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Backward- versus Forward-Looking Feedback Interest Rate Rules[1]

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Abstract

This paper proposes conditions for the existence and uniqueness of solutions to systems of differential equations with delays or advances in which some variables are non-predetermined. An application to the issue of optimal interest rate policy is then developed in a flexible-price model where money enters the utility function. Central banks have the choice between a rule that depends on past inflation rates or one that depends on predicted interest rates. When inflation rates are selected over a bounded time interval, the problem is characterized by a system of delay or advanced differential equations. We then prove that if the central bank's forecast horizon is not too long, an active and forward-looking monetary policy is not too destabilizing: the equilibrium trajectory is unique and monotonic.

JEL Classification: E52, E31, E63.

Keywords: Interest Rate Rules, Indeterminacy, Functionnal Differential Equations.

1 Introduction

In his pioneer paper, Taylor (1993) suggested that monetary authorities should fix their nominal interest rates based on inflation recorded over the last four quarters. Since then, many authors have put forward arguments in favor of a rule that depends on expected inflation rates instead[1]. Recently, Bernanke (2011) has underlined the necessity for central banks to announce their future policies and commit themselves to achieving an objective of medium-term inflation. In both cases, the central bank's interest rate policy is not defined according to the current value of inflation, but on its values over a finite horizon, which may be either backward- or forward-looking. The objective of this paper is to carry out an analytical comparison between the effects of a backward-looking rule and those of a forward-looking one in an extension of a model developed by Benhabib, Schmitt-Grohé and Uribe (2001).

The first difficulty in this analysis lies in the fact that we are considering a bounded time interval. This implies that if the dynamics of the model were to be represented in discrete time, their dimensions could be too great to be analyzed analytically. Thus, we will follow Benhabib (2004) and instead study a continuous-time version of the model. In continuous time, the dynamics are described by a differential equation with discrete delays in the case of a backward-looking policy, and discrete advances in the case of a forward-looking policy. A second difficulty has to do with the fact that the initial value of some variables is unknown. This is a common problem in many economic models that can be solved by using certain asymptotic properties, such as convergence toward a steady state. Mathematically, these are boundary value problems. The resolution method consists in projecting the trajectory onto the stable manifold of the dynamic system. By comparing the dimensions of the space of the non-

[1]See Bernanke and Woodford (1997), Clarida, Galí and Gertler (2000), and Orphanides (2001).

predetermined variables and those of the space of the unstable manifold, we can deduce the properties for the existence and determinacy of a solution to the system being considered (Blanchard and Khan, 1980). Equilibrium is said to be indeterminate when there is more than one solution, and sunspot fluctuations may appear (Azariadis, 1981, Benhabib and Farmer, 1999). However, the mathematical theorems that characterize these properties were only established for systems of finite dimensions comprising ordinary differential equations (ODEs) or difference equations. In this paper, we generalize these theorems to include some systems of delay or advanced differential equations (DDEs or ADEs) based on results that we obtained in a previous paper (d'Albis, Augeraud-Véron and Hupkes, 2011).

DDE systems, which are characterized by a stable manifold of infinite dimensions, have generated an abundance of mathematical literature. However, the existing theorems are only valid for systems where all the variables are predetermined and defined as continuous function[2]. We extend these theorems to cases where some variables are non-predetermined – their past values are given but their value when the system is initiated is unknown – and to cases where some predetermined variables are discontinuous. To do so, we rewrite the spectral projection formula according to the initial conditions and the jump made by non-predetermined variables. Next, we set the projection on the unstable manifold to zero and deduce the magnitude of the jump that nullifies the projection on the unstable manifold. The spectral projection formula then enables us to establish the conditions for the existence and uniqueness of a solution. Most notably, we prove that it is possible to come to a conclusion by comparing the dimensions of the space of the unknown initial conditions and those of the space of the roots with positive real parts. Our results also apply to systems of

[2]See Bellman and Cooke (1963) and the recent textbook by Diekmann, van Gils, Verduyn-Lunel and Walther (1995).

algebraic equations with delays, if their nth derivative is a DDE. In this case, the constraints imposed by such equations must be taken into account in the conditions for existence and uniqueness.

Next, we will study systems of ADEs. Such systems are more similar to ODE systems as they have a stable manifold of finite dimensions. We will demonstrate that the solution is generated by a finite number of eigenvalues simply by projecting the trajectory onto the stable manifold. Conditions for existence and determinacy are obtained by comparing the number of roots with negative real parts and the number of missing initial conditions. We will also study the case of systems that include algebraic equations and define the additional constraints that must be taken into consideration.

The main contribution of our paper is methodological in nature as our theorems may be applied to other economic problems. As Burger (1956) has pointed out, many dynamic systems in economics can be written in the form of differential equations with delays. However, for want of a theorem, up until now, authors have had to confine their work to very specific cases where the stability properties of the dynamics may be proven[3]. We have chosen to apply our theorems to the problem of the optimal interest rate rule studied by Benhabib, Schmitt-Grohé and Uribe (2001) and Benhabib (2004), as it allows us to illustrate all our mathematical results. It also allows us to demonstrate the simplicity with which our theorems may be applied as well as the succinctness of the proofs. Since McCallum (1981), monetary feedback rules have been studied to reestablish determinacy in monetary models (Woodford, 2003). But the choices pertaining to the modeling of the variables' timing are particularly important in those models (Carlstrom and Fuerst, 2000, 2001), and recommendations may

[3]See, among others, Gray and Turnovsky (1979), Boucekkine, de la Croix and Licandro (2004), Boucekkine, Licandro, Puch and del Rio (2005), Bambi (2008), Augeraud-Véron and Bambi (2011), d'Albis, Augeraud-Véron and Venditti (2012). Alternatively, some authors use partial differential equations or optimal control to solve these issues.

vary depending on whether a discrete- or continuous-time representation is chosen (Dupor, 2001, Carlstrom and Fuerst, 2003, 2005). In order to highlight the assumption of a bounded horizon, we will study a case where all the other assumptions are standard: prices are flexible, money, which enters the utility function, complements consumption, and the fiscal policy is Ricardian. The results of our study are as follows. When the interest rate rule is a function of past inflation rates, equilibrium is indeterminate if the policy is passive and unique if the policy is active. This result is not affected by the length of the interval for which inflation rates are taken into account in the interest rate rule. Even when equilibrium is determinate, the dynamics are characterized by short-term fluctuations. When the interest rate rule is a function of future inflation rates, equilibrium remains indeterminate if the policy is passive; on the other hand, if the policy is active, equilibrium is unique provided that the central bank's forecast horizon is not too distant. This implies that fixing a short-term horizon may be an alternative to an aggressive interest rates policy.

Our study does, however, have certain limitations. First of all, we do not investigate the global dynamics of the system despite the fact that several studies have demonstrated its importance in interest rate policies (Benhabib, Schmitt-Grohé and Uribe, 2003, Eurepi, 2005, Cochrane, 2011). Similarly, we do not study permanent oscillations, especially those generated by Hopf bifurcations, despite the fact they may appear in our approach. In both instances, we are limited by the fact that there are no general theorems for the type of equations being considered. Nonetheless, this points to some very promising avenues for research.

Section 2 contains all our mathematical theorems. They establish the conditions for the existence and uniqueness of solutions to systems of delay or advanced differential equations. In Section 3, we present an interest rate policy model whose solution is studied in Sections 4 and 5, where we make the distinction between backward-looking and forward-looking policies. The conclusion is presented in Section 6.

2 Multiple Solutions in Systems of Functional Differential Equations

In order to fix matters, let us consider a delay differential equation (DDE, hereafter). Letting $t \in \mathbb{R}_+$ denote time, the dynamic problem can be written as follows:

$$\begin{cases} x'(t) = \int_{t-1}^{t} d\mu\,(u-t)\,x\,(u)\,, \\ x\,(\theta) = \bar{x}\,(\theta) \text{ given for } \theta \in [-1,0]\,, \end{cases} \tag{1}$$

where x is a variable whose initial value is given by a continuous function over the interval $[-1, 0]$, where x' denotes its derivative with respect to time and μ is a measure on $[-1, 0]$. The equation in (1) features continuous delays with the largest one being normalized to one.[4] Classical results for such dynamics are presented in Diekmann, van Gils, Verduyn-Lunel and Walther (1995).

In economic models, other kind of systems may appear. We will consider three dynamics that differ from (1). First, we study algebraic equations with delay that reduce to DDEs upon (a finite number of) differentiations with respect to time. The dynamic problem now writes:

$$\begin{cases} x(t) = \int_{t-1}^{t} d\mu\,(u-t)\,x\,(u)\,, \\ x\,(\theta) = \bar{x}\,(\theta) \text{ given for } \theta \in [-1,0]\,. \end{cases} \tag{2}$$

The main difference with the DDE presented above comes from a discontinuity that is allowed at time $t = 0$: $x\,(0^+)$ is given but may be different from $x\,(0^-)$. Indeed, $x\,(0^+)$ is given through the algebraic equation:

$$x(0^+) = \int_{-1}^{0} d\mu\,(u)\,x\,(u)\,. \tag{3}$$

[4] It could have been any positive real number; however, we do not consider systems with infinite delays, whose characteristic roots may not be isolated.

To summarize, the initial value is thus given by a continuous function over the interval $[-1, 0)$ where $x(0^-)$ exists, and a given value $x(0^+)$. In both problems (1) and (2), the variable is predetermined, and is usually called backward. The second kind of dynamics we consider allows for forward variables whose initial value at time $t = 0$ is not given. The dynamic problem can be written in the case of a DDE as follows:

$$\begin{cases} x'(t) = \int_{t-1}^{t} d\mu\,(u-t)\,x\,(u)\,, \\ x\,(\theta) = \bar{x}\,(\theta) \;\text{ given for } \theta \in [-1, 0). \end{cases} \tag{4}$$

Finally, the third dynamics aim at considering equations with advances rather than delays. For instance, a differential equation with a continuum of advances (ADE, hereafter) can be written as:

$$x'(t) = \int_{t}^{t+1} d\mu\,(u-t)\,x\,(u)\,, \tag{5}$$

Depending on whether $x(0)$ is given or not, the dynamics characterize a backward or a forward variable. Below, we study functional differential-algebraic systems with delays separately from those with advances.

2.1 Functional systems with delays

Let us consider a linear system that writes as:

$$\begin{cases} \mathbf{x}_0'(t) = \int_{t-1}^{t} d\bar{\mu}_1\,(u-t)\,W\,(u)\,, \\ \mathbf{x}_1(t) = \int_{t-1}^{t} d\bar{\mu}_2\,(u-t)\,W\,(u)\,, \\ \mathbf{y}'(t) = \int_{t-1}^{t} d\bar{\mu}_3\,(u-t)\,W\,(u)\,, \\ \mathbf{x}_i\,(\theta) = \bar{\mathbf{x}}_i\,(\theta) \;\text{ given for } \theta \in [-1, 0] \;\text{ and } i = \{0, 1\}\,, \\ \mathbf{y}\,(\theta) = \bar{\mathbf{y}}\,(\theta) \;\text{ given for } \theta \in [-1, 0). \end{cases} \tag{6}$$

The details of the system list as follows: $\mathbf{x}_0 \in \mathbb{R}^{n^b}$ is a vector of n^b backward variables whose dynamics are characterized by DDEs and $\mathbf{x}_0'$ denotes its gradient; $\mathbf{x}_1 \in \mathbb{R}^{n_1^b}$ is a vector of n_1^b backward variables characterized by a algebraic equation with delays; $\mathbf{y} \in \mathbb{R}^{n^f}$ is a vector of n^f forward variables characterized by DDE and $\mathbf{y}'$ denotes its gradient. Moreover, $W = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{y})$ is a vectorial function.

We assume there exists a steady-state normalized to zero and define a solution to system (6) as a function $W(t)$ whose restriction for positive time belongs to $C([0, +\infty))$, satisfies (6) and is such that $\lim_{t \to +\infty} W(t) = 0$.

Let n^+ denote the number of eigenvalues with positive real parts of the characteristic function of system (6) and s the number of independent adjoint eigenvectors of the characteristic function generated by the n^+ eigenvalues. By definition, $s \leq n^b + n_1^b + n^f$. Further:

Assumption H1. There are no eigenvalues with real parts equal to zero and eigenvalues are simple.

These restrictions are often assumed for ordinary differential equations: the absence of pure imaginary roots excludes a central manifold, simple roots imply a one dimensional Jordan block. The system (6) displays a configuration with a stable manifold of infinite dimension and an unstable manifold of dimension s. Hence, provided that $s \geq 1$, the configuration is saddle point but multiple solutions may emerge. By multiple solutions, we implicitly mean an infinity of solutions since it features a continuum of initial values for forward variables that initiate a trajectory satisfying system (6) and converging to the steady-state. Further:

Assumption H2. The stable manifold is not transverse to the (x_0, x_1) coordinates.

This second assumption implies that the projection of initial conditions on the unstable manifold encounters the stable manifold. Using it, we conclude that $s \leq \min\left\{n^+, n^f\right\}$. Then, we obtain the following result.

Theorem 1. *Let H1 and H2 prevail. There exists a solution to system (6) if $n^+ = s$ and there may be no solution if $n^+ > s$. Upon existence, a solution is unique if and only if $n^f = s$.*

Proof. See the Appendix.

Corollary 1. *Provided that adjoint eigenvectors are linearly independent, the system (6) may have no solution if $n^f < n^+$, always has a unique solution if $n^f = n^+$, and always has multiple solutions if $n^f > n^+$.*

To establish a rule for existence and uniqueness, the proof of Theorem 1 aims at finding initial conditions for forward variables, i.e. for $\mathbf{y}(0^+)$, such that the projection of the dynamics on the unstable manifold is the null vector. In our case, the number of unknowns has the same dimension as $\mathbf{y}$. The number of forward variables is hence compared with the number of conditions obtained by setting the considered projection to zero; those conditions are linked to the number of eigenvalues with positive real parts. Conversely, as the dimensions of the stable manifold and the set of initial conditions are infinite, the information on the number of backward variables is not involved in the argument.

2.2 Functional systems with advances

Let us now study a linear system that writes:

$$\begin{cases} \mathbf{x}'(t) = \int_t^{t+1} d\bar{\mu}_1 (u-t) \, W(u), \\ \mathbf{y}_0'(t) = \int_t^{t+1} d\bar{\mu}_2 (u-t) \, W(u), \\ \mathbf{y}_1(t) = \int_t^{t+1} d\bar{\mu}_3 (u-t) \, W(u), \\ \mathbf{x}(0) = \bar{\mathbf{x}}(0) \ \text{ given.} \end{cases} \tag{7}$$

where $\mathbf{x} \in \mathbb{R}^{n^b}$ is a vector of n^b backward variables whose dynamics are characterized by ADEs and where $\mathbf{x}'$ denotes its gradient; where $\mathbf{y}_0 \in \mathbb{R}^{n^f}$ and $\mathbf{y}_1 \in \mathbb{R}^{n_1^f}$ are vectors of n^f and n_1^f forward variables characterized, respectively, by differential and algebraic equations with advances. Moreover, $W = (\mathbf{x}_0, \mathbf{y}_0, \mathbf{y}_1)$ is a vectorial function. A solution is defined as in the previous subsection.

Let n^- denote the number of eigenvalues with negative real parts of the characteristic function of system (7) and s the number of independent eigenvectors of the characteristic function generated by the n^- eigenvalues. Assuming H1 and provided that $s \geq 1$, the system (7) displays a saddle point configuration with an unstable manifold of infinite dimension and a stable manifold of dimension s. Further:

Assumption H3. The unstable manifold is not transverse to the (y_0, y_1) coordinates.

We obtain the following result.

Theorem 2. *Let H1 and H3 prevail. There exists a solution to system (7) if* $n^b = s$ *and there may be no solution if* $n^b > s$*. Upon existence, a solution is unique if and only if* $n^- = s$*.*

Proof. See the Appendix.

Corollary 2. *Provided that eigenvectors are linearly independent, the system (7) may have no solution if* $n^- < n^b$*, always has a unique solution if* $n^- = n^b$*, and always has multiple solutions if* $n^- > n^b$*.*

We see that the rule that permits to establish the existence and uniqueness of solutions is different from the one presented in Theorem 1. With advances, as the dimension of the unstable manifold is infinite, the idea is to find initial conditions for forward variables that permit to write the dynamics on the stable manifold. This is why we use the number of eigenvalues with negative real parts to state whether the solution exists and is unique. Since we rewrite the system as a finite dimensional system, the proof of Theorem 2 is similar to what can be found for ordinary differential equations.

3 A Model with Bounded Backward- or Forward-Looking Feedback Rules

We consider a model that is similar to those studied by Benhabib, Schmitt-Grohé and Uribe (2001) and Benhabib (2004). This is a flexible-price model where nominal interest rates are set by the Central Bank as a function of past or forecasted inflation rates. The novelty is to consider that backward and forward horizons of the Central Bank are bounded.

Time is continuous and is denoted by $t \in \mathbb{R}_+$. Let $c(t)$, $m(t)$ and $a(t)$ be respectively the real consumption, the real balances held for non-production purposes and the real financial wealth. The household's problem is:

$$\max_{\{c,m,a\}} \int_0^{\infty} e^{-rt} U(c(t), m(t))\, dt$$

$$s.t. \left| \begin{array}{l} a'(t) = [R(t) - \pi(t)]\, a(t) - R(t)\, m(t) + Y - c(t) - \tau(t), \\ a(0) > 0 \text{ given}, \\ \lim_{t \to +\infty} a(t)\, e^{-\int_0^t [R(z) - \pi(z)] dz} \geq 0, \end{array} \right. \tag{8}$$

where $r > 0$ and $Y > 0$ denote the rate of time preference and the output, respectively. $R(t)$, $\pi(t)$ and $\tau(t)$ are perfectly anticipated by the household and denote the trajectories of the nominal interest rate, the inflation rate and the real lump-sum taxes. The instant utility function $U(c(t), m(t))$ is strictly increasing ($U_c > 0$, $U_m > 0$) and strictly concave ($U_{cc} > 0$, $U_{mm} > 0$) in both arguments. Moreover, consumption and real balances are supposed to be complementary ($U_{cm} > 0$), which implies that they are both normal goods. The cases where real balances are substitutable with consumption or are productive are immediate extensions of the present work. Moreover, we assume that the fiscal policy is Ricardian.

Letting $\lambda(t)$ be the Lagrange multiplier associated with the household's instant budget constraint, the first order conditions are:

$$\lambda(t) = U_c(c(t), m(t)), \tag{9}$$

$$R(t) = \frac{U_m(c(t), m(t))}{U_c(c(t), m(t))}, \tag{10}$$

$$\lambda'(t) = \lambda(t)[r + \pi(t) - R(t)], \tag{11}$$

together with the household's instant budget constraint given in problem (8) and the transversality condition:

$$\lim_{t\to+\infty} a(t) e^{-\int_0^t [R(z)-\pi(z)]dz} = 0. \tag{12}$$

We assume that nominal interest rates are set by the Central Bank according to the following rule:

$$R(t) = \rho\left(\left(1 - \chi^b - \chi^f\right)\pi(t) + \chi^b \pi^b(t) + \chi^f \pi^f(t)\right), \tag{13}$$

where $\pi^b(t)$ and $\pi^f(t)$ respectively denote backward and forward indicators of inflation defined by the weighted averages of, respectively, past and expected future rates of inflation. The indicators write:

$$\pi^b(t) = \frac{\int_{t-\Omega^b}^t e^{\beta^b(u-t)}\pi(u)\,du}{\int_{-\Omega^b}^0 e^{\beta^b u}dz} \text{ and } \pi^f(t) = \frac{\int_t^{t+\Omega^f} e^{-\beta^f(u-t)}\pi(u)\,du}{\int_0^{\Omega^f} e^{-\beta^f u}du}, \tag{14}$$

where $\Omega^b > 0$ and $\Omega^f > 0$ respectively denote the bounded backward and forward horizon of the Central Bank, where $\beta^b > 0$ and $\beta^f > 0$ are the weights associated to inflation rates within the indicators, and where $\chi^b \geq 0$ and $\chi^f \geq 0$, which satisfy $\chi^b + \chi^f < 1$, are the weights given by the Central Bank to the backward and forward indicators. Below, we consider either a backward-looking feedback rule, for $\chi^f = 0$, or a forward-looking one, for $\chi^b = 0$. Finally, we assume that $\rho'(x) > 0$ for all $x \in \mathbb{R}_+$. As in Leeper (1991), the policy is considered to be active for $\rho' > 1$ and passive for $\rho' < 1$. The interest rate rule we consider is more general than the one of Benhabib, Schmitt-Grohé and Uribe

(2001) who study the limit cases $\Omega^b \to +\infty$ and $\Omega^f \to +\infty$. It is, moreover, similar to Benhabib (2004) who supposes $\chi^f = 0$, $\chi^b = 1$ and considers the following indicator:

$$\pi^b(t) = \frac{\int_{-\infty}^{t-\Omega^b} e^{\beta^b(u-t)}\pi(u)\,du}{\int_{-\infty}^{-\Omega^b} e^{\beta^b u}du}. \tag{15}$$

Benhabib (2004) hence supposes that information about past inflation rates is obtained by the Central Bank after a delay while we suppose that, after some time, information conveyed by past inflation rate is not considered as relevant by the Central Bank. Formally, the advantage of (15) is that the functional equation may reduce to a difference equation, which makes the comparison with discrete time models easier.

In equilibrium, the goods market must clear, which writes: $c(t) = Y$. By replacing this equilibrium condition in (9) and (10), we obtain: $\lambda(t) = U_c(Y, m(t))$, and $R(t) = \frac{U_m(Y,m(t))}{U_c(Y,m(t))}$. Let us use the latter to define the implicit function $m(t) = M(R(t))$ and replace it in the former. We differentiate with respect to time the new equation and rearrange using (11) to obtain:

$$R'(t) = \Lambda(R(t))\left[R(t) - r - \pi(t)\right], \tag{16}$$

with

$$\Lambda(R(t)) \equiv \frac{U_m(Y, M(R(t)))}{U_c(Y, M(R(t)))} - \frac{U_{mm}(Y, M(R(t)))}{U_{cm}(Y, M(R(t)))}. \tag{17}$$

The dynamics of the variables $R(t)$, $\pi(t)$, $\pi^b(t)$, $\pi^f(t)$ for all $t \in \mathbb{R}_+$ are characterized by a system composed of two algebraic equations given in (14), a static equation (13), and a differential equation (16). We remark that the algebraic equations reduce to differential equations with a discrete delay or a discrete advance when differentiated once with respect to time. $R(t)$ and $\pi(t)$ are forward variables, with $R(0^+)$ and $\pi(0^+)$ that are not given. Initial conditions for these two variables hence write: $R(\theta) = \bar{R}(\theta) \in C\left([-\Omega^b, 0), \mathbb{R}_+\right)$ given and $\pi(\theta) = \bar{\pi}(\theta) \in C\left([-\Omega^b, 0), \mathbb{R}_+\right)$ given. In principle, we also have that $\pi^b(\theta) = \bar{\pi}^b(\theta) \in C\left([-\Omega^b, 0), \mathbb{R}_+\right)$, $R\left(0^-\right)$ given. However, using the first equation of (14) computed for $t = 0$, we see that $\pi^b\left(t\right)$ is a backward variable whose initial condition is now: $\pi^b(\theta) = \bar{\pi}^b(\theta)$ given for $\theta \in \left[-\Omega^b, 0\right]$ and where $\pi^b(0^+)$ is given by the algebraic equation. This implies that $\bar{\pi}^b(\theta)$ may be discontinuous at $\theta = 0$. Conversely, using (13) computed at $t = 0$, we see that $\pi^f\left(t\right)$ is a forward variable with the following initial condition: $\pi^f(\theta) = \bar{\pi}^f(\theta) \in C\left([-\Omega^b, 0), \mathbb{R}_+\right)$ given.

For the perfect-foresight equilibrium we consider, it is implicitly assumed that the initial price level is given. More precisely, for all $\theta \in [-\Omega^b, 0)$, the price level, denoted $P\left(\theta\right)$, solves:

$$P\left(\theta\right) = P\left(-\Omega^b\right) e^{\int_{-\Omega^b}^{0} \pi(u)du}. \tag{18}$$

Hence, this is $P\left(0^+\right)$ (which is allowed to be different from $P\left(0^-\right)$) that is arbitrary chosen in our framework. As it is well known since Sargent and Wallace (1975), this type of model cannot determine the initial price level.

As in Benhabib, Schmitt-Grohé and Uribe (2001), we are going to study the trajectories for which the inflation rate converges to a constant. We provide a local analysis of these trajectories and, therefore, restrict ourselves to neighborhoods of a steady-state defined as a collection $(R_*, \pi_*, \pi_*^b, \pi_*^f)$ that solves $\pi_* = \pi_*^b = \pi_*^f = R_* - r$, and

$$\rho\left(\pi_*\right) - \pi_* - r = 0. \tag{19}$$

We assume[5] there exists a steady-state that satisfies: $\rho'\left(\pi_*\right) \neq \left(1 - \chi^b - \chi^f\right)^{-1}$. It is also important (Benhabib, Schmitt-Grohé and Uribe, 2002 and Cochrane, 2011) to assume the uniqueness of the steady-state.

Since we consider functional equations, the Hartman-Grobman theorem does

[5] Equivalently, we could have given straightforward conditions on the limits of function ρ that would be sufficient for existence of a real solution to (19). Note also that since $\rho' > 0$, existence implies uniqueness.

not apply to our problem, and we have to prove that studying the linearized system is not misleading. This is done by establishing the following result.

Lemma 1. *In the neighborhood of the steady-state, the dynamics of the system of equations (13), (14) and (16) behave similarly to those of its linearized counterpart, provided that the latter is hyperbolic.*

Proof. See the Appendix.

The condition in Lemma 1 will be satisfied below. We now compare the dynamics induced by the choice of the Central Bank to follow either a backward-looking interest rate rule or a forward-looking one.

4 Bounded Backward-Looking Feedback Rules

We consider the case where nominal interest rates are set by the Central Bank as a function of past and present inflation rates. For $\chi^f = 0$, the perfect-foresight equilibrium satisfies:

$$\begin{cases} R'(t) = \Lambda\left(R(t)\right)\left[R(t) - r - \frac{\rho^{-1}(R(t)) - \chi^b \pi^b(t)}{(1-\chi^b)}\right], \\ \pi^b(t) = \frac{\int_{t-\Omega^b}^{t} e^{\beta^b(u-t)}\left[\rho^{-1}(R(u)) - \chi^b \pi^b(u)\right] du}{(1-\chi^b)\int_{-\Omega^b}^{0} e^{\beta^b u} du}, \\ \pi^b(\theta) = \bar{\pi}^b(\theta) \text{ given for } \theta \in \left[-\Omega^b, 0\right], \\ \lim_{t\to+\infty} \pi^b(t) = \pi_*, \ \lim_{t\to+\infty} R(t) = R_*. \end{cases} \tag{20}$$

The system (20) is defined as a two dimensional system composed by a ordinary differential equation and an algebraic equation with a continuum of delays, which reduces to a DDE when differentiated once with respect to time. There is one forward variable, $R(t)$, and one backward variable, $\pi^b(t)$, for which a discontinuity is allowed at $t = 0$, as $\pi^b(0^+)$ is given by the algebraic equation. System (20) constitutes a particular case of system (6) where $n^b = 0$ and $n^f = n_1^b = 1$.

We are going to study the local existence and uniqueness of solutions a such a system by applying Theorem 1. As in Section 2, let us denote by n^{+} the number of roots with positive real parts of the characteristic function of the linearized counterpart of system (20). We remark that if $n^{+} > 0$, then $0 < s \leq n^{+}$.

Lemma 2. *The characteristic equation associated with the linearized counterpart of system (20) has* $n^{+} = 0$ *if* $\rho'(\pi_*) \in (0,1)$*, and* $n^{+} = 1$ *if* $\rho'(\pi_*) > 1$.

Proof. See the Appendix.

By applying Theorem 1, an immediate corollary of Lemma 2 is that the equilibrium is locally indeterminate if the monetary policy is passive while it is locally unique if the policy is active. Said differently, if the policy is passive the steady-state is stable in the sense that there is a continuum of $R(0^{+})$ that initiate a converging trajectory. If the policy is active, there is a unique solution for $R(0^{+})$ and the steady-state is saddle-point stable. This confirms and extends the results obtained by Benhabib (2004) for the case where recent and contemporaneous inflation rates are not included in the rule. He finds that following a passive policy is a sufficient condition for local indeterminacy whereas an active policy is a necessary condition for uniqueness. By using Theorem 1 and studying complex roots, we are able to show that an active policy is also a sufficient condition for local uniqueness.

To see the importance of the bounded backward horizon on this result, we now consider the case where $\Omega^{b} \rightarrow +\infty$, which is also studied in Benhabib (2004) who considers the limit case of no informational delay. The system (20) reduces to a system of two ordinary differential equations. The next Lemma studies the roots of the corresponding characteristic function.

Lemma 3. *Suppose that* $\Omega^{b} \rightarrow +\infty$*. The characteristic equation has* $n^{+} = 0$ *if* $\rho'(\pi_*) \in (0,1)$*, and* $n^{+} = 1$ *if* $\rho'(\pi_*) > 1$.

Proof. See the Appendix.

We see that the determinacy property is the same as in Lemma 2: there are multiple equilibria if the policy is passive but the equilibrium is unique if the policy is active. The finite delay has, thus, no impact on the determinacy of the equilibrium. However, the dynamics are qualitatively quite different. In the case where the policy is active, the dynamics converge with exponentially decreasing fluctuations toward the steady-state if Ω^b is finite whereas they are monotonic if Ω^b is infinite. This difference is due to the complex roots that generically emerge when using delay differential equations.

There is an assumption that is important in our setting. We suppose that contemporaneous inflation is necessarily included in the interest rate rule, which formally comes from assumption: $\chi^b < 1$. Conversely, if it is supposed that $\chi^b = 1$, the rule (13) becomes $R(t) = \rho\left(\pi^b(t)\right)$ and the problem is significantly modified as the nominal interest rate becomes a backward variable. Moreover, using (14) and (16), the dynamics reduce to a single equation that writes:

$$R(t) = \rho\left(\frac{\int_{t-\Omega^b}^{t} e^{\beta^b(u-t)}\left[R(u) - \frac{R'(u)}{\Lambda(R(u))}\right]du}{\int_{-\Omega^b}^{0} e^{\beta^b u}dz} - r\right). \tag{21}$$

This is a delay differential equation of neutral type and, unfortunately, our Theorem 1 does not cover this kind of equation.

5 Bounded Forward-Looking Feedback Rules

We now consider the case where nominal interest rates are set by the Central Bank as a function of expected and present inflation rates. For $\chi^b = 0$, the perfect-foresight equilibrium satisfies:

$$\begin{cases} R'(t) = \Lambda(R(t))\left[R(t) - r - \frac{\rho^{-1}(R(t)) - \chi^f \pi^f(t)}{(1-\chi^f)}\right], \\ \pi^f(t) = \frac{\int_t^{t+\Omega^f} e^{-\beta^f(u-t)}\left[\rho^{-1}(R(u)) - \chi^f \pi^f(u)\right]du}{(1-\chi^f)\int_0^{\Omega^f} e^{-\beta^f u}du}, \\ \lim_{t\to+\infty} \pi^f(t) = \pi_*,\ \lim_{t\to+\infty} R(t) = R_*. \end{cases} \tag{22}$$

The system (22) is similar to system (20) except that the algebraic equation includes advances rather than delays. Moreover, the two variables, $R(t)$ and $\pi^f(t)$, are forward. To study the local existence and uniqueness of solutions of (22), Theorem 2 can be easily applied to its linearized counterpart. It constitutes a particular case of system (7) where $n^b = 0$ and $n^f = n_1^f = 1$. Using Corollary 2, we conclude that there exists at least one solution to system (22). The next Lemma studies local uniqueness of the equilibrium. As in section 2, we denote by n^- the number of roots of the characteristic function that have negative real parts.

Lemma 4. *Let $\bar{\beta}^f \equiv \Lambda(R_*)$ and assume that $\beta^f > \bar{\beta}^f$. The characteristic equation associated with the linearized counterpart of system (22) has $n^- = 1$ if $\rho'(\pi_*) \in (0,1)$, and $n^- = 0$ if $\rho'(\pi_*) > 1$.*

Proof. See the Appendix.

Lemma 4 states that if the weight associated to future inflation rates is sufficiently low the determinacy typology of the equilibrium is the same as the one obtained with backward-looking feedback rules. There exists a unique equilibrium if the policy is active (the steady-state is unstable and both $R(t)$ and $\pi^f(t)$ jump on their long-run value) while the equilibrium is locally indeterminate if the policy is passive. For low discount factors (i.e. for $\beta^f < \bar{\beta}^f$), the characterization of the roots is tedious but other kind of situations may arise. For instance, it can be shown that there exist sets of parameters such that the equilibrium is indeterminate whatever the value of $\rho'(\pi_*)$ and others that are such that pure imaginary roots appear. This would confirm previous results on the destabilizing effects of forward-looking feedback rules (Woodford, 1999, and Benhabib, Schmitt-Grohé, and Uribe, 2001). Nevertheless, in our setting, an natural way to reduce the weight associated to future inflation rates is obtained by modifying the upper bound of the integral, Ω^f. This is studied in the next Lemma.

Lemma 5. *There exists* $\bar{\Omega}^f > 0$ *such that for* $\Omega^f < \bar{\Omega}^f$, *the characteristic equation associated with the linearized counterpart of system (22) has* $n^- \geq 1$ *if* $\rho'(\pi_*) \in (0,1)$, *and* $n^- = 0$ *if* $\rho'(\pi_*) > 1$.

Proof. See the Appendix.

Lemma 5 implies that local indeterminacy of active monetary policies with a forward-looking feedback rule are ruled out by choosing a not too long forecasting horizon. Conversely, the case of an infinite horizon (which was analyzed by Benhabib, Schmitt-Grohé, and Uribe, 2001) may imply multiple equilibria. This is studied in the next Lemma.

Lemma 6. *Let* $\hat{\beta}^f \equiv \bar{\beta}^f \left[\frac{1}{\rho'_*(\pi_*)} - \left(1-\chi^f\right)\right]$ *and assume that* $\Omega^f \to +\infty$. *For* $\rho'(\pi_*) \in (0,1)$, *the characteristic equation has* $n^- = 1$. *For* $\rho'(\pi_*) > 1$, *the characteristic equation has* $n^- = 2$ *if* $\beta^f < \hat{\beta}^f$, *and* $n^- = 0$ *if* $\beta^f > \hat{\beta}^f$.

Proof. See the Appendix.

To eliminate local indeterminacy, Lemma 6 suggests that the Central Bank should follow a very active (i.e. choose a large $\rho'_*(\pi_*)$) monetary policy. With Lemma 5, we conclude that the reduction of the forecasting horizon can be an alternative to an aggressive monetary policy.

6 Conclusion

This paper proposes theorems for the existence and uniqueness of solutions to systems of differential or algebraic equations with delays or advances. These theorems propose conditions that link the space of unknown initial conditions to the sign of the roots of the characteristic equation, just like the well-known

Blanchard-Kahn conditions. An optimal interest rate policy model allows us to apply our theorems and demonstrate how easy they are to use. They could therefore contribute to the development of the use of delay differential equations in economics, which would enable the analytical study of many phenomena. Sometimes, certain economic dynamics are characterized by differential equations that have both delays and advances. In such cases, both the stable and unstable manifolds are of infinite dimensions. Therefore, the existence and uniqueness of their solution cannot be analyzed using the theorems developed in this paper. This problem has been noted for further study.

7 Proofs

Proof of Theorem 1. Since it has been assumed that algebraic equations reduce to DDEs when differentiated a finite number of times, system (6) can be rewritten as:

$$\begin{cases} \mathbf{x}'(t) = \int_{t-1}^{t} d\mu_1 (t-u) V(u), \\ \mathbf{y}'(t) = \int_{t-1}^{t} d\mu_2 (t-u) V(u), \\ \mathbf{x}(\theta) = \bar{\mathbf{x}}(\theta) \text{ given for } \theta \in [-1,0], \\ \mathbf{y}(\theta) = \bar{\mathbf{y}}(\theta) \text{ given for } \theta \in [-1,0), \end{cases} \tag{23}$$

where $\mathbf{x} \in \mathbb{R}^{n-n^f}$ is a vector of backward variables (with $n \equiv n^b + n_1^b + n^f$) and $\mathbf{y} \in \mathbb{R}^{n^f}$ is a vector of forward variables, and where $V = (\mathbf{x}, \mathbf{y})$. Let us first rewrite system (23) in a compact way using the linear operator L_-, acting on $C([-1,0], \mathbb{R}^n)$ and defined as follows:

$$L_-(V(t)) = \int_0^1 d\mu(u) V(t-u).$$

To be able to study a system like (23) that incorporates forward variables, d'Albis, Augeraud-Véron and Hupkes (2011) suggest to extend the set of initial conditions to $C([-1,0])$. A solution to (23) is defined as a function $V(t) \in T$

where:

$$T = C\left([-1,0], \mathbb{R}^n\right) \times \{V \in C\left([0,\infty), \mathbb{R}^n\right) : \|V\|_\infty < \infty\},$$

with initial conditions $(\bar{\mathbf{x}}(\theta), \bar{\mathbf{y}}(\theta))$ defined on $C([-1,0], \mathbb{R}^n)$ by $\bar{\mathbf{y}}(0) = \bar{\mathbf{y}}(0^-)$ and such that $V(t)$ satisfies (23). Let us note that the solution may be multivalued at $t = 0$, which is due to the fact that $\mathbf{y}(0^+)$ may be different from $\bar{\mathbf{y}}(0^-)$. In order to deal with a possible jump at time $t = 0$ and to be able to computed it, we modify the definition of L_- in such a way that L_- is now acting on $C([-1,0], \mathbb{R}^n) \times \mathbb{R}^n$ and is defined as follows:

$$L_-\left(V(t), v\right) = \int_0^1 d\mu(u) V(t-u) + \left(\mu\left(0^+\right) - \mu\left(0^-\right)\right) v.$$

Moreover, the initial conditions $X = (\bar{\mathbf{x}}(\theta), \bar{\mathbf{y}}(\theta), (\mathbf{x}(0^+), \mathbf{y}(0^+)))$ now belong to $C([-1,0], \mathbb{R}^n) \times \mathbb{R}^n$.

Informations concerning the local existence and multiplicity of solutions are contained in the characteristic function. Let us denote by $\Delta_{L_-}(\lambda) = \lambda I - \int_{-1}^{0} d\mu(u) e^{\lambda u}$, the characteristic function of (23). It can be computed as follows:

$$\Delta_{L_-}(\lambda) = \prod_{i=1}^{N_1^b} (\lambda - \alpha_i) \delta_{L_-}(\lambda),$$

where $\delta_{L_-}(\lambda)$ is the characteristic function of system (6) and where $(\alpha_i)_{1 \leq i \leq N_1^b}$ denote the N_1^b roots that appear as a consequence of the differentiation of the algebraic equations of system (6). If algebraic equations reduce to differential equation when differentiated once with respect to time, $N_1^b = n_1^b$. If this reduction needs more than one differentiation, $N_1^b > n_1^b$ but N_1^b conditions are now provided at time $t = 0$.

Let us denote by $Q_{\alpha_i}(X)$ the spectral projection on the vector space spanned by $e^{\alpha_i t}$. We have: $Q_{\alpha_i}(X) = e^{\alpha_i t} H_{\alpha_i} R_{\alpha_i}(X)$, where:

$$R_{\alpha_i}(X) = (\mathbf{x}(0^+), \mathbf{y}(0^+)) + \sum_{j=1}^{3} \int_{-1}^{0} d\bar{\mu}_j(u) e^{\alpha_i u} \int_u^0 e^{-\alpha_i s} d\bar{\mu}_j(s) (\bar{\mathbf{x}}(s), \bar{\mathbf{y}}(s))$$

and where H_{α_i} is a matrix such that: $\Delta_{L_-}(\alpha_i) H_{\alpha_i} = H_{\alpha_i}\Delta_{L_-}(\alpha_i) = 0$. The computation of $H_{\alpha_i}R_{\alpha_i}(X)$ (see Theorem 3.16 in d'Albis, Augeraud-Véron and Hupkes, 2011) permits to see that it is proportional to

$$\mathbf{x}_1(0) - \int_{-1}^{0} d\bar{\mu}_2(u) W(u),$$

which implies that: $H_{\alpha_i}R_{\alpha_i}(X) = 0$.

Let us assume in the following that $\Delta_{L_-}(\lambda) = 0$ has no roots with real part equal to 0 and let us denote by n^+ the number of roots with positive real parts and which are distinct to any α_i.

If $n^+ = 0$, there is no unstable manifold, which implies that the set of initial conditions leading to a solution is $C([-1,0],\mathbb{R}^n) \times \mathbb{R}^n$. For any initial condition $(\mathbf{x}(\theta), \mathbf{y}(\theta)) \in C([-1,0],\mathbb{R}^n)$ with $\mathbf{y}(0) = \mathbf{y}(0^-)$, and any $(\mathbf{x}(0^+), \mathbf{y}(0^+))$, a continuous and bounded solution can be found.

If $n^+ > 0$, there exists an unstable manifold and one need to use the spectral projection formula to describe the solutions to system (23). Let $(\lambda_j)_{1\leq j\leq n^+}$ be the characteristic roots with positive real part of $\delta_{L_-}(\lambda) = 0$. The spectral projection $Q_{\lambda_j}(X)$ on the vector space spanned by $e^{\lambda_j t}$ is $Q_{\lambda_j}(X) = e^{\lambda_j t} H_{\lambda_j} R_{\lambda_j}(X)$ where:

$$R_{\lambda_j}(X) = (\mathbf{x}(0^+), \mathbf{y}(0^+)) + \sum_{j=1}^{2} \int_{-1}^{0} d\mu_j(u) e^{\lambda_j u} \int_{u}^{0} e^{-\lambda_j s} d\mu_j(s) (\bar{\mathbf{x}}(s), \bar{\mathbf{y}}(s))$$

and where H_{λ_j} is such that: $\Delta_{L_-}(\lambda_j) H_{\lambda_j} = H_{\lambda_j}\Delta_{L_-}(\lambda_j) = 0$. As the dynamics belong to the stable manifold, the projection on the unstable manifold should be null, which formally writes:

$$Q_{\lambda_j}(X) = 0. \tag{24}$$

We thus obtain a system of n^+ equations with n^f unknowns, which are given by $\mathbf{y}(0^+)$. Since eigenvectors may be linearly dependent, system (24) can be decomposed in two parts: a system of s equations with n^f unknowns, and

$(n^{+}-s)$ conditions on the initial known conditions $(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.))$, which are such that $\bar{\mathbf{x}}(0^{-}), \bar{\mathbf{x}}(0^{+})$ and $\bar{\mathbf{y}}(0^{-})$ are given. As the adjoint eigenvectors, denoted $(W_i^*)_{1\leq i\leq s}$, are linearly independent, we can write this formally as follows:

$$W_i^*\left(0, \mathbf{y}\left(0^{+}\right)-\bar{\mathbf{y}}\left(0^{-}\right)\right)=M_i(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.)) \text{ for } 1\leq i\leq s,$$

and

$$0=M_i(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.)) \text{ for } s+1\leq i\leq n^{+},$$

where $M_i(\bar{\mathbf{x}}(.), \bar{\mathbf{y}}(.))$ is an operator acting on the initial conditions, which is defined using the fact that the spectral projection on the unstable manifold has to be null. We notice that the first equation implies that W_i^* should not be colinear to the x axis if we want to avoid degeneracies. As the W_i^* are orthogonal to the stable manifold, the stable manifold should not be orthogonal to the x axis. If $s<n^f$, there are multiple solutions: some components of $y(0^{+})$ can be freely chosen to have a solution. If $n^{+}>s$, there is no solution generically: whatever $y(0^{+})$, the system of n^{+} equations with n^f unknowns cannot be solved unless the initial condition happens to satisfy the conditions, which is not guaranteed. If $s=n^f$, the system for $\mathbf{y}(0^{+})-\bar{\mathbf{y}}(0^{-})$ has the same number of equations and unknowns, which implies, as the W_i^* are linearly independent, that upon existence the solution is unique. □

Proof of Theorem 2. Since algebraic equations reduce to ADE when differentiated a finite number of time, system (7) can be rewritten as:

$$\begin{cases} \mathbf{x}'(t)=\int_t^{t+1} d\mu_1(t-u)\, V(u), \\ \mathbf{y}'(t)=\int_t^{t+1} d\mu_2(t-u)\, V(u), \\ x(0)=\bar{x}(0) \text{ given.} \end{cases}$$

where $\mathbf{x}\in\mathbb{R}^{n^b}$ is a vector of backward variables and $\mathbf{y}\in\mathbb{R}^{n-n^b}$ a vector of forward variables (with $n\equiv n^b+n_1^b+n^f$), and where $V=(\mathbf{x}, \mathbf{y})$. Let n^{-} be the number of eigenvalues with negative real parts, and s be the number of linearly independent eigenvectors. Any element of the stable space can be written as:

$$V(t) = \sum_{j=0}^{n^-} \alpha_j v_j e^{\lambda_j t}, \tag{25}$$

where the $(\lambda_j)_{1 \leq j \leq n^-}$ are the eigenvalues with negative real parts, the $(v_j)_{1 \leq j \leq n^-}$ are the eigenvectors, and the $(\alpha_j)_{1 \leq j \leq n^-}$ are the residues.

Evaluating the system (25) implies to solve a system with n^- unknowns and n^b equations. Since the eigenvectors $(v_j)_{1 \leq j \leq n^-}$ may be linearly dependant, the system splits in two parts. Let us denote with $(w_j)_{1 \leq j \leq s}$ the family of linearly independent eigenvectors. The first subsystem we obtain rewrites:

$$\sum_{j=0}^{n^-} \beta_j w_j = \bar{\mathbf{x}}(0),$$

which gives a system of s unknown $(\beta_j)_{1 \leq j \leq s}$ and n^b constraints. And, when the $(\beta_j)_{1 \leq j \leq s}$ are defined, we obtain a second system that rewrites:

$$\sum_{j=0}^{n^-} \alpha_j v_j = \sum_{j=0}^{s} \beta_j w_j,$$

which leads to a system of s equations and n^- unknowns, namely the $(\alpha_j)_{1 \leq j \leq n^-}$.

□

Proof of Lemma 1. The system composed by equations (13), (14) and (16) rewrites as follows:

$$\begin{cases} R'(t) = \Lambda(R(t)) \left[R(t) - r - \frac{\rho^{-1}(R(t)) - \chi^b \pi^b(t) - \chi^f \pi^f(t)}{1 - \chi^b - \chi^f} \right], \\ \pi^b(t) = \frac{\int_{t-\Omega^b}^{t} e^{\beta^b(u-t)} \left[\rho^{-1}(R(u)) - \chi^b \pi^b(u) - \chi^f \pi^f(u) \right] du}{(1-\chi^b-\chi^f) \int_{-\Omega^b}^{0} e^{\beta^b u} du}, \\ \pi^f(t) = \frac{\int_{t}^{t+\Omega^f} e^{-\beta^f(u-t)} \left[\rho^{-1}(R(u)) - \chi^b \pi^b(u) - \chi^f \pi^f(u) \right] du}{(1-\chi^b-\chi^f) \int_{0}^{\Omega^f} e^{-\beta^f u} du}. \end{cases} \tag{26}$$

A Taylor approximation on the neighborhood of the steady-state transforms the previous system into:

$$\left\{\begin{array}{rcl} R'(t) & = & \Lambda_*\left[\left(1-\frac{1}{\rho'_*(1-\chi^b-\chi^f)}\right)R(t)+\frac{\chi^b\pi^b(t)+\chi^f\pi^f(t)}{1-\chi^b-\chi^f}\right] \\ & & +\mathcal{M}\left(R(t),\pi^b(t),\pi^f(t)\right), \\ \pi^b(t) & = & \frac{\int_{t-\Omega^b}^{t}e^{\beta^b(u-t)}\left[\frac{R(u)}{\rho'_*}-\chi^b\pi^b(u)-\chi^f\pi^f(u)\right]du}{(1-\chi^b-\chi^f)\int_{-\Omega^b}^{0}e^{\beta^b u}du}+\frac{\int_{t-\Omega^b}^{t}e^{\beta^b(u-t)}\mathcal{N}(R(u))du}{(1-\chi^b-\chi^f)\int_{-\Omega^b}^{0}e^{\beta^b u}du}, \\ \pi^f(t) & = & \frac{\int_{t}^{t+\Omega^f}e^{-\beta^f(u-t)}\left[\frac{R(u)}{\rho'_*}-\chi^b\pi^b(u)-\chi^f\pi^f(u)\right]du}{(1-\chi^b-\chi^f)\int_{0}^{\Omega^f}e^{-\beta^f u}du}+\frac{\int_{t}^{t+\Omega^f}e^{-\beta^f(u-t)}\mathcal{N}(R(u))du}{(1-\chi^b-\chi^f)\int_{0}^{\Omega^f}e^{-\beta^f u}du}, \end{array}\right. \tag{27}$$

where $\Lambda_* \equiv \Lambda(R_*)$, and $\rho'_* \equiv \rho'(\pi_*)$, and where the nonlinearities write:

$$\begin{array}{rcl} \mathcal{M}\left(R,\pi^b,\pi^f\right) & = & \left[\Lambda(R+R_*)-\Lambda(R_*)\right]\left[R-\frac{\frac{R}{\rho'_*}-\chi^b\pi^b-\chi^f\pi^f}{1-\chi^b-\chi^f}\right] \\ & & -\frac{\Lambda(R+R_*)\mathcal{N}(R)}{1-\chi^b-\chi^f}, \\ \mathcal{N}(R) & = & \rho^{-1}(R_*+R)-\rho^{-1}(R_*)-\frac{R}{\rho'_*}. \end{array}$$

D'Albis, Augeraud-Véron and Hupkes (2011) provide a linearization theorem for hyperbolic system of differential-algebraic equations (Theorem 3.17 page 19). Some first conditions are standard and obviously satisfied for system (26). Second, both the linear and the non linear parts of the algebraic equations should reduce to functional differential equations when differentiated a finite number of times, which can easily be checked here by differentiating once with respect to time. Third, the non linear part and its first derivatives with respect to $\left(R,\pi^b,\pi^f\right)$ should vanish for $\left(R,\pi^b,\pi^f\right)=(0,0,0)$, which is also satisfied. □

Proof of Lemma 2. The linearized counterpart of system (20) is obtained by substituting $\chi^f=0$ in the linear parts of the first two equations of system (27). We obtain:

$$\left\{\begin{array}{l} R'(t)=\Lambda_*\left[\left(1-\frac{1}{\rho'_*(1-\chi^b)}\right)R(t)+\frac{\chi^b}{1-\chi^b}\pi^b(t)\right], \\ \pi^b(t)=\kappa^b\int_{t-\Omega^b}^{t}e^{\beta^b(u-t)}\left[\frac{R(u)}{\rho'_*}-\chi^b\pi^b(u)\right]du, \end{array}\right.$$

where we recall that $\Lambda_* \equiv \Lambda(R_*)$, and $\rho'_* \equiv \rho'(\pi_*)$ and where we introduce $\kappa^b \equiv 1/\left[\left(1-\chi^b\right)\int_{-\Omega^b}^{0}e^{\beta^b u}du\right]$. The characteristic function of this system, denoted

$\delta(z)$, is defined such that $\delta(z) = \det(\mathcal{I}(z))$ where:

$$\mathcal{I}(z) = \begin{bmatrix} z - \Lambda_* \left(1 - \frac{1}{\rho'_*(1-\chi^b)}\right) & -\Lambda_* \frac{\chi^b}{1-\chi^b} \\ -\frac{\kappa^b}{\rho'_*} \int_{-\Omega^b}^0 e^{\left(\beta^b+z\right)\sigma} d\sigma & 1 + \chi^b \kappa^b \int_{-\Omega^b}^0 e^{\left(\beta^b+z\right)\sigma} d\sigma \end{bmatrix}$$

Thus:

$$\delta(z) = (z - \Lambda_*) \left(\chi^b \kappa^b \int_{-\Omega^b}^0 e^{\left(\beta^b+z\right)\sigma} d\sigma + 1\right) + \frac{\Lambda_*}{\rho'_*(1-\chi^b)}. \tag{28}$$

To prove the lemma, we proceed in two steps. 1/ we show there exists a unique positive real root if $\rho'_* > 1$ and that there is no positive real root if $\rho'_* \in (0, 1)$. 2/ we show there is no complex root with positive real parts.

1/ Real roots of $\delta(z) = 0$. Observe first that if $z - \Lambda_* \geq 0$, one has $\delta(z) > 0$ and that if $z - \Lambda_* < 0$, one has $\delta'(z) > 0$. Moreover, for $z \in (0, \Lambda_*)$, one has $\delta(0) < \delta(z) < \delta(\Lambda_*)$ with $\delta(0) = \Lambda_*(1 - \rho'_*)/\rho'_*(1-\chi^b)$ and $\delta(\Lambda_*) = \Lambda_*/\rho'_*(1-\chi^b)$. Hence, if $1 - \rho'_* < 0$, there exists a real root $z \in (0, \Lambda_*)$ such that $\delta(z) = 0$ and if $1 - \rho'_* > 0$, there is no real root $z \in (0, \Lambda_*)$ such that $\delta(z) = 0$.

2/ Complex roots of $\delta(z) = 0$. Let us denote the complex roots by $z = p + iq$. We first prove that there are no complex roots with positive real part that satisfy $p > \Lambda_*$ by showing that $|\delta(z)| > 0$. For $p > \Lambda_*$, one has:

$$|\delta(z)| > \left| \frac{\Lambda_*}{\rho'_*(1-\chi^b)} - |(\Lambda_* - z)| \left| \left(\chi^b \kappa^b \int_{-\Omega^b}^0 e^{\left(\beta^b+z\right)\sigma} d\sigma + 1\right) \right| \right|.$$

Then, it is sufficient to observe that the right-hand-side of the above inequality is greater than $\delta(p) > 0$ to conclude. Let us now consider the roots whose real parts belong to $(0, \Lambda_*)$. We are going to show that in this case: $\mathrm{Im}(\delta(z)) > q > 0$. One has:

$$\begin{aligned} \mathrm{Im}(\delta(z)) &= q \left[\chi^b \kappa^b \int_{-\Omega^b}^0 e^{(\beta^b+p)\sigma} \cos(q\sigma) d\sigma + 1\right] \\ &\quad -(p - \Lambda_*)\chi^b \kappa^b \int_0^{-\Omega^b} e^{(\beta^b+p)\sigma} \sin(q\sigma) d\sigma. \end{aligned}$$

Thus:

$$\mathrm{Im}(\delta(z)) > q \left[\chi^b \kappa^b \int_{-\Omega^b}^0 e^{(\beta^b+p)\sigma} \left[q \cos(q\sigma) + p \sin(q\sigma)\right] d\sigma + 1\right].$$

Since:

$$q\cos(q\sigma)+p\sin(q\sigma)=\left[\left(p^2+q^2\right)\int_\sigma^0 e^{pu}\sin\left(qu\right)du-q\right]e^{-p\sigma},$$

one has:

$$\mathrm{Im}\left(\delta\left(z\right)\right)>q\left[\chi^b\kappa^b\int_{-\Omega^b}^0 e^{\beta^b\sigma}\left[\left(p^2+q^2\right)\int_\sigma^0 e^{pu}\sin\left(qu\right)du-q\right]d\sigma+1\right].$$

Using the fact that $\int_0^\sigma e^{(\beta^b+p)u}\sin(qu)du>0$ for $\sigma<0$, suffices to complete the proof. $\square$

Proof of Lemma 3. When $\Omega^b\to+\infty$, the characteristic function (28) becomes

$$\delta\left(z\right)=z^2+z\left[\frac{\beta^b}{(1-\chi^b)}+\Lambda_*\left(\frac{1-\rho_*'(1-\chi^b)}{\rho_*'(1-\chi^b)}\right)\right]+\frac{\Lambda_*\beta^b}{(1-\chi^b)}\left(\frac{1-\rho_*'}{\rho_*'}\right),$$

for all $z>-\beta^b$. One has:

$$\begin{aligned}\delta\left(-\beta^b\right)&=-\frac{\chi^b\beta^b}{(1-\chi^b)}\left(\beta^b+\Lambda_*\right)<0\\ \delta\left(0\right)&=\frac{\Lambda_*\beta^b}{(1-\chi^b)}\left(\frac{1-\rho_*'}{\rho_*'}\right)\end{aligned}$$

Hence, there is one real root that belongs to $\left(-\beta^b,0\right)$ if $\rho_*'<1$ and that is positive if $\rho_*'>1$. The other real root is lower than $-\beta^b$, but the projection on the eigenvector related to this latter root is zero according to the definition of $\pi^b\left(0^+\right)$. $\square$

Proof of Lemma 4. The linearized counterpart of system (20) is obtained by substituting $\chi^b=0$ in the linear parts of the first and the third equations of system (27). We obtain:

$$\left\{\begin{array}{l}R'\left(t\right)=\Lambda_*\left[\left(1-\frac{1}{\rho_*'(1-\chi^f)}\right)R\left(t\right)+\frac{\chi^f}{1-\chi^f}\pi^f\left(t\right)\right]\\ \pi^f\left(t\right)=\kappa^f\int_t^{t+\Omega^f}e^{-\beta^f(u-t)}\left[\frac{R(u)}{\rho_*'}-\chi^f\pi^f\left(u\right)\right]du,\end{array}\right.$$

where we recall that $\Lambda_*\equiv\Lambda\left(R_*\right)$, and $\rho_*'\equiv\rho'\left(\pi_*\right)$ and where we introduce $\kappa^f\equiv1/\left[\left(1-\chi^f\right)\int_0^{\Omega^f}e^{-\beta^fu}du\right]$. The characteristic function of this system, denoted $\delta(z)$, is defined such that $\delta(z) = \det(\mathcal{I}(z))$ where:

$$\mathcal{I}(z) = \begin{pmatrix} z - \Lambda_* \left(1 - \frac{1}{\rho'_*(1-\chi^f)}\right) & -\Lambda_* \frac{\chi^f}{1-\chi^f} \\ -\frac{\kappa^f}{\rho'_*} \int_0^{\Omega^f} e^{(z-\beta^f)\sigma} d\sigma & 1 + \chi^f \kappa^f \int_0^{\Omega^f} e^{(z-\beta^f)\sigma} d\sigma \end{pmatrix},$$

which gives:

$$\delta(z) = (z - \Lambda_*) \left(\chi^f \kappa^f \int_0^{\Omega^f} e^{(z-\beta^f)\sigma} d\sigma + 1 \right) + \frac{\Lambda_*}{\rho'_* (1 - \chi^f)}. \qquad (29)$$

Let us define $\bar{\beta}^f \equiv \Lambda_*$. We restrict to the case $\beta^f \geq \bar{\beta}^f$ and proceed in two steps by showing: 1/ there exists one negative real root if $\rho'_* \in (0,1)$ and no negative real root if $\rho'_* > 1$; 2/ there are no complex roots with negative real parts.

1/ Real roots of $\delta(z) = 0$. Let us compute the derivative of (29):

$$\delta'(z) = 1 + \chi^f \kappa^f \Omega^f e^{(z-\beta^f)\Omega^f} + \left(\beta^f - \Lambda_*\right) \chi^f \kappa^f \int_0^{\Omega^f} \sigma e^{(z-\beta^f)\sigma} d\sigma. \qquad (30)$$

For $\beta^f \geq \bar{\beta}^f$, one has $\delta'(z) > 0$. Since $\lim_{z \to -\infty} \delta(z) = -\infty$ and $\delta(0) = \frac{\Lambda_*(1-\rho'_*)}{\rho'_*(1-\chi^f)}$, we conclude that there exists a unique negative real root if $\rho'_* \in (0,1)$ and no negative real root if $\rho'_* > 1$.

2/ Complex roots of $\delta(z) = 0$. Let us denote the complex roots by $z = p + iq$. One has:

$$|\delta(z)| > \left| \left(1 - \chi^f \kappa^f \left(1 + \frac{\beta^f - \Lambda_*}{z - \beta^f} \right) \left(1 - e^{(z-\beta^f)\Omega^f} \right) \right) + \frac{\Lambda_*}{\rho'_*(1 - \chi^f)} \right|.$$

We conclude that for $\beta^f \geq \bar{\beta}^f$, one has: $|\delta(z)| > |\delta(p)| > 0$. □

Proof of Lemma 5. We show that for Ω^f small enough there exists one negative real root if $\rho'_* \in (0,1)$ and no root with negative real part if $\rho'_* > 1$. We proceed in three steps:

1/ There is a unique real root to $\delta(z) = 0$ that is negative if $\rho'_* \in (0,1)$ and positive if $\rho'_* > 1$. For Ω^f small, $\delta'(z) > 0$ (where $\delta'(z)$ is given by (30)). We conclude with $\lim_{z \to -\infty} \delta(z) = -\infty$, $\lim_{z \to +\infty} \delta(z) = +\infty$, and $\delta(0) = \frac{\Lambda_*(1-\rho'_*)}{\rho'_*(1-\chi^f)}$.

2/ For $\rho'_* > 1$, the positive real root, denoted z_1, is smaller than Λ_*. For $\Omega^f = 0$, the characteristic function (29) rewrites (by applying l'Hôpital's Rule):

$$\delta(z) = \frac{1}{1-\chi^f}\left[z - \Lambda_*\left(1 - \frac{1}{\rho'_*}\right)\right].$$

For Ω^f small, z_1 is close to $\Lambda_*\left(1 - \frac{1}{\rho'_*}\right)$ and is thus smaller than Λ_* for $\rho'_* > 1$.

3/ For $\rho'_* > 1$, there is no complex root $z = p + iq$, with real part smaller than z_1. One has:

$$\begin{aligned}
\mathrm{Re}(\delta(z)) &= \mathrm{Re}\left((z - z_1)\chi^f\kappa^f\int_0^{\Omega^f}\left(e^{(z-\beta^f)\sigma} - e^{(z_1-\beta^f)\sigma}\right)d\sigma\right) \\
&\quad + \mathrm{Re}\left((z_1 - \Lambda_*)\chi^f\kappa^f\int_0^{\Omega^f} e^{(z-\beta^f)\sigma}d\sigma\right) + \mathrm{Re}((z - z_1)).
\end{aligned}$$

Using the fact that $z_1 < \Lambda_*$, we conclude that, for Ω^f small, $\mathrm{Re}(\delta(z)) < 0$ for all $z < z_1$. $\square$

Proof of Lemma 6. For $\Omega^f \to +\infty$, the characteristic function (29) is defined for $z \in \left(-\infty, \beta^f\right)$ and writes:

$$\delta(z) = -z^2 + \frac{z}{(1-\chi^f)}\left[\beta^f - \Lambda_*\left[\frac{1}{\rho'_*} - \left(1-\chi^f\right)\right]\right] + \left(\frac{1}{\rho'_*} - 1\right)\frac{\Lambda_*\beta^f}{(1-\chi^f)},$$

Since $\lim_{z\to-\infty}\delta(z) = -\infty$ and $\delta(0) = \left(\frac{1-\rho'_*}{\rho'_*}\right)\frac{\Lambda_*\beta^f}{(1-\chi^f)}$ we conclude there is one negative real root if $\rho'_* \in (0,1)$ and either zero or two roots with negative real parts if $\rho'_* > 1$. The condition that excludes roots with negative real parts is $\delta'(0) > 0$, or equivalently $\beta^f > \Lambda_*\left[\frac{1}{\rho'_*} - \left(1-\chi^f\right)\right]$. $\square$

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