. (a) A comonad $\mathcal{S} = (\mathcal{S}, \delta, \epsilon)$ in $\mathcal{C}$ is a functor $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ equiped with two natural transformations $\delta : \mathcal{S} \to \mathcal{S} \circ \mathcal{S}$ and $\epsilon : \mathcal{S} \to \mathcal{I}$ satisfying the usual comonoid axioms:

-*coassociativity*:

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathcal{S}\circ\mathcal{S} \arrow[d, "\mathcal{S}\delta"] \\
\mathcal{S}\circ\mathcal{S} \arrow[r, "\delta\mathcal{S}"'] & \mathcal{S}\circ\mathcal{S}\circ\mathcal{S}
\end{tikzcd}$$

-*counitarity*:

$$\begin{tikzcd}
 & \mathcal{S} \arrow[dl, "="'] \arrow[d, "\delta"] \arrow[dr, "="] & \\
\mathcal{I}\circ\mathcal{S} & \mathcal{S}\circ\mathcal{S} \arrow[l, "\epsilon\mathcal{S}"] \arrow[r, "\mathcal{S}\epsilon"'] & \mathcal{S}\circ\mathcal{I}
\end{tikzcd} .$$

(b) A $\mathcal{S}$-coalgebra (C,c) is an object C of $\mathcal{C}$ equiped with a morphism $c : C \to \mathcal{S}(C)$ such that the following diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "c"'] & \mathcal{S}(C) \arrow[d, "\delta(c)"] \\
\mathcal{S}(C) \arrow[r, "\mathcal{S}(c)"'] & \mathcal{S}(\mathcal{S}(C))
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[dr, "="'] & \mathcal{S}(C) \arrow[d, "\epsilon(c)"] \\
 & C
\end{tikzcd} .$$

(c) A morphism of $\mathcal{S}$-coalgebras $(C,c) \to (D,d)$ is a morphism $f : C \to D$ of $\mathcal{C}$ such that the following diagram commute:

$$\begin{tikzcd}
(C \arrow[r, "c"] \arrow[d, "f"'] & \mathcal{S}(C) \arrow[d, "\mathcal{S}(f)"] \\
D \arrow[r, "d"'] & \mathcal{S}(D)
\end{tikzcd}$$

We denote $\mathcal{S} - Coalg$ the category of $\mathcal{S}$-coalgebras.

Again, we see that cooperads and their coalgebras are special cases of comonads and their coalgebras.

Now, suppose we have in our category $\mathcal{C}$ a monad $(\mathcal{T},\gamma,\iota)$ and a comonad $(\mathcal{S},\delta,\epsilon)$. We would like to make $\mathcal{T}$ and $\mathcal{S}$ compatible, that is to define $\mathcal{S}$-coalgebras in $\mathcal{T}$-algebras or conversely $\mathcal{T}$-algebras in $\mathcal{S}$-coalgebras. This compatibility is formalized by the notion of mixed distributive law:

Definition 1.24. A mixed distributive law $\lambda : \mathcal{T}\mathcal{S} \to \mathcal{S}\mathcal{T}$ between $\mathcal{T}$ and $\mathcal{S}$ is a natural transformation satisfying the following conditions:
(i)$\Lambda \circ \gamma\mathcal{S} = \mathcal{S}\gamma \circ \Lambda$
(ii)$\delta\mathcal{T} \circ \Lambda = \Lambda \circ \mathcal{T}\delta$
(iii)$\lambda \circ \iota\mathcal{S} = \mathcal{S}\iota$
(iv)$\epsilon\mathcal{T} \circ \lambda = \mathcal{T}\epsilon$

where the $\Lambda : \mathcal{T}^m\mathcal{S}^n \rightarrow \mathcal{S}^n\mathcal{T}^m$, for every natural integers m and n, are the natural transformations obtained by iterating λ. For instance, for $m = 2$ and $n = 3$ we have

$$\mathcal{T}^2\mathcal{S}^3 \xrightarrow{\mathcal{T}\lambda\mathcal{S}^2} \mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S}^2 \xrightarrow{\lambda^2\mathcal{S}} \mathcal{S}\mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S} \xrightarrow{\mathcal{S}\lambda^2} \mathcal{S}^2\mathcal{T}\mathcal{S}\mathcal{T} \xrightarrow{\mathcal{S}^2\lambda\mathcal{T}} \mathcal{S}^3\mathcal{T}^2$$

This conditions allow us to lift $\mathcal{T}$ as an endofunctor of $\mathcal{S} - Coalg$ and $\mathcal{S}$ as an endofunctor of $\mathcal{T} - Alg$.

Finally we introduce the notion of bialgebra over a pair (monad,comonad) endowed with a mixed distributive law:

Definition 1.25. (a) Given a monad $\mathcal{T}$, a comonad $\mathcal{S}$ and a mixed distributive law $\lambda : \mathcal{T}\mathcal{S} \rightarrow \mathcal{S}\mathcal{T}$, a $(\mathcal{T},\mathcal{S})$-bialgebra (B,β,b) is an object B of $\mathcal{C}$ equiped with two morphisms $\beta : \mathcal{T}(B) \rightarrow B$ and $b : B \rightarrow \mathcal{S}(B)$ defining respectively a $\mathcal{T}$-algebra structure and a $\mathcal{S}$-coalgebra structure. Furthermore, the maps β and b satisfy a compatibility condition expressed through the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{T}(\mathcal{S}(B)) \arrow[d, "\lambda(B)"'] & \mathcal{T}(B) \arrow[l, "\mathcal{T}(b)"'] \arrow[dd, "\beta"] \\
\mathcal{S}(\mathcal{T}(B)) \arrow[d, "\mathcal{S}(\beta)"'] & \\
\mathcal{S}(B) & B \arrow[l, "b"]
\end{tikzcd}$$

(b) A morphism of $(\mathcal{T},\mathcal{S})$-bialgebras is a morphism of $\mathcal{C}$ which is both a morphism of $\mathcal{T}$-algebras and a morphism of $\mathcal{S}$-coalgebras.

The category of $(\mathcal{T},\mathcal{S})$-bialgebras is denoted $(\mathcal{T},\mathcal{S}) - Bialg$.

Remark 1.26. The application $\mathcal{S}(\beta)\circ\lambda(B)$ endows $\mathcal{S}(B)$ with a $\mathcal{T}$-algebra structure, and the application $\lambda(B)\circ\mathcal{T}(b)$ endows $\mathcal{T}(B)$ with a $\mathcal{S}$-coalgebra structure. Moreover, given these two structures, the compatibility diagram of definition 1.25 shows that β is a morphism of $\mathcal{S}$-coalgebras and b a morphism $\mathcal{T}$-algebras. The $(\mathcal{T},\mathcal{S})$-bialgebras can therefore be considered as $\mathcal{S}$-coalgebras in $\mathcal{T} - Alg$ or as $\mathcal{T}$-algebras in $\mathcal{S} - Coalg$.

In the particular case of operads, the mixed distributive laws can be defined by explicite formulae:

Definition 1.27. Let P and Q be two operads. A mixed distributive law between P and Q is a family of applications $\{M(m,n)\}_{m,n\geq 1}$ where

$$M(m,n) : P(m)\otimes Q(n) \rightarrow \bigoplus (Q(t_1)\otimes ...\otimes Q(t_m))\otimes_{\Sigma_{t_1}\times...\times\Sigma_{t_m}} \mathbb{K}(\Sigma_N)\otimes_{\Sigma_{s_1}\times...\times\Sigma_{s_n}} (P(s_1)\otimes ...\otimes P(s_n))$$

where the direct sum is indexed by every $N \geq 1$ and $t_1+...+t_m = s_1+...+s_n = N$. Moreover, these applications have to be compatible with operadic compositions and symmetric groups actions at the inputs and the outputs. The detailed axioms can be found in [9].

Suppose that $\mathbb{K}$ is of characteristic zero and that every $Q(n)$ is finite dimensional. Then we know that the notions of Q-coalgebras and Q^*-coalgebras coincide. The notion of Q^*-coalgebra is exactly the definition of a coalgebra over a comonad.

We can therefore define a (P,Q)-bialgebra in the sense of definition 1.25, with a P-algebra structure, a Q-coalgebra structure and compatibilities with respect to the distributive law. The operadic distributive law as defined in definition 1.26 formalizes the interplay between algebraic operations and coalgebraic cooperations of the bialgebra.

Theorem 1.28. *(cf. [9], theorem 11.10) Let B be a (P,Q)-coalgebra. Then the free P-algebra $P(B)$ has a natural structure of Q-coalgebra and the cofree Q-coalgebra $Q^*(B)$ has a natural structure of P-algebra.*

1.3. **Model categories and the small object argument.** Model categories were introduced in [27] as an effective approach to do localization with respect to a particular class of morphisms called the weak equivalences. The original motivation was to transpose ideas related to homotopies, fibrations and cofibrations in topological spaces in a more general and categorical framework. Model categories are therefore the natural setting to do homotopical algebra. This means that they encode well defined notions of cylinder objects and path objects, homotopy classes, non-abelian cohomology theories and non abelian functor derivation (Quillen's derived functors). We will just recall here some basic facts about model categories, cofibrantly generated model categories and the small object argument. We will not review the construction of the associated homotopy category and its properties. We refer the reader to the classical reference [27], but also to [8] for a well-written and detailed account on basis of model categories and their homotopy theories, as well as [17] to push the analysis further.

Definition 1.29. A (closed) model category is a category $\mathcal{M}$ with the data of three classes of morphisms: the weak equivalences $\xrightarrow{\sim}$, the cofibrations $\rightarrowtail$ and the fibrations $\twoheadrightarrow$. Each of theses classes is stable by composition and contains the identity morphisms. A morphism which is both a cofibration and a weak equivalence is called an acyclic cofibration, and a morphism which is both a fibration and a weak equivalence is called an acyclic fibration. The following axioms hold:

MC1. Small limits and colimits exist in $\mathcal{M}$ (*completeness axiom*).

MC2. Weak equivalences satisfy the "two-out-of-three" property: if f and g are two composable morphisms such that two among f, g and $f\circ g$ are weak equivalences, then so is the third (*two-out-of-three axiom*).

MC3. If f is a retract of g and g belongs to one of the three aforementioned classes, then so does f (*retract axiom*).

MC4. Given a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

a lifting exists in the two following situations :

(i) i is cofibration and p an acyclic fibration. One says that cofibrations have the left lifting property with respect to acyclic fibrations.

(ii)i is an acyclic cofibration and p a fibration. One says that fibrations have the right lifting property with respect to acyclic cofibrations. These are the *lifting axioms*.

MC5. Every morphism f admits the two following factorizations:

(i)$f = p \circ i$ where p is a fibration and i an acyclic cofibration.
(ii)$f = p \circ i$ where p is an acyclic fibration and i a cofibration.
These are the *factorization axioms.*

An object is cofibrant if its initial morphism is a cofibration, and fibrant if its final morphism is a fibration. Cofibrations and fibrations satisfy the following properties:

Proposition 1.30. *(cf.* [8]*, proposition 3.13) Let $\mathcal{M}$ be a model category.*
(i)The cofibrations of $\mathcal{M}$ are the morphisms which have the left lifting property with respect to the acyclic fibrations.
(ii) The acyclic cofibrations of $\mathcal{M}$ are the morphisms which have the left lifting property with respect to the fibrations.
(iii)The fibrations of $\mathcal{M}$ are the morphisms which have the right lifting property with respect to the acyclic cofibrations.
(iv)The acyclic fibrations of $\mathcal{M}$ are the morphisms which have the right lifting property with respect to the cofibrations.

Remark 1.31. According to this proposition, it is sufficient to define the weak equivalences and the cofibrations to get the fibrations, or to define the weak equivalences and the fibrations to get the cofibrations.

Proposition 1.32. *(cf.* [8]*, proposition 3.14) Let $\mathcal{M}$ be a model category.*
(i)The class of cofibrations is stable by cobase change.
(ii)The class of acyclic cofibrations is stable by cobase change.
(iii)The class of fibrations is stable by base change.
(iv)The class of acyclic fibrations is stable by base change.

In the most common cases, the core of the model category structure consists of the class of weak equivalences and two sets of generators for the cofibrations and acyclic cofibrations. Any (acyclic) cofibration is obtained by retracts and pushouts of these generators, and (acyclic) fibrations are obtained by their right lifting property. Such a model category is said to be cofibrantly generated. Moreover, when one has found the sets of generators, there is a classical construction called the small object argument which produces the factorization axioms needed to satisfy axiom MC5. Let us define more precisely these notions. We start with the small object argument, which is a general and useful way to produce factorizations with lifting properties with respect to a given class of morphisms. We just sum up the construction given in [8] without detailing the process. It is important to note that, although the sequential colimits used here run over the natural integers, the small object argument works for higher ordinals. We refer the reader to [17] for a detailed treatment in full generality.

Definition 1.33. An object A of $\mathcal{M}$ is sequentially small if for every functor $F : \mathbb{N} \to \mathcal{M}$, the canonical map

Remark 1.34. A $\mathbb{K}$-module is sequentially small if and only if it admits a finite presentation, i.e it is isomorphic to the cokernel of a morphism of finitely generated free $\mathbb{K}$-modules. A chain complex M is sequentially small if and only if a finite number of M_n are non trivial and each M_n has a finite presentation.

Let $\mathcal{F} = \{f_i : A_i \to B_i\}_{i \in I}$ a set of morphisms of $\mathcal{M}$. We consider a morphism $p : X \to Y$ of $\mathcal{C}$ for which we want to produce a factorization $X \to X' \to Y$, such that $X' \to Y$ has the right lifting property with respect to the morphisms of $\mathcal{F}$.

We do not consider the trivial case $X' = Y$. Then there is a recursive construction providing the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "p"] & G^1(\mathcal{F},p) \arrow[r, "i_2"] \arrow[d, "p_1"] & ... \arrow[r, "i_k"] & G^k(\mathcal{F},p) \arrow[r, "i_{k+1}"] \arrow[d, "p_k"] & ... . \\ Y \arrow[r, equal] & Y \arrow[r, equal] & ... \arrow[r, equal] & Y \arrow[r, equal] & ... \end{tikzcd}$$

In this recursive procedure, each i_k is obtained by a pushout of the form

$$\begin{tikzcd} \bigoplus_\alpha A_\alpha \arrow[r] \arrow[d, "\bigoplus_\alpha f_\alpha"'] & G^{k-1}(\mathcal{F},p) \arrow[d, "i_k"] \\ \bigoplus_\alpha B_\alpha \arrow[r] & G^k(\mathcal{F},p) \end{tikzcd}$$

where the f_α are morphisms of $\mathcal{F}$. The category $\mathcal{M}$ is supposed to admit small colimits, so we can consider the infinite composite $i_\infty : X \rightarrow G^\infty(\mathcal{F},p)$ of the sequence of maps

$$X \xrightarrow{i_1} G^1(\mathcal{F},p) \xrightarrow{i_2} ... \xrightarrow{i_k} G^k(\mathcal{F},p) \xrightarrow{i_{k+1}} ... \longrightarrow G^\infty(\mathcal{F},p)$$

where $G^\infty(\mathcal{F},p)$ is the sequential colimit of this system. The morphism $i_\infty : X \rightarrow G^\infty(\mathcal{F},p)$ is called a relative $\mathcal{F}$-cell complex. By universal property of the colimit, the morphism p has a factorization $p = p_\infty \circ i_\infty$ where $p_\infty : G^\infty(\mathcal{F},p) \rightarrow Y$.

Proposition 1.35. *(cf. [8], proposition 7.17) In the preceding situation, suppose that for every $i \in I$, the object A_i is sequentially small in $\mathcal{M}$. Then the morphism p_∞ has the right lifting property with respect to the morphisms of $\mathcal{F}$.*

Now we can define the notion of cofibrantly generated model category:

Definition 1.36. The model category $\mathcal{M}$ is said to be cofibrantly generated if there exists two sets of morphisms $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{M}$ such that:
(i) The domains of the morphisms of $\mathcal{I}$ are small.
(ii) The domains of the morphisms of $\mathcal{J}$ are small.
(iii) The class of fibrations is the class of morphisms having the right lifting property with respect to the morphisms of $\mathcal{J}$.
(iv) The class of acyclic fibrations is the class of morphisms having the right lifting property with respect to the morphisms of $\mathcal{I}$.
One says that $\mathcal{I}$ is the set of generating cofibrations and $\mathcal{J}$ the set of generating acyclic cofibrations.

A usual way to construct a cofibrantly generated model category is the following: start by defining these two sets of generators. Then use the small object argument in order to obtain for any morphism f a factorization $f = f_\infty \circ i_\infty$ such that f_∞ has the right lifting property with respect to $\mathcal{I}$ (resp. $\mathcal{J}$). Define the acyclic fibrations (resp. fibrations) as the morphisms having the right lifting property with respect to the morphisms of $\mathcal{I}$ (resp. $\mathcal{J}$). The application f_∞ forms therefore an acyclic fibration (resp. a fibration). Afterwards, define cofibrations as relative $\mathcal{I}$-cell complexes and acyclic cofibrations as relative $\mathcal{J}$-cell complexes. In particular, this implies that i_∞ is a cofibration (resp. acyclic cofibration), and thus the factorization axioms MC5 hold.

In the remaining sections of our paper, in order to deal with applications of the form of i_∞ we will need the two following lemmas:

Lemma 1.37. *Let us consider a pushout of the form*

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i"'] & K' \arrow[d, "j"] \\ L \arrow[r, "g"'] & L' \end{tikzcd}$$

in a category $\mathcal{C}$ admitting small colimits. Suppose that i has the left lifting property with respect to a given family $\mathcal{F}$ of morphisms of $\mathcal{C}$. Then j has also the left lifting property with respect to $\mathcal{F}$. Another way to state this result is to say that the left lifting property with respect to a given family of morphisms is invariant under cobase change.

Proof. Let $p : E \to B$ be a morphism of $\mathcal{F}$. Let us consider a commutative square

$$\begin{tikzcd} K' \arrow[r, "a"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ L' \arrow[r, "b"'] & B \end{tikzcd} .$$

We obtain a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i"'] & K' \arrow[r, "a"] & E \arrow[d, "p"] \\ L \arrow[r, "g"'] & L' \arrow[r, "b"'] & B \end{tikzcd} .$$

in which a lifting $h : L \to E$ exists by property of i. We then obtain a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i"'] & K' \arrow[d, "j"] \arrow[ddr, "a"] & \\ L \arrow[r, "g"] \arrow[drr, "h"'] & L' \arrow[dr, dashed, "\varphi"] & \\ & & E \end{tikzcd}$$

and thus a morphism $\varphi : L' \to E$ such that $\varphi \circ j = a$ and $\varphi \circ g = h$ by universal property of the pushout. We have a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i"'] & K' \arrow[d, "j"] \arrow[ddr, "p\circ a"] & \\ L \arrow[r, "g"] \arrow[drr, "b\circ g"'] & L' \arrow[dr, "b"] & \\ & & B \end{tikzcd}$$

and also

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "i"'] & K' \arrow[d, "j"] \arrow[ddr, "a"] & \\
L \arrow[r, "g"'] \arrow[drr, "h"'] & L' \arrow[dr, "p\circ\varphi"] & \\
& & E
\end{tikzcd} .$$

The unicity property of morphism factorizations through a pushout implies that $p \circ \varphi = b$, so ϕ is the desired lifting. ☐

Lemma 1.38. *Let $\mathcal{C}$ be a category admitting small colimits. Let us consider a sequential direct system*

$$G^0 \xrightarrow{i_1} G^1 \xrightarrow{i_2} ... \xrightarrow{i_k} G^k \xrightarrow{i_{k+1}} ... \longrightarrow colim_k G^k = G^\infty .$$

Let us note $i_\infty : G^0 \to G^\infty$ the transfinite composite of the i_k. If for every $k \geq 0$, the morphism i_k has the left lifting property with respect to a given family $\mathcal{F}$ of morphisms of $\mathcal{C}$, then so does i_∞.

Proof. Let us consider a commutative square

$$\begin{tikzcd}
G^0 \arrow[r, "u"] \arrow[d, "i_1"'] & X \arrow[ddd, "f\in\mathcal{F}"] \\
G^1 \arrow[d, "i_2"] & \\
\vdots \arrow[d, dashed] & \\
G^\infty \arrow[r, "v"'] & Y
\end{tikzcd} .$$

We obtain a diagram

$$\begin{tikzcd}
G^0 \arrow[rrr, "u"] \arrow[d, "i_1"'] & & & X \arrow[d, "f"] \\
G^1 \arrow[r, "i_2"'] & ... \arrow[r] & G^\infty \arrow[r, "v"'] & Y
\end{tikzcd} .$$

By hypothesis there exists a lifting $\omega_1 : G^1 \to X$ in this diagram. We construct the ω_k recursively: let $k \geq 1$, suppose the ω_i are constructed for $1 \leq i \leq k-1$, then in the commutative square

$$\begin{tikzcd}
G^{k-1} \arrow[rrr, "\omega_{k-1}"] \arrow[d, "i_k"'] & & & X \arrow[d, "f"] \\
G^k \arrow[r, "i_{k+1}"'] & ... \arrow[r] & G^\infty \arrow[r, "v"'] & Y
\end{tikzcd} .$$

there exists a lifting ω_k. We then obtain the diagram

$$\begin{tikzcd} \cdots \arrow[r] \arrow[drr] & G^k \arrow[rr, "i_{k+1}"] \arrow[dr, "\omega_k"] & & G^{k+1} \arrow[r] \arrow[dl, "\omega_{k+1}"'] & \cdots . \arrow[dll] \\ & & X & & \end{tikzcd}$$

By universal property of the colimit, the ω_k factorize via G^∞ and thus give rise to a morphism $\omega_\infty : G^\infty \rightarrow X$ such that $\omega_\infty \circ \varphi_k = \omega_k$, where φ_k is the transfinite composite of the ω_l for $l \geq k$. We conclude that ω_∞ is the desired lifting. □

It is time now to give a concrete example of model category. Of course, topological spaces provide the initial example from which the theory of model categories arised. However, the example we will use to illustrate these notions is that of chain complexes. This choice is motivated by two reasons. Firstly, this will be the base category for the remaining part of our paper. Secondly, the model category structures of algebras and coalgebras over operads will be transfered from this one via adjunctions.

Theorem 1.39. *(cf.* [8]*, theorem 7.2) The category $Ch_{\mathbb{K}}$ of chain complexes over a field $\mathbb{K}$ forms a cofibrantly generated model category such that a morphism f of $Ch_{\mathbb{K}}$ is*

(i) a weak equivalence if for every $n \geq 0$, the induced map $H_n(f)$ in homology is an isomorphism.

(ii) a fibration if for every $n > 0$, the map f_n is surjective.

(iii) a cofibration if for every $n \geq 0$, the map f_n is injective.

For $n \geq 1$, the chain complex D^n is defined by

$$D^n_k = \begin{cases} 0 & k \neq n, n-1 \\ \mathbb{K}b_{n-1} & k = n-1 \\ \mathbb{K}e_n & k = n \end{cases}$$

with $deg(b_{n-1} = n-1$, $deg(e_n) = n$, and a differential δ satisfying $\delta(e_n) = b_{n-1}$. The chain complex S^n is defined by

$$S^n_k = \begin{cases} 0 & k \neq n \\ \mathbb{K}b_n & k = n \end{cases}$$

with $deg(b_n) = n$. We have for every $n \geq 1$ an obvious inclusion $j_n : S^{n-1} \rightarrow D^n$ which is the identity on $\mathbb{K}b_{n-1}$. The sets of generating cofibrations and generating acyclic cofibrations are given by the following proposition:

Proposition 1.40. *(cf.* [8]*, proposition 7.19) A morphism f of $Ch_{\mathbb{K}}$ is*

(i) a fibration if and only if for every $n \geq 1$, it has the right lifting property with respect to the inclusions $i_n : 0 \rightarrow D^n$.

(ii)an acyclic fibration if and only if for every $n \geq 1$, it has the right lifting property with respect to the inclusions $j_n : S^{n-1} \rightarrow D^n$.

2. The model category of algebras over an operad

The most general statement about model categories of algebras over operads holds in any cofibrantly generated symmetric monoidal model category: algebras over a Σ-cofibrant operad (i.e the underlying Σ-module is cofibrant in the model category of Σ-modules) form a cofibrantly generated semi-model category, see [12] theorem 12.3.A. The semi-model structure is a weakened model structure in which the lifting and factorization axioms can be applied only on morphisms with cofibrant domains.

We fix an operad P in the category of non-negatively graded chain complexes $Ch_{\mathbb{K}}$ over a field $\mathcal{K}$ of characteristic zero. In this case we construct a full model category structure. Moreover, the proof of theorem 3.1 illustrates fundamental methods for the remaining part of this paper. The central theorem of this section is the following:

Theorem 2.1. *The category of P-algebras ${}_PCh_{\mathbb{K}}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}_PCh_{\mathbb{K}}$ is*

(i)a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}$, where U is the forgetful functor;

(ii)a fibration if $U(f)$ is a fibration in $Ch_{\mathbb{K}}$;

(iii)a cofibration if it has the left lifting property with respect to acyclic fibrations.

We can also say that cofibrations are relative cell complexes with respect to the generating cofibrations.

We will make the generating cofibrations and generating acyclic cofibrations explicit in 2.3. The three classes defined above contain identity and are clearly stable by composition. Thus it remains to prove the MC axioms. Axioms MC2 and MC3 are clear and easily proved as in the case of chain complexes (theorem 1.39). Axiom MC4(i) is obvious by definition of the cofibrations.

Acutally, what we have obtained is a transfer of cofibrantly generated model category structure via the adjunction $P : Ch_{\mathbb{K}} \rightleftarrows_P Ch_{\mathbb{K}} : U$. The forgetful functor creates fibrations and weak equivalences. The free P-algebra functor P preserves generating (acyclic) cofibrations as we will see in 2.3, by definition of the generating (acyclic) cofibrations of ${}_PCh_{\mathbb{K}}$. Moreover, it preserves colimits as a left adjoint (it is a general property of adjunctions, see [23] for instance). Thus it preserves all (acyclic) cofibrations, which are relative cell complexes with respect to the generating (acyclic) cofibrations. Such a pair of functors is called a Quillen adjunction, and induces an adjunction at the level of the associated homotopy categories.

2.1. **Small limits and colimits.**

2.1.1. *The small limits.* The forgetful functor creates the small limits in ${}_PCh_{\mathbb{K}}$. Indeed, let us consider a diagram $\{A_i\}_{i\in I}$ of P-algebras. We obtain a diagram $\{U(A_i)\}_{i\in I}$ in $Ch_{\mathbb{K}}$ via the forgetful functor U. The category $Ch_{\mathbb{K}}$ admits small limits so $lim_iU(A_i)$ exists.

The structure of P-algebra on A_i is the data of $\mathbb{K}$-linear maps $\alpha_n^i : P(n)\otimes A_i^{\otimes n} \rightarrow A_i$ satisfying the appropriate properties of associativity, equivariance and unitarity. The limit $lim_iU(A_i)$ is equiped with projections $\pi_i : lim_iU(A_i) \rightarrow U(A_i)$, thus we get a linear map

$$P(n)\otimes (lim_iU(A_i))^{\otimes n} \overset{id\otimes\pi_i^{\otimes n}}{\rightarrow} P(n)\otimes U(A_i)^{\otimes n} \overset{\alpha_n^i}{\rightarrow} U(A_i)$$

which factorizes via $lim_iU(A_i)$ by universal property of the limit, hence a commutative square for every i and n

$$\begin{tikzcd} P(n)\otimes U(A_i)^{\otimes n} \arrow[r, "\alpha_n^i"] & U(A_i) \\ P(n)\otimes(lim_iU(A_i))^{\otimes n} \arrow[u, "id\otimes\pi_i^{\otimes n}"] \arrow[r, "\alpha_n^{\infty}"'] & lim_iU(A_i) \arrow[u, "\pi_i"'] \end{tikzcd} .$$

The α_n^{∞} endow $lim_iU(A_i)$ with a structure of P-algebra. The structure morphism $\gamma_{A_i} : P(A_i) \rightarrow A_i$ is the sum of the α_n^i in each arity n, so the following diagram commutes for every i:

$$\begin{tikzcd} P(A_i) \arrow[r, "\gamma_{A_i}"] & A_i \\ P(lim_iU(A_i)) \arrow[u, "P(\pi_i)"] \arrow[r, "\gamma_{\infty}"'] & lim_iA_i \arrow[u, "\pi_i"'] \end{tikzcd}$$

where γ_{∞} is the sum of the α_n^{∞} in each arity n and constitutes the structure morphism of P-algebra of lim_iA_i. It proves that the π_i are morphisms of P-algebras. We conclude that $lim_iU(A_i)$ endowed with the P-algebra structure defined by the α_n^{∞} is the limit of $\{A_i\}$ in ${}_PCh_{\mathbb{K}}$.

2.1.2. *The small colimits.* First recall the definition of a reflexive coequalizer:

Definition 2.2. Let $\mathcal{C}$ be a category and $f,g : A \rightarrow B$ two arrows $\mathcal{C}$. A coequalizer of (f,g) is an arrow $u : B \rightarrow E$ such that
(i) $u\circ f = u\circ g$
(ii) if $h : B \rightarrow C$ satisfies $h\circ f = h\circ g$, then h admits a unique factorization $h = h'\circ u$.

Definition 2.3. A pair of morphisms $f,g : A \rightarrow B$ is reflexive if there exists a morphism $s : B \rightarrow A$ such that $f\circ s = g\circ s = id_B$. The coequalizer of a reflexive pair is called a reflexive coequalizer.

The reflexive coequalizers allow us to build all the small colimits. This arises from the following theorem:

Theorem 2.4. *Let $\mathcal{C}$ be a category. If $\mathcal{C}$ contains the reflexive coequalizers of every pairs of arrows and all small coproducts, then $\mathcal{C}$ contains all small colimits.*

Proof. Let $\{X_i\}_{i\in I}$ be a diagram in $\mathcal{C}$, i.e a functor $X : I \rightarrow \mathcal{C}$ where I is a small category. We consider the following pair of morphisms:

$$\bigvee_{u:i\rightarrow j\in Mor(I)} X_i \rightrightarrows^{-} d_{0-}d_1 \bigvee_{k\in ob(I)} X_k$$

where

$$d_0 : (X_i, u : i\rightarrow j) \mapsto (X_i, i) \hookrightarrow \bigvee_k X_k$$

and

$$d_0 : (X_i, u : i\rightarrow j) \mapsto (X_j, j) \hookrightarrow_{-} u^* \bigvee_k X_k.$$

We define $s_0 : \bigvee_{k\in ob(I)} X_k \to \bigvee_{u:i\to j\in Mor(I)} X_i$ by $s_0(X_k) = (X_k, id_k : k \to k)$. We see clearly that $d_0 \circ s_0 = d_1 \circ s_0 = id_{\bigvee_{k\in ob(I)} X_k}$. By hypothesis, there exists a reflexive coequalizer for the reflexive pair (d_0, d_1):

$$\bigvee_{u:i\to j\in Mor(I)} X_i \rightrightarrows^{-} d_0 {}_{-} d_1 \bigvee_{k\in ob(I)} X_k \to_{-} \pi Y.$$

It remains to show that this coequalizer Y satisfies the universal property of a colimit of $\{X_i\}_{i\in I}$. Let $\{f_i : X_i \to Y\}_{i\in I}$ a familiy of morphisms of $\mathcal{C}$. The f_i admit a unique factorization via $\bigvee_{k\in ob(I)} X_k$ through a map $f : \bigvee_{k\in ob(I)} X_k \to Y$. The fact that $f \circ d_0 = f \circ d_1$ is equivalent to the fact that the f_i commute with the arrows of the diagram $\{X_i\}_{i\in I}$. Indeed, it means that for every $u : i \to j \in Mor(I)$, $(f \circ d_0) |_(X_i, u) = (f \circ d_1) |_(X_i, u)$, i.e that $f_i = f_j \circ u^*$. In this case, the morphism f admit a unique factorization via the reflexive coequalizer Y of (d_0, d_1), so the f_i admits a unique factorization via Y. We conclude that $Y = colim_I X_i$. $\square$

Now we verify that ${}_PCh_{\mathbb{K}}$ contains the reflexive coequalizers of every pairs of morphisms and all small coproducts, which conclude our proof of the existence of small colimits and thus our proof of axiom MC1.

Lemma 2.5. *Let us consider a reflexive pair $(d_0, d_1 : A \rightrightarrows B, s_0 : B \to A)$ in ${}_PCh_{\mathbb{K}}$. Then $coker(d_0 - d_1)$ has a structure of P-algebra and forms the reflexive coequalizer of (d_0, d_1) in ${}_PCh_{\mathbb{K}}$.*

Proof. We recall briefly that an ideal of the P-algebra B is a sub-chain complex $I \subset U(B)$ (where U is the forgetful functor) such that for every $\mu \in P(n), r_1, ..., r_{n-1} \in B, x \in I$, $\mu(r_i, ..., r_{n-1}) \in I$. Let us show that $im(d_0 - d_1)$ is an ideal of B. Let $\mu \in P(n), b_1, ..., b_{n-1} \in B, a \in A$. Then

$$\begin{cases} \mu(b_1, ..., b_{n-1}, d_0(a)) = \mu((d_0 \circ s_0)(b_1), ..., (d_0 \circ s_0)(b_{n-1}), d_0(a)) \\ d_0(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) \\ (d_0 - d_1)(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) + d_1(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) \\ (d_0 - d_1)(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) + \mu(b_1, ..., b_{n-1}, d_1(a)). \end{cases}$$

In this series of equalities, we use that $d_0 \circ s_0 = d_1 \circ s_0 = id_B$ and that these are morphisms of P-algebras, therefore they commute with the operations. We deduce that

$$\mu(b_1, ..., b_{n-1}, (d_0 - d_1)(a)) = (d_0 - d_1)(\mu(s_0(b_1), ..., s_0(b_{n-1}), a)) \in im(d_0 - d_1).$$

Then we can equip $coker(d_0 - d_1) = B/im(d_0 - d_1)$ with the structure of P-algebra induced by that of B via the projection $\pi : B \to B/im(d_0 - d_1)$:

$$\begin{tikzcd} P(n) \otimes B^{\otimes n} \arrow[r, "\alpha_n^B"] \arrow[d, "id\otimes\pi^{\otimes n}"'] & B \arrow[d, "\pi"] \\ P(n) \otimes coker(d_0 - d_1)^{\otimes n} \arrow[r, "\alpha_n^{coker}"'] & coker(d_0 - d_1) \end{tikzcd}$$

where α_n^B and α_n^{coker} denote the P-algebras structures respectively of B and $coker(d_0 - d_1)$. We deduce the following commutative square:

$$\begin{tikzcd} P(B) \arrow[r, "\gamma_B"] \arrow[d, "P(\pi)"'] & B \arrow[d, "\pi"] \\ P(coker(d_0-d_1)) \arrow[r, "\gamma_{coker}"'] & coker(d_0-d_1) \end{tikzcd}$$

so π is a morphism of P-algebras. Furthermore, the space $coker(d_0 - d_1)$ is the reflexive coequalizer of (d_0, d_1) in $Ch_{\mathbb{K}}$. We conclude that it is their coequalizer in ${}_PCh_{\mathbb{K}}$. □

Lemma 2.6. *Let $\{R_i\}_{i\in I}$ be a set of P-algebras. We set*

$$d_0 = P(\bigoplus \gamma_{R_i}) : P(\bigoplus P(R_i)) \to P(\bigoplus R_i)$$

and

$$d_1 = \gamma(\bigoplus R_i) \circ P(+_i P(i_{R_i})) : P(\bigoplus P(R_i)) \hookrightarrow P(P(\bigoplus_{R_i})) \to P(\bigoplus R_i)$$

where γ is the composition product of the monad (P, γ, ι) and the $i_{R_j} : R_j \to \bigoplus R_i$ are inclusions. Then $\bigvee R_i = coker(d_0 - d_1)$ is the coproduct of the R_i in ${}_PCh_{\mathbb{K}}$.

Proof. We detail the proof in the case of two P-algebras R and S. The method is the same in the general case. Let us consider $s_0 = P(\iota(R) \oplus \iota(S)) : P(R \oplus S) \to P(P(R) \oplus P(S))$. Then

$$d_0 \circ s_0 = P(\gamma_R \oplus \gamma_S) \circ P(\iota(R) \oplus \iota(S)) = P((\gamma_R \circ \iota(R)) \oplus (\gamma_S \circ \iota(S))) = P(id_{R\oplus S}) = id_{P(R\oplus S)}$$

by definition of γ_R, γ_S and the functoriality of P. We also get

$$d_1 \circ s_0 = \gamma(R \oplus S) \circ P(P(i_R), P(i_S)) \circ P(\iota(R) \oplus \iota(S)) = id_{P(R\oplus S)}$$

by unitality of ι. According to the preceding lemma, the space $coker(d_0, d_1)$ is the reflexive coequalizer of (d_0, d_1) in ${}_PCh_{\mathbb{K}}$. Now let X be a P-algebra. Two linear maps $u : R \to X$, $v : S \to X$ induce a unique application $u + v : R \oplus S \to X$, which admits a unique factorization through a morphism of P-algebras $\varphi_{(u,v)} : P(R\oplus S) \to X$ by universal property of the free P-algebra. It remains to prove that $\varphi_{(u,v)}$ factorizes via $coker(d_0 - d_1)$ if and only if u and v are morphisms of P-algebras. The map $\varphi_{(u,v)}$ factorizes via $coker(d_0 - d_1)$ if and only if $\varphi_{(u,v)} \circ d_0 = \varphi_{(u,v)} \circ d_1$, or

$$\begin{aligned} \varphi_{(u,v)\circ d_0} = \varphi_{(u,v)} \circ d_1 \quad &\Leftrightarrow \quad \begin{cases} \varphi_{(u,v)\circ d_0} \mid_{P(R)} = \varphi_{(u,v)} \circ d_1 \mid_{P(R)} \\ \varphi_{(u,v)\circ d_0} \mid_{P(S)} = \varphi_{(u,v)} \circ d_1 \mid_{P(S)} \end{cases} \\ &\Leftrightarrow \quad \begin{cases} u \circ \gamma_R = \gamma_X \circ P(u) \\ v \circ \gamma_S = \gamma_X \circ P(v) \end{cases} \end{aligned}$$

which is by definition equivalent to the fact that u and v are morphims of P-algebras. □

2.2. **Enveloping operad.** Let A be a P-algebra. We can associate to it a particular operad called the enveloping operad of A. First, let us note that for every natural integers r and n, we have an obvious group injection $\Sigma_r \hookrightarrow \Sigma_{n+r}$: every permutation of Σ_r can be extended to a permutation of Σ_{n+r} by fixing the elements $\{r+1,...,n+r\}$. We then consider the Σ-module $P[A]$ given by

$$P[A](n) = \bigoplus_{r=1}^{\infty} P(n+r)\otimes_{\Sigma_r} A^{\otimes r}.$$

We need the following lemma:

Lemma 2.7. *Let A be a P-algebra. For every chain complex C of $Ch_{\mathbb{K}}$, we have $P[A](C)\cong P(A\oplus C)$.*

Proof. Let us note

$$Sh_{p,q} = \{\sigma\in\Sigma_{p+q} \mid \sigma(1)<...<\sigma(p), \sigma(p+1)<...<\sigma(p+q)\}$$

the set of (p,q)-shuffles, then

$$P(n)\otimes_{\Sigma_n}(A\oplus C)^{\otimes n} = \left(\bigoplus_{Sh_{p,q},p+q=n} P(n)\otimes A^{\otimes p}\otimes C^{\otimes q}\right)_{\Sigma_n} = \bigoplus_{p+q=n} P(p+q)\otimes_{\Sigma_p\times\Sigma_q}(A^{\otimes p}\otimes C^{\otimes q})$$

hence

$$\begin{aligned}
P(A\otimes C) &= \bigoplus_{n\geq 0} P(n)\otimes_{\Sigma}(A\oplus C)^{\otimes n}\\
&= \bigoplus_{n}\bigoplus_{p+q=n} P(p+q)\otimes_{\Sigma_p\times\Sigma_q}(A^{\otimes p}\otimes C^{\otimes q})\\
&= \bigoplus_{n}\bigoplus_{p+q=n} (P(p+q)\otimes_{\Sigma_p} A^{\otimes p})\otimes_{\Sigma_q} C^{\otimes q})\\
&= \bigoplus_{q} P[A](q)\otimes_q C^{otimesq}\\
&= P[A](C).
\end{aligned}$$

□

The P-algebra structure morphism $\gamma_A : P(A)\rightarrow A$ induces a morphism of Σ-modules $d_0 : P[P(A)]\rightarrow P[A]$ defined by

$$d_0(n) = \bigoplus_{r=1}^{\infty}(id\otimes\gamma_A^{\otimes r}) : \bigoplus_{r=1}^{\infty} P(n+r)\otimes_{\Sigma_r} P(A)^{\otimes r}\rightarrow \bigoplus_{r=1}^{\infty} P(n+r)\otimes_{\Sigma_r} A^{\otimes r}.$$

The operadic composition product $\gamma : P\circ P\rightarrow P$ induces another morphism of Σ-modules $d_1 : P[P(A)]\rightarrow P[A]$: for every chain complex C of $Ch_{\mathbb{K}}$, there is a map

$$P[P(A)](C)\cong P(P(A)\oplus C) \overset{P(P(i_A),i_C)}{\rightarrow} P(P(A\oplus C)) \overset{\gamma(A\oplus C)}{\rightarrow} P(A\oplus C)\cong P[A](C)$$

where $i_A : A\hookrightarrow A\oplus C$ and $i_C : C\hookrightarrow P(A\oplus C)$. We know that we have a faithful embedding of the category of Σ-modules in the category of endofunctors (the one which associates to every Σ-module its Schur functor, see remark 1.6). Therefore the morphism of Schur functors above corresponds to a unique morphism of Σ-modules $d_1 : P[P(A)]\rightarrow P[A]$.

The unit $\iota : I \to P$ induces a morphism of Σ-modules $s_0 : P[A] \to P[P(A)]$ with $s_0(0) = P(\iota(A))$, obtained by the following morphism of Schur functors: for every chain complex C, define

$$P[A](C) \cong P(A \oplus C) \overset{P(\iota(A\oplus C))}{\to} P(P(A \oplus C)) \overset{P(P(pr_A), \pi\circ P(pr_C))}{\to} P(P(A) \oplus C) \cong P[P(A)](C)$$

where π is the projection on the component of arity 1 and $pr_A : A \oplus C \to A$, $pr_C : A \circ C \to C$ the obvious projections. Thus we finally get a reflexive pair (d_0, d_1) of morphisms of Σ-modules induced by a reflexive pair of morphisms of Schur functors. The enveloping operad of A is then the reflexive coequalizer $U_P(A) = coker(d_0 - d_1)$ in the Σ-modules, endowed with the operad structure induced by that of $P[A]$.

Now we want to prove that for every chain complex C, there is an isomorphism $U_P(A)(C) \cong A \vee P(C)$ where $\vee$ is the coproduct in ${}_PCh_{\mathbb{K}}$. We need the following expression of such a coproduct:

Lemma 2.8. *Let A be a P-algebra and C be a chain complex of $Ch_{\mathbb{K}}$. The following coequalizer defines the coproduct $A \vee P(C)$ in the P-algebras:*

$$\begin{tikzcd} P(P(A) \oplus C) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & P(A \oplus C) \arrow[l, bend right, "s_0"'] \arrow[r] & coker(d_0 - d_1) = A \vee P(C) \end{tikzcd}$$

where $d_0 \mid_A = \gamma_A$, $d_0 \mid_C = id_C$, $d_1 \mid_A = \gamma(A)$, $d_1 \mid_C = id_C$, $s_0 \mid_A = \iota(A)$, $s_0 \mid_C = id_C$.

Proof. We clearly have $d_0 \circ s_0 = d_1 \circ s_0 = id$ hence a reflexive pair in ${}_PCh_{\mathbb{K}}$. The cokernel $coker(d_0 - d_1)$ is thus the reflexive coequalizer of (d_0, d_1) in ${}_PCh_{\mathbb{K}}$ according to lemma 2.6. Let X be a P-algebra, $u : A \to X$ a morphism of P-algebras and $v : C \to X$ a linear map. These two maps induce an application $(u, v) : A \oplus C \to X$ hence an application $\varphi_{(u,v)} : P(A \oplus C) \to X$ by universal property of the free P-algebra. The proof ends by noting that $\varphi_{(u,v)}$ admits a unique factorization through $coker(d_0 - d_1)$. $\square$

The reflexive coequalizer defining the enveloping operad induces a reflexive coequalizer in P-algebras

$$\begin{tikzcd} P[P(A)](C) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & P[A] \arrow[l, bend right, "s_0"'] \arrow[r] & U_P(A)(C) \end{tikzcd}$$

where $P[A](C) \cong P(A \oplus C)$, $P[P(A)](C) \cong P(P(A) \oplus C)$ and d_0, d_1, s_0 turn out to be the morphisms of the lemma above. By unicity of the colimit, we have proved the following result:

Proposition 2.9. *Let A be a P-algebra and C a chain complex of $Ch_{\mathbb{K}}$, then*

$$U_P(A)(C) \cong A \vee P(C).$$

2.3. **Generating (acyclic) cofibrations, proofs of MC4 and MC5.**

2.3.1. *Generating (acyclic) cofibrations.* The generating (acyclic) cofibrations are, as expected, the images of the generating (acyclic) cofibrations of $Ch_{\mathbb{K}}$ under the free P-algebra functor P. Recall that the $j_n : S^{n-1} \hookrightarrow D^n$ and the $i_n : 0 \hookrightarrow D^n$ are respectively the generating cofibrations and the generating acyclic cofibrations of $Ch_{\mathbb{K}}$.

Proposition 2.10. *Let $f : A \to B$ be a morphism of P-algebras.*

(i)It is a fibration if and only if it has the right lifting property with respect to the $P(i_n)$ for every $n \geq 1$, i.e the $P(i_n)$ are the generating acyclic cofibrations.

(ii)It is an acyclic fibration if and only if it has the right lifting property with respect to the $P(j_n)$ for every $n \geq 1$, i.e the $P(j_n)$ are the generating cofibrations.

Proof. Part (ii) can be proved in the same way than part (i), so we only give the details for part (i). Suppose that $f : A \to B$ is a fibration and consider a commutative square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "P(i_n)"] & A \arrow[d, "f"] \\ P(D^n) \arrow[r, "v"'] & B \end{tikzcd}$$

in ${}_PCh_{\mathbb{K}}$. Via the forgetful functor we obtain in $Ch_{\mathbb{B}}$ a commutative square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "(U\circ P)(i_n)"] & U(A) \arrow[d, "U(f)"] \\ (U\circ P)(D^n) \arrow[r, "U(v)"'] & U(B) \end{tikzcd} .$$

The unit $\eta : id_{Ch_{\mathbb{K}}} \to U \circ P$ associated to the adjunction between P and U provides a commutative diagram

$$\begin{tikzcd} 0 \arrow[rr] \arrow[d, "i_n"'] & & U(A) \arrow[d, "U(f)"] \\ D^n \arrow[r, "\eta(D^n)"'] \arrow[urr, dashed] & (U\circ P)(D^n) \arrow[r, "U(v)"'] & U(B) \end{tikzcd} .$$

A lifting $\hat{v} : D^n \to U(A)$ exits in this diagram, given that $U(f)$ is a fibration and has therefore the right lifting property with respect to i_n. By applying P we obtain a new commutative diagram

$$\begin{tikzcd} 0 \arrow[rr] \arrow[d, "P(i_n)"'] & & (P\circ U)(A) \arrow[d, "(P\circ U)(f)"] \\ P(D^n) \arrow[r, "P(\eta(D^n))"'] \arrow[urr, "P(\hat{v})"] & (P\circ U\circ P)(D^n) \arrow[r, "(P\circ U)(v)"'] & (P\circ U)(B) \end{tikzcd}$$

in ${}_PCh_{\mathbb{K}}$. The counity $\epsilon : P \circ U \to id_{{}_PCh_{\mathbb{K}}}$ associated to the adjunction gives rise to the commutative diagram

$$\begin{tikzcd} 0 \arrow[rr] \arrow[d, "P(i_n)"'] & & (P\circ U)(A) \arrow[r, "\epsilon(A)"] \arrow[d, "(P\circ U)(f)"] & A \arrow[d, "f"] \\ P(D^n) \arrow[r, "P(\eta(D^n))"'] \arrow[urr, "P(\hat{v})"] & (P\circ U\circ P)(D^n) \arrow[r, "(P\circ U)(v)"'] & (P\circ U)(B) \arrow[r, "\epsilon(B)"] & B \end{tikzcd} .$$

Moreover, the following diagrams commute:

$$\begin{tikzcd}
(P\circ U)(P(D^n)) \arrow[r, "(P\circ U)(v)"] \arrow[d, "\epsilon(P(D^n))"'] & (P\circ U)(B) \arrow[d, "\epsilon(B)"] \\
P(D^n) \arrow[r, "v"'] & B
\end{tikzcd}$$

by naturality of ϵ, and

$$\begin{tikzcd}
P \arrow[r, "P\eta"] \arrow[dr, equal] & P\circ U\circ P \arrow[d, "\epsilon P"] \\
& P
\end{tikzcd}$$

which is a property associated to any adjunction, see [23] for more details. We conclude that $\epsilon(B)\circ(P\circ U)(v)\circ P(\eta(D^n)) = v\circ(\epsilon P\circ P\eta)(D^n) = v\circ id_{P(D^n)} = v$. Thus $\epsilon(A)\circ P(\hat{v}) : P(D^n)\rightarrow A$ is the desired lifting.

We have to prove the other direction of the equivalence. Let us suppose that f has the right lifting property with respect to the $P(i_n)$ and consider the commutative square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "i_n"'] & U(A) \arrow[d, "U(f)"] \\
D^n \arrow[r, "v"'] & U(B)
\end{tikzcd} .$$

By applying P we obtain

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "P(i_n)"'] & (P\circ U)(A) \arrow[d, "(P\circ U)(f)"] \\
P(D^n) \arrow[r, "P(v)"'] & (P\circ U)(B)
\end{tikzcd} ,$$

hence via the counity ϵ of the adjunction

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "P(i_n)"'] & (P\circ U)(A) \arrow[r, "\epsilon(A)"] & A \arrow[d, "f"] \\
P(D^n) \arrow[r, "P(v)"'] \arrow[urr, "h"] & (P\circ U)(B) \arrow[r, "\epsilon(B)"'] & B
\end{tikzcd}$$

where h exists by hypothesis about f. We apply U:

$$\begin{tikzcd}
0 \arrow[rr] \arrow[d, "(U\circ P)(i_n)"'] & & U(A) \arrow[d, "U(f)"] \\
(U\circ P)(D^n) \arrow[r, "(U\circ P)(v)"'] \arrow[urr, "U(h)"] & (U\circ P\circ U)(B) \arrow[r, "U(\epsilon(B))"'] & U(B)
\end{tikzcd}$$

hence via the unity η of the adjunction:

$$\begin{tikzcd} 0 \arrow[rrr] \arrow[d, "i_n"'] & & & U(A) \arrow[d, "U(f)"] \\ D^n \arrow[r, "\eta(D^n)"'] & (U\circ P)(D^n) \arrow[r, "(U\circ P)(v)"'] \arrow[urr, "U(h)"] & (U\circ P\circ U)(B) \arrow[r, "U(\epsilon(B))"'] & U(B) \end{tikzcd} .$$

Morevoer the following diagrams commute:

$$\begin{tikzcd} D^n \arrow[r, "\eta(D^n)"] \arrow[d, "v"'] & (U\circ P)(D^n) \arrow[d, "(U\circ P)(v)"] \\ U(B) \arrow[r, "\eta U(B)"'] & (U\circ P\circ U)(B) \end{tikzcd}$$

by naturality of η, and

$$\begin{tikzcd} U \arrow[r, "\eta U"] \arrow[dr, equal] & U\circ P\circ U \arrow[d, "U\epsilon"] \\ & U \end{tikzcd}$$

which is a property associated to any adjunction. We deduce that $U(\epsilon(B))\circ(U\circ P)(v)\circ\eta(D^n) = (U\epsilon\circ\eta U)(B)\circ v = id_{U(B)}\circ v = v$. Therefore $U(h)\circ\eta(D^n) : D^n \to U(A)$ is the desired lifting: the morphism $U(f)$ forms a fibration in $Ch_{\mathbb{K}}$, which implies by definition that f forms a fibration in ${}_PCh_{\mathbb{K}}$. $\square$

2.3.2. *Axioms MC4 and MC5.* **MC5 (i).**In order to apply the small object argument to the $P(j_n)$ and consequently obtain MC5 (i), we need the following lemma:

Lemma 2.11. *Let C be a chain complex of $Ch_{\mathbb{K}}$. If C is sequentially small in $Ch_{\mathbb{K}}$, then $P(C)$ is sequentially small in ${}_PCh_{\mathbb{K}}$.*

Proof. Let us suppose that C is sequentially small in $Ch_{\mathbb{K}}$, and let $F : \mathbb{N} \to_P Ch_{\mathbb{K}}$ be a functor. For every $n \in \mathbb{N}$,

$$Hom_{{}_PCh_{\mathbb{K}}}(P(C), F(n)) \cong Hom_{Ch_{\mathbb{K}}}(C, (U\circ F)(n))$$

hence

$$\begin{aligned} colim_n Hom_{{}_PCh_{\mathbb{K}}}(P(C), F(n)) &\cong colim_n Hom_{Ch_{\mathbb{K}}}(C, (U\circ F)(n)) \\ &\cong Hom_{Ch_{\mathbb{K}}}(C, colim_n (U\circ F)(n)) \end{aligned}$$

because $U\circ F : \mathbb{N} \to Ch_{\mathbb{K}}$ and C is sequentially small. We can equip $colim_n(U\circ F)(n)$ with a structure of P-algebra, such that with this structure it forms the colimit of the $F(n)$ in ${}_PCh_{\mathbb{K}}$. Indeed, we have $colim_n(U\circ F)(n) = \{[a], a \in F(n)\}/\sim$ where $a\sim b$ (i.e $[a]=[b]$), $a\in F(n), b\in F(m)$, $n\leq m$, if the application $F(n)\to F(m)$ in the sequential system sends a to b. Let $[a_1],...,[a_r]\in colim_n(U\circ F)(n)$ such that $a_1\in F(n_1),...,a_r\in F(n_r)$. We consider $F(n)$ for a given $n\geq max(n_1,...,n_r)$ and we set, for $\mu\in P(n)$, $\mu([a_1],...,[a_r]) = \mu(a_1',...,a_r')$ where $a_1',...,a_r'$ are representing elements of $[a_1],...,[a_r]$ in $F(n)$. We then obtain a P-algebra structure on $colim_n(U\circ F)(n)$ (one says that the forgetful functor creates the sequential colimits). We can finally write

$$\begin{aligned} colim_n Hom_{{}_PCh_{\mathbb{K}}}(P(C),F(n)) &\cong Hom_{Ch_{\mathbb{K}}}(C,U(colim_n F(n))) \\ &\cong Hom_{{}_PCh_{\mathbb{K}}}(P(C),colim_n F(n)). \end{aligned}$$

□

The S^{n-1} are sequentially small in $Ch_{\mathbb{K}}$, so the $P(S^{n-1})$ are sequentially small in ${}_PCh_{\mathbb{K}}$. We can then apply the small object argument to a given morphism $f:A\rightarrow B$ of ${}_PCh_{\mathbb{K}}$ and the family of morphisms $\mathcal{F}=\{P(j_n)\}_{n\geq 1}$. We obtain a factorization $f=p_\infty\circ i_\infty$ where $i_\infty:A\rightarrow G^\infty(\mathcal{F},f)$, $p_\infty:G^\infty(\mathcal{F},f)\rightarrow B$ and p_∞ has the right lifting property with respect to the $P(j_n)$. According to proposition 2.10, the morphism p_∞ is an acyclic fibration. According to lemmas 1.37 and 1.38, the morphism i_∞ has the right lifting property and forms therefore a cofibration. We have the desired factorization.

MC5 (ii). In order to prove MC5 (ii), we need two general results about Σ-modules:

Proposition 2.12. *Let M be a Σ-module and C a chain complex. If $H_*(C)=0$ then $H_*(M(C))=H_*(M(0))$.*

Proof. Recall that we work over a field $\mathbb{K}$ of characteristic 0. We use the norm map $N:M(n)\otimes_{\Sigma_n}C^{\otimes n}\rightarrow M(n)\otimes C^{\otimes n}$ defined by

$$N(c\otimes v_1\otimes ...\otimes v_n)=\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\sigma.c\otimes v_{\sigma(1)}\otimes ...\otimes v_{\sigma(n)}.$$

If we denote $p:M(n)\otimes C^{\otimes n}\rightarrow M(n)\otimes_{\Sigma_n}C^{\otimes n}$ the projection, then

$$\begin{aligned} (p\circ N)(c\otimes v_1\otimes ...\otimes v_n) &= \frac{1}{n!}\sum_{\sigma\in\Sigma_n}p(\sigma.c\otimes v_{\sigma(1)}\otimes ...\otimes v_{\sigma(n)}) \\ &= \frac{1}{n!}\mid\Sigma_n\mid c\otimes v_1\otimes ...\otimes v_n \\ &= c\otimes v_1\otimes ...\otimes v_n \end{aligned}$$

so $p\circ N=id$. Therefore $M(n)\otimes_{\Sigma_n}C^{\otimes n}$ is a retract of $M(n)\otimes C^{\otimes n}$. For $n\geq 1$, the Künneth formula gives us for every $k\geq 0$

$$H_k(M(n)\otimes C^{\otimes n})=\bigoplus_{p+q=k}H_p(M(n)\otimes C)\otimes H_q(C^{\otimes n-1}).$$

This is equal to 0 for $n>1$ because the fact that $H_*(C)=0$ implies recursively that $H_*(C^{\otimes n})=0$ by the Künneth formula. This is also equal to 0 for $n=1$ because the fact that $H_k(C)=0$ implies that $H_k(M(1)\otimes C)=0$. For $n=0$, we have $H_k(M(0))$. We conclude that $H_k(M(C))=H_k(M(0))$. □

Lemma 2.13. *Let M be a Σ-module. An injection $A\hookrightarrow B$ in $Ch_{\mathbb{K}}$ induces an injection $M(A)\hookrightarrow M(B)$.*

Proof. We know that $M(n)\otimes_{\Sigma_n}A^{\otimes n}$ is a retract of $M(n)\otimes A^{\otimes n}$ by using the projection and the norm maps. Moreover, the tensor product over a field preserves the injections so if $A\hookrightarrow B$ is an injection, then we obtain for every $n\geq 1$ the injections $u_n:M(n)\otimes A^{\otimes n}\hookrightarrow M(n)\otimes B^{\otimes n}$. We deduce injections $\tilde{u_n}=p\circ u_n\circ N:M(n)\otimes_{\Sigma_n}A^{\otimes n}\hookrightarrow M(n)\otimes_{\Sigma_n}B^{\otimes n}$, hence an injection $\bigoplus_n\tilde{u_n}:M(A)\hookrightarrow M(B)$. □

Now we can start the proof of MC5 (ii). Recall that the family of generating acyclic cofibrations of ${}_PCh_{\mathbb{K}}$ is given by $\mathcal{F} = \{P(i_n) : P(0) \rightarrow P(D^n)\}_{n\geq 1}$. Let $f : X \rightarrow Y$ be a morphism of P-algebras. For every $k > 0$, the construction of $G^k(\mathcal{F}, f)$ (see section 1.3 about the small object argument) follows from a pushout of the form:

$$\begin{tikzcd} \bigvee P(0) \arrow[r] \arrow[d, "\bigvee P(i_n)"'] & G^{k-1}(\mathcal{F}, f) \arrow[d, "i_k"] \\ \bigvee P(D^n) \arrow[r] & G^k(\mathcal{F}, f) \end{tikzcd} .$$

According to lemma 1.37, the left lifting property with respect to a given family of morphisms is invariant under cobase change. The coproduct $\bigvee P(i_n)$ has the left lifting property with respect to fibrations, so i_k has the same property, in particular i_k has the left lifting property with respect to acyclic fibrations. According to lemma 1.38, the transfinite composite i_∞ of the i_k inherits such a property and forms therefore a cofibration. The small object argument provides a factorization $f = p_\infty \circ i_\infty$ where $p_\infty : G^\infty(\mathcal{F}, f) \rightarrow Y$ has the right lifting property with respect to generating acyclic cofibrations and forms therefore a fibration. We have seen that i_∞ is a cofibration. It remains to prove that it is acyclic. For this aim, we use the following classical lemma:

Lemma 2.14. *Let $\{C_n\}$ be a sequential direct system of chain complexes, then there is an isomorphism $colim_n H_*(C_n) \cong H_*(colim_n C_n)$.*

Suppose that all the i_k are weak equivalences, i.e induce isomorphisms in homology. If we apply the homology functor H_* to the sequential direct system of the i_k, we obtain a sequential direct system in which every arrow is an isomorphism, so the transfinite composite of these arrows is also an isomorphism. By composing it with the isomorphism of the lemma above we obtain the isomorphism $H_*(i_\infty)$, i.e i_∞ is acyclic. Therefore we just have to prove that the i_k are acyclic.

The chain complex 0 is the initial object of $Ch_{\mathbb{K}}$, so via the adjunction $P : Ch_{\mathbb{K}} \rightleftarrows_P Ch_{\mathbb{K}} : U$ the object $P(0)$ is initial in ${}_PCh_{\mathbb{K}}$. The coproduct of any object A with the initial object is isomorphic to A. Another general categorical fact is that in any category endowed with an initial object I and admitting the coproduct of two objects A and B, this coproduct $A \vee B$ corresponds to the pushout

$$\begin{tikzcd} I \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \vee B \end{tikzcd} .$$

We deduce from these facts that $\bigvee P(0) \cong P(0)$, $G^{k-1}(\mathcal{F}, f) \cong G^{k-1}(\mathcal{F}, f) \vee P(0)$ and therefore $G^k(\mathcal{F}, f) = G^{k-1}(\mathcal{F}, f) \vee (\bigvee P(D^n))$. Furthermore, $\bigvee P(D^n) \cong P(\bigoplus D^n)$ and proposition 2.9 implies that

$$G^{k-1}(\mathcal{F}, f) \vee P(0) \cong U_P(G^{k-1}(\mathcal{F}, f))(0)$$

and

$$G^k(\mathcal{F}, f) \cong G^{k-1}(\mathcal{F}, f) \vee P(\bigoplus D^n) \cong U_P(G^{k-1}(\mathcal{F}, f))(\bigoplus D^n).$$

We obtain consequently the following pushout:

$$\begin{tikzcd} P(0) \arrow[r] \arrow[d] & U_P(G^{k-1}(\mathcal{F},f))(0) \arrow[d, "i_k"] \\ P(\bigoplus D^n) \arrow[r] & U_P(G^{k-1}(\mathcal{F},f))(\bigoplus D^n) \end{tikzcd} .$$

Lemma 2.13 applied to the Σ-module $U_P(G^{k-1}(\mathcal{F},f))$ implies that i_k is an injection. Given that $H_*(\bigoplus D^n)=0$, proposition 2.12 implies that $H_*U_P(G^{k-1}(\mathcal{F},f))(\bigoplus D^n)\cong U_P(G^{k-1}(\mathcal{F},f))(0)$ and i_k is acyclic. We conclude that i_∞ is acyclic, which achieve our proof of MC5 (ii).

MC4 (i). Obvious by definition of the cofibrations.

MC4 (ii). We have to use axiom MC5 (ii), which will be proved below. Let $f : X\to Y$ be an acyclic cofibration, according to MC5 (ii) f admits a factorization $f=p_\infty\circ i_\infty$ where $i_\infty : X\to G^\infty(\mathcal{F},f)$ is an acyclic cofibration and $p_\infty : G^\infty(\mathcal{F},f)\to Y$ a fibration. The morphisms f and i_∞ are weak equivalences, therefore p_∞ is also a weak equivalence according to MC2. We obtain a commutative square

$$\begin{tikzcd} X \arrow[r, "i_\infty"] \arrow[d, "f"'] & G^\infty(\mathcal{F},f) \arrow[d, "p_\infty"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

where a lifting h exists because f is a cofibration and has thus the left lifting property with respect to acyclic fibrations. The morphism f is therefore a retract of i_∞ via the retraction diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i_\infty"'] & X \arrow[d, "f"'] \\ Y \arrow[r, "h"'] & G^\infty(\mathcal{F},f) \arrow[r, "p_\infty"'] & Y \end{tikzcd} .$$

The $P(0)\to P(D^n)$ have the left lifting property with respect to fibrations, so by lemma 1.37 the maps i_k inherit this property, and so does i_∞ by lemma 1.38. Now consider a commutative square

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & A \arrow[d, "p"] \\ Y \arrow[r, "\beta"'] & B \end{tikzcd}$$

where p is a fibration. Combined with the retraction diagram it gives rise to the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i_\infty"'] & X \arrow[r, "\alpha"] \arrow[d, "f"'] & A \arrow[d, "p"] \\ Y \arrow[r, "h"'] & G^\infty(\mathcal{F},f) \arrow[r, "p_\infty"'] & Y \arrow[r, "\beta"'] & B \end{tikzcd} .$$

In the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_\infty"'] & A \arrow[d, "p"] \\ G^\infty(\mathcal{F}, f) \arrow[r, "\beta\circ p_\infty"'] & B \end{tikzcd}$$

there exists a lifting $\tilde{h}$. Now let us define $\hat{h} = \tilde{h} \circ h$. Then

$$\hat{h} \circ f = \tilde{h} \circ h \circ f = \tilde{h} \circ i_\infty = \alpha$$

and

$$p \circ \hat{h} = p \circ \tilde{h} \circ h = \beta \circ p_\infty \circ h = \beta$$

so h is the desired lifting : the acyclic cofibration f has the left lifting property with respect to fibrations.

3. The model category of coalgebras over an operad

The previous results are available in the category $Ch_{\mathbb{K}}$ of non-negatively graded chain complexes over a field $\mathbb{K}$ of characteristic zero. Roughly speaking, we will follow in this section a combination of the methods of section 2 and that of [14]. However, we also need the following additional assumptions.We work in the full subcategory $Ch_{\mathbb{K}}^+$ of $Ch_{\mathbb{K}}$ whose objects are the chain complexes C such that $C_0 = 0$, i.e the connective chain complexes. The category $Ch_{\mathbb{K}}^+$ is actually a model subcategory of $Ch_{\mathbb{K}}$. We suppose that P is an operad in $Ch_{\mathbb{K}}^+$ such that the $P(n)$ are finite dimensional, $P(0) = 0$ and $P(1) = \mathbb{K}$. Note that the commonly used operads satisty this hypothesis, for instance As (for the associative algebras), Com (for the commutative associative algebras), Lie (for the Lie algebras), $Pois$ (for the Poisson algebras). The model category structure on coalgebras is given by the following theorem:

Theorem 3.1. *The category of P-coalgebras ${}^P Ch_{\mathbb{K}}^+$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^P Ch_{\mathbb{K}}^+$ is*
(i) a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}^+$;
(ii) a cofibration if $U(f)$ is a cofibration in $Ch_{\mathbb{K}}^+$;
(iii) a fibration if f has the right lifting property with respect to acyclic cofibrations.

The three class of morphisms defined in this theorem are clearly stable by composition and contain the identity maps. Axioms MC2 and MC3 are clear, and MC4 (ii) is obvious by definition of the fibrations. It remains to prove axioms MC1, MC4 (i) and MC5. We first need a description of the cofree P-coalgebra functor, and the notion of enveloping cooperad.

3.1. **Cofree coalgebra over an operad.** There exists a cofree P-algebra functor $P^* : Ch_{\mathbb{K}}^+ \rightarrow_P Ch_{\mathbb{K}}^+$, which is by definition the right adjoint to the forgetful functor and is given by the following formula:

Theorem 3.2. *Let V be an object of $Ch_{\mathbb{K}}^+$. Then*

$$P^*(V) = \bigoplus_{r=1}^{\infty} P(r)^* \otimes_{\Sigma_r} V^{\otimes r}$$

inherits a P-coalgebra structure and forms the cofree P-coalgebra.

For the needs of the proof we give the following definition:

Definition 3.3. Let I be a finite set of cardinal k, then we define $P(I)$ by $P(I) = Bij(\underline{k}, I) \otimes_{\Sigma_k} P(k)$ where $\underline{k} = 1, ..., k$. The elements of $P(I)$ of the form $u \otimes \mu$, $u \in Bij(\underline{k}, I)$ and $\mu \in P(k)$, satisfy $u \otimes \sigma.\mu = (u \circ \sigma) \otimes \mu$ for any permutation $\sigma \in \Sigma_k$.

The classical operadic compositions

$$\gamma(n_1, ..., n_r) : P(r) \otimes P(n_1) \otimes ... \otimes P(n_r) \rightarrow P(n_1 + ... + n_r)$$

extend to these objects:

$$\gamma(I_1, ..., I_r) : P(r) \otimes P(I_1) \otimes ... \otimes P(I_r) \rightarrow P(I_1 \amalg ... \amalg I_r)$$

is defined by

$$\gamma(I_1, ..., I_r)(\mu \otimes (q_1, v_1) \otimes ... \otimes (q_r, v_r)) = (\mu(q_1, ..., q_r), v_1 \amalg ... \amalg v_r)$$

where $q_i \in P(n_i)$, the map $v_i : \underline{n_i} \rightarrow I_i$ is a bijection, and $\mu(q_1, ..., q_r) = \gamma(n_1, ..., n_r)(\mu, q_1, ..., q_r)$. We have a bijection $\underline{n_1} \amalg ... \amalg \underline{n_r} \cong \underline{n_1 + ... + n_r}$ by renumbering the elements, so $v_1 \amalg ... \amalg v_r : \underline{n_1 + ... + n_r} \rightarrow I_1 \amalg ... \amalg I_r$ is a well defined bijection.

Proof. We want to equip $P^*(V)$ with a P-coalgebra structure, i.e linear applications $\rho_r : P(r) \otimes P^*(V) \rightarrow P^*(V)^{\otimes r}$ with the adequate properties. Let us first compute explicitely $P(r) \otimes P^*(V)$ and $P^*(V)^{\otimes r}$:

$$P(r) \otimes P^*(V) = \bigoplus_{n=1}^{\infty} P(r) \otimes P(n)^* \otimes_{\Sigma_n} V^{\otimes n}.$$

In arity n, we have

$$(P^*(V)^{\otimes r})_n = \bigoplus_{i_1 + ... + i_r = n} P^*(V)_{i_1} \otimes ... \otimes P^*(V)_{i_r}$$

so

$$\begin{aligned} P^*(V)^{\otimes r} &= \bigoplus_n \bigoplus_{i_1 + ... + i_r = n} P^*(V)_{i_1} \otimes ... \otimes P^*(V)_{i_r} \\ &= \bigoplus_n \bigoplus_{I_1 \amalg ... \amalg I_r = \underline{n}} (P(I_1)^* \otimes_{\Sigma_{I_1}} V^{\otimes I_1}) \otimes ... \otimes (P(I_n)^* \otimes_{\Sigma_{I_n}} V^{\otimes I_n}) \end{aligned}$$

. We want now to define maps

$$diag_{r,n} : P(r) \otimes P(n)^* \otimes V^{\otimes n} \rightarrow \bigoplus_{I_1 \amalg ... \amalg I_r = \underline{n}} (P(I_1)^* \otimes_{\Sigma_{I_1}} V^{\otimes I_1}) \otimes ... \otimes (P(I_n)^* \otimes_{\Sigma_{I_n}} V^{\otimes I_n}).$$

For $I_1 \amalg ... \amalg I_r = \underline{n}$, the map

$$\gamma(I_1, ..., I_r) : P(r) \otimes P(I_1) \otimes ... \otimes P(I_r) \rightarrow P(I_1 \amalg ... \amalg I_r) = P(\underline{n}) \cong P(n)$$

induces a map $\overline{\gamma}(I_1, ..., I_r) : P(r) \otimes P(n)^* \rightarrow P(I_1)^* \otimes ... \otimes P(I_r)^*$. Indeed, the $P(I_k), k = 1, ..., r$ are finite dimensional so we can use the following sequence of isomorphisms:

$$\begin{aligned} Hom_{\mathbb{K}}(P(r) \otimes P(I_1) \otimes ... \otimes P(I_r), P(n)) &\cong Hom_{\mathbb{K}}(P(r) \otimes P(I_1) \otimes ... \otimes P(I_r), Hom_{\mathbb{K}}(P(n)^*, \mathbb{K})) \\ &\cong Hom_{\mathbb{K}}(P(r) \otimes P(n)^* \otimes P(I_1) \otimes ... \otimes P(I_r), \mathbb{K}) \\ &\cong Hom_{\mathbb{K}}(P(r) \otimes P(n)^*, Hom_{\mathbb{K}}(P(I_1) \otimes ... \otimes P(I_r), \mathbb{K})) \\ &\cong Hom_{\mathbb{K}}(P(r) \otimes P(n)^*, P(I_1)^* \otimes ... \otimes P(I_r)^*). \end{aligned}$$

We define
$$diag_{r,n}=\sum_{I_1\amalg...\amalg I_r=\underline{n}}\overline{\gamma}(I_1,...,I_r)\otimes id$$
which is defined on $P(r)\otimes P(n)^*\otimes_{\Sigma_n}V^{\otimes n}$ and take its values in
$$\begin{aligned} & \bigoplus_{I_1\amalg...\amalg I_r=\underline{n}}(P(I_1)^*\otimes...\otimes P(I_r)^*)\otimes_{\Sigma_n}V^{\otimes n} \\ \cong\ & \bigoplus_{I_1\amalg...\amalg I_r=\underline{n}}(P(I_1)\otimes_{\Sigma_{I_1}}V^{\otimes I_1})\otimes...\otimes(P(I_r)\otimes_{Sigma_{I_r}}V^{\otimes I_r}).\end{aligned}$$
The hypothesis $P(0)=0$ ensures that only a finite number of $\overline{\gamma}(I_1,...,I_r)$ such that $I_1\amalg...\amalg I_r=\underline{n}$ are non zero, so the application $diag_{r,n}$ is well defined. Then we can set
$$\rho_r=\bigoplus_n diag_{r,n}:P(r)\otimes P^*(V)\rightarrow P^*(V)^{\otimes r}.$$
By construction, the ρ_n make the appropriate diagram commute and are Σ_n-equivariant, so they equip $P^*(V)$ with a structure of P-coalgebra.

Now we have to prove that $P^*(V)$ is cofree. It means that for every morphism $f:C\rightarrow V$ of $Ch_{\mathbb{K}}$ where C is a P-coalgebra and V any chain complex, there exists a unique factorization
$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\tilde{f}"'] & V \\ P^*(V) \arrow[ur, "\pi"'] & \end{tikzcd}$$
where $\tilde{f}$ is a morphism of P-coalgebras and $\pi:\bigoplus_{r=1}^{\infty}P(r)^*\otimes_{\Sigma_r}V^{\otimes r}\rightarrow P(1)^*\otimes V\cong V$ (recall that $P(1)=\mathbb{K}$) is the projection on the component of arity 1.

The structure of P-coalgebra on C is given by morphisms $\rho_r:C\rightarrow P(r)^*\otimes_{\Sigma_r}V^{\otimes r}$, hence in degree n the maps $(\rho_r)_n:C_n\rightarrow(P(r)^*\otimes_{\Sigma_r}V^{\otimes r})_n$. We have $C_0=0$ so $(C^{\otimes r})_n=\bigoplus_{i_1+...+i_r=n}C_{i_1}\otimes...\otimes C_{i_r}=0$ if $r>n$. We deduce that for a fixed degree n, only a finite number of $(\rho_r)_n$ are non zero. We can then set
$$\phi_n=\sum_r(\rho_r)_n:C_n\rightarrow\bigoplus_{r=1}^{\infty}(P(r)^*\otimes_{\Sigma_r}C^{\otimes r})_n$$
hence
$$\phi_C=\bigoplus_n\phi_n:C\rightarrow P^*(C).$$
Consequently we set $\tilde{f}=P^*(f)\circ\phi_C:C\rightarrow P^*(V)$. Given that π is the projection on the component of arity 1, in order to show that $\pi\circ\tilde{f}=f$ we look after what it gives in arity 1:
$$C\overset{\phi|_{r=1}}{\rightarrow}P(1)^*\otimes C\overset{id\otimes f}{\rightarrow}P(1)^*\otimes V\overset{\pi|_{r=1}}{\rightarrow}V.$$
We obtain f as expected. It remains to prove the unicity of $\tilde{f}$. For every n we have a commutative square
$$\begin{tikzcd} P^*(V) \arrow[r, "\rho_n^{P^*(V)}"] \arrow[d, "projection"'] & P(n)^*\otimes P^*(V)^{\otimes n} \arrow[d, "id\otimes\pi^{\otimes n}"] \\ P(n)^*\otimes_{\Sigma_n}V^{\otimes n} \arrow[r, "N"'] & P(n)^*\otimes V^{\otimes n} \end{tikzcd}$$

where N is the norm map. The map $\tilde{f}$ is a morphism P-coalgebras, so we have for every n another commutative square

$$\begin{tikzcd} C \arrow[r, "\tilde{f}"] \arrow[d, "\rho_n^C"'] & P^*(V) \arrow[d, "\rho_n^{P^*(V)}"] \\ P(n)^* \otimes C^{\otimes n} \arrow[r, "id\otimes \tilde{f}^{\otimes n}"'] & P(n)^* \otimes P^*(V)^{\otimes n} \end{tikzcd} .$$

Combining these two squares we obtain a new commutative diagram

$$\begin{tikzcd} C \arrow[r, "\tilde{f}"] \arrow[d, "\rho_n^C"'] & P^*(V) \arrow[r, "projection"] \arrow[d, "\rho_n^{P^*(V)}"] & P(n)^* \otimes_{\Sigma_n} V^{\otimes n} \arrow[d, "N"] \\ P(n)^* \otimes C^{\otimes n} \arrow[r, "id\otimes \tilde{f}^{\otimes n}"'] & P(n)^* \otimes P^*(V)^{\otimes n} \arrow[r, "id\otimes \pi^{\otimes n}"'] & P(n)^* \otimes V^{\otimes n} \end{tikzcd} .$$

We have $(id\otimes \pi^{\otimes n})\circ(id\otimes \tilde{f}^{\otimes n}) = id\otimes f^{\otimes n}$, so $\tilde{f}$ is determined in a unique way by f in each arity n. Indeed, according to the commutative diagram above, we have $N\circ(\tilde{f})_n = id\otimes f^{\otimes n}$. Recall that $p\circ N = id$, where $p : P(n)\otimes V^{\otimes n} \to P(n)\otimes_{\Sigma_n} V^{\otimes n}$ is the projection. It implies that $(\tilde{f})_n = p\circ(id\otimes f^{\otimes n})$. $\square$

3.2. **Enveloping cooperad.** Let A be a P-coalgebra. We want to construct a particular cooperad associated to A and called the enveloping cooperad of A. Then we will prove a result linking this construction with the coproduct of ${}^P Ch_{\mathbb{K}}^+$ which will be crucial in the proof of MC5 (i).

We consider the Σ-module $P^*[A]$ defined by

$$P^*[A](n) = \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} A^{\otimes r}.$$

We need the following lemma:

Lemma 3.4. *Let A be a P-coalgebra. For every chain complex C of $Ch_{\mathbb{K}}^+$ we have $P^*[A](C) \cong P^*(A\oplus C)$.*

Proof. This is exactly the same computations than in the proof of lemma 2.7. $\square$

The P-coalgebra structure morphism $\rho_A : A \to P^*(A)$ of A induces a Σ-modules morphism

$$d_0 : P^*[A] \to P^*[P^*(A)]$$

where

$$d_0(n) = \bigoplus_{r=1}^{\infty} id\otimes \rho_A^{\otimes r} : \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} A^{\otimes r} \to \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} P^*(A)^{\otimes r}.$$

The coproduct $\Delta : P^* \to P^*\circ P^*$ associated to the comonad (P^*, Δ, η) induces another morphism of Σ-modules

$$d_1 : P^*[A] \to P^*[P^*(A)]$$

(where $d_1(0) = \Delta(A)$) defined in the following way: for every chain complex C we have an application

$$P^*[A](C) \cong P^*(A \oplus C) \overset{\Delta(A\oplus C)}{\to} P^*(P^*(A \oplus C)) \overset{P^*(P^*(pr_A),\pi\circ P^*(pr_C))}{\to} P^*(P^*(A) \oplus C) \cong P^*[P^*(A)](C)$$

where π is the projection on the component of arity 1, hence the associated unique morphism of Σ-modules $d_1 : P^*[A] \to P^*[P^*(A)]$.

The counity $\eta : P^* \to Id$ induces a morphism of Σ-modules

$$s_0 : P^*[P^*(A)] \to P^*[A]$$

(where $s_0(0) = P^*(\eta(A))$) defined in the following way: for every chain complex C, we have an application

$$P^*[P^*(A)](C) \cong P^*(P^*(A) \oplus C) \overset{P^*(P^*(i_A),i_C)}{\to} P^*(P^*(A \oplus C)) \overset{P^*(\eta(A\oplus C))}{\to} P^*(A \oplus C) \cong P^*[A](C)$$

where $i_A : A \to A \oplus C$ and $i_C : C \to A \oplus C$, hence the unique associated morphism of Σ-modules $s_0 : P^*[P^*(A)] \to P^*[A]$. We finally obtain a reflexive pair (d_0, d_1) of morphisms of Σ-modules induced by the associated reflexive pair of morphisms of Schur functors. The enveloping cooperad of A is the coreflexive equalizer

$$\begin{tikzcd} U_{P^*}(A) = ker(d_0 - d_1) \arrow[r] & P^*[A] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & P^*[P^*(A)] \arrow[l, bend right, "s_0"'] \end{tikzcd}$$

in Σ-modules endowed with the cooperad structure induced by that of $P^*[A]$.

Now we want to prove that for every P-coalgebra A and every chain complex C, we have an isomorphism $U_{P^*}(A)(C) \cong A \times P^*(C)$ where $\times$ is the product in ${}^PCh_{\mathbb{K}}^+$. For this aim we need the following lemma:

Lemma 3.5. *Let A be a P-coalgebra and C be a chain complex. The following equalizer defines the product $A \times P^*(C)$ in the category of P-coalgebras:*

$$\begin{tikzcd} A \times P^*(C) = ker(d_0 - d_1) \arrow[r] & P^*(A \oplus C) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & P^*(P^*(A) \oplus C) \arrow[l, bend right, "s_0"'] \end{tikzcd}$$

where $d_0 \mid_A = \rho_A$, $d_0 \mid_C = id_C$, $d_1 \mid_A = \Delta(A)$, $d_l \mid_C = id_C$, $s_0 \mid_A = \eta(A)$, $s_0 \mid_C = id_C$.

Proof. We clearly have $d_0 \circ s_0 = d_1 \circ s_0 = id$ so (d_0, d_1) is a reflexive pair in ${}^PCh_{\mathbb{K}}^+$. The space $ker(d_0 - d_1)$ is the coreflexive equalizer of (d_0, d_1) in $Ch_{\mathbb{K}}^+$ and is a sub-P-coalgebra of $P^*(A \oplus C)$, so it is the coreflexive equalizer of (d_0, d_1) in ${}^PCh_{\mathbb{K}}^+$. Let X be a P-coalgebra, $u : X \to A$ a morphism of P-coalgebras and $v : X \to C$ a linear map. They induce a map $(u, v) : X \to A \oplus C$, hence a morphism of P-coalgebras $\varphi_{(u,v)} : X \to P^*(A \oplus C)$ obtained by the universal property of the cofree P-coalgebra. The proof ends by seeing that $\varphi_{(u,v)}$ admits a unique factorization through $ker(d_0 - d_1)$. $\square$

The coreflexive equalizer in Σ-modules defining the enveloping cooperad induces a coreflexive equalizer in P-coalgebras

$$
\begin{tikzcd}
U_{P^*}(A)(C) \arrow[r] & P^*[A](C) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & P^*[P^*(A)](C) \arrow[l, bend right, "s_0"']
\end{tikzcd}
$$

where $P^*[A](C) \cong P^*(A \oplus C)$, $P^*[P^*(A)](C) \cong P^*(P^*(A) \oplus C)$ and d_0, d_1, s_0 turn out to be the morphisms of the lemma above. By unicity of the limit, we have proved the following result:

Proposition 3.6. *Let A be a P-coalgebra and C be a chain complex, then $U_{P^*}(A)(C) \cong A \times P^*(C)$.*

We finally reach the crucial result of this section:

Corollary 3.7. Let A be a P-coalgebra and C be a chain complex. If $H_*(C) = 0$ then the canonical projection $A \times P^*(C) \to A$ is a weak equivalence in ${}^P Ch_{\mathbb{K}}^+$.

Proof. According to proposition 3.6, we have $U_{P^*}(A)(C) \cong A \times P^*(C)$. We can apply proposition 2.12 to the Σ-module $U_{P^*}(A)$ since $H_*(C) = 0$ by hypothesis, so

$$H_*(A \times P^*(C)) = H_*(U_{P^*}(A)(C)) = H_*(U_{P^*}(A)(0)).$$

It remains to prove that $H_*(U_{P^*}(A)(0)) = H_*(A)$. For this aim we show that $U_{P^*}(A)(0) \cong A$. It comes from a categorical result: in any category with a final object and admitting products, the product of any object A with the final object is isomorphic to A. We apply this fact to $U_{P^*}(A)(0) \cong A \times P^*(0)$. Indeed, the chain complex 0 is final in $Ch_{\mathbb{K}}^+$ so $P^*(0)$ is final in ${}^P Ch_{\mathbb{K}}^+$. □

3.3. **Proof of MC1.** The forgetful functor creates the small colimits. The proof of this fact is exactly the same as the proof of the existence of small limits in the P-algebras case. To prove the existence of small limits in ${}^P Ch_{\mathbb{K}}^+$, we use a method dual to that of the proof of the existence of small colimits in the P-algebras case.

Theorem 3.8. *Let $\mathcal{C}$ be a category. If $\mathcal{C}$ admits the coreflexive equalizers of every pair of arrows and all small coproducts, then $\mathcal{C}$ admits all the small limits.*

Proof. This is exactly the same proof than the one of theorem 2.4. One has just to replace the small coproducts by the small products and a reflexive coequalizer by a coreflexive equalizer. □

Now let us prove that ${}^P Ch_{\mathbb{K}}^+$ admits the coreflexive equalizers and the small products.

Lemma 3.9. *Let $(d_0, d_1 : A \to B, s_0 : B \to A)$ be a reflexive pair in ${}^P Ch_{\mathbb{K}}^+$. Then $ker(d_0 - d_1)$ is the coreflexive equalizer of (d_0, d_1) in ${}^P Ch_{\mathbb{K}}^+$.*

Proof. The subspace $ker(d_0 - d_1) \subset A$ is the coreflexive equalizer of (d_0, d_1) in $Ch_{\mathbb{K}}^+$. Moreover, it is a sub-P-coalgebra of A and the inclusion is obviously a P-coalgebras morphism, hence the result. □

Lemma 3.10. *Let $\{R_i\}_{i \in I}$ be a set of P-coalgebras. Let us set*

$$d_0 = P^*(\bigoplus \rho_{R_i}) : P^*(\bigoplus R_i) \to P^*(\bigoplus P^*(R_i))$$

and

$$d_1 = \pi \circ \Delta(\bigoplus R_i) : P^*(\bigoplus R_i) \to P^*(\bigoplus P^*(R_i))$$

where $\pi : P^*(P^*(\bigoplus R_i)) \rightarrow P^*(\bigoplus P^*(R_i))$ *is the canonical projection and* Δ *the comultiplication of the comonad* (P^*, Δ, η). *Then* $\times R_i = ker(d_0 - d_1)$ *is the product of the* R_i *in* ${}^P Ch_{\mathbb{K}}^+$.

Proof. We prove the lemma in the case of two P-coalgebras R and S. The proof is the same in the general case. Let us set

$$s_0 = P^*(\eta(R) \oplus \eta(S)) : P^*(P^*(R) \oplus P^*(S)) \rightarrow P^*(R \oplus S),$$

then $d_0 \circ s_0 = d_1 \circ s_0 = id$. According to lemma 3.9, the space $ker(d_0 - d_1)$ is the coreflexive equalizer of (d_0, d_1) in ${}^P Ch_{\mathbb{K}}^+$. Let X be a P-coalgebra. Two linear maps $u : X \rightarrow R$ and $v : X \rightarrow S$ induce a map $(u, v) : X \rightarrow R \oplus S$. This map admits a unique factorization through $P^*(R \oplus S)$ to give a P-coalgebras morphism $\varphi_{(u,v)} : X \rightarrow P^*(R \oplus S)$ by the universal property of the cofree P^*-coalgebra. This morphism admits a unique factorization through $ker(d_0 - d_1)$ if and only if u and v are morphisms of P-coalgebras. By unicity of the limit this concludes our proof, since $ker(d_0 - d_1)$ satisfies the same universal property than $R \times S$. □

3.4. **Generating (acyclic) cofibrations, proofs of MC4 and MC5.** Before specifying the families of generating cofibrations and generating acyclic cofibrations, we prove axioms MC4 (i) and MC5 (i). The cofibrantly generated structure will then be used to prove MC5 (ii) by means of a small object argument, slightly different from the preceding one since the sequential colimits will run over a higher ordinal (the one of $\mathbb{R}$). The general strategy is strongly inspired from that of [14], with the necessary modifications to adapt it to our setting.

MC5 (i). We first need a preliminary lemma:

Lemma 3.11. *Every chain complex* X *of* $Ch_{\mathbb{K}}^+$ *can be embedded in a chain complex* V *satisfying* $H_*(V) = 0$.

Proof. Let us set $V_0 = 0$ and $V_n = X_n \oplus X_{n-1}$ for every $n \geq 1$. We define the differential of V by $\partial_n^V : X_n \oplus X_{n-1} \rightarrow X_{n-1} \hookrightarrow X_{n-1} \oplus X_{n-2} = V_{n-1}$ which is the projection followed by the inclusion for every $n \geq 2$, and $\partial_1^V = 0$. We have $\partial_{n+1}^V \circ \partial_n^V = 0$ so $(V, \partial^V) \in Ch_{\mathbb{K}}^+$. Moreover, for every $n \in \mathbb{N}$, we have $X_n \subset V_n$ so X is injected into V. Finally, for every $n \in \mathbb{N}$, $H_n(V) = ker(\partial_n^V)/im(\partial_{n+1}^V) \cong X_n/X_n = 0$. □

This lemma helps us to prove the following result:

Proposition 3.12. *(i) Let* C *be a* P*-coalgebra and* V *be a chain complex such that* $H_*(V) = 0$. *Then the projection* $C \times P^*(V) \rightarrow C$ *is an acyclic fibration with the right lifting property with respect to all cofibrations.*

(ii) Every P*-coalgebras morphism* $f : D \rightarrow C$ *admits a factorization*

$$D \xrightarrow{j} X \xrightarrow{q} C$$

where j *is a cofibration and* q *an acyclic fibration with the right lifting property with respect to all cofibrations (in particular we obtain axiom MC5 (i)).*

Proof. (i) According to corollary 3.7, the map $C \times P^*(V) \rightarrow C$ is a weak equivalence so it remains to prove that it has the right lifting property with respect to all cofibrations (which implies in particular that it is a fibration). Let us consider the following commutative square in ${}^P Ch_{\mathbb{K}}$:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & C \times P^*(V) \arrow[d] \\ B \arrow[r, "b"'] & C \end{tikzcd}$$

where i is a cofibration. A lifting in this square is equivalent to a lifting in each of the two squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, equal] \\ B \arrow[r] & C \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & P^*(V) \arrow[d] \\ B \arrow[r] & 0 \end{tikzcd} .$$

In the first square this is obvious, just take the bottom map $B \to C$ as a lifting. In the second square, via the adjunction $U : {}^P Ch_{\mathbb{K}}^+ \rightleftarrows Ch_{\mathbb{K}}^+ : P^*$, the lifting problem is equivalent to a lifting problem in the following square of $Ch_{\mathbb{K}}^+$:

$$\begin{tikzcd} U(A) \arrow[r] \arrow[d, "U(i)"'] & V \arrow[d] \\ U(B) \arrow[r] & 0 \end{tikzcd} .$$

The map $V \to 0$ is degreewise surjective so it is a fibration of $Ch_{\mathbb{K}}^+$, which is acyclic because $H_*(V) = 0$. The map i is a cofibration, so $U(i)$ is a cofibration by definition and has therefore the left lifting property with respect to acyclic fibrations.

(ii) According to lemma 3.11, there exists an injection $i : U(D) \hookrightarrow V$ in $Ch_{\mathbb{K}}^+$ where V is such that $H_*(V) = 0$. Let us set $X = C \times P^*(V)$, $q : X \to C$ the projection and

$$j = (f, \tilde{i}) : D \to C \times P^*(V)$$

where $\tilde{i} : D \to P^*(V)$ is the factorization of i by universal property of the cofree P-coalgebra. We have $q \circ j = f$. According to (i), the map q is an acyclic fibration with the right lifting property with respect to all cofibrations. It remains to prove that j is a cofibration. Let us consider the composite

$$D \stackrel{j}{\to} C \times P^*(V) \stackrel{pr_2}{\to} P^*(V) \stackrel{\pi}{\to} V$$

where pr_2 is the projection on the second component and π the projection associated to the cofree P-coalgebra on V. We have $\pi \circ pr_2 \circ j = \pi \circ \tilde{i} = i$ by definition of $\tilde{i}$. The map i is injective so j is also injective, which implies that $U(j)$ is a cofibration in $Ch_{\mathbb{K}}^+$. By definition it means that j is a cofibration in ${}^P Ch_{\mathbb{K}}^+$. □

MC4 (i). Let $p : X \to Y$ be an acyclic cofibration, let us consider the commutative square

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ D \arrow[r, "b"'] & Y \end{tikzcd}$$

where i is a cofibration. According to proposition 3.12, the map p admits a factorization $p = q \circ j$ where $j : X \to T$ is a cofibration and $q : T \to Y$ an acyclic fibration with the right lifting property with respect to all cofibrations. Axiom MC2 implies that j is a weak equivalence. Let us consider the commutative square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "j"'] & X \arrow[d, "p"] \\ T \arrow[r, "q"'] & Y \end{tikzcd} .$$

According to axiom MC4 (ii), there exists a lifting $r : T \to X$ in this square and p is consequently a retract of q via the following retraction diagram:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & T \arrow[r, "r"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd} .$$

A reasoning similar to that of the proof of MC4 (ii) for P-algebras concludes the proof: the map f inherits the property of right lifting property with respect to cofibrations.

Generating (acyclic) cofibrations We first need two preliminary lemmas:

Lemma 3.13. *Let C be a P-coalgebra. For every homogeneous element $x \in C$ there exists a sub-P-coalgebra $D \subset C$ of finite dimension such that $x \in D$ and $D_k = 0$ for every $k > deg(x)$.*

Proof. Suppose that $x \in C_n$. Let us note $\Delta : C \to C \otimes C$ the coproduct. We have

$$\Delta(x) = \sum_{i+j=n} (\sum x'_i \otimes x''_j)$$

where $x'_i \in C_i$ and $x''_j \in C_j$. Using Sweedler's notation we have

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

where $x_{(1)'} \in C_i$, $x_{(2)} \in C_j$ and the sum is indexed by the integers $0 \leq i \leq n$, $0 \leq j \leq n$ such that $i + j = n$.

We do a recursive reasoning on the degree n of x. For $n = 1$, the element x belongs to $C_1 = \mathbb{K}$ which is a sub-P-coalgebra of finite dimension. Now suppose the lemma true for every $k < n$. Let $x \in C_n$ and $\Delta(x)$ as above. By hypothesis, there exists sub-P-coalgebras of finite dimension $D_{(1)}$ and $D_{(2)}$ satisfying the following conditions:

-$x_{(1)} \in D_{(1)}$, $x_{(2)} \in D_{(2)}$;

-$(D_{(1)})_j = 0$ if $j > deg(x_{(1)})$ and $(D_{(2)})_j = 0$ if $j > deg(x_{(2)})$.

Then we set

$$D = \mathbb{K}.x \oplus (\sum D_{(1)} + \sum D_{(2)}).$$

This is a finite sum of finite dimensional sub-P-coalgebras so D is a finite dimensional sub-P-coalgebra containing x. Furthermore, since $deg(x) > deg(x_{(1)})$ and $deg(x) > deg(x_{(2)})$, this construction implies that $D_j = 0$ if $j > deg(x)$. □

Lemma 3.14. *Let $j : C \to D$ be an acyclic cofibration and $x \in D$ a homogeneous element. Then there exists a sub-P-coalgebra $B \subseteq D$ such that:*
(i) $x \in B$;
(ii) B is finite dimensional;
(iii) the injection $C \cap B \hookrightarrow B$ is an acyclic cofibration in ${}^{P}Ch_{\mathbb{K}}^{+}$ (we denote also by C the image of C under j, since j is injective and thus $j(C) \cong C$).

Proof. We want to define recursively sub-P-coalgebras

$$B(1) \subseteq B(2) \subseteq ... \subseteq D$$

such that $x \in B(1)$, each $B(n)$ is finite dimensional and the induced map

$$\frac{B(n-1)}{C \cap B(n-1)} \to \frac{B(n)}{C \cap B(n)}$$

is zero in homology. This map is well defined, since we do the quotient by an intersection of two sub-P-coalgebras which is still a sub-P-coalgebra.

The P-coalgebra $B(1)$ is given by lemma 3.13. Now suppose that for some integer $n \geq 1$ the coalgebra $B(n-1)$ has been well constructed. The space $B(n-1)$ is of finite dimension, so we can choose a finite set of homogeneous cycles $z_i \in B(n-1)$, $\overline{z_i} \in \frac{B(n-1)}{C\cap B(n-1)}$, such that the homology classes of the $\overline{z_i}$ span $H_*(\frac{B(n-1)}{C\cap B(n-1)})$. For every i, lemma 3.13 provides us a finite dimensional sub-P-coalgebra $A(z_i) \subseteq D$ containing z_i. We can then define

$$B(n) = B(n-1) + \sum_i A(z_i).$$

The sub-P-coalgebra $B(n)$ is of finite dimension because it is the sum of finite dimensional sub-p-coalgebras. Moreover, the induced map in homology

$$H_*(\frac{B(n-1)}{C \cap B(n-1)}) \to H_*(\frac{B(n)}{C \cap B(n)})$$

is zero because it sends the homology classes of the $\overline{z_i}$ to 0.

Let us define $B = \bigcup B(n)$ and prove that $C \cap B \hookrightarrow B$ is an acyclic cofibration. Firs it is injective so it is a cofibration. To prove its acyclicity, let us consider the following short exact sequence:

$$0 \to B \cap C \hookrightarrow B \to \frac{B}{C \cap B} \to 0.$$

It is sufficient to consider the long exact sequence induced by this sequence in homology and to prove that $H_*(\frac{B}{C\cap B}) = 0$. Let $z \in B$ such that $\partial(\overline{z}) = 0$ in $H_*(\frac{B}{C\cap B})$, where ∂ is the differential of $\frac{B}{C\cap B}$. We have $\partial(z) \in B \cap C = \bigcup B(n) \cap C$ and $B(1) \subseteq ... \subseteq D$ so there exists an integer n such that $z \in B(n-1)$ and $\partial(z) \in B(n-1) \cap C$. It implies that $[\overline{z}] \in H_*(\frac{B(n-1)}{C\cap B(n-1)})$, where $[\overline{z}]$ is the homology class of $\overline{z}$. Thus $[\overline{z}] = 0$ in $H_*(\frac{B(n)}{C\cap B(n)})$, since the map $H_*(\frac{B(n-1)}{C\cap B(n-1)}) \to H_*(\frac{B(n)}{C\cap B(n)})$ is zero in homology. We deduce that $z = \partial(b) + B(n) \cap C$ for a certain $b \in B(n)$, so

$\overline{z} = \partial(\overline{b})$ in $\frac{B}{B\cap C}$ (the projection $x \mapsto \overline{x}$ commutes with the differentials). Finally, it means that every cycle of $\frac{B}{B\cap C}$ is a boundary, i.e that $H_*(\frac{B}{C\cap B}) = 0$.

It remains to prove that B is finite dimensional. According to lemma 3.13, the chain complex $B(1)$ is concentrated in degrees $k \leq deg(x)$. Let us suppose that $B(n-1)$ is concentrated in degrees $k \leq deg(x)$, then so does $\frac{B}{B\cap C}$. The $\overline{z_i}$ are then of degree $deg(\overline{z_i}) \leq deg(x)$, and so do the z_i. The $A(z_i)$ are obtained by lemma 3.13 and thus concentrated in degrees $k \leq deg(\overline{z_i}) \leq deg(x)$. The chain complex $B(n) = B(n-1) + \sum_i A(z_i)$ is consequently also concentrated in degrees $k \leq deg(x)$. The sequence $B(1) \subseteq B(2) \subseteq ...$ is stationnary, which allows us to conclude that $B = \bigcup B(n)$ is finite dimensional. □

Now we can give a characterization of generating cofibrations and generating acyclic cofibrations.

Proposition 3.15. *A morphism $p : X \rightarrow Y$ of ${}^P Ch_{\mathbb{K}}^+$ is*

(i) a fibration if and only if it has the right lifting property with respect to the acyclic cofibrations $A \hookrightarrow B$ where B is of finite dimension;

(ii) an acyclic fibration if and only if it has the right lifting property with respect to the cofibrations $A \hookrightarrow B$ where B is of finite dimension.

Proof. (i) One of the two implications is obvious. Indeed, if p is a fibration then it has the right lifting property with respect to acyclic cofibrations by definition. Conversely, suppose that p has the right lifting property with respect to the acyclic cofibrations $A \hookrightarrow B$ where B is finite dimensional. We consider the following lifting problem:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ D \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where j is an acyclic cofibration. Let us define Ω as the set of pairs $(\overline{D}, g)$, where $\overline{D}$ fits in the composite of two acyclic cofibrations

$$C \hookrightarrow \overline{D} \hookrightarrow D$$

such that this composite is equal to j. The map $g : \overline{D} \rightarrow X$ is a lifting in

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[d] & & X \arrow[d, "p"] \\ \overline{D}ar[r] & D \arrow[r] & Y \end{tikzcd} .$$

Recall that cofibrations are injective P-coalgebras morphisms. We endow Ω with a partial order defined by $(\overline{D_1}, g_1) \leq (\overline{D_2}, g_2)$ if $\overline{D_1} \subseteq \overline{D_2}$ and $g_2 \mid_{\overline{D_1}} = g_1$. The commutative square

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[d, equal] & & X \arrow[d, "p"] \\ C \arrow[r, "j"'] & D \arrow[r] & Y \end{tikzcd}$$

admits f as an obvious lifting, so $(C, f) \in \Omega$ and thus Ω is not empty. Moreover, any totally ordered subset of Ω admits an upper bound, just take the sum of its elements. We can therefore apply Zorn lemma. Let $(E,g)\in\Omega$ be a maximal element. We know that E is injected in D by definition, and we want to prove that D is injected in E in order to obtain $E=D$.

Let $x\in D$ be a homogeneous element. According to lemma 3.14 applied to the acyclic cofibration $E\hookrightarrow D$, there exists a finite dimensional sub-P-coalgebra $B\subseteq D$ such that $x\in B$ and $E\cap B\hookrightarrow B$ is an acyclic cofibration. The lifting problem

$$\begin{tikzcd} E\cap B \arrow[r] \arrow[d] & E \arrow[r, "g"] & X \arrow[d] \\ B \arrow[r] \arrow[urr, dashed, "h"] & D \arrow[r] & Y \end{tikzcd}$$

admits a solution h by hypothesis about p. We therefore extend g into a map $\tilde{g}:E+B\to X$ such that $\tilde{g}\mid_E=g$, $\tilde{g}\mid_B=h$. According to the diagram above, we have $h\mid_{E\cap B}=g\mid_{E\cap B}$ so $\tilde{g}$ is well defined.The short exact sequences

$$0\to E\cap B\to B\to\frac{B}{E\cap B}\to 0$$

and

$$0\to E\to E+B\to\frac{E+B}{B}\to 0$$

induce long exact sequences in homology

$$...\to H_{n+1}(\frac{B}{E\cap B})\to H_n(E\cap B)\to H_n(B)\to H_n(\frac{B}{E\cap B}\to ...$$

and

$$...\to H_{n+1}(\frac{E+B}{E})\to H_n(E)\to H_n(E+B)\to H_n(\frac{E+B}{E}\to ...$$

But $E\cap B\hookrightarrow B$ induces an isomorphism in homology so in the first exact sequence $H_*(\frac{B}{E\cap B})=0$. Furthermore, the isomorphism $\frac{B}{E\cap B}\cong\frac{E+B}{E}$ implies that $H_*(\frac{E+B}{E})=0$. Accordingly, the map $E\hookrightarrow E+B$ in the second exact sequence induces an isomorphism in homology, i.e $E\hookrightarrow E+B$ is an acyclic cofibration. It means that $(E+B,\tilde{g})\in\Omega$, and by definition of $\tilde{g}$ the inequality $(E,g)\leq(E+B,\tilde{g})$ holds in Ω. Given that (E,g) is supposed to be maximal, we conclude that $E=E+B$, hence $x\in E$ and $E=D$. The map g is the desired lifting. The map p is a fibration.

(ii) If p is an acyclic fibration, then p has the right lifting property with respect to cofibrations according to axiom MC4 (i). Conversely, let us suppose that p has the right lifting property with respect to cofibrations $A\hookrightarrow B$ where B is finite dimensional. The proof is similar to that of (i) with a slight change in the definition of Ω. Indeed, we consider the lifting problem

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & X \arrow[d, "p"] \\ D \arrow[r] & Y \end{tikzcd}$$

where j is a cofibration. We define Ω as the set of pairs $(\overline{D},g)$ where $\overline{D}$ fits in a composite of cofibrations $C\hookrightarrow\overline{D}\hookrightarrow D$ such that this composite is equal to j. We define the same partial order on Ω than in (i), and Ω is clearly not empty since $(C,f)\in\Omega$. The set Ω is inductive so we can apply Zorn's lemma. Let (E,g) be a maximal element of Ω, as before E is injected in D and we want to prove that D is injected in E. Let $x\in D$ be a homogeneous element, according to lemma 3.14

there exists a finite dimensional sub-P-coalgebra $B \subseteq D$ containing x. The map p has the right lifting property with respect to $E \cap B \hookrightarrow B$ by hypothesis, so the method of (i) works here. We extend g to $\tilde{g} : E + B \to X$, we have $(E + B, \tilde{g}) \in \Omega$ and $(E, g) \leq (E + B, \tilde{g})$. The maximality of (E, g) implies that $E = E + B$ and $g : E = D \to X$ is the desired lifting. □

MC5 (ii). We need here to use a slightly refined version of the small object argument. We will consider smallness only with respect to injections systems. Suppose that $\mathcal{C}$ is a category admitting small colimits. A direct system of injections

$$... \hookrightarrow B(n) \hookrightarrow B(n+1) \hookrightarrow ...$$

indexed by a set has $\bigcup_n B(n)$ as colimit. For any object A of $\mathcal{C}$, the functor $Hom(A, -)$ gives a commutative diagram of injections

$$\begin{tikzcd} ... \arrow[r] \arrow[drr] & Hom(A,B(n)) \arrow[rr] \arrow[dr] & & Hom(A,B(n+1)) \arrow[r] \arrow[dl] & ... . \\ & & Hom(A,\bigcup_n B(n)) & & \end{tikzcd}$$

By universal property of the colimit, this diagram induces a canonical map

$$colim_n Hom(A, B(n)) = \bigcup_n Hom(A, B(n)) \to Hom(A, \bigcup_n B(n)).$$

We say that A is small with respect to direct systems of injections if this map is a bijection. Consider a morphism f of $\mathcal{C}$ and a family of morphisms $\mathcal{F} = \{f_i : A_i \to B_i\}_{i \in I}$ such that the A_i are small with respect to injections systems. If we can prove that the i_k obtained in the construction of the $G^k(\mathcal{F}, f)$ are injections, then we can use this refined version of the small object argument. We then obtain a factorization $f = f_\infty \circ i_\infty$ where f_∞ has the right lifting property with respect to the morphisms of $\mathcal{F}$ and i_∞ is an injection (the injectivity passes to the transfinite composite). This is the argument we are going to use to prove axiom MC5 (ii) in ${}^P Ch_{\mathbb{K}}^+$.

Recall that the generating acyclic cofibrations of ${}^P Ch_{\mathbb{K}}^+$ are the acyclic injections $j_i : A_i \hookrightarrow B_i$ of P-coalgebras such that the B_i are finite dimensional. In order to apply the refined small object argument explained above, we need the following lemma:

Lemma 3.16. *Let C be a object of ${}^P Ch_{\mathbb{K}}^+$. If $U(C)$ is small with respect to injections systems, then so does C in ${}^P Ch_{\mathbb{K}}^+$.*

Proof. Let us consider a system of injections

$$... \hookrightarrow B(n) \hookrightarrow B(n+1) \hookrightarrow ...$$

of P-coalgebras, and let $f : C \to \bigcup_n B(n)$ be a morphism of P-coalgebras. The chain complex $U(C)$ is small with respect to injections systems, so there exists an integer N such that we have a unique factorization in $Ch_{\mathbb{K}}^+$

$$f : C \xrightarrow{\tilde{f}} B(N) \hookrightarrow \bigcup_n B(n).$$

The map f is a morphism of P-coalgebras and so does $B(N) \hookrightarrow \bigcup_n B(n)$, thus $\tilde{f}$ is a morphism of P-coalgebras. We have the desired factorization in ${}^P Ch_{\mathbb{K}}^+$. □

In the family of generating acyclic cofibrations $\mathcal{F} = \{j_i : A_i \hookrightarrow B_i\}_{i\in I}$, the B_i are finite dimensional so the A_i too, thus the $U(A_i)$ are small. In particular, they are small with respect to injections systems. Lemma 3.16 implies that the A_i are small with respect to injection systems. Now, let $f : X \to Y$ be a morphism of P-coalgebras. Recall that the construction of $G^k(\mathcal{F}, f)$ is given by a pushout

$$\begin{tikzcd} \bigvee_i A_i \arrow[r] \arrow[d, "\bigvee_i j_i"'] & G^{k-1}(\mathcal{F}, f) \arrow[d, "i_k"] \\ \bigvee_i B_i \arrow[r] & G^k(\mathcal{F}, f) \end{tikzcd} .$$

The forgetful functor creates the small colimits, so we obtain the same pushout in $Ch_{\mathbb{K}}^+$ by forgetting P-coalgebras structures. By definition of cofibrations and weak equivalences in $Ch_{\mathbb{K}}^+$, given that $\bigvee_i j_i$ is an acyclic cofibration, the map $U(\bigvee_i j_i)$ is an acyclic cofibration in $Ch_{\mathbb{K}}^+$. In any model category, acyclic cofibrations are stable by pushouts, so the $U(i_k)$ are acyclic cofibrations. By definition, it means that the i_k are acyclic cofibrations, i.e in our case acyclic injections of P-coalgebras. We use our refined version of the small object argument to obtain a factorization $f = f_\infty \circ i_\infty$. Injectivity and acyclicity are two properties which passes to the transfinite composite i_∞, so i_∞ is an acyclic cofibration of ${}^P Ch_{\mathbb{K}}^+$. Moreover, the map f_∞ has by construction the right lifting property with respect to the generating acyclic cofibrations and forms consequently a fibration. Our proof is now complete.

Remark 3.17. This method provides us another way to prove MC5 (i), by using this time the family of generating cofibrations.

4. The model category of bialgebras over a pair of operads in distribution

Let P be an operad in $Ch_{\mathbb{K}}^+$. Let Q be an operad in $Ch_{\mathbb{K}}^+$ such that $Q(0) = 0$, $Q(1) = \mathbb{K}$ and the $Q(n)$ are of finite dimension for every $n \in \mathbb{K}$. We suppose that there exists a mixed distributive law between P and Q (see definition 1.27). In the following, the operad P will encode the operations of our bialgebras and the operad Q will encode the cooperations.

According to theorem 1.28, we can define the free P-algebra functor on the category of Q-coalgebras, so the adjunction

$$P : Ch_{\mathbb{K}}^+ \rightleftarrows_P Ch_{\mathbb{K}}^+ : U$$

becomes an adjunction

$$P :^Q Ch_{\mathbb{K}}^+ \rightleftarrows_P^Q Ch_{\mathbb{K}}^+ : U.$$

Similarly, the adjunction

$$U :^Q Ch_{\mathbb{K}}^+ \rightleftarrows Ch_{\mathbb{K}}^+ : Q^*$$

becomes an adjunction

$$U :_P^Q Ch_{\mathbb{K}}^+ \rightleftarrows_P Ch_{\mathbb{K}}^+ : Q^*.$$

The model category structure on (P, Q)-bialgebras is then given by the following theorem:

Theorem 4.1. *The category ${}_P^Q Ch_{\mathbb{K}}^+$ inherits a cofibrantly generated model category structure such that a morphism f of ${}_P^Q Ch_{\mathbb{K}}^+$ is*

(i) a weak equivalence if $U(f)$ is a weak equivalence in ${}^Q Ch_{\mathbb{K}}^+$ (i.e a weak equivalence in $Ch_{\mathbb{K}}^+$ by definition of the model structure on ${}^Q Ch_{\mathbb{K}}^+$);
(ii) a fibration if $U(f)$ is a fibration in ${}^Q Ch_{\mathbb{K}}^+$;
(iii) a cofibration if f has the left lifting property with respect to acyclic fibrations.

It is clear that this three classes of morphisms are stable by composition and contain the identity morphisms. Axioms MC2 and MC3 are clear, axiom MC4 (i) is obvious by definition of the cofibrations. It remains to prove axioms MC1, MC4 (ii) and MC5.

MC1. The forgetful functor $U :{}_P^Q Ch_{\mathbb{K}}^+ \rightarrow {}^Q Ch_{\mathbb{K}}^+$ creates the small limits. The proof is the same than in the case of P-algebras, see section 2.1. The forgetful functor : $U_P^Q Ch_{\mathbb{K}}^+ \rightarrow_P Ch_{\mathbb{K}}^+$ creates the small colimits. The proof is the same than in the case of P-coalgebras, see section 3.3.

Generating (acyclic) cofibrations. The treatment is similar to the case of P-algebras. Let us note $\{j : A \hookrightarrow B\}$ the family of generating cofibrations and $\{i : A \hookrightarrow B\}$ the family of generating acyclic cofibrations. Then the $P(j)$ form the generating cofibrations of ${}_P^Q Ch_{\mathbb{K}}^+$ and the $P(i)$ form the generating acyclic cofibrations:

Proposition 4.2. *Let f be a morphism of ${}_P^Q Ch_{\mathbb{K}}^+$. Then*
(i) f is a fibration if and only if it has the right lifting property with respect to the $P(i)$, where $i : A \hookrightarrow B$ an acyclic injection of Q-coalgebras such that B is finite dimensional;
(ii) f is an acyclic fibration if and only if it has the right lifting property with respect to the $P(j)$, where $j : A \hookrightarrow B$ is an injection of Q-coalgebras such that B is finite dimensional.

Proof. We apply the same reasoning on adjunction properties than the one used for the proof of proposition 2.10. □

MC4 (ii). If MC5 (ii) is proved, then MC4 (ii) follows from the same proof than MC4 (ii) in the case of P-algebras.

MC5. The main difficulty here is to prove axiom MC5. Let f be a morphism of ${}_P^Q Ch_{\mathbb{K}}^+$. Let us note $\mathcal{F} = \{P(j_i), j_i : A_i \hookrightarrow B_i\}_{i\in I}$ the family of generating cofibrations. Recall that the A_i are sequentially small with respect to injections systems. The same reasoning on adjunctions than the one of the proof of lemma 2.11 ensures that the $P(A_i)$ are also sequentially small with respect to injections systems. We want to apply the small object argument to obtain a factorization $f = f_\infty \circ i_\infty$ of f. Recall that for every $k > 0$, the space $G^k(\mathcal{F}, f)$ is obtained by a pushout

$$\begin{tikzcd} \bigvee_i P(A_i) \arrow[r] \arrow[d, "\bigvee_i P(j_i)"'] & G^{k-1}(\mathcal{F}, f) \arrow[d, "i_k"] \\ \bigvee_i P(B_i) \arrow[r] & G^k(\mathcal{F}, f) \end{tikzcd} .$$

The forgetful functor $U :{}_P^Q Ch_{\mathbb{K}}^+ \rightarrow_P Ch_{\mathbb{K}}^+$ creates small colimits, so we obtain the same pushout diagram in ${}_P Ch_{\mathbb{K}}^+$ by forgetting the Q-coalgebras structures. The j_i are cofibrations of ${}^Q Ch_{\mathbb{K}}^+$, so the underlying chain complexes morphisms are cofibrations of $Ch_{\mathbb{K}}^+$. Thus, via the adjunction $P : Ch_{\mathbb{K}}^+ \rightleftarrows_P Ch_{\mathbb{K}}^+ : U$, the $P(j_i)$ are cofibrations of ${}_P Ch_{\mathbb{K}}^+$ and so does $\bigvee_i P(j_i)$. In any model category, cofibrations are stable by pushouts, so the i_k are cofibrations of ${}_PCh_{\mathbb{K}}^+$. By definition of cofibrations in ${}_PCh_{\mathbb{K}}^+$, we can apply lemma 1.38 to i_∞ to deduce that i_∞ forms a cofibration in ${}_PCh_{\mathbb{K}}^+$. We now use the following proposition:

Proposition 4.3. *An (acyclic) cofibration of ${}_PCh_{\mathbb{K}}^+$ forms an (acyclic) cofibration in $Ch_{\mathbb{K}}^+$.*

Proof. See section 4.6.3 in [16] (Note that for a base field of characteristic zero, every operad is Σ-split in the sense defined by Hinich). □

The maps i_k (and thus i_∞) forms therefore cofibrations in $Ch_{\mathbb{K}}^+$, i.e injections. This is crucial to apply our version of the small object argument, since the $P(A_i)$ are small only with respect to injections systems. Finally, i_∞ forms a cofibration in ${}_P^QCh_{\mathbb{K}}^+$. The map f_∞ has the right lifting property with respect to the generating cofibrations and forms thus an acyclic fibration. Axiom MC5 (i) is proved.

The method to prove MC5 (ii) is the same up to two minor changes: we consider the family of generating acyclic cofibrations, and use the stability of acyclic cofibrations under pushouts.

References

[1] J. Boardman, R. Vogt, *Homotopy-everything H-spaces*, Bull. Amer. Math. Soc. 74 (1968), 1117-1122.

[2] J. Boardman, R. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics 347, Springer-Verlag, 1973.

[3] M. Chas, D. Sullivan, *String topology*, arxiv preprint math/9911159.

[4] M. Chas, D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, arxiv preprint math/0212358.

[5] , M. Chas, *Combinatorial Lie bialgebras of curves on surfaces*, Topology, volume 43, issue 3 (2004), 543-568.

[6] V. G. Drinfeld, *Quantum groups*, Proceedings of the International Congress of Mathematicians 986, Vol. 1, 798Ű820, AMS 1987.

[7] V.G. Drinfeld, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $Gal(\overline{Q}/Q)$*, Algebra i Analiz 2 (1990), 149-181. English translation in Leningrad Math. J. 2 (1991), 829-860.

[8] W.G Dwyer, J. Spalinski, *Homotopy theory and model categories*, Handbook of algebraic topology (1995), 73-126.

[9] T. F. Fox, M. Markl, *Distributive laws, bialgebras, and cohomology*, Contemp. Math. 202 (1997), 167-205.

[10] B. Fresse, *Cogroups in algebras over an operad are free algebras*, Comment. Math. Helv. 73 (1998), 637-676.

[11] B. Fresse, *Lie theory of formal groups over an operad*, Journal of Algebra 202 (1998), 455-511.

[12] B. Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics 1967, Springer-Verlag (2009).

[13] B. Fresse, *Props in model categories and homotopy invariance of structures*, Georgian Math. J. 17 (2010), pp. 79-160.

[14] E. Getzler, P. Goerss, *A model category structure for differential graded coalgebras*, preprint (1999).

[15] V. Ginzburg, M. Kapranov, *Koszul duality for operads*, Duke Math. J. 76 (1994), 203-272.

[16] V. Hinich, *Homological algebra of homotopy algebras*, Communications in algebra, 25(10) (1997), 3291-3323.

[17] M. Hovey, *Model categories*, Mathematical Surveys and Monographs volume 63, AMS (1999).

[18] M. W. Johnson, D. Yau, *On homotopy invariance for algebras over colored PROPs*, Journal of Homotopy and Related Structures 4 (2009), 275-315.

[19] M. Livernet, *Homotopie rationnelle des algèbres sur une opérade*, thesis at Strasbourg University, IRMA prepublication (1998).

[20] J-L. Loday, B. Vallette, *Algebraic Operads*, Grundlehren der mathematischen Wissenschaften, Volume 346, Springer-Verlag (2012).

[21] I. G. MacDonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, Clarendon Press, second edition (1995).

[22] S. MacLane, *Categorical algebra*, Bull. Amer. Math. Soc. Volume 71, Number 1 (1965), 40-106.

[23] Saunders MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, second edition, Springer (1998).

[24] M. Markl. *Models for operads*, Comm. Algebra, 24(4) (1996), 1471Ũ1500.

[25] M. Markl, S. Schnider, J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs volume 96, AMS (2002).

[26] P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics 271, Springer-Verlag, 1972.

[27] D. G. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics 43, Springer- Verlag (1967).

[28] S. Yalin, *Classifying spaces and moduli spaces of algebras over a prop*, preprint (2012). Submitted. Available at http://arxiv.org/abs/1207.2964v2.

Laboratoire Paul Painlevé, Université de Lille 1, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France

E-mail address: Sinan.Yalin@ed.univ-lille1.fr