SOME REMARKS ON THE L^2-CRITICAL NONLINEAR SCHRÖDINGER EQUATION WITH A NONLINEAR DAMPING.

Abstract. We consider the Cauchy problem for the L^2-critical nonlinear Schrödinger equation with a nonlinear damping. According to the power of the damping term, we prove the global existence or the existence of finite time blowup dynamics with the log-log blow-up speed for $\|\nabla u(t)\|_{L^2}$.

1. Introduction

In this paper, we study the blowup and the global existence of solutions for the focusing NLS equation with a nonlinear damping:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + ia|u|^p u = 0, (t,x) \in [0,\infty[\times\mathbb{R}^d, d = 1,2,3,4. \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \tag{1.1}$$

with initial data $u(0) = u_0 \in H^1(\mathbb{R}^d)$ where $a > 0$ is the coefficient of friction and $p \in N^*$. Note that if we replace $+|u|^{\frac{4}{d}}u$ by $-|u|^{\frac{4}{d}}u$, (1.1) becomes the defocusing NLS equation.

Equation (1.1) arises in various areas of nonlinear optics, plasma physics and fluid mechanics. Fibich [6] noted that in the nonlinear optics context, the origin of nonlinear damping is multiphoton absorption. For example, in the case of solids the number p corresponds to the number of photons it takes to make a transition from the valence band to the conduction band. Similar behavior can occur with free atoms, in this case p corresponds to the number of photons needed to make a transition from the ground state to some excited state or to the continuum.

The Cauchy problem for (1.1) was studied by Kato [9] and Cazenave [2] and it is known that if $p < \frac{4}{d-2}$, then the problem is locally well-posed in $H^1(\mathbb{R}^d)$: For any $u_0 \in H^1(\mathbb{R}^d)$, there exist $T \in (0,\infty]$ and a unique solution $u(t)$ of (1.1) with $u(0) = u_0$ such that $u \in C([[0,T]); H^1(\mathbb{R}^d))$. Moreover, T is the maximal existence time of the solution $u(t)$ in the sense that if $T < \infty$ then $\lim\limits_{t\to T} \|u(t)\|_{H^1(\mathbb{R}^d)} = \infty$.

Let us notice that for $a = 0$ (1.1) becomes the L^2-critical nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \tag{1.2}$$

For $u_0 \in H^1$, a sharp criterion for global existence for (1.2) has been exhibited by Weinstein [22]:

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For $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the solution of (1.2) is global in H^1. This follows from the conservation of the energy and the L^2 norm and the sharp Gagliardo-Nirenberg inequality:

$$\forall u \in H^1, E(u) \geq \frac{1}{2}(\int |\nabla u|^2)\Big(1 - (\frac{\int |u|^2}{\int |Q|^2})^{\frac{2}{d}}\Big).$$

On the other hand, there exists explicit solutions with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up in finite time.
In the series of papers [12, 20], Merle and Raphael have studied the blowup for (1.2) with $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \delta$, δ small and have proven the existence of the blowup regime corresponding to the log-log law:

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{1.3}$$

Darwich [5] has proved in case of the linear damping ($p = 0$), the global existence in H^1 for $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, and has showed that the log-log regime is stable by perturbations and the solutions blows up in finite time with the log-log law.
Numerical observations suggest that this finite time blowup phenomena persists in the case of the nonlinear damping for $p < \frac{4}{d}$ (see Fibich [6] and [18]). Passot and Sulem [18] have proved that the solutions are global in $H^1(\mathbb{R}^2)$ for $p > 2$. Therefore, it is interesting now to study for which values of p, we have either the blow-up or the global existence of solutions, it is our aim in this paper and we will show that:
If $p \geq \frac{4}{d}$, then the solution of (1.1) is global in H^1.
The global existence for small data in H^1, for $p \leq 2$.
The existence of finite time blowup solutions.

More precisely, we have the following theorem:

Theorem 1.1. *Let u_0 in $H^1(\mathbb{R}^d)$ with $d = 1, 2, 3, 4$:*
1. *if $\frac{4}{d-2} > p \geq \frac{4}{d}$, then the solution of (1.1) is global in H^1.*
2. *For $p = 1$ and $d = 1, 2, 3$ or $p = 2$ and $d = 2$, the solution is global in H^1, for $u_0 \in H^1$ such that $\|u_0\|_{L^2} < \alpha$ with $\alpha > 0$ small enough.*
3. *let $p < \frac{4}{d}$, then there exists $\delta_0 > 0$ such that $\forall a > 0$ and $\forall \delta \in]0, \delta_0[$, there exists $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, such that the solution of (1.1) blows up in finite time in the log-log regime.*

In the case $p = \frac{4}{d}$, we have the following theorem:

Theorem 1.2. *Let $p = \frac{4}{d}$, then the initial-value probem (1.1) is globally well posed in $H^s(R^d)$, $s \geq 0$.*

Remark 1.1. *Theorem 1.2 and part (1) and (2) of Theorem 1.1 still true in the defocusing case.*

In this paper we will prove the existence of log-log explosive solutions in the case $p < \frac{4}{d}$. We etablish the global existence of the solutions as soon as $\frac{4}{d} \leq p < \frac{4}{d-2}$, where $\frac{4}{d-2}$ is the H^1-critical exponent.

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2. Global existence.

In this section, we prove assertion (1) and (2) of Theorem 1.1 and Theorem 1.2. To prove part (1), we will prove that the H^1-norm of u is bounded for any time. To prove part (2), we use generalised Gagliardo-Nirenberg inequality to show that the energy is non increasing. Finally to prove Theorem 1.2, we etablish an a priori estimate on the critical Strichartz norm.

Theorem 2.1. *Let $p \leq \frac{4}{d-2}$(for $d > 2$), then the initial-value probem (1.1) is locally well posed in $H^1(R^d)$(If $p < \frac{4}{d-2}$ the minimal time of the existence depends on $\|u_0\|_{H^1}$.) .*

Proof: See [2] page 93 Theorem 4.4.1.

To prove the following proposition, we will proceed in the same way as in the section 3.1 in [18].

Proposition 2.1. *Let u be a solution of (1.1) and $\frac{4}{d-2} \geqslant p > \frac{4}{d}$ then*

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \leq \|\nabla u(0)\|_{L^2(\mathbb{R}^d)} e^{a^{(\frac{-4t}{pd-4})}}.$$

Proof: Multiply Eq. (1.1) by $\Delta\overline{u}$, integrate and take the imaginary part, this gives

$$\frac{1}{2}\frac{d}{dt}\int|\nabla u|^2dx + a\int|u|^p|\nabla u|^2 + a\Re\int u\nabla|u|^p\nabla\overline{u}dx = -\frac{4}{d}\Im\int u\nabla\overline{u}\Re(u\nabla u)|u|^{\frac{4}{d}-2}. \tag{2.1}$$

In the l.h.s, a simple calculation shows that the third term rewrites in the form $\frac{p}{4}\int|u|^{p-2}(\nabla|u|^2)^2$.Equation (2.1) becomes:

$$\frac{1}{2}\frac{d}{dt}\int|\nabla u|^2dx + a\int|u|^p|\nabla u|^2 + a\frac{p}{4}\int|u|^{p-2}(\nabla|u|^2)^2 \leq \frac{2}{d}\int|u|^{\frac{4}{d}}|\nabla u|^2. \tag{2.2}$$

To estimate the r.h.s of (2.2), we rewrite it as ($p > \frac{4}{d}$)

$$\int|u|^{\frac{4}{d}}|\nabla u|^2 = \int|u|^{\frac{4}{d}}|\nabla u|^{\frac{8}{pd}}|\nabla u|^{2-\frac{8}{pd}}.$$

Now by Hölder inequality we obtain that

$$\int|u|^{\frac{4}{d}}|\nabla u|^2 \leq (\int|u|^p|\nabla u|^2)^{\frac{4}{pd}}(\int|\nabla u|^2)^{1-\frac{4}{pd}}.$$

Then inequality (2.2) takes the form:

$$\frac{d}{dt}w(t) + 2av(t) \leq \frac{4}{d}v(t)^{\frac{4}{pd}}w(t)^{1-\frac{4}{pd}}.$$

where $w(t) = \int|\nabla u|^2$ and $v(t) = \int|u|^p|\nabla u|^2$.

Using Young's inequality $ab \leq \epsilon a^q + C\epsilon^{-\frac{1}{q-1}}b^{q'}$, $\frac{1}{q}+\frac{1}{q'} = 1$, with $q = \frac{pd}{4}$ and $\epsilon = \frac{ad}{2}$ we obtain :

$$\frac{d}{dt}w(t) \leq a^{-\frac{1}{\frac{pd}{4}-1}} w(t).$$

This ensures that:

$$w(t) \leq w(0)e^{a^{\left(-\frac{4t}{pd-4}\right)}}. \square$$

This show that, the H^1-norm of u is bounded for any time and gives directly the proof of part one of Theorem 1.1.
Now we will prove the global existence for small data, for this we will use the following lemma, for the proof see [7]:

Lemma 2.1. *Let q, r be any real numbers satisying $1 \leq q$, $r \leq \infty$, and let j, m be any integers satisfying $0 \leq j < m$. If u is any functions in $C_0^m(R^n)$, then*

$$|D^j u|_{L^s} \leq C|D^m u|_r^a |u|_q^{1-a}$$

where

$$\frac{1}{s} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \leq a \leq 1,$$

where C is a constant depending only on n, m, j,q,r and a.

Now we have the following one:

Lemma 2.2. *Let $p \leq 2$ and $v \in C_0^\infty(R^n)$ then:*

$$\int |v|^{\frac{4}{d}+2+p} \leq C(\int |\nabla(|v|^{\frac{p+2}{2}})|^2) \times (\int |v|^2)^{\frac{2}{d}}.$$

where $c > 0$ depending only on d and p.

Proof: Take $s = \frac{\frac{4}{d}+2+p}{1+\frac{p}{2}}$, $q = \frac{2}{1+\frac{p}{2}}$ $r = 2$, $j = 0$ and $m = 1$,then by Lemma 2.1 we obtain that:

$$|u|_{L^{\frac{\frac{4}{d}+2+p}{1+\frac{p}{2}}}} \leq C|\nabla u|_{L^2}^{\frac{4+2p}{\frac{8}{d}+4+2p}} |u|_{L^{\frac{2}{1+\frac{p}{2}}}}^{\frac{\frac{8}{d}}{\frac{8}{d}+4+2p}}.$$

Now take $u = |v|^{1+\frac{p}{2}}$, we obtain our lemma.
Now we can prove the following proposition:

Proposition 2.2. *let $p \leq 2$. There exists $0 < \alpha = \alpha(p,d) < \|Q\|_{L^2}$, such that for any $u_0 \in H^1$ with$\|u_0\|_{L^2} < \alpha$, it holds*

$$\frac{d}{dt}E(u(t)) \leq 0, \ \forall t > 0.$$

Proof: It is a easy to prove that :

$$\frac{d}{dt}E(u(t)) = a(\int |u|^{\frac{4}{d}+p+2} - C_p \int |\nabla(|u|^{\frac{p+2}{2}})|^2),$$

then by Lemma 2.2 we obtain that:

$$\frac{d}{dt}E(u(t)) \leq a(\int |\nabla(|u|^{\frac{p+2}{2}})|^2)(C(\int |u|^2)^{\frac{2}{d}} - C_p)$$

Choosing $\alpha^{\frac{2}{d}} < \frac{C_p}{C}$, and using (3.2) we get the result.
Now we will prove part (2) of Theorem 1.1: We have by the sharp Gagliardo-Nirenberg inequality :

$$\forall u \in H^1, E(u) \geq \frac{1}{2}(\int |\nabla u|^2)\Big(1 - (\frac{\int |u|^2}{\int |Q|^2})^{\frac{2}{d}}\Big).$$

Note that $E(u) \leq E(u_0)$, then by this inequality we can found a bound for the H^1-norm and we obtain the global existence.

2.1. **Critical case ($p = \frac{4}{d}$).** Now we will treat the critical case. First let us prove that, if the solution blows up in finite time T, then $\|u\|_{L^{\frac{4}{d}+2}([0,T];L^{\frac{4}{d}+2}(R^d))} = +\infty$.

Proposition 2.3. *Let u be the unique maximal solution of (1.1) in $[0, T^*)$; if $T^* < \infty$, then $\|u\|_{L^\sigma([0,T],L^\sigma)} = \infty$ where $\sigma = \frac{4}{d} + 2$.*

For this we will need the following proposition:

Proposition 2.4. *There exists $\delta > 0$ with the following property. If $u_0 \in L^2(R^d)$ and $T \in (0, \infty]$ are such that $\|S(.)u_0\|_{L^\sigma([0,T],L^\sigma)} < \delta$, there exists a unique solution $u \in C([0, T], L^2(R^d)) \cap L^\sigma([0, T], L^\sigma(R^n))$ of (1.1). In addition, $u \in L^q([0, T], L^r(R^d))$ for every admissible pair (q, r); for $t \in [0, T]$.Finally, u depends continuously in $C([0, T], L^2(R^d))\cap L^\sigma([0, T], L^\sigma(R^d))$ on $u_0 \in L^2(R^n)$. If $u_0 \in H^1(R^d)$, then $u \in C([0, T], H^1(R^d))$.*

See [3] for the proof.

We need also the following lemma (see [3]):

Lemma 2.3. *Let $T \in (0, \infty]$, let $\sigma = \frac{4}{d} + 2$, and let (q, r) be an admissible pair. Then, whenever $u \in L^\sigma([0, T], L^\sigma(R^d))$, it follows that $F(u) = -i\int_0^t S(t - s)(|u|^{\frac{4}{d}} u + ia|u|^p u)ds \in C([0, T], H^{-1}(R^d)) \cap L^q(0, T, L^r(R^d))$. Furthermore, there exists K, independent of T, such that*

$$\|Fv-Fu\|_{L^q(]0,T[,L^r)} < K(\|u\|^{\frac{4}{d}}_{L^\sigma(]0,T[,L^\sigma(R^d))}+\|v\|^{\frac{4}{d}}_{L^\sigma(]0,T[,L^\sigma(R^d))})\|u-v\|_{L^\sigma(]0,T[,L^\sigma(R^d))} \tag{2.3}$$

for every u, $v \in L^\sigma(]0, T[, L^\sigma(R^d))$.

Proof of prop 2.3:

Let $u_0 \in L^2(R^d)$. Observe that$\|S(.)u_0\|_{L^\sigma(0,T.L^\sigma)} \longrightarrow 0$,as $T \longrightarrow 0$. Thus for sufficiently small T, the hypotheses of Proposition 2.4 are satisfied. Applying iteratively this proposition, we can construct the maximal solution $u \in C([0, T^*), L^2(R^d))) \cap L^\sigma([0, T^*), L^\sigma(R^d))$ of (1.1). We proceed by contradiction, assuming that $T^* < \infty$, and $\|u\|_{L^\sigma(]0,T[,L^\sigma)} < \infty$. Let $t \in [0, T^*)$. For every $s \in [0, T^* - t)$, we have

$$S(s)u(t) = u(t + s) - F(u(t+\cdot))(s).$$

From (2.3), we obtain

$$\|S(.)u(t)\|_{L^\sigma([0,T^*-t),L^\sigma(R^d))} \leq \|u\|_{L^\sigma(]t,T^*[,L^\sigma)} + K(\|u\|_{L^\sigma(]t,T^*[,L^\sigma)})^{\frac{4}{d}+1}$$

Therefore, for t fixed close enough to T^*, it follows that

$$\|S(.)u(t)\|_{L^\sigma([0,T^*-t),L^\sigma(R^d))} \leq \delta.$$

Applying Proposition 2.4, we find that u can be extended after T^*, which contradicts the maximality.

Corollary 2.1. *For $p = \frac{4}{d}$, the solution of (1.1) is global.*

Proof Multiply equation (1.1) by $\overline{u}$, and take the imaginary part to obtain:

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2a\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d+2}} = 0.$$

Hence $\forall t \in R_+$

$$\|u\|_{L^{\frac{4}{d}+2}[0,t[L^{\frac{4}{d}+2}(R^d)} \leq \frac{1}{2a}\|u_0\|_{L^2}^2.$$

Theorem 1.2 follows then directly from Propostion 2.3.

3. Blow up solution.

Now we will prove the existence of the explosive solutions in the regime log-log.
We look for a solution of (1.1) such that for t close enough to blowup time, we shall have the following decomposition:

$$u(t,x) = \frac{1}{\lambda^{\frac{d}{2}}(t)}(Q_{b(t)} + \epsilon)(t, \frac{x - x(t)}{\lambda(t)})e^{i\gamma(t)}, \tag{3.1}$$

for some geometrical parameters $(b(t), \lambda(t), x(t), \gamma(t)) \in (0,\infty) \times (0,\infty) \times \mathbb{R}^d \times \mathbb{R}$, here $\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}}$, and the profiles Q_b are suitable deformations of Q related to some extra degeneracy of the problem.

Now we define the following quantities:

L^2-norm : $\|u(t,x)\|_{L^2} = \displaystyle\int |u(t,x)|^2 dx.$

Energy : $E(u(t,x)) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{d}{4+2d}\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$

Kinetic momentum : $P(u(t)) = Im(\displaystyle\int \nabla u\overline{u}(t,x)).$

Remark 3.1. *It is easy to prove that if u is a solution of (1.1) then:*

$$\frac{d}{dt}\|u(t)\|_{L^2} = -2a\int |u|^{p+2}, t \in [0,T), \tag{3.2}$$

$$\frac{d}{dt}E(u(t)) = -a(\|u^{\frac{p}{2}}\nabla u\|_{L^2}^2 - \|u\|_{L^{\frac{4}{d}+2+p}}^{\frac{4}{d}+2+p}) \tag{3.3}$$

and

$$\frac{d}{dt}P(u(t)) = -2aIm\int \overline{u}|u|^p\nabla u, t \in [0,T). \tag{3.4}$$

Now we have the following theorem:

Theorem 3.1. *There exist a set of initial data Ω in H^1, such that for $0 < a < a_0$ with a_0 small enough, the corresponding solution $u(t)$ of (1.1) blows up in finite time in the log-log regime.*

The set of initial data Ω is open in H^1, using the continuity with regard to the initial data and the parameters, we can prove the following corollary:

Corollary 3.1. *Let $u_0 \in H^1$ be an initial data such that the corresponding solution $u(t)$ of (1.2) blows up in the loglog regime. There exist $\beta_0 > 0$ and $a_0 > 0$ such that if $v_0 = u_0 + h_0$, $\|h_0\|_{H^1} \leq \beta_0$ and $a \leq a_0$, the solution $v(t)$ for (1.1) with the initial data v_0 blowup in finite time.*

Proof of Corollary 3.1: Let $S(t)$ be the propagator for the linear equation:
$$i\partial_t u + \Delta u = 0, \quad (t,x) \in [0,\infty[\times R^d.$$
The Cauchy problem for (1.1) with $u(0) = u_0 \in H^1(\mathbb{R}^d)$ is equivalent to the integral equation:
$$u(t) = S(t)u_0 + i\int_0^t S(t-s)(|u(s)|^{\frac{4}{d}}u(s) + ia|u(s)|^p u)ds.$$
We know from Lemma 3.2: there exist $T(\|u_0\|_{H^1(\mathbb{R}^d)}) > 0$ such that:
$$\forall 0 \leq a \leq 1,\ \|u\|_{L^\infty([0,T];H^1)} \leq 2\|u_0\|_{H^1(\mathbb{R}^d)}.$$
Let u a solution for (1.1) and v solution for (1.2) we have:
$$u-v = S(t)(u_0-v_0)+i\int_0^t S(t-s)(|u(s)|^{\frac{4}{d}}u(s)-|v(s)|^{\frac{4}{d}}v(s))ds+ia\int_0^t S(t-t')u(t')|u(t')|^p dt'.$$
By Strichartz we obtain (see the proof of Lemma 3.2):
$$\begin{aligned}\|u-v\|_{L^\infty([0,T];H^1)} &\leq \|u_0-v_0\|_{H^1(\mathbb{R}^d)}\\ &+CT^\gamma\big(\|u\|^{\frac{4}{d}}_{L^\infty([0,T];H^1)} + \|v\|^{\frac{4}{d}}_{L^\infty([0,T];H^1)}\big)\|u-v\|_{L^\infty([0,T];H^1)}\\ &+CaT\|u\|^{p+1}_{L^\infty([0,T];H^1)}.\end{aligned}$$
Thus for $T_1 = \mathrm{Min}\big(T(\|u_0\|_{H^1(\mathbb{R}^d)}), T(\|v_0\|_{H^1(\mathbb{R}^d)})\big)$ we obtain $\forall\ 0 \leq t \leq T_1$:
$$\begin{aligned}\|u-v\|_{L^\infty([0,T];H^1)} &\leq \|u_0-v_0\|_{H^1(\mathbb{R}^d)}\\ &+CT^\gamma\big(\|u_0\|^{\frac{4}{d}}_{H^1} + \|v_0\|^{\frac{4}{d}}_{H^1}\big)\|u-v\|_{L^\infty([0,T];H^1)}\\ &+CaT\|u\|^{p+1}_{L^\infty([0,T];H^1)}.\end{aligned}$$
Now for $T_2 = \frac{1}{2}\mathrm{Min}\big(\mathrm{Max}^{-\frac{1}{\gamma}}(\|u_0\|^p_{H^1}, \|v_0\|^p_{H^1}), T_1\big)$:
$$\begin{aligned}\|u-v\|_{L^\infty([0,t];H^1)} &\leq \|u_0-v_0\|_{H^1(\mathbb{R}^d)} + \tfrac{1}{2}\|u-v\|_{L^\infty([0,t];H^1)}\\ &+a,\end{aligned}$$
thus
$$\|u-v\|_{L^\infty([0,t];H^1)} \leq C\|u_0-v_0\|_{H^1(\mathbb{R}^d)} + a.\ \forall\ 0 < t < T_2$$
.

Thus the map $(a,\phi) \to u(\cdot,a,\phi)$ is continuous in $(0,u_0)$ from $\mathbb{R}\times H^1(\mathbb{R}^d)$ to $C([0,T_2],H^1(\mathbb{R}^d))$. Since T_2 only depends on $\|u_0\|_{H^1(\mathbb{R}^d)}$, this continuity extends to any interval $[0,T]$ in the maximal interval of existence of u.

Note that we will abbreviated our proof because it is very very close to the case of linear damping ($p = 0$ see Darwich [5]). Actually, as noticed in [19], we only need to prove that in the log-log regime the L^2 norm does not grow, and the growth of the energy(resp the momentum) is below $\frac{1}{\lambda^2}$ (resp $\frac{1}{\lambda}$) . In this paper, we will prove that in the log-log regime, the growth of the energy and the momentum are bounded by:

$$E(u(t)) \lesssim log(\lambda(t))\lambda(t)^{-\frac{pd}{2}},\ P(u(t)) \lesssim log(\lambda(t))\lambda(t)^{1-\frac{pd}{4}}.$$

Let us recall that a fonction u :$[0,T] \longmapsto H^1$ follows the log-log regime if the following uniform controls on the decomposition (3.1) hold on $[0,T]$:

- Control of $b(t)$
$$b(t) > 0,\ b(t) < 10b(0). \tag{3.5}$$
- Control of λ:
$$\lambda(t) \leq e^{-e^{\frac{\pi}{100b(t)}}} \tag{3.6}$$
and the monotonicity of λ:
$$\lambda(t_2) \leq \frac{3}{2}\lambda(t_1), \forall\ 0 \leq t_1 \leq t_2 \leq T. \tag{3.7}$$
Let $k_0 \leq k_+$ be integers and $T^+ \in [0,T]$ such that
$$\frac{1}{2^{k_0}} \leq \lambda(0) \leq \frac{1}{2^{k_0-1}}, \frac{1}{2^{k_+}} \leq \lambda(T^+) \leq \frac{1}{2^{k_+-1}} \tag{3.8}$$
and for $k_0 \leq k \leq k_+$, let t_k be a time such that
$$\lambda(t_k) = \frac{1}{2^k}, \tag{3.9}$$
then we assume the control of the doubling time interval:
$$t_{k+1} - t_k \leq k\lambda^2(t_k). \tag{3.10}$$
- control of the excess of mass:
$$\int |\nabla\epsilon(t)|^2 + \int |\epsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{4}}. \tag{3.11}$$

Remark 3.2. *We will use a boostrap argument, in fact the set of initial data Ω will be to initialize the bootstrap.*

3.1. **Control of the energy and the kinetic momentum in the log-log regime.** We recall the Strichartz estimates. An ordered pair (q,r) is called admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $2 < q \leq \infty$. We define the Strichartz norm of functions $u : [0,T] \times R^d \longmapsto C$ by:

$$\|u\|_{S^0([0,T]\times\mathbb{R}^d)} = \sup_{(q,r) admissible} \|u\|_{L_t^q L_x^r([0,T]\times\mathbb{R}^d)} \tag{3.12}$$

and

$$\|u\|_{S^1([0,T]\times\mathbb{R}^d)} = \sup_{(q,r) admissible} \|\nabla u\|_{L_t^q L_x^r([0,T]\times\mathbb{R}^d)} \tag{3.13}$$

We will sometimes abbreviate $S^i([0,T]\times\mathbb{R}^2)$ with S_T^i or $S^i[0,T]$, $i = 1,2$. Let us denote the Hölder dual exponent of q by q' so that $\frac{1}{q} + \frac{1}{q'} = 1$. The Strichartz estimates may be expressed as:

$$\|u\|_{S_T^0} \leq \|u_0\|_{L^2} + \|(i\partial_t + \Delta)u\|_{L_t^{q'} L_x^{r'}} \tag{3.14}$$

where (q, r) is any admissible pair. Now we will derive an estimate on the energy, to check that it remains small with respect to λ^{-2}:

Lemma 3.1. *Assuming that (3.5)-(3.11) hold, then the energy and kinetic momentum of the solution u to (1.1) are controlled on* $[0, T]$ *by:*

$$|E(u(t))| \leq C(log(\lambda(t))\lambda(t)^{-\frac{pd}{4}}), \tag{3.15}$$

$$|P(u(t))| \leq C(log(\lambda(t))\lambda(t)^{1-\frac{pd}{4}}). \tag{3.16}$$

To prove this lemma, we shall need the following one:

Lemma 3.2. *Let u be a solution of* (1.1) *emanating for* u_0 *in* H^1*. Then u* $\in C([0, \Delta T], H^1)$ *where* $\Delta T = \|u_0\|_{L^2}^{\frac{d-4}{d}} \|u_0\|_{H^1}^{-2}$*, and we have the following control*

$$\|u\|_{S^0[t,t+\Delta T]} \leq 2\|u_0\|_{L^2}\ ,\ \|u\|_{S^1[t,t+\Delta T]} \leq 2\|u_0\|_{H^1(\mathbb{R}^d)}\,.$$

Proof of **Lemma 3.1:** According to (3.10) each interval $[t_k, t_{k+1}]$, can be divided into k intervals, $[\tau_k^j, \tau_k^{j+1}]$ such that the estimates of the previous lemma are true. From (3.3) and the Gagliardo-Nirenberg inequality, we obtain that:

$$\frac{d}{dt}E(u(t)) \leq \|u\|_{L^2}^{\frac{4}{d}+p-\frac{pd}{2}} \|\nabla u\|_{L^2}^{2+\frac{pd}{2}}$$

(3.2) Then

$$\int_{\tau_k^j}^{\tau_{k^{j+1}}} \frac{d}{dt}E(u(t))dt \leq C\int_{\tau_{k^j}}^{\tau_k^{j+1}} \|\nabla u(t)\|_{L^2}^{2+\frac{pd}{2}}$$

Then by Lemma 3.2, we obtain that:

$$\int_{\tau_k^j}^{\tau_{k^{j+1}}} \frac{d}{dt}E(u(t))dt \leq C(\tau_{k^{j+1}} - \tau_{k^j})\lambda^{-2-\frac{pd}{2}}(\tau_{k^j})$$

Note that $\tau_k^{j+1} - \tau_k^j \sim \lambda^2(\tau_k^j) \sim \lambda^2(t_k)$, then

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt}E(u(t))dt \leq C\lambda^{-\frac{pd}{2}}(t_k)$$

Summing from $j = 1$ to $J_k \leq CK$, we obtain that:

$$\sum_{j=1}^{J_k}\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt}E(u(t))dt \leq Ck\lambda^{-\frac{pd}{2}}(t_k)$$

Now taking $T^+ = T$ and summing from K_0 to K^+, we obtain:

$$\int_0^{T^+} \frac{d}{dt}E(u(t))dt \leq CK^+\lambda^{-\frac{pd}{2}}(T^+) \lesssim Clog(\lambda(T))\lambda^{-\frac{pd}{2}}(T).$$

Note that $\log\,(\lambda(T))\lambda^{-\frac{pd}{2}}(T)$ is small with to respect $\frac{1}{\lambda^2}$ because $p < \frac{4}{d}$.

Now we prove **(3.16)**: From (3.4) we have :

$$|\frac{d}{dt}P(u(t))| \leq \int |\overline{u}||\nabla u||u|^p$$

By Gagliardo-Nirenberg inequality we have:

$$\|u\|_{L^{2p+2}}^{2p+2} \leq \|u\|_{L^2}^{2p+2-dp} \|\nabla u\|_{L^2}^{dp}$$

then

$$\frac{d}{dt}P(u(t)) \leq (\int |u|^{2(p+1)})^{\frac{1}{2}} \|\nabla u\|_{L^2} \leq C \|\nabla u\|_{L^2}^{1+\frac{pd}{2}}.$$

Then:

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt}P(u(t)) \leq C(\tau_k^{j+1} - \tau_k^j) \left\|\nabla u(\tau_k^j)\right\|_{L^2}^{1+\frac{pd}{2}} \leq C \|\nabla u(t_k)\|_{L^2}^{-1+\frac{pd}{2}}$$

Summing successively into j and k we obtain that:

$$\int_0^{T^+} \frac{d}{dt}P(u(t)) \lesssim \log(\lambda(T^+))\lambda^{1-\frac{pd}{2}}(T^+).$$

Remark that this quantity is small with to respect $\frac{1}{\lambda}$ because $p < \frac{4}{d}$. $\square$

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