THE CLOSED KNIGHT TOUR PROBLEM IN HIGHER DIMENSIONS

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Abstract. The closed knight tour problem for rectangular chessboards was solved by Schwenk in 1991. Last year, in 2011, DeMaio and Mathew provide an extension of this result for 3-dimensional rectangular boards. In this article, we give the solution for n-dimensional rectangular boards, for $n \geq 4$.

Contents

1. Introduction

On a chessboard, a knight moves by two squares in one direction and by one square in the other one (like a L). A classical challenge is the so-called *knight tour*. The knight is placed on the empty board and, moving according to the rules of chess, must visit each square exactly once. A knight's tour is called a *closed tour* if the knight ends on a square attacking the square from which it began.

(1.1)

1	30	33	16	3	24
32	17	2	23	34	15
29	36	31	14	25	4
18	9	6	35	22	13
7	28	11	20	5	26
10	19	8	27	12	21

A 6×6 closed tour

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Some early solutions were given by Euler, see [Eul] and also by De Moivre (we refer to Mark R. Keen for historical remarks, see [Kee]). The problem was recently considered for various types of chessboards: as a cylinder [Wat1], a torus [Wat2], a sphere [Cai], the exterior of the cube [QiW], the interior of the cube [De]... It represents also an active field of research in computer science, e.g., [Pab] (see references therein). In this paper, we shall focus on rectangular boards.

In 1991, Schwenk considers the question of the closed knight tour problem in a 2-dimensional rectangular chessboard. He provided a necessary and sufficient condition on the size of the board in order to have a closed knight tour. He obtained:

Theorem 1.1 (Schwenk)**.** *Let $1 \leq m \leq n$. The $m \times n$ chessboard has* no *closed knight tour if and only if one of the following assumption holds:*

(a) *m and n are both odd,*
(b) *$m \in \{1, 2, 4\}$,*
(c) *$m = 3$ and $n \in \{4, 6, 8\}$.*

We refer to [Sch] (see also [Wat]) for a proof. When conditions (a),(b), and (c) are not fulfilled, he is reducing the problem by studying a finite number of elementary boards. On each of them, he is exhibiting a closed tour. Then, he is explaining how to "glue" the elementary boards together, in order to make one the closed tour for the union out of the disjoint ones from the elementary blocks. We explain the latter on the example (1.1). Say we want a closed tour for a 12×6 board. Write, side by side, two copies of (1.1). Delete the connection between 21 and 22 for the left board and the connection between 28 and 29 for the right one. Then link 21 with 28 and 22 with 29. The Hamiltonian cycle goes as follows:

$$\begin{aligned} 1[L] &\to 2[L] \to \ldots \to 21[L] \to 28[R] \to 27[R] \to \ldots \to 1[R] \to \\ &\to 36[R] \to \ldots \to 29[R] \to 22[L] \to 23[L] \to \cdots \to 36[L] \to 1[L], \end{aligned}$$

where $[L]$ and $[R]$ stand for left and right, respectively.

We turn to the question for higher dimensions. In dimension 3 or above, a knight is moving by two steps along one coordinate and by one step along a different one. We refer to Section 2 for a mathematical definition. In 2011, in [DeM], DeMaio and Mathew extend Theorem 1.1 to 3-dimensional rectangular chessboards.

Theorem 1.2 (DeMaio and Mathew)**.** *Let $2 \leq m \leq n \leq p$. The $m \times n \times p$ chessboard has* no *closed knight tour if and only if one of the following assumption holds:*

(a) *m, n, and p are all odd,*
(b) *$m = n = 2$,*
(c) *$m = 2$ and $n = p = 3$.*

The strategy is the same as in Theorem 1.1. We refer to Appendix A for some comments.

In this paper extend the previous results to higher dimensional boards. We do not give an alternate approach to the case $n = 3$. However, we rely strongly on the proof of [DeM] in order to treat the case $n \geq 4$. We proceed by induction. Before giving the main statement, we explain the key idea with (1.1). We first extract four *cross-patterns*.

1	30
·	·
29	36

A wocp

10	19
·	·
18	9

A wocp

23	34
·	·
35	22

A nwocp

27	20
·	·
20	19

A nwocp

We represented them up to some rotation. The two first ones are called *well-oriented cross-patterns* (wocp) and the other ones *non-well-oriented cross-patterns*(nwocp). Note that the two first are with disjoint support and so are the two last.

We construct a tour for a $6 \times 6 \times 2$ board first. We work with coordinates. The tour given in (1.1) is given by $(a_i)_{i \in [\![1,36]\!]}$, with $a_1 := (1,6)$, $a_2 := (3,5)$, ... We take two copy of the tour given in (1.1) and denote them by $(a_i,1)_{i\in[\![1,36]\!]}$ and $(a_i,2)_{i\in[\![1,36]\!]}$ for the first and second copy, respectively. We can construct a tour as follows:

$$(2,4,1) \to (2,6,2) \to (3,4,2) \to \ldots \to (1,4,2) \to (1,6,1) \to \ldots \to (2,4,1).$$

To enhance the idea we rewrite it abusively by

$$(\text{“36”},1) \to (\text{“30”},2) \to (\text{“31”},2) \to \ldots \to (\text{“29”},2) \to (\text{“1”},1) \to \ldots \to (\text{“36”},1).$$

We explain how to treat a $6 \times 6 \times k$ board for $k \geq 3$. As the proof is the same, we write the case $k = 3$. We use the two wocp as follows:

$$\begin{aligned} (\text{“36”},1) &\to (\text{“30”},2) \to (\text{“31”},2) \to \ldots \to (\text{“9”},2) \to \\ &\to (\text{“19”},3) \to (\text{“20”},3) \to \ldots \to (\text{“18”},3) \to \\ &\to (\text{“10”},2) \to (\text{“11”},2) \to \ldots \to (\text{“29”},2) \to \\ &\to (\text{“1”},1) \to \ldots \to (\text{“36”},1). \end{aligned} \tag{1.2}$$

Note that we have use only one wocp on the first copy and one on last one. Two of them are still free. We can therefore repeat the procedure and add inductively further dimensions. For instance, we can go from a tour for a $6 \times 6 \times k$ board to one of a $6 \times 6 \times k \times l$ board, with $k, l \geq 2$. One takes l copies of the tour. By noticing that there are two wocp on the initial tour, we proceed as in (1.2) for the tour. Thus, we will get a tour for the $6 \times 6 \times k \times l$ board, which contains in turn two wocp. To prove all these facts, one can use coordinates. We refer to Section 3.2 for more details.

We mention that one could alternatively use two nwocp. The tour that will result from that is a bit different, the orientation of the cycle on the copy has to be reversed, see Example 3.1.

The strategy is now clear. We shall study the structure of the elementary boards, which are obtained in [DeM] and seek for cross-patterns into them. Then, we shall conclude by induction on the dimension. We obtain:

Theorem 1.3. *Let $2 \leq n_1 \leq n_2 \leq \ldots \leq n_k$, with $k \geq 3$. The $n_1 \times \ldots \times n_k$ chessboard has* no *closed knight tour if and only if one of the following assumption holds:*

(a) *For all i, n_i is odd,*
(b) $n_{k-1} = 2$,
(c) $n_k = 3$.

Note that the hypotheses are the same as the ones given in Theorem 1.2 when $k = 3$. We mention that a conjecture for this theorem was given in [Kum].

The paper is organized as follows. In Section 2 we set the notation and precise the graph structure induced by a knight on a n dimensional board. In Section 3.1, we start the proof of Theorem 1.3 and provide all the negative answers. Then, in Section 3.2 we introduce the notion of *cross-patterns* and explain how to use them in order to gain a dimension. After that, in Section 3.3, we finish the proof of Theorem 1.3. Finally in Appendix A, we give a fast idea of the paper of [DeM].

2. Notation

Let $\underline{n} := (n_1, \ldots, n_k)$ be a multi-index where n_i are with values in $\mathbb{N} \setminus \{0\}$. We denote by $|\underline{n}| := k$, the size of the multi-index $\underline{n}$. Set also $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$. The chessboard associated to $\underline{n}$ is defined by

$$\mathcal{B}_{\underline{n}} := [\![1, n_1]\!] \times \ldots \times [\![1, n_{|\underline{n}|}]\!].$$

We endow it with a graph structure. We turn to the definition of the *moves of the knight.* We set:

$$\begin{aligned}\mathcal{C}_{|\underline{n}|} :=& \{(a_1, \ldots, a_{\underline{n}}) \in \mathbb{Z}^{|\underline{n}|}, \quad \text{such that} \\ & |\{i, a_i = 0\}| = \underline{n} - 2, \quad |\{i, a_i \in \{\pm 1\}\}| = 1, \quad \text{and} \quad |\{i, a_i \in \{\pm 2\}\}| = 1\}.\end{aligned}$$

We endow $\mathcal{B}_{\underline{n}}$ with a graph structure, as follows. We set $\mathcal{E}_{\underline{n}} : \mathcal{B}_{\underline{n}} \times \mathcal{B}_{\underline{n}} \to \{0, 1\}$ be a symmetric function defined as follows:

$$\mathcal{E}_{\underline{n}}(a, b) := 1, \quad \text{if} \quad a - b := (a_1 - b_1, \ldots, a_{\underline{n}} - b_{\underline{n}}) \in \mathcal{C}_{|\underline{n}|} \tag{2.1}$$

and 0 otherwise. In other words, a is linked to b by a knight move, if and only if $\mathcal{E}_{\underline{n}}(a, b) := 1$. The couple $\mathcal{G}_{\underline{n}} := (\mathcal{B}_{\underline{n}}, \mathcal{E}_{\underline{n}})$ is the *non-oriented graph* corresponding to all the possible paths of a knight on the chessboard associated to $\underline{n}$.

Set $\phi : \mathcal{B}_{\underline{n}} \to \{-1, 1\}$ given by $\phi(a_1, \ldots, a_n) := (-1)^{a_1 \times \ldots \times a_n}$, i.e., we assign the color black or white to each square. Then note that given $b \in \mathcal{B}_n$ such that $\mathcal{E}_n(a, b) = 1$, one has $\phi(a) \times \phi(b) = -1$. Therefore the graph $\mathcal{G}_{\underline{n}} := (\mathcal{B}_{\underline{n}}, \mathcal{E}_{\underline{n}})$ is *bipartite.*

Let $\{a^i\}_{i=1,\ldots,|\mathcal{B}_{\underline{n}}|}$ be some elements of $\mathcal{B}_{\underline{n}}$. We say that $\{a^i\}_i$ is a *Hamiltonian cycle* if the elements are two by two distinct and if $\mathcal{E}_{\underline{n}}(a^i, a^{i+1}) := 1$ for all $i \in \{1, \ldots, |\mathcal{B}_{\underline{n}}|\}$ and if $\mathcal{E}_{\underline{n}}(a^1, a^{|\mathcal{B}_{\underline{n}}|}) := 1$. Note that because of (2.1), $\mathcal{G}_n$ has a closed knight tour is just rephrasing that fact that $\mathcal{G}_n$ has a Hamiltonian cycle.

Remark 2.1. *Given a multi-index $\underline{n}$ and ψ a bijection from $[\![1, |\underline{n}|]\!]$ onto itself, we set $\underline{m}$ of size $|\underline{n}|$ given by $m_i := n_{\psi(i)}$ for all $i \in [\![1, |n|]\!]$. Since $\mathcal{C}_{|\underline{n}|} = \mathcal{C}_{|\underline{m}|}$ is invariant under permutation, $\mathcal{G}_{\underline{n}}$ has a Hamiltonian cycle if and only $\mathcal{G}_{\underline{m}}$ has one.*

3. Into the proof

In this section we give the proof of Theorem 1.3 and consider boards associated to a multi-index $\underline{n} := (n_1, \ldots, n_{|\underline{n}|})$, where $n_i \geq 2$ and $|\underline{n}| \geq 3$.

3.1. **Forbidden boards.** We start with the sufficient part of Theorem 1.3. It is a straightforward generalization of the dimension 3. There are three main cases:

a) $n_i \in (2\mathbb{N} + 1)$, for all i. A bipartite graph cannot have a Hamiltonian cycle if its cardinality is odd.

b) $\underline{n} := (2, 2, \ldots, k)$, with $k \geq 2$. Set $a^0 := (2, 2, \ldots, 2) \in \mathcal{B}_n$. Let $(a^i)_{i\geq 0}$, be the elements of a Hamiltonian cycle. By noting that $(a^0 - a^i) \in (\{1, 2\}, \ldots, \{1, 2\}, 2\mathbb{Z})$, for all $i \geq 0$. Then, $(1, 1, \ldots, 1)$ is never reached. Contradiction.

c) $\underline{n} := (n_1, \ldots, n_{|\underline{n}|})$, with $2 \leq n_1 \leq \ldots \leq n_{|\underline{n}|} \leq 3$. Set $a := (2, \ldots, 2) \in \mathcal{B}_n$ and note that there is no $b \in \mathcal{C}_{|\underline{n}|}$, such that $a + b \in \mathcal{B}_n$. Therefore, the graph $\mathcal{G}_n := (\mathcal{B}_n, \mathcal{E}_n)$ has no closed tour.

3.2. **Looking for cross-patterns.** To each element $c := (a_1, \ldots, a_{\underline{n}}) \in \mathcal{C}_{|\underline{n}|}$, we associate two elements $e_{[c,1]}$ and $e_{[c,2]}$ of $\mathbb{Z}^{|\underline{n}|}$, by setting

$$e_{[c,1]}(i) := \delta_{a_i \in \{\pm 1\}}, \text{ and } e_{[c,2]}(i) := \delta_{a_i \in \{\pm 2\}},$$

where $\delta_X := 0$ if X is empty and $\delta_X := 1$ otherwise. Note that:

$$c \in \{\pm 1 e_{[c,1]} + \pm 2 e_{[c,2]}\}.$$

We now flip c with respect to $e_{[c,1]}$. More precisely, we set

$$\tilde{c} := -\langle c, e_{[c,1]} \rangle e_{[c,1]} + \langle c, e_{[c,2]} \rangle e_{[c,2]},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{|\underline{n}|}$.

Definition 3.1. *Given a Hamiltonian cycle $(a^i)_{i\in I}$ for $\mathcal{B}_n$, we say that it contains the* well-oriented cross-pattern *(wocp) if there are $c \in \mathcal{C}_{|\underline{n}|}$ and $n, m \in I$, so that:*

$$a^{n+1} = a^n + c \text{ and } a^{m+1} = a^m + \tilde{c} \tag{3.1}$$
$$a^m - a^n = \langle c, e_{[c,1]} \rangle e_{[c,1]}.$$

We denote this wocp by $(a^n, a^{n+1}, a^m, a^{m+1})$.

We say that it contains the non-well-oriented cross-pattern *(nwocp) if there are $c \in \mathcal{C}_{|\underline{n}|}$ and $n, m \in I$, so that:*

$$a^{n+1} = a^n + c \text{ and } a^m = a^{m+1} + \tilde{c} \tag{3.2}$$
$$a^{m+1} - a^n = \langle c, e_{[c,1]} \rangle e_{[c,1]}.$$

We denote this nwocp by $(a^n, a^{n+1}, a^{m+1}, a^m)$.

We give an example in a 2-dimensional board:

$\cdot$	a^{n+1}	a^{m+1}	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	a^m	a^n	$\cdot$

A wocp

$\cdot$	a^{n+1}	a^m	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	a^{m+1}	a^n	$\cdot$

A nwocp

We present an important feature of these patterns. It would be extensively used in Example 3.1 and in the Proposition 3.1.

Remark 3.1. *In the case of the wocp* (3.1)*, we have*

$$a^{n+1} - a^m = a^{m+1} - a^n = \langle c, e_{[c,2]} \rangle e_{[c,2]}.$$

In particular, $a^{n+1} - a^m$ and $a^{m+1} - a^n$ are in $\{\pm 2 e_{[c,2]}\}$.

In the case of the nwocp (3.2)*, we get*

$$a^{n+1} - a^{m+1} = a^m - a^n = \langle c, e_{[c,2]} \rangle e_{[c,2]}.$$

This also yields that $a^{n+1} - a^{m+1}$ and $a^m - a^n$ are in $\{\pm 2 e_{[c,2]}\}$.

We now explain how to connect two Hamiltonian cycles and gain one dimension with the help of cross-patterns.

Example 3.1. *Take $\mathcal{B}_{\underline{n}}$, with $|\underline{n}| = 2$, such that it contains a Hamiltonian cycle $(a^i)_{i\in I}$, see Theorem 1.1. Take now $\underline{m} := (n, 2)$. Note that $\mathcal{B}_{\underline{m}}$ contains two cycles:*

$$(a^i, 1)_{i\in I} \text{ and } (a^i, 2)_{i\in I}.$$

Suppose there is a well-oriented cross-pattern $(a^n, a^{n+1}, a^m, a^{m+1})$. Using Remark (3.1), we see that $\mathcal{E}_{\underline{m}}((a^n, 1), (a^{m+1}, 2)) = 1$ and $\mathcal{E}_{\underline{m}}((a^m, 2), (a^{n+1}, 1)) = 1$. We can construct a Hamiltonian cycle as follows:

$$(a^n, 1) \to (a^{m+1}, 2) \to (a^{m+2}, 2) \to \ldots \to (a^m, 2) \to (a^{n+1}, 1) \to \ldots \to (a^n, 1).$$

Suppose now there is a non-well-oriented cross-pattern $(a^n, a^{n+1}, a^{m+1}, a^m)$. Similarly, We construct a Hamiltonian cycle as follows:

$$(a^n, 1) \to (a^m, 2) \to (a^{m-1}, 2) \to \ldots \to (a^{m+1}, 2) \to (a^{n+1}, 1) \to \ldots \to (a^n, 1).$$

Note that we went backward on the second copy.

It is obvious that one can gain as many dimension as one has disjoint cross-patterns. However, the situation is much better: having solely two of them are enough to gain as many dimension as we want. We rely on the next Proposition.

Proposition 3.1. *Take $\mathcal{B}_{\underline{n}}$ such that it contains a Hamiltonian cycle. Suppose that there are two disjoint well-oriented cross-patterns (non-well-oriented resp.). Take now $\underline{m} := (n, k)$, with $k \geq 2$. Then $\mathcal{B}_{\underline{m}}$ contains a Hamiltonian cycle with two disjoint well-oriented cross-patterns (non-well-oriented resp.).*

We stress that here, disjoint means with disjoint support.

Proof. Set $(a^i)_{i\in I}$ the Hamiltonian cycle. Note that $(a^i, j)_{i\in I}$, for $j \in [\![1, k]\!]$, are k disjoint cycle in $\mathcal{B}_{\underline{m}}$. Take two well-oriented cross-patterns $(a^{n_1}, a^{n_1+1}, a^{m_1}, a^{m_1+1})$ and $(a^{n_2}, a^{n_2+1}, a^{m_2}, a^{m_2+1})$. Let k be *even.* Using Remark 3.1, we get

$$\begin{aligned}
(a^{n_1}, 1) &\to (a^{m_1+1}, 2) \to (a^{m_1+2}, 2) \to \ldots \to (a^{n_2}, 2) \to \\
&\to (a^{m_2+1}, 3) \to (a^{m_2+2}, 3) \to \ldots \to (a^{n_1}, 3) \to \\
&\to (a^{m_1+1}, 4) \to (a^{m_1+2}, 4) \to \ldots \to (a^{n_2}, 4) \to \ldots \\
&\to (a^{m_2+1}, k-1) \to (a^{m_2+2}, k-1) \to \ldots \to (a^{n_1}, k-1) \to \qquad (3.3)\\
&\to (a^{m_1+1}, k) \to (a^{m_1+2}, k) \to \ldots \to (a^{m_1}, k) \to \qquad (3.4)\\
&\to (a^{n_1+1}, k-1) \to (a^{n_1+2}, k-1) \to \ldots \to (a^{m_2}, k-1) \to \ldots \qquad (3.5)\\
&\to (a^{n_2+1}, 2) \to (a^{n_2+2}, 2) \to \ldots \to (a^{m_1}, 2) \\
&\to (a^{n_1+1}, 1) \to (a^{n_1+2}, 1) \to \ldots \to (a^{n_1}, 1).
\end{aligned}$$

Finally note that $((a^{n_2}, j), (a^{n_2+1}, j), (a^{m_2}, j), (a^{m_2+1}, j))$ with $j \in \{1, k\}$ are well-oriented cross-patterns for this Hamiltonian cycle in $\mathcal{B}_{\underline{m}}$. When k is odd, we replace the lines (3.3), (3.4), and (3.5) by

$$\begin{aligned}
&\to (a^{m_1+1}, k-1) \to (a^{m_1+2}, k-1) \to \ldots \to (a^{n_2}, k-1) \to \\
&\to (a^{m_2+1}, k) \to (a^{m_2+2}, k) \to \ldots \to (a^{m_2}, k) \to \\
&\to (a^{n_2+1}, k-1) \to (a^{n_2+2}, k-1) \to \ldots \to (a^{m_1}, k-1) \to \ldots
\end{aligned}$$

In this case, we obtain that $((a^{n_2},1),(a^{n_2+1},1),(a^{m_2},1),(a^{m_2+1},1))$ and also $((a^{n_1},k),(a^{n_1+1},k),(a^{m_1},k),(a^{m_1+1},k))$ are well-oriented cross-patterns associated to the Hamiltonian cycle in $\mathcal{B}_{\underline{m}}$.

We turn to the non-well-oriented cross-patterns. Take $(a^{n_1},a^{n_1+1},a^{m_1+1},a^{m_1})$ and $(a^{n_2},a^{n_2+1},a^{m_2+1},a^{m_2})$. Let k be *even*. Using Remark 3.1, we get

$$\begin{aligned}
(a^{n_1+1},1) &\to (a^{m_1+1},2) \to (a^{m_1+2},2) \to \ldots \to (a^{n_2},2) \to \\
&\to (a^{m_2},3) \to (a^{m_2-1},3) \to \ldots \to (a^{n_1+1},3) \to \\
&\to (a^{m_1+1},4) \to (a^{m_1+2},4) \to \ldots \to (a^{n_2},4) \to \ldots \\
&\to (a^{m_2},k-1) \to (a^{m_2-1},k-1) \to \ldots \to (a^{n_1+1},k-1) \to \qquad (3.6)\\
&\to (a^{m_1+1},k) \to (a^{m_1+2},k) \to \ldots \to (a^{m_1},k) \to \qquad (3.7)\\
&\to (a^{n_1},k-1) \to (a^{n_1-1},k-1) \to \ldots \to (a^{m_2+1},k-1) \to \ldots \qquad (3.8)\\
&\to (a^{n_2+1},2) \to (a^{n_2+2},2) \to \ldots \to (a^{m_1},2) \\
&\to (a^{n_1},1) \to (a^{n_1-1},1) \to \ldots \to (a^{n_1+1},1).
\end{aligned}$$

When k is odd, we replace the lines (3.6), (3.7), and (3.8) by

$$\begin{aligned}
&\to (a^{m_1+1},k-1) \to (a^{m_1+2},k-1) \to \ldots \to (a^{n_2},k-1) \to \\
&\to (a^{m_2},k) \to (a^{m_2-1},k) \to \ldots \to (a^{m_2+1},k) \to \\
&\to (a^{n_2+1},k-1) \to (a^{n_2+2},k-1) \to \ldots \to (a^{m_1},k-1) \to \ldots
\end{aligned}$$

In this case, we obtain that $((a^{n_2},1),(a^{n_2+1},1),(a^{m_2},1),(a^{m_2+1},1))$ and also $((a^{n_1},k),(a^{n_1+1},k),(a^{m_1},k),(a^{m_1+1},k))$ are non-well-oriented cross-patterns associated to the Hamiltonian cycle in $\mathcal{B}_{\underline{m}}$. $\square$

3.3. **Proof of the main result.** We finish the proof of Theorem 1.3 and deal with the necessary part by an induction on:

$\mathcal{P}_k :=$"Given a multi-index $\underline{n} = (n_1, n_2, \ldots, n_k)$, where $2 \leq n_1 \leq n_2 \leq \ldots \leq n_k$ and such that:

(a) There is $i_0 \in [\![1,k]\!]$ such that n_{i_0} is even,
(b) $n_{k-1} \geq 3$,
(c) $n_k \geq 4$,

then, $\mathcal{B}_{\underline{n}}$ contains a Hamiltonian cycle which has at least two disjoint well-oriented or two disjoint non-well-oriented cross-patterns."

for all $k \geq 3$.

Basis: First, we study the elementary blocks that are exhibited in [DeM]. For each of them, we give the list of well-oriented cross-patterns (wocp), non-well-oriented cross-patterns (nwocp) and also the edges that are used to combine two different blocks, in order to create a Hamiltonian cycle for the union. For the latter, we precise the other figure that is glued and denote it between square brackets. We also strike out all the wocp and nwocp which are incompatible with the gluing operation of [DeM]. For instance, in Figure 6, we stroke $(2,3,21,20)$ out, because the edge $(3,4)$ is already used when one glues Figure 35.

Wocp	Nwocp	Used edges [with Figure]
$(4,5,11,12)$ $(7,8,28,29)$	$(1,2,22,21)$ $\cancel{(2,3,21,20)}$ $(13,14,18,17)$	$(3,4)[6,11,14,17], (8,9)[6]$ $(10,11)[6], (14,15)[6],$ $(20,21)[35], (25,26)[6],$ $(27,28)[6]$

Figure 6 : size $2 \times 4 \times 4$

Wocp	Nwocp	Used edges [with Figure]
$(1,2,30,31)$ $\cancel{(4,5,31,32)}$ $\cancel{(12,13,39,40)}$ $(19,20,24,25)$	$(14,15,35,34)$ $\cancel{(16,17,23,22)}$	$(4,5)[20], (8,9)[20],$ $(10,11)[11], (11,12)[23],$ $(16,17)[11], (17,18)[26],$ $(31,32)[11], (37,38)[11], (39,40)[6]$

Figure 11 : size $2 \times 4 \times 5$

Wocp	Nwocp	Used edges [with Figure]
$(1,2,3,4)$ $(7,8,26,27)$ $(8,9,25,26)$ $(10,11,31,32)$ $(11,12,44,45)$ $(13,14,32,33)$ $(15,16,34,35)$ $\cancel{(18,19,39,40)}$ $(48,1,37,38)$	$(14,15,47,46)$	$(4,5)[29], (9,10)[32],$ $(28,29)[14], (29,30)[14],$ $(30,31)[14], (36,37)[6],$ $(39,40)[14], (43,44)[29],$ $(47,48)[23]$

Figure 14 : size $2 \times 4 \times 6$

Wocp	Nwocp	Used edges [with Figure]
$(8,9,21,22)$ $(9,10,16,17)$ $(10,11,15,16)$	$(2,3,7,6)$	$(4,5)[17], (5,6)[17],$ $(7,8)[17], (11,12)[6],$ $(20,21)[17,26,32,35]$

Figure 17 : size $2 \times 4 \times 3$

Wocp	Nwocp	Used edges [with Figure]
$(3,4,38,39)$ $(5,6,28,29)$ $(8,9,25,26)$ $(12,13,45,46)$ $\cancel{(14,15,47,48)}$ $(17,18,30,31)$	$(9,10,24,23)$ $(18,19,33,32)$ $\cancel{(19,20,50,49)}$ $\cancel{(20,21,43,42)}$	$(20,21)[11], (43,44)[20],$ $(47,48)[11], (49,50)[20]$

Figure 20 : size $2 \times 5 \times 5$

Here, as explained in Appendix A, we will use the Figure 23′ instead of the Figure 23 of [DeM].

Wocp	Nwocp	Used edges [with Figure]
$(6,7,53,54)$	$(3,4,52,51)$	$(1,2)[23], (12,13)[11]$,
$(13,14,30,31)$	$(10,11,29,28)$	$(32,33)[14], (44,45)[23]$
	$(16,17,23,22)$	
	$(18,19,35,34)$	
	$(19,20,36,35)$	
	$(20,21,37,36)$	
	$(39,40,46,45)$	
	$(41,42,58,57)$	
	$(42,43,59,58)$	
	$(43,44,60,59)$	

Figure 23′ : size $2 \times 5 \times 6$

Wocp	Nwocp	Used edges [with Figure]
$\cancel{(6,7,23,24)}$	$(3,4,22,21)$	$(14,15)[11], (23,24)[17]$,
	$(9,10,16,15)$	$(27,28)[26], (29,30)[26]$
	$\cancel{(11,12,28,27)}$	
	$(12,13,29,28)$	
	$\cancel{(13,14,30,29)}$	

Figure 26 : size $2 \times 5 \times 3$

Wocp	Nwocp	Used edges [with Figure]
$\cancel{(15,16,62,63)}$	$(2,3,69,68)$	$(15,16)[14], (22,23)[29]$,
$(16,17,57,58)$	$(4,5,47,46)$	$(29,30)[14], (32,33)[29]$
$(47,48,72,1)$	$(6,7,37,36)$	
	$(7,8,36,35)$	
	$(10,11,51,50)$	
	$(12,13,39,38)$	
	$(20,21,53,52)$	
	$(23,24,70,69)$	
	$(27,28,64,63)$	
	$(30,31,43,42)$	
	$(31,32,42,41)$	

Figure 29 : size $2 \times 6 \times 6$

Wocp	Nwocp	Used edges [with Figure]
$(6,7,15,16)$	$(3,4,14,13)$	$(11,12)[14], (18,19)[32]$,
$(8,9,17,18)$	$(4,5,15,14)$	$(22,23)[32], (26,27)[17]$
$\cancel{(18,19,31,32)}$	$(25,26,34,33)$	
$(19,20,28,29)$	$\cancel{(26,27,35,34)}$	

Figure 32 : size $2 \times 6 \times 3$

Wocp	Nwocp	Used edges [with Figure]
$(2,3,25,26)$	$(3,4,24,23)$	$(7,8)[6], (12,13)[35],$
$(6,7,19,20)$	$(5,6,22,21)$	$(13,14)[17], (14,15)[35]$
$\cancel{(7,8,16,17)}$	$(32,33,39,38)$	
$(8,9,15,16)$	$(33,34,40,39)$	
$\cancel{(13,14,37,38)}$		

Figure 35 : size $2 \times 7 \times 3$

Wocp	Nwocp	Used edges [with Figure]
	$(3,4,36,35)$	$(9,10)[39], (28,29)[39]$
	$\cancel{(9,10,20,19)}$	
	$(11,12,16,15)$	
	$\cancel{(28,29,1,36)}$	

Figure 39 : size $4 \times 3 \times 3$

Wocp	Nwocp	Used edges [with Figure]
$(2,3,5,6)$	$(4,5,9,8)$	$(21,22)[40], (28,29)[40]$
$(10,11,15,16)$	$\cancel{(18,19,29,28)}$	
$(11,12,14,15)$	$(32,33,43,42)$	
$(13,14,16,17)$		
$(19,20,22,23)$		
$(20,21,25,26)$		
$\cancel{(21,22,24,25)}$		
$(36,37,49,50)$		

Figure 40 : size $6 \times 3 \times 3$

In every case, there is at least a pair of wocp or of nwocp which have disjoint supports.

Take $\underline{n}$ such that $|\underline{n}| = 3$ and such that a), b) and c) of $\mathcal{P}_3$ hold true. Thanks to the proof of Theorem 1.2, we get there is a Hamiltonian cycle in $\mathcal{G}_{\underline{n}}$ which is obtained by gluing together Hamiltonian cycles coming the elementary blocks that we have just discussed. Therefore, the Hamiltonian cycle contains at a pair of wocp or of nwocp which have disjoint supports. This yields $\mathcal{P}_3$.

Inductive step: Suppose that $\mathcal{P}_k$ holds true for some $k \geq 3$. Take $\underline{n} = (n_1, n_2, \ldots, n_{\mathbf{k+1}})$, where $2 \leq n_1 \leq n_2 \leq \ldots \leq n_{k+1}$, such that a), b) and c) hold true.

If all n_i but one are odd, then, up to a permutation, we can always suppose that n_1 is odd. We set

$$\underline{n'} = (n_2, \ldots, n_{k+1}). \tag{3.9}$$

Otherwise, without operating a permutation, we set (3.9).

Note that $\mathcal{B}_{\underline{n'}}$ satisfies the hypothesis a), b) and c) of $\mathcal{P}_k$ and therefore, by $\mathcal{P}_k$, we get if contains a Hamiltonian cycle and at least well-oriented or two disjoint non-well-oriented cross-patterns. We apply Proposition 3.1 and get $\mathcal{P}_{k+1}$ is true.

Therefore, we have proved by induction that $\mathcal{P}_k$ is true for all $k \geq 3$. In particular, this proves the theorem. $\square$

Appendix A. About the construction in 3 dimension

Without going into details, we give a rough idea about the approach of [DeM]. Take $n \times m \times p$, that is not satisfying the hypotheses a), b) and c) of Theorem 1.2. Note that at least one of them is even. Then one basically works modulus 4: A $n \times m \times p$ block can be written as a union of the following ones: $2\times4\times4$, $2\times4\times5$, $2\times4\times6$, $2\times4\times3$, $2\times5\times5$, $2\times5\times6$, $2\times5\times3$, $2\times6\times6$, $2\times6\times3$, $2\times7\times3$, $4\times3\times3$, $6\times3\times3$. The list is rather long since one has no Hamiltonian cycle for $2\times2\times3$ and $2\times3\times3$ boards. Subsequently, the authors construct some Hamiltonian cycle for each elementary blocks. After that, given two compatible blocks, by deleting one edge from each block and by creating two edges that are "gluing" these blocks together, they construct a Hamiltonian cycle for the union, starting with the two disjoint ones.

However, one has to be careful with the block $2\times5\times6$ which is given in Figure 23 of [DeM]. Indeed, one sees that 1 cannot be reached from 60, therefore this is no Hamiltonian cycle but just a path. To fix this, we propose to take:

35	18	25	32	13
28	23	10	17	26
19	34	27	24	31
58	41	48	55	6
51	46	3	40	49
42	57	50	47	54

Layer 1

20	37	12	15	30
11	16	29	22	9
36	21	38	33	14
43	60	5	8	53
4	39	52	45	2
59	44	1	56	7

Layer 2

Figure 23′

We replace the table following the board, i.e., the table on the top of page 9. By

	Delete edges	Create edges
Vertical	$11-12$ top board, $12-13$ bottom board	$11-12$, $12-13$
Horizontal	$32-33$ left board, $47-48$ right board	$32-47$, $33-48$
Front	$1-2$ front board, $44-45$ back board	$1-44$, $2-45$

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