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A Hurwitz Like Characterization of GUAS Planar Switched System

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Abstract

This paper deals with the stability problem for the planar linear switched system $\dot{x}(t) = u(t)\, A_1\, x(t) + (1 - u(t))\, A_2\, x(t)\,, x = (x_1, x_2) \in \mathbb{R}^2$ where the real matrices $A_1, A_2 \in \mathbb{R}^2$ are Hurwitz and $u(.)\colon [0, +\infty) \longrightarrow \{0, 1\}$ is a measurable function. We give a Hurwitz like characterization of globally uniformly asymptotically stable planar switched systems. Another contribution of this paper is a practical version of the main result in [NS] using real algebraic geometry tools. This new approach give a Hurwitz like characterization of switched systems which share a same strict or large common quadratic Lyapunov function and simplify the main result in [BBM].

1 Introduction

Let A_1 and A_2 be two 2×2 real Hurwitz matrices. This paper deal with the problem of finding necessary and sufficient conditions on A_1 and A_2 under which the switched system $(\mathcal{S})$:

$$\dot{x}(t) = u(t)\, A_1\, x(t) + (1 - u(t))\, A_2\, x(t)\,, x = (x_1, x_2) \in \mathbb{R}^2$$

is globally uniformly asymptotically stable (GUAS), with respect to a measurable switching functions $u(.) : [0, \infty) \longrightarrow \{0, 1\}$.

This problem has been studied in [B] when both A_1 and A_2 are diagonalizable in $\mathbb{C}$ and in [MB] when at least one among A_1 and A_2 is not diagonalizable. In both cases the stability conditions are given in terms of coordinate-invariant parameters. Unfortunately the parameters used in the diagonalizable case become singular in the nondiagonalizable one and therefore the two cases were studied separately.

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More recently, in [BBM], the authors unify the previous studies by reformulating them in terms of new invariants that permit to treat all cases at the same time. Hence, they reduce the cases to be studied from 24 to 6 cases.

In this paper we reduce such cases to be studied from 6 to 1, therefore give a Hurwitz like characterization of globally uniformly asymptotically stable planar switched systems. Another contribution of our paper is a practical version of the main result in [NS] using real algebraic geometry tools. This new approach give a Hurwitz like characterization of switched systems which share a same strict or large common quadratic Lyapunov function and simplify the main result given in [BBM]. In section 2 of this paper, we remind some classical notions concerning stability of switched systems and introduce the Sturm-Habitcht theorem which will be very useful. In section 3, we state our main result and present its proof thereafter.

2 Mathematical preliminaries

2.1 Stability notions

Notation 1. *For $\delta > 0$, let $\boldsymbol{B_\delta} \subset \mathbb{R}^2$ be the unit ball of radius δ, centered in the origin. We denote by $\boldsymbol{\mathcal{U}}$ the set of measurable functions defined on $[0, +\infty)$ with values in $\{0,1\}$. Given $x_0 \in \mathbb{R}^2$, we denote by $\boldsymbol{\gamma_{x_0,u(.)}(.)}$ the trajectory of $(\mathcal{S})$ starting at x_0 and corresponding to the control $u(.)$.*

Definition 2. ***[Unboundness]****. The switched system $(\mathcal{S})$ is unbounded at the origin if there exist a initial position x_0 and a control $u(.)$ such that the trajectory $\gamma_{x_0,u(.)}(t)$ goes to infinity as t goes to infinity.*

Definition 3. ***[Uniform Stability]*** *The switched system $(\mathcal{S})$ is uniformly stable at the origin if and only if for any ball B_ε one can find a ball B_δ such that for any control $u(.) \in \mathcal{U}$ and any initial position $x_0 \in B_\delta$, the trajectory $\gamma_{x_0,u(.)}(.)$ stay in the ball B_ε.*

Definition 4. ***[Global Uniform Asymptotical Stability]****. The switched system $(\mathcal{S})$ is globally uniformly asymptotically stable at the origin (**GUAS** for short) if it is uniformly stable at the origin and globally uniformly attractive, i.e, for any balls B_ε and B_δ, for any initial position $x_0 \in B_\delta$, for any $u(.) \in \mathcal{U}$, there exist a time $T > 0$ such that $\gamma_{x_0,u(.)}(t)$ stay in the ball B_ε for every $t \geqslant T$.*

According to the following proposition, the stability properties of the switched system $(\mathcal{S})$ do not change if we allow measurable switching functions taking values in $[0, 1]$ instead of $\{0, 1\}$. We will name **convexified** switched system, the switched system $(\mathcal{S})$ with $u(.)$ taking values in $[0, 1]$.

Proposition 5. ***[MBC]****. The switched system $(\mathcal{S})$ and its convexified has the same stability behavior. More precisely, the switched system $(\mathcal{S})$ is **GUAS** (resp. uniformly stable or unbounded) if and only if its convexified is **GUAS** (resp. uniformly stable or unbounded).*

Definition 6. ***[Common Lyapunov Function]***. *$V\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ is a common Lyapunov function (**CLF** for short) for the switched system $(\mathcal{S})$ if V is continuous, positive definite and strictly decreasing along non-constant trajectories of $(\mathcal{S})$. If V is a positive definite, continuous and non-increasing along non-constant trajectories of $(\mathcal{S})$ function then V is called a non-strict common Lyapunov function.*

Definition 7. ***[Common Quadratic Lyapunov Function]***. *$V\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ is a common quadratic Lyapunov function (**CQLF** for short) for the switched system $(\mathcal{S})$, if it exists a 2×2 positive definite symmetric matrix P such that :*

1. *$V(x) = x^T P x$;*
2. *$A_1^T P + P A_1$ and $A_2^T P + P A_2$ are negative definite.*

We recall that, for the switched system $(\mathcal{S})$, the existence of a common Lyapunov function is equivalent to **GUAS** (see for instance [DM]). Moreover the existence of a non-strict Lyapunov function guarantees the uniform stability of $(\mathcal{S})$.

2.2 A useful real algebraic geometry tool

Hereafter we recall some classical results from real algebraic geometry namely the Sturm-Habitcht theorem which allow to determine precisely the number of real roots, in any interval $[a,b]\subset\mathbb{R}$ of a given polynomial $P\in\mathbb{R}[X]$. This real algebraic classical tool will be very useful to characterize switched systems which share a common quadratic Lyapunov function.

Definition 8. ***[Sturm Sequence]***. *Let $P \in \mathbb{R}[X]$, $(f_i(X))_{i\in[\![0,s]\!]}$ a sequence of element of $\mathbb{R}[X]$ and $[a,b]$ an interval of $\mathbb{R}$. By definition, $(f_i(X))_{i\in[\![0,s]\!]}$ is a Sturm sequence associated to P on $[a,b]$ if the following conditions are satisfied:*

1. *f_s does not vanish on the interval $(a,b]$;*
2. *For any $i\in[\![0,s]\!], \alpha\in[a,b]$ such that $f_i(\alpha)=0$, we have $f_{i-1}(\alpha)\, f_{i+1}(\alpha)<0$;*
3. *For any $\alpha \in [a,\, b]$ such that $f_0(\alpha) = 0$, there exist $\varepsilon > 0$ such that $f_0(\alpha + \varepsilon)\, f_1(\alpha+\varepsilon)<0$ and $f_0(\alpha-\varepsilon)\, f_1(\alpha-\varepsilon)<0$*

Definition 9. ***[Sign Variations]***. *Let $A := (a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$. We call variation of the sequence A, denoted by $V(A)$ or $V(a_0, \ldots, a_n)$, the number of pairs $(i, i+k)$ with $k \geqslant 1$ such that $a_i\, a_{i+k} < 0$ and $a_{i+r} = 0$ for $r \in [\![1, k-1]\!]$.*

Example 10. Consider the sequence $A = (1,\ -1,\ 0,\ 5,\ 3,\ 3,\ -7)$, its variation corresponds to the number of its sign variations hence $V(A)=3$.

Let $P\in\mathbb{R}[X]$ and $(f_i(X))_{i\in[\![0,s]\!]}$ a Sturm sequence associated to P on $[a,b]$ and consider the following function :

$$w\colon \begin{cases} \mathbb{R} \longrightarrow \mathbb{N} \\ y \longmapsto V(f_0(y), \ldots, f_s(y)) \end{cases}$$

Theorem 11. *[BPR]The number of distinct real roots of P in $(a, b]$ is equal to $w(a) - w(b)$.*

For the effective computation of a Sturm sequence associated to P, the following theorem provides an algorithm, based on the euclidean division algorithm, to compute it.

Theorem 12. ***[Sturm-Habicht]****[BPR]. Let $[a, b]$ be an interval of $\mathbb{R}$, $P \in \mathbb{R}[X]$ a square-free polynomial, $(f_i(X))_{i \in [\![0,s]\!]}$ a sequence of element of $\mathbb{R}[X]$ defined as follow : $f_0 = P$, $f_1 = P'$ and for $i \geqslant 2$, $f_{i-2} = f_{i-1} g_i - f_i$ with $\deg(f_i) < \deg(f_{i-1})$ ($-f_i$ is the remainder of the euclidean division of f_{i-2} by f_{i-1}) and s is the smallest integer such that f_s does not vanish in $[a, b]$. Then the sequence $(f_i(X))_{i \in [\![0,s]\!]}$ is a Sturm sequence associated to P on $[a, b]$.*

3 Stability behavior of a switched planar system

Consider A_1 and A_2 two 2×2 real Hurwitz matrices. The paper deal with the problem of finding necessary and sufficient conditions on A_1 and A_2 under which the switched system $(\mathcal{S})$:

$$\dot{x}(t) = u(t)\, A_1\, x(t) + (1 - u(t))\, A_2\, x(t)\,, x = (x_1, x_2) \in \mathbb{R}^2$$

is globally uniformly asymptotically stable (GUAS) with respect to measurable switching functions $u(.) : [0, \infty) \longrightarrow \{0, 1\}$.

In this section we remind the notations and objects that was used in [BBM] to state their stability result. We will use the same invariants to state our stability results which extend the ones given in [BBM]. In the following the word invariant will indicate any object which is invariant with respect to coordinate transformations. As usual, we denote by $\det(X)$ and $\operatorname{tr}(X)$ the determinant and the trace of a matrix X. If $X \in \mathbb{R}^2 \times \mathbb{R}^2$ the discriminant is defined as:

$$\delta_X := \operatorname{tr}(X)^2 - 4 \det(X) \tag{1}$$

Given a pair of matrices X and Y, we define the following invariant:

$$\Gamma(X, Y) := \frac{1}{2}(\operatorname{tr}(X) \operatorname{tr}(Y) - \operatorname{tr}(XY)) \tag{2}$$

Since A_1 and A_2 are suppose to be two real Hurwitz matrices, so we have:

$$\det(A_1), \det(A_2) \in \mathbb{R}_+^* \quad \text{and} \quad \operatorname{tr}(A_1), \operatorname{tr}(A_2) \in \mathbb{R}_-^* \tag{3}$$

By means of these invariants the following invariants associated to $(\mathcal{S})$ are defined:

$$\tau_i := \begin{cases} \frac{\operatorname{tr}(A_i)}{\sqrt{|\delta_{A_i}|}}\,, \text{if } \delta_{A_1}\delta_{A_2} \neq 0 \\ \frac{\operatorname{tr}(A_i)}{\sqrt{|\delta_{A_j}|}}\,, \text{if } \delta_{A_1}\delta_{A_2} = 0, \text{but } \delta_{A_j} \neq 0 \\ \frac{\operatorname{tr}(A_i)}{2}\,, \text{if } \delta_{A_1} = \delta_{A_2} = 0 \end{cases} \tag{4}$$

$$k := \frac{\tau_1\,\tau_2}{\mathrm{tr}(A_1)\mathrm{tr}(A_2)}(2\mathrm{tr}(A_1\mathrm{tr}(A_2)) - \mathrm{tr}(A_1)\mathrm{tr}(A_2)) \tag{5}$$

$$\Delta := 4(\Gamma(A_1, A_2)^2 - \Gamma(A_1, A_1)\Gamma(A_2, A_2)) \tag{6}$$

$$\mathcal{R} := \frac{2\Gamma(A_1, A_2) + \sqrt{\Delta}}{2\sqrt{\det(A_1)\det(A_2)}}\, e^{\tau_1 t_1 + \tau_2 t_2} \tag{7}$$

where for $i = 1, 2$

$$t_i := \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{\mathrm{tr}(A_1)\mathrm{tr}(A_2)(k\tau_i + \tau_{3-i})}{2\sqrt{\Delta}}\right), \text{if } \delta_{A_i} < 0 \\ \mathrm{arctanh}\left(\frac{2\,\tau_1\tau_2\sqrt{\Delta}}{\mathrm{tr}(A_1)\mathrm{tr}(A_2)(k\tau_i - \tau_{3-i})}\right), \text{if } \delta_{A_i} > 0 \\ \frac{2\sqrt{\Delta}}{\left(\mathrm{tr}(A_1A_2) - \frac{1}{2}\mathrm{tr}(A_1)\mathrm{tr}(A_2)\right)\tau_i}, \text{if } \delta_{A_i} = 0 \end{cases} \tag{8}$$

We remind now the main stability result in [BBM] of Baldé, Boscain and Mason.

Theorem 13. ***[Balde-Boscain-Mason]*** *Consider the switched system* $(\mathcal{S})$*. Then:*

1. *if* $\Gamma(A_1, A_2) > -\sqrt{\det(A_1)\det(A_2)}$ *and* $\mathrm{tr}(A_1A_2) > -2\sqrt{\det(A_1)\det(A_2)}$ *then the switched system* $(\mathcal{S})$ *admits a CQLF.*

   If $-\sqrt{\det(A_1)\det(A_2)} < \Gamma(A_1, A_2) \leqslant \sqrt{\det(A_1)\det(A_2)}$ *then the condition* $\mathrm{tr}(A_1A_2) > -2\sqrt{\det(A_1)\det(A_2)}$ *is automatically satisfied. As a consequence the system admits a CQLF.*

2. *if* $\Gamma(A_1, A_2) < \sqrt{\det(A_1)\det(A_2)}$ *then the switched system* $(\mathcal{S})$ *is unbounded.*

3. $\Gamma(A_1, A_2) = -\sqrt{\det(A_1)\det(A_2)}$ *then the switched system* $(\mathcal{S})$ *is uniformly stable but not GUAS.*

4. *if* $\Gamma(A_1, A_2) > \sqrt{\det(A_1)\det(A_2)}$ *and* $\mathrm{tr}(A_1A_2) \leqslant -2\sqrt{\det(A_1)\det(A_2)}$ *then the switched system* $(\mathcal{S})$ *is GUAS uniformly stable (but not GUAS) or unbounded respectively if*

$$\mathcal{R} < 1, \;\; \mathcal{R} = 1, \;\; \mathcal{R} > 1.$$

3.1 Statement of our main result

In this sub-section we state our mains results which characterizes completely *GUAS* two-dimensional bilinear switched systems like $(\mathcal{S})$. Note that the Theorem 15 Show that the first condition of Theorem 13 is not only sufficient but also necessary.

Theorem 14. ***[Characterization of GUAS switched systems]****.The switched system* $(\mathcal{S})$ *is globally uniformly asymptotically stable (GUAS) if and only if:*

$$\Gamma(A_1, A_2) \in \left(-\sqrt{\det(A_1)\det(A_2)}, \sqrt{\det(A_1)\det(A_2)}\cosh(\tau_1 t_1 + \tau_2 t_2)\right).$$

Theorem 15. ***[Characterization of switched systems with a CQLF]***. *The switched system* $(\mathcal{S})$ *admits a common quadratic Lyapunov function if and only if*

$$\Gamma(A_1, A_2) \in \left(-\sqrt{\det(A_1)\det(A_2)}, \frac{1}{2}\operatorname{tr}(A_1)\operatorname{tr}(A_2) + \sqrt{\det(A_1)\det(A_2)} \right)$$

Along the next two sub-sections, we will give, among other, the complete proof of our main result.

3.2 Strict or large common quadratic Lyapunov function

For shortness we will denote by:

$$\begin{cases} d_1 := \det(A_1), d_2 := \det(A_2) \\ T_1 := \operatorname{tr}(A_1), T_2 := \operatorname{tr}(A_2) \\ n := \operatorname{tr}(A_1 A_2) \\ m := \Gamma(A_1, A_2) \end{cases}$$

Since we have suppose A_1 and A_2 to be two real Hurwitz matrices, so we have:

$$d_1, d_2 \in \mathbb{R}_+^* \quad \text{and} \quad T_1, T_2 \in \mathbb{R}_-^* \tag{9}$$

By means of these invariants we can define the following invariants associated to $(\mathcal{S})$:

$$\tau_i := \begin{cases} \frac{T_i}{\sqrt{|\delta_{A_i}|}}, \text{if } \delta_{A_1}\delta_{A_2} \neq 0 \\ \frac{T_i}{\sqrt{|\delta_{A_j}|}}, \text{if } \delta_{A_1}\delta_{A_2} = 0, \text{but } \delta_{A_j} \neq 0 \\ \frac{T_i}{2}, \text{if } \delta_{A_1} = \delta_{A_2} = 0 \end{cases} \tag{10}$$

$$k := \frac{\tau_1 \tau_2}{T_1 T_2}(n - 2m) \quad \text{and} \quad \lambda := t_1 \tau_1 + t_2 \tau_2 \tag{11}$$

$$\mathcal{R} := \frac{m + \sqrt{m^2 - d_1 d_2}}{\sqrt{d_1 d_2}} e^{\lambda} \tag{12}$$

where for $i = 1, 2$

$$t_i := \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{T_1 T_2 (k\tau_i + \tau_{3-i})}{4\sqrt{m^2 - d_1 d_2}} \right), \text{if } \delta_{A_i} < 0 \\ \operatorname{arctanh}\left(\frac{4\tau_1\tau_2\sqrt{m^2 - d_1 d_2}}{T_1 T_2 (k\tau_i - \tau_{3-i})} \right), \text{if } \delta_{A_i} > 0 \\ \frac{4\sqrt{m^2 - d_1 d_2}}{(n - 2m)\tau_i}, \text{if } \delta_{A_i} = 0 \end{cases} \tag{13}$$

For any $u \in [0, 1]$, let $D(u)$ and $N(u)$ be the following expressions :

$$D(u) := \det(u A_1 + (1-u)A_2) \quad \text{and} \quad N(u) := \det(u A_1 + (1-u)A_2^{-1}) \times d_2$$

3.2.1 Systems with a strict common quadratic Lyapunov function

The following theorem of Narenda and Shorten, from [NS], provides a necessary and sufficient condition for the existence of common quadratic Lyapunov function for the switched system $(\mathcal{S})$.

Theorem 16. ***[Narenda and Shorten]***. *The switched system* $(\mathcal{S})$ *admits a common quadratic Lyapunov function if and only if for any* $u \in [0,\ 1]$, $D(u) > 0$ and $N(u) > 0$.

Lemma 17. *[BBM] For any* $u \in [0,1]$, *we have :*

$$D(u) = (d_1 + d_2 - 2\,m)u^2 + 2(m - d_2)u + d_2$$
$$N(u) = (d_1 d_2 + 1 - n)u^2 + (n-2)u + 1$$

Thanks to the Theorem 16, we would like to express the necessary and sufficient condition for the existence of a common quadratic Lyapunov function "for any $u \in [0,\ 1]$, $D(u) > 0$ and $N(u) > 0$" into necessary and sufficient condition with respect to our chosen invariants. Since from Lemma 17 it appears that $D(u)$ and $N(u)$ are polynomials, so one can use the Sturm-Habitcht Theorem 12 to study the sign variations of $D(u)$ and $N(u)$ on $[0,1]$ and hence achieves our goal.

Study of the sign variations of $D(u)$, $u \in [0,1]$:

It is clear that $D(0) = d_2$ and $D(1) = d_1$. Since A_1 and A_2 are real Hurwitz matrices, it appears that $D(0), D(1) \in \mathbb{R}_+^*$, so one can deduce directly the sign of D on $[0,1]$ by studying its number of real roots in $[0,1]$. The main rule is the following: "$D(u) > 0$ on $[0,1]$ if and only if D has no root in $[0,1]$". And we use Sturm Habicht theorem to count the number of real roots of $D(u) = (d_1 + d_2 - 2\,m)u^2 + 2(m - d_2)u + d_2$, $u \in [0,1]$. Before we build a Sturm sequence associated to $D(u)$ on $[0,1]$, let us point out some important remarks.

Remark 18. If $d_1 + d_2 - 2m = 0$ *i.e* $m = \frac{d_1 + d_2}{2}$, then $D(u) = 2(m - d_2) + d_2$. As $D(0)$, $D(1) \in \mathbb{R}_+^*$ and D is affine, therefore we have $D(u) > 0$ for all $u \in [0,1]$.

Suppose now that $d_1 + d_2 - 2m \neq 0$ and let $f_1(u) = D(u)$, $f_2(u) = D'(u) = 2(d_1 + d_2 - 2\,m)u + 2(m - d_2)$. Using the euclidean division algorithm we obtain, according to the Sturm-Habicht Theorem 12, $f_3(u) = \frac{m^2 - d_1 d_2}{d_1 + d_2 - 2m}$ as the opposite of the remainder of the euclidean division of $f_1(u)$ by $f_2(u)$. Since $f_3(u)$ does not depend on u, so in our case $s = 3$.

Remark 19. The sign variations of $D(u)$ in $[0,1]$ depends on the number of roots of D in $[0,\ 1]$ which depends, according to Theorem 11, on $w(0) - w(1)$ where $w(y) = V(f_1(y), f_2(y), f_3(y))$.

- $f_1(0) = d_2,\ f_2(0) = 2(m - d_2),\ f_3(0) = \frac{m^2 - d_1 d_2}{d_1 + d_2 - 2m}$
- $f_1(1) = d_1,\ f_2(1) = 2(d_1 - m),\ f_3(1) = \frac{m^2 - d_1 d_2}{d_1 + d_2 - 2m}$

We know that $d_1, d_2 \in \mathbb{R}_+^*$. We we are going to construct a sign table that will give us the sign of $f_2(0), f_2(1)$ and $f_3(0) = f_3(1)$ in terms of m. The particular values of m that should appear in the sign table are:

$$d_1, d_2, \frac{d_1 + d_2}{2}, -\sqrt{d_1 d_2}, \sqrt{d_1 d_2}$$

It will be necessary to order these values before giving the sign table. That is the purpose of the next lemma.

Lemma 20.

1. *If* $d_1 < d_2$ *then* $-\sqrt{d_1 d_2} < d_1 < \sqrt{d_1 d_2} < \frac{d_1+d_2}{2} < d_2$.

2. *If* $d_2 < d_1$ *then* $-\sqrt{d_1 d_2} < d_2 < \sqrt{d_1 d_2} < \frac{d_1+d_2}{2} < d_1$.

3. *if* $d_1 = d_2$ *then* $-\sqrt{d_1 d_2} < d_2 = \sqrt{d_1 d_2} = \frac{d_1+d_2}{2} = d_1$.

Proof. Suppose $d_1 < d_2$. Then $(\sqrt{d_1} - \sqrt{d_2})^2 = d_1 + d_2 - 2\sqrt{d_1 d_2} > 0$, hence $\sqrt{d_1 d_2} < \frac{d_1+d_2}{2}$. Since $0 < d_1 < d_2$, we have $d_1 + d_2 < 2d_2$ and $d_1^2 < d_1 d_2$, hence $\frac{d_1+d_2}{2} < d_2$ and $d_1 < \sqrt{d_1 d_2}$. The same arguments prove the second case. The third point is obvious. □

For the sign table of D, we will consider the previous three cases for making sure that we will not neglect or forget any case.

m	$-\infty$		-$\sqrt{d_1 d_2}$		d_1		$\sqrt{d_1 d_2}$		$\frac{d_1+d_2}{2}$		d_2		$+\infty$
d_1+d_2-2m		$+$		$+$		$+$		$+$	0	$-$		$-$	
$m^2-d_1d_2$		$+$	0	$-$		$-$	0	$+$		$+$		$+$	
f_3		$+$	0	$-$		$-$	0	$+$		$-$		$-$	
$f_2(0)$		$-$		$-$		$-$		$-$		$-$	0	$+$	
$f_2(1)$		$+$		$+$	0	$-$		$-$		$-$		$-$	
f_1		$+$		$+$		$+$		$+$		$+$		$+$	
$w(0)$	2		1	1	1	1	1	2	1	1	1	1	
$w(1)$	0		0	1	1	1	1	2	1	1	1	1	
$w(0)-w(1)$	2		1	0	0	0	0	0	0	0	0	0	

Figure 1. The sign table of D when $d_1 < d_2$

m	$-\infty$		$-\sqrt{d_1 d_2}$		d_2		$\sqrt{d_1 d_2}$		$\frac{d_1+d_2}{2}$		d_1		$+\infty$
d_1+d_2-2m		$+$		$+$		$+$		$+$	0	$-$		$-$	
$m^2-d_1d_2$		$+$	0	$-$		$-$	0	$+$		$+$		$+$	
f_3		$+$	0	$-$		$-$	0	$+$		$-$		$-$	
$f_2(1)$		$+$		$+$		$+$		$+$		$+$	0	$-$	
$f_2(0)$		$-$		$-$	0	$+$		$+$		$+$		$+$	
f_1		$+$		$+$		$+$		$+$		$+$		$+$	
$w(0)$		2	1	1	1	1	0	0	1	1	1	1	
$w(1)$		0	0	1	1	1	0	0	1	1	1	1	
$w(0)-w(1)$		2	1	0	0	0	0	0	0	0	0	0	

Figure 2. The sign table of D when $d_2 < d_1$

m	$-\infty$		$-d_1$		d_1		$+\infty$
$2(d_1-m)$		$+$		$+$	0	$-$	
$m^2-d_1^2$		$+$	0	$-$	0	$+$	
f_3		$+$	0	$-$		$-$	
$f_2(0)$		$-$		$-$	0	$+$	
$f_2(1)$		$+$		$+$	0	$-$	
f_1		$+$		$+$		$+$	
$w(0)$		2	1	1	1	1	
$w(1)$		0	0	1	1	1	
$w(0)-w(1)$		2	1	0	0	0	

Figure 3. The sign table of D when $d_1=d_2$

From the three previous sign tables we obtain the following proposition.

Proposition 21.

1. *For all $u\in[0,1]$, $D(u)>0$ if and only if $m>-\sqrt{d_1d_2}$.*

2. *For all $u\in[0,1]$, $D(u)\geqslant 0$ and there exists $u_0\in(0,1)$ such that $D(u_0)=0$ if and only if $m=-\sqrt{d_1d_2}$.*

3. *D changes sign twice in $[0,1]$ if and only if $m<-\sqrt{d_1d_2}$.*

Proof. From the previous sign tables, it appears that D has respectively, in $[0,1]$ zero, one and two distinct roots when $m>-\sqrt{d_1d_2}$, $m=-\sqrt{d_1d_2}$ and $m<-\sqrt{d_1d_2}$. Since $D(0),D(1)\in\mathbb{R}_+^*$, it comes the announced results. □

Study of the sign variations of $N(u)$, $u\in[0,1]$:

In this paragraph, we achieve the sign variations study of $N(u)=(d_1d_2+1-$

$n)u^2+(n-2)u+1$ in $[0,1]$, by using the same strategy as in the previous sub-section. Before we build the Sturm sequence associated to $N(u)$ on $[0,1]$, let us point out some important remarks.

Remark 22. If $d_1d_2-n+1=0$ *i.e* $n=d_1d_2+1$, then $N(u)=(d_1d_2-1)u+1$. As $N(0)=1>0$, $N(1)=d_1d_2>0$ and N is affine, therefore we have $N(u)>0$ for all $u\in[0,1]$.

Suppose now that $d_1d_2-n+1\neq 0$ and let $g_1(u)=N(u)$, $g_2(u)=N'(u)=2(d_1d_2-n+1)u+(n-2)$. Using the euclidean division algorithm we obtain, according to the Sturm-Habicht Theorem 12, $g_3(u)=\frac{n^2-4d_1d_2}{4(d_1d_2-n+1)}$ as the opposite of the remainder of the euclidean division of $g_1(u)$ by $g_2(u)$. Since $g_3(u)$ does not depend on u, so in our case $s=3$.

Remark 23. The sign variations of $N(u)$ in $[0,1]$ depends on the number of roots of N in $[0,\ 1]$ which depends, according to Theorem 11, on $w(0)-w(1)$ where $w(y)=V(g_1(y),g_2(y),g_3(y))$.

- $g_1(0)=1,\ g_2(0)=(n-2),\ g_3(0)=\frac{n^2-4d_1d_2}{4(d_1d_2-n+1)}$
- $g_1(1)=d_1d_2,\ g_2(1)=2d_1d_2-n,\ g_3(1)=\frac{n^2-4d_1d_2}{4(d_1d_2-n+1)}$

We know that $d_1,d_2\in\mathbb{R}_+^*$. We are going to construct a sign table that will give us the sign of $g_2(0)$, $g_2(1)$ and $g_3(0)=g_3(1)$ in terms of n. The particular values of n that should appear in the sign table are:

$$2,2d_1d_2,d_1d_2+1,-2\sqrt{d_1d_2},2\sqrt{d_1d_2}$$

It will be necessary to order these values before giving the sign table. That is the purpose of the next lemma.

Lemma 24.

1. *If* $d_1d_2<1$ *then* $-2\sqrt{d_1d_2}<2d_1d_2<2\sqrt{d_1d_2}<d_1d_2+1<2$.
2. *If* $d_1d_2>1$ *then* $-2\sqrt{d_1d_2}<2<2\sqrt{d_1d_2}<d_1d_2+1<2d_1d_2$.
3. *If* $d_1d_2=1$ *then* $2=2\sqrt{d_1d_2}=d_1d_2+1=2d_1d_2$.

Proof. Suppose $d_1d_2<1$. Since $d_1d_2>0$ then $(d_1d_2)^2<d_1d_2$, hence $2d_1d_2<2\sqrt{d_1d_2}$.

Since $(\sqrt{d_1d_2}-1)^2 = d_1d_2 - 2\sqrt{d_1d_2}+1 > 0$ and $d_1d_2 < 1$, hence $2\sqrt{d_1d_2} < d_1d_2+1 < 2$. The same arguments prove the second case. The third point is obvious. $\square$

For the sign table of N, we will consider the previous three cases for making sure that we will not neglect or forget any case.

n	$-\infty$		-$2\sqrt{d_1d_2}$		$2d_1d_2$		$2\sqrt{d_1d_2}$		d_1d_2+1		2		$+\infty$
d_1d_2-n+1		$+$		$+$		$+$		$+$	0	$-$		$-$	
$n^2-4d_1d_2$		$+$	0	$-$		$-$	0	$+$		$+$		$+$	
g_3		$+$	0	$-$		$-$	0	$+$		$-$		$-$	
$g_2(0)$		$-$		$-$		$-$		$-$		$-$	0	$+$	
$g_2(1)$		$+$		$+$	0	$-$		$-$		$-$		$-$	
g_1		$+$		$+$		$+$		$+$		$+$		$+$	
$w(0)$		2	1	1	1	1	1	2	0	1	1	1	
$w(1)$		0	0	1	1	1	1	2	0	1	1	1	
$w(0)-w(1)$		2	1	0	0	0	0	0	0	0	0	0	

Figure 4. The sign table of N when $d_1d_2 < 1$

n	$-\infty$		-$2\sqrt{d_1d_2}$		2		$2\sqrt{d_1d_2}$		d_1d_2+1		$2d_1d_2$		$+\infty$
d_1d_2-n+1		$+$		$+$		$+$		$+$	0	$-$		$-$	
$n^2-4d_1d_2$		$+$	0	$-$		$-$	0	$+$		$+$		$+$	
g_3		$+$	0	$-$		$-$	0	$+$		$-$		$-$	
$g_2(1)$		$+$		$+$		$+$		$+$		$+$	0	$-$	
$g_2(0)$		$-$		$-$	0	$+$		$+$		$+$		$+$	
g_1		$+$		$+$		$+$		$+$		$+$		$+$	
$w(0)$		2	1	1	1	1	0	0	0	1	1	1	
$w(1)$		0	0	1	1	1	0	0	0	1	1	1	
$w(0)-w(1)$		2	1	0	0	0	0	0	0	0	0	0	

Figure 5. The sign table of N when $d_1d_2>1$

n	$-\infty$		-2		2		$+\infty$
$2-n$		$+$		$+$	0	$-$	
n^2-4		$+$	0	$-$	0	$+$	
g_3		$+$	0	$-$		$-$	
$g_2(0)$		$-$		$-$	0	$+$	
$g_2(1)$		$+$		$+$	0	$-$	
g_1		$+$		$+$		$+$	
$w(0)$		2	1	1	0	1	
$w(1)$		0	0	1	0	1	
$w(0)-w(1)$		2	1	0	0	0	

Figure 6. The sign table of D when $d_1d_2=1$

From the three previous sign tables we obtain the following proposition.

Proposition 25.

1. *For all $u\in[0,1]$, $N(u)>0$ if and only if $n>-2\sqrt{d_1d_2}$.*
2. *For all $u\in[0,1]$, $N(u)\geqslant 0$ and there exists $u_1\in(0,1)$ such that $N(u_1)=0$ if and only if $n=-2\sqrt{d_1d_2}$.*
3. *N changes sign twice in $[0,1]$ if and only if $n<-2\sqrt{d_1d_2}$.*

Proof. From the previous sign tables, it appears that N has respectively, in $[0,1]$ zero, one and two distinct roots when $n>-2\sqrt{d_1d_2}$, $n=-2\sqrt{d_1d_2}$ and $n<-2\sqrt{d_1d_2}$. Since $N(0),N(1)\in\mathbb{R}_+^*$, it comes the announced results. □

We remind in the next proposition the expression of the Theorem 15 and give its proof.

Proposition 26. ***[Characterization of switched systems with a CQLF]****. The switched system $(\mathcal{S})$ admits a common quadratic Lyapunov function if and only if*

$$m\in\left(-\sqrt{d_1d_2},\frac{1}{2}T_1T_2+\sqrt{d_1d_2}\right)$$

Proof. From Theorem 16, the switched system $(\mathcal{S})$ admits a common quadratic Lyapunov function if and only if for all $u \in [0, 1]$, $D(u) > 0$ and $N(u) > 0$. From Proposition 21 and Proposition 25:

$$\begin{aligned} &\text{for all } u \in [0,1]\ D(u) > 0 \quad \text{if and only if} \quad m > -\sqrt{d_1 d_2} \\ &\text{for all } u \in [0,1]\ N(u) > 0 \quad \text{if and only if} \quad n > -2\sqrt{d_1 d_2} \end{aligned}$$

Since $m = \frac{1}{2}(T_1 T_2 - n)$, it comes the announced result. □

3.2.2 Systems with a large common quadratic Lyapunov function

We will need the following lemma, from [NS], to prove the Proposition 28 which characterize switched system $(\mathcal{S})$ with a non-strict common quadratic Lyapunov function.

Lemma 27. *[NS]. Let $A \in \mathbb{R}^{2\times 2}$ be a Hurwitz matrice and consider the linear time invariant systems :*

$$\begin{aligned} \mathcal{S}_A &: \dot{x} = A\,x \\ \mathcal{S}_{A^{-1}} &: \dot{x} = A^{-1}\,x \end{aligned}$$

Then, any Lyapunov function for $\mathcal{S}_A$ of the form $V(x) = x^T P x$, is also a Lyapunov function for $\mathcal{S}_{A^{-1}}$.

Proposition 28. ***[Switched systems with a NCQLF]**. The switched system $(\mathcal{S})$ admits a non-strict common quadratic Lyapunov function if*

$$m = -\sqrt{d_1\, d_2} \quad \text{or} \quad m = \frac{1}{2} T_1 T_2 + \sqrt{d_1 d_2}$$

Moreover, the switched system $(\mathcal{S})$ is GUAS when $m = \frac{1}{2}T_1T_2 + \sqrt{d_1 d_2}$ and not GUAS when $m = -\sqrt{d_1 d_2}$.

Proof. Suppose $m = -\sqrt{d_1 d_2}$. According to Proposition 21, we have: for all $u \in [0,1]$, $D(u) \geqslant 0$ and there exists an unique $u_0 \in (0,1)$ such that $D(u_0) = 0$. Therefore the system $(\mathcal{S})$ cannot be GUAS. Because $D(u_0) = \det\,(u_0 A_1 + (1-u_0)A_2) = 0$ implies one of the eigenvalues of $B = u_0 A_1 + (1-u_0)A_2$ is equal to zero and the other one is equal to $\mathrm{tr}(B) = \mathrm{tr}(u_0 A_1 + (1-u_0)A_2) = u_0\,\mathrm{tr}(A_1) + (1-u_0)\,\mathrm{tr}(A_2) < 0$ since A_1 and A_2 are real Hurwitz matrices. Hence, if we consider the control function $u(t) := u_0$, for all $t \in \mathbb{R}_+$, it appears that the convexified of the switched system $(\mathcal{S})$ is not GUAS, so according to Proposition 5 the switched system $(\mathcal{S})$ is not GUAS. It remains now to prove that, for $m = -\sqrt{d_1 d_2}$, the switched system $(\mathcal{S})$ admits a non-strict common quadratic Lyapunov function. For this task we refer the reader to [BBM] where such a non-strict common quadratic Lyapunov function is given.

Suppose now $m = \frac{1}{2}T_1T_2 + \sqrt{d_1 d_2}$. The trick here is to consider the following switched system $(\mathcal{S}^*)$:

$$\dot{x}(t) = u(t)\,A_1\,x(t) + (1-u(t))\,A_2^{-1}\,x(t), x = (x_1, x_2) \in \mathbb{R}^2$$

If we plane to analyze $(\mathcal{S}^*)$ as we did with $(\mathcal{S})$ in the previous sub-section, we shall study the sign variations of $D^*(u) := \det\,(u\,A_1 + (1-u)A_2^{-1})$ and $N^*(u) := \det\,(u\,A_1 + (1-u)A_2) \times d_2$. The key point is that for all $u \in [0,1]$:

$$D^*(u) = d_2^{-1} \times N(u)$$
$$N^*(u) = d_2 \times D(u)$$

Since $m = \frac{1}{2}T_1T_2 + \sqrt{d_1d_2}$ corresponds, according to Proposition 25, to the existence of an unique $u_1 \in (0,1)$ such that $N(u_1) = 0$, so for $m = \frac{1}{2}T_1T_2 + \sqrt{d_1d_2}$ we have $D^*(u_1) = 0$. So by the same analyze as in the first part of the current proof we obtain that, when $m = \frac{1}{2}T_1T_2 + \sqrt{d_1d_2}$, the switched system $(\mathcal{S}^*)$ is not GUAS but admits a non-strict common quadratic Lyapunov function (the one given in [BBM] adapted to the invariants of $(\mathcal{S}^*)$). Let us call V such a non-strict common quadratic Lyapunov function. It comes that V is a non-strict quadratic Lyapunov function for $\mathcal{S}_{A_1}$ and for $\mathcal{S}_{A_2^{-1}}$. Hence, according to Lemma 27, V is non-strict quadratic Lyapunov function for $\mathcal{S}_{A_2}$. In conclusion if V is a non-strict quadratic Lyapunov function for $\mathcal{S}_{A_1}$ and for $\mathcal{S}_{A_2}$, so when $m = \frac{1}{2}T_1T_2 + \sqrt{d_1d_2}$, V is a non-strict common quadratic Lyapunov function for the switched system $(\mathcal{S})$. It remain to prove that when $m = \frac{1}{2}T_1T_2 + \sqrt{d_1d_2}$ the switched system $(\mathcal{S})$ is GUAS to finish the proof. This task will be achieve in the next sub-section (more precisely in Remark 32). □

3.3 Global uniform asymptotic stable switched systems

The object of this sub-section is to establish the results announced in Theorem 14 We recall partially Theorem 13 from [BBM] in the next proposition.

Proposition 29. ***[Balde-Boscain-Mason]*** *Consider the switched system $(\mathcal{S})$. Then:*

1. *if $m < -\sqrt{d_1d_2}$, then the switched system $(\mathcal{S})$ is unbounded,*
2. *if $m > \sqrt{d_1d_2}$ and $n \leqslant -2\sqrt{d_1d_2}$ then the switched system $(\mathcal{S})$ is GUAS, uniformly stable (but not GUAS) or unbounded respectively if*

$$\mathcal{R} < 1, \;\; \mathcal{R} = 1, \;\; \mathcal{R} > 1.$$

We recall that $m = \frac{1}{2}(T_1T_2 - n)$. So $n \leqslant -2\sqrt{d_1d_2}$ if and only if $m \geqslant \frac{1}{2}T_1T_2 + \sqrt{d_1d_2}$. Since $T_1, T_2 \in \mathbb{R}_-^*$ (because $A_1$and A_2 are real Hurwitz matrices), the second point of the previous proposition can be reformulate simply as follow:

if $m \geqslant \frac{1}{2}T_1T_2 + \sqrt{d_1d_2}$ then the switched system $(\mathcal{S})$ is GUAS, uniformly stable (but not GUAS) or unbounded respectively if

$$\mathcal{R} < 1, \;\; \mathcal{R} = 1, \;\; \mathcal{R} > 1.$$

We recall that $\mathcal{R} := \frac{m + \sqrt{m^2 - d_1d_2}}{\sqrt{d_1d_2}}\, e^{\lambda}$. Hence $\sqrt{m^2 - d_1d_2} = e^{-\lambda}\mathcal{R}\sqrt{d_1d_2} - m$ and by squaring this last equality we obtain :

$$m^2 - d_1d_2 = d_1d_2\, e^{-2\lambda}\mathcal{R}^2 - 2e^{-\lambda}\mathcal{R}m\sqrt{d_1d_2} + m^2$$

Therefore

$$m = \sqrt{d_1 d_2}\left(\frac{e^{-\lambda}\mathcal{R}^2 + e^{\lambda}}{2\mathcal{R}}\right) \tag{14}$$

In the followings, we study the variations of m with respect to the parameter $\mathcal{R}$. Before we start this study, let us proof the following lemma which will be very useful.

Lemma 30. *For $m \geqslant \frac{1}{2}t_1 t_2 + \sqrt{d_1 d_2}$ we have the following relations:*

1. $\lambda := \tau_1 t_1 + \tau_2 t_2 \in \mathbb{R}_-$.
2. $\mathcal{R} \geqslant e^{\lambda}$.

Proof. From the relations (3) and (4), it appears $\tau_1, \tau_2 \in \mathbb{R}_-$. Let us now prove that $t_1 \geqslant 0$ and $t_2 \geqslant 0$. We recall that:

$$t_i := \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{T_1 T_2 (k\tau_i + \tau_{3-i})}{4\sqrt{m^2 - d_1 d_2}}\right), \text{if } \delta_{A_i} < 0 \\ \operatorname{arctanh}\left(\frac{4\tau_1\tau_2\sqrt{m^2 - d_1 d_2}}{T_1 T_2 (k\tau_i - \tau_{3-i})}\right), \text{if } \delta_{A_i} > 0 \\ \frac{4\sqrt{m^2 - d_1 d_2}}{(n - 2m)\tau_i}, \text{if } \delta_{A_i} = 0 \end{cases}$$

$$k := \frac{\tau_1 \tau_2}{T_1 T_2}(n - 2m)$$

Since for any $x \in \mathbb{R}$, $\arctan(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it comes $t_i \geqslant 0$ if $\delta_{A_i} < 0$. As $m \geqslant \frac{1}{2}T_1 T_2 + \sqrt{d_1 d_2}$, so $n - 2m \leqslant n - T_1 T_2 - 2\sqrt{d_1 d_2}$. We have $m = \frac{1}{2}(T_1 T_2 - n)$, hence $n - 2m \leqslant -2(m + \sqrt{d_1 d_2}) < 0$. Since $\tau_i \leqslant 0$, it comes that $t_i \geqslant 0$ if $\delta_{A_i} = 0$. Since $n - 2m < 0$, it appears that $k < 0$, hence $(k\tau_i - \tau_{3-i}) > 0$. Knowing that $\operatorname{arctanh}(x) \geqslant 0$ when $x \geqslant 0$, it comes $t_i \geqslant 0$ if $\delta_{A_i} > 0$. This finish the proof of $\lambda := \tau_1 t_1 + \tau_2 t_2 \in \mathbb{R}_-$ when $m \geqslant \frac{1}{2}T_1 T_2 + \sqrt{d_1 d_2}$. Now let us prove by contradiction that $\mathcal{R} \geqslant e^{\lambda}$. If we had $\mathcal{R} < e^{\lambda}$, we would have $0 < \frac{m + \sqrt{m^2 - d_1 d_2}}{\sqrt{d_1 d_2}} e^{\lambda} < e^{\lambda}$. Thus $\left(m + \sqrt{m^2 - d_1 d_2}\right)^2 < d_1 d_2$. Therefore $(m^2 - d_1 d_2) < -m\sqrt{m^2 - d_1 d_2}$, so $\sqrt{m^2 - d_1 d_2} < m$ that is impossible because $m \geqslant \frac{1}{2}T_1 T_2 + \sqrt{d_1 d_2} > 0$. □

Now let us study the sign variations of m with respect to $\mathcal{R}$. As $\mathcal{R}$ is use in [BBM] to inspect the behavior when $m \geqslant \frac{1}{2}T_1 T_2 + \sqrt{d_1 d_2} > 0$, then we can consider deservedly that $\mathcal{R} \geqslant e^{\lambda}$, according to Lemma 30. Therefore the sign variations of m with respect to $\mathcal{R}$ will be studied in $[e^{\lambda}, +\infty)$. If we differentiate m with respect to $\mathcal{R}$ we obtain:

$$\frac{dm}{d\mathcal{R}} = \sqrt{d_1 d_2}\left(\frac{e^{-\lambda}\mathcal{R}^2 - e^{\lambda}}{2\mathcal{R}^2}\right) \tag{15}$$

Hence $\frac{dm}{d\mathcal{R}} = 0$ if and only if $\mathcal{R} = -e^{\lambda}$ or $\mathcal{R} = e^{\lambda}$. According to Lemma 30 $\lambda \in \mathbb{R}_-$, so we obtain the following table:

$\mathcal{R}$	e^{λ}	1	$+\infty$
$\frac{dm}{d\mathcal{R}}$	0	$+$	
m	$\sqrt{d_1 d_2}$ $\nearrow$	$\sqrt{d_1 d_2}\cosh(\lambda)$ $\nearrow$	$+\infty$

Figure 7. Variations of m with respect to $\mathcal{R}$

It comes out from this table that for $m \geqslant \frac{1}{2}T_1T_2 + \sqrt{d_1 d_2}$:

$$\begin{aligned} \mathcal{R} < 1 &\Leftrightarrow m < \sqrt{d_1 d_2}\cosh(\lambda) \\ \mathcal{R} = 1 &\Leftrightarrow m = \sqrt{d_1 d_2}\cosh(\lambda) \\ \mathcal{R} > 1 &\Leftrightarrow m > \sqrt{d_1 d_2}\cosh(\lambda) \end{aligned}$$

So we can reformulate the second point of the Proposition 29 into the following proposition.

Proposition 31. *Consider the switched system* $(\mathcal{S})$*. Then:*

1. *If* $m \in \left[\frac{1}{2}T_1T_2 + \sqrt{d_1 d_2},\ \sqrt{d_1 d_2}\cosh(\lambda)\right)$ *then the switched system* $(\mathcal{S})$ *is GUAS.*
2. *If* $m = \sqrt{d_1 d_2}\cosh(\lambda)$ *then the switched system* $(\mathcal{S})$ *is uniformly stable (but not GUAS).*
3. *If* $m > \sqrt{d_1 d_2}\cosh(\lambda)$ *then the switched system* $(\mathcal{S})$ *is unbounded.*

Remark 32. The first point of the previous proposition, in the particular case when $m = \frac{1}{2}T_1T_2 + \sqrt{d_1 d_2}$, complete the remaining part of the proof of the Proposition 28.

If we gather the Theorem 15, Proposition 16 and Proposition 31 we obtain our main result and its corollaries:

Theorem 33. ***[Characterization of GUAS switched systems]****.The switched system* $(\mathcal{S})$ *is globally uniformly asymptotically stable (GUAS) if and only if:*

$$\Gamma(A_1, A_2) \in \left(-\sqrt{\det(A_1)\det(A_2)},\ \sqrt{\det(A_1)\det(A_2)}\cosh(\tau_1 t_1 + \tau_2 t_2)\right).$$

Corollary 34. *The switched system* $(\mathcal{S})$ *is uniformly stable not GUAS if and only if:* $\Gamma(A_1, A_2) = -\sqrt{\det(A_1)\det(A_2)}$ *or* $\Gamma(A_1, A_2) = \sqrt{\det(A_1)\det(A_2)}\cosh(\tau_1 t_1 + \tau_2 t_2)$.

Corollary 35. *The switched system* $(\mathcal{S})$ *is unbounded if and only if:* $\Gamma(A_1, A_2) < -\sqrt{\det(A_1)\det(A_2)}$ *or* $\Gamma(A_1, A_2) > \sqrt{\det(A_1)\det(A_2)}\cosh(\tau_1 t_1 + \tau_2 t_2)$

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