Particles approximations of Vlasov equations with singular forces : Propagation of chaos

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Abstract. We obtain the mean field limit and the propagation of chaos for a system of particles interacting with a singular interaction force of the type $1/|x|^\alpha$, with $\alpha < 1$ in dimension $d \geq 3$. We also provides results for forces with singularity up to $\alpha < d-1$ but with large enough cut-off. This last result thus almost includes the most interesting case of Coulombian or gravitationnal interaction.
Key words. Derivation of kinetic equations. Particle methods. Vlasov equation. Propagation of chaos.

1 Introduction

The N particles system. The starting point is the classical Newton dynamics for point-particles. We denote by $(X_1, \ldots, X_N)$ the position of the particles in $\mathbb{R}^d$, and by $(V_1, \ldots, V_N)$ their velocities in $\mathbb{R}^d$. Assuming that particles interact two by two with the interaction kernel $F(x)$, one finds the usual

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = E_N(X_i) = \displaystyle\sum_{j \neq i} \frac{1}{N} F(X_i - X_j). \end{cases} \tag{1.1}$$ eq:ODE

We use the so-called mean-field scaling which consist in keeping the total mass (or charge) of order 1 thus formally enabling us to pass to the limit. This explains the $1/N$ factor in front of the force terms. This implies corresponding rescaling in position, velocity and time.
There are many examples of physical systems following (eq:ODE 1.1). The best known concerns Coulombian force $F(x) = C\, x/|x|^{d-1}$, which serves as a guiding example and reference. Those describe a plasma, or for $C < 0$ gravitational interactions, in which case the system under study may be a galaxy, a cloud of star or galaxies (and thus particles can be "stars" or even "galaxies"). For simplicity we consider here only a basic form for the interaction. However the same techniques would apply to more complex models, for instance with several species (electons and ions in a plasma), 3-particles (or more) interactions, models where the force depends also on the speed has in swarming models like the Cucker-Smale one CarCanBol [CDPJ08a]...
For convenience, we also use the notation $Z_i = (X_i, V_i)$ for the solution and
$Z^0 = (X_1^0, V_1^0, \ldots, X_n^0, V_n^0)$ for the given initial conditions.
Finally let us mention that sometimes the kernel F in fact depend on the number of particles. This might seem quite strange from the physical point of view but is in fact

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very common for numerical simulations (to regularize the interactions). When such a cut-off is used, we will denote by F_N (some N dependent mollification of F) the force field.

The Jeans-Vlasov equation. At first glance, the system (1.1) might seem quite reasonable However many problems arise when one tries to use it for practical applications. But in our case, the main issue is the number of particles, *i.e.* the dimension of the system. For example a plasma or a galaxy usually contains a very large number of "particles", typically from 10^{10} to 10^{23}, which makes solving (1.1) impossible.
As usual in this kind of situation, one would like to replace the discrete system (1.1) by a continuous model. In that case this model is posed in the phase space, *i.e.* it involves the distribution function $f(t,x,v)$ in time, position and velocity. The evolution of that function $f(t,x,v)$ is given by the Jeans-Vlasov equation

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + E(x)\cdot\nabla_v f = 0\,, \\ E(x) = \int_{\mathbb{R}^d} \rho(t,y)\, F(x-y)\, dy, \\ \rho(t,x) = \int_v f(t,x,v)\, dv, \end{cases} \tag{1.2}$$

where here ρ is the spatial density and the initial density f^0 is given.
Our whole concern in this article is to understand when and in which sense, Eq. (1.2) can be seen as a limit of system (1.1). This question is of obvious importance for theoretical reasons, to justify the validity of the Vlasov equation for example.
It also plays a role for numerical simulation, and especially Particles in Cells methods which introduce a large number (roughly around 10^6 or 10^8, to compare with the order 10^{10} to 10^{23} mentioned above) of "virtual" particles in order to obtain a particles system solvable numerically. The problem in that case is to explain why it is possible to correctly approximate the system by using much fewer particles. This would of course be ensured by the convergence of (1.1) to (1.2).

Formal derivation of Eq. (1.2) from (1.1). One of the simplest way to understand formally how to derive Eq. (1.2) is to introduce the empirical measure

$$\mu_N^z(t) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t),V_i(t)}.$$

Then if (X_i,V_i) is solution to (1.1), and if there is no self-interaction : $F(0)=0$, then μ_N^z solves (1.2) in the sense of distribution. Formally one may then expect that any limit of μ_N^z still satisfies the same equation.

The question of convergence. The previous formal argument suggests a first way of rigorously deriving the Vlasov equation (1.2). Take a sequence of initial conditions Z^0 (to be given for every number N or a sequence of such numbers) and assume that the corresponding empirical measures converges (in the usual weak-$*$ topology)

$$\mu_N^z(0) \longrightarrow f^0(x,v).$$

One would then try to prove that the empirical measures at later times $\mu_N^z(t)$ weakly converges to a solution $f(t,x,v)$ to (1.2) with initial data f^0.

In other words, is the following diagram commutative?

$$\begin{tikzcd}
\mu_N^Z(0) \arrow[r, squiggly, "\text{cvg}"] \arrow[d, "Npart"'] & f(0) \arrow[d, "VP"] \\
\mu_N^Z(t) \arrow[r, squiggly, "\text{cvg ?}"] & f(t)
\end{tikzcd}$$

Note that for singular kernels F (among which Coulombian and gravitationnal interactions), one does not expect to be able to do that for any initial conditions. First of all solutions to (1.2) do not exist in general if the initial data f^0 is only a measure. And even if f^0 is smooth but a small amount of the particles are initially concentrated in a small region, problems will likely occur (as the interactions blow up)
Of course it is not obvious what are admissible initial conditions or how to precise that. The simplest way is to give some properties that the initial conditions must satisfy. For instance one could ask the initial positions and velocities to be uniformly distributed (on a grid for numerical simulations). However from the point of view of statistical physics, this is too restricitve. This leads us to the notion of propagation of chaos.

Propagation of chaos. In most physical settings, one expects the initial positions and velocities to be selected randomly and typically independently. In that case the law is initially given by $(f^0)^{\otimes N}$ (i.e. randomly and independently with profile f^0). Note that the empirical measure at time 0 is then close to f^0 with large probability (in some weak norm, See the Proposition 3 for a precise version).
The propagation of chaos was formalized by Kac's in [Kac56] and goes back to Boltzmann and its "Stosszahl ansatz". A standard reference is the famous famous course by Sznitman [Szn91]. It is mainly use in probabilistic system, where some randomness is introduced in the dynamics of the particle, but is also relevant in our context.
Denoting by $f^N(t, x_1, v_1, \dots, x_N, v_N)$ the image by the dynamics (1.1) of the initial law $(f^0)^{\otimes N}$, one may define the k-marginals

$$f_k^N(t, z_1, \dots, z_k) = \int_{R^{2d(N-k)}} f^N(t, z_1, \dots, z_N)\, dz_{k+1} \dots dz_N.$$

According to the general definition, the propagation of chaos means that for any fixed k, $f_k^N(t)$ will converge weakly to $(f)^{\otimes k}$ as $N \to \infty$. In fact it is sufficient that the convergence holds for only one $k \geq 2$.
As Sznitman shows, it is also equivalent to say that the empirical measures $\mu_N^Z(t)$ converge in law towards the constant variable $f(t)$. In fact we will give in theorem 1 and 2 a quantified version of the convergence in probability of $\mu_N^Z(t)$ towards $f(t)$.
Shortly, this is possible because, the marginals can be recovered from the expectations of moments of the empirical measure

$$f_k^N = \mathbb{E}(\mu_N^Z(t, x_1, v_1) \dots \mu_N^Z(t, x_k, v_K)) + O\left(\frac{k^2}{N}\right),$$

a result sometimes called Grunbaum lemma. For detailed explanations about quantification of the equivalence between convergence of the f_k^N and the convergence of the law of the empirical distributions, we refer to [HM12]. This quantified equivalence was for instance used in the recent and important work of Mischler and Mouhot about Kac's program in kinetic theory [MM11].
Of course we do not expect propagation of chaos to hold for any initial distribution f^0, for instance if f^0 is too singular for (1.2) to have a solution. Hence we limit ourselves here to $f^0 \in L^\infty$.

Well posedness for System (1.1). We have not mentioned yet the most basic question for System (1.1) with a singular force kernel, namely whether one can even expect to have solutions to the system for a fixed number of particles. Indeed, because of the singularity, the usual Cauchy-Lipschitz theory cannot be applied.
First of all for the type of singularity that we will handle here, the answer is relatively easy as it would be simple to show that velocities remain bounded and that particles cannot collide for almost all initial data. We do not give a specific proof as that result is a simple consequence of our analysis.
However in more singular cases (especially repulsive ones), the question remains. The classical approach to this problem uses the so-called theory of renormalized solutions developed by DiPerna-Lions. We refer to [DL89] and to Hauray [Hau04] for this specific problem.

Previous results with cut-off or for smooth interactions. The convergence and the propagation of chaos are known to hold for smooth interaction forces ($F \in C^1$ in general or at least $W^{1,\infty}$) since the end of the seventies and the works of Braun and Hepp [BH77], Neunzert and Wick [NW80] and Dobrushin [Dob79]. Those articles introduces the main ideas and the formalism behind mean field limits, we also refer to the nice book by Spohn [Spo91].
Their proofs however rely on Gronwall type estimates and are connected to the fact that Gronwall estimates are actually true for (1.1) uniformly in N if $F \in W^{1,\infty}$. Those makes them impossible to generalize to any case where F is singular (including Coulombian interactions and many other physically interesting models).
Instead, by keeping the same general approach, it is possible to deal with singular interactions with cut-off. For instance for Coulombian interactions, one could consider

$$F_N(x) = C\,\frac{x}{(|x| + \varepsilon(N))^d},$$

or other type of regularization at the scale $\varepsilon(N)$.
The system (1.1) does not have much physical meaning but the corresponding studies are crucial to understand the convergence of numerical methods.
For particles initially on a mesh, we refer to the works of Ganguly and Victory [GV89], Wollman [Wol00] and Batt [Bat01] (the later gives a simpler proof, but valid only for larger cut-off). Unfortunately they had to impose that $\lim_{N\to\infty} \varepsilon(N)/N^{-1/d} = +\infty$, meaning that the cut-off for convergence results is usually larger than the one used in practical numerical simulations . Note that the scale $N^{-1/d}$ is the average distance between two neighboring particles in position.
Of course propagation of chaos cannot be proved in those cases as the particles are on a mesh initially and hence cannot be taken randomly. Moreover, we emphasize that the two problems with particles initially on a mesh, or with particles not equally distributed seems to be very different. In the last case, the previously mentioned results do not apply, and Ganguly, Lee and Victory [GLV91] are only able to prove the convergence for a very large cut-off $\varepsilon \approx (\ln N)^{-1}$.

Previous results for $2d$ Euler or other macroscopic equations. A well known case, very similar at first sight with the question here, is the vortices system for the $2d$ incompressible Euler equation. One replaces (1.1) by

$$\dot X_i = \frac{1}{N}\sum_{j\neq i} \alpha_i\,\alpha_j\,F(X_i - X_j), \tag{1.3}$$

where F is still the Coulombian kernel (in 2 dimensions here) and $\alpha_i = \pm 1$. One expects this system to converge to the Euler equation in vorticity formulation

$$\partial_t \omega + \operatorname{div}(u\,\omega) = 0, \qquad \operatorname{div} u = 0, \qquad \operatorname{curl} u = \omega. \tag{1.4}$$

The same questions of convergence and propagation of chaos can be asked in this setting. Two results without regularization for the true kernel are already known. The work of Goodman, Hou and Lowengrub, [GHL90] and [GH91], has a numerical point of view but use the true singular kernel in a interesting way. The work of Schochet [Sch96] uses the weak formulation of Delort of the Euler equation and prove that empirical measure with bounded energy converges towards weak measures solution to (1.4). Unfortunately, the possible lack of uniqueness of Euler equation in the class of measures do not allows to deduce the propagation of chaos.

As equations like (1.4) are notoriously harder to deal with than kinetic equations like (1.2), one could expect similar results for our problem. Unfortunately, the mean field limits are more difficult in the phase space. There are several reasons for that, in particular the fact that system (1.1) is second order while (1.3) is first order. This implies that collisions or near collisions (in physical space) between particles are very common for (1.1) even for repulsive interactions and rare for (1.3) (at least for vortices of the same sign).

For example, the references mentioned above use the symmetry of the forces in the vortex case, a symmetry which does not exist in our kinetic problem. The force is still symmetric with respect to the space variable, but there is now a velocity variable (second order again) which breaks the argument used in the vortices case. For a more complete description of the vortices system, we refer to the references already quoted or to [Hau09], which introduces in that case techniques somewhat similar to the one used here.

Previous results in singular cases without cut-off. Let us first mention that the equation (1.2) at the limit is now well understood, even when the interaction F is singular, including the Coulombian case. The existence of weak solutions goes back to [Dob79] or [Ars75]. Existence of global classical solutions is proved in [Pfa92], [Sch91] (see also [Hor93]) and at the same time in [LP91]. Of course those results require some smoothness on the initial data f^0 (for instance compactly support and bounded).

To our knowledge however, the only mean field limit result available up to now is [HJ07]. This proves the convergence (not the propagation of chaos) provided that

- The interaction kernel F behaves like $|x|^{-\alpha}$ with $\alpha < 1$.
- The particles are initially well distributed, meaning that their minimal distance in phase space is of the same order as the average distance between neighboring particles $N^{-1/2d}$.

The second assumption is all right for numerical purposes but does not allow to consider physically realistic initial conditions (as per the propagation of chaos property). This assumption is indeed not generic for empirical measures chosen with law $(f^0)^{\otimes N}$ (*i.e.* it is satisfied with probability going to 0 in the large N limit).

Our result without cut-off. In the present article, we keep the same conditions on the inteaction kernel, but require only a much weaker assumption on the minimal distance between particles. This allows us to prove the propagation of chaos, for forces satisfying a (S^α)-condition :

$$(S^\alpha) \qquad \forall x \in \mathbb{R}^d, \quad |F(x)| \le \frac{C}{|x|^\alpha}, \quad |\nabla F(x)| \le \frac{C}{|x|^{\alpha+1}}, \tag{1.5}$$

with $\alpha < 1$. Our precise result without cut-off is the following

Theorem 1. *Assume that $d \geq 3$ and that F satisfies a (S^α)-condition with $\alpha < 1$. Choose any initial condition $f^0 \in L^\infty$ with compact support and total mass one for the Vlasov equation (1.2). For each $N \in \mathbb{N}^*$, consider at the particles system (1.1) with initial positions $(X_i, V_i)_{i \leq N}$ chosen randomly according to the probability $(f^0)^{\otimes N}$. Then for all $T \geq 0$, all*

$$\frac{2+2\alpha}{d+\alpha} < \gamma < 1 \text{ and } \quad 0 < s < \frac{\gamma d - (2-\gamma)\alpha - 2}{2(1+\alpha)},$$

there exists positive constants $C_0(f, F)$, $C_s(\gamma, s, f, F)$ such that for N large enough ($\ln N \gtrsim T$)

$$\mathbb{P}\left(\exists t \in [0,T],\ W_1(\mu_N(t), f(t)) \geq \frac{3e^{C_0 t}}{N^{\gamma/(2d)}}\right) \leq \frac{C_s}{N^s}, \tag{1.6}$$

where $f(t)$ is the unique strong solution of the Vlasov equation (1.2) with initial condition f^0, (the constant C_s blows up when s approaches its maximum value) and W_1 denotes the 1 Monge-Kantorovitch-Wasserstein distance.

The notation $C(f)$ means that the constant depends on the function f (essentially via conserved quantities like $\|f\|_\infty$ and also the size of its support) on the whole interval of time under consideration, here $[0, T]$.
The conditions on γ and s are not completely obvious, but it can be checked that if $\alpha < 1$ and $d \geq 3$, $\frac{2+2\alpha}{d+\alpha} < 1$ so that admissible γ exists. And for an admissible γ, the quantity $\frac{\gamma d-(2-\gamma)\alpha-2}{2(1+\alpha)}$ is also positive, so that admissible s also exists.
Roughly speaking, under the assumption of Theorem 1, the probability of finding a deviation strictly larger than the average inter-particle distance $N^{-1/2d}$ is small.

Remark 1. *Unfortunately, we are not able to provide a similar result for $d = 2$, even if α is very small. It can be seen that the condition on γ in theorem 1 is empty in that case.*

Even if the theorem is stated probabilistically in terms of propagation of chaos, the core of the proof is a deterministic theorem (See the Theorem 3 stated in the second section) which has generic assumptions with respect to the law $(f^0)^{\otimes N}$. Thanks to the deterministic result we can also construct explicit sequences of initial conditions for which the convergence towards the Vlasov equation will holds (for instance, particles well choosen on a mesh, but not only).

The improvements with respect to [HJ07]. The major improvement is of course the much weaker condition on the inital distribution of positions and velocities. We are hence able to show the propagation of chaos, which is again the crucial property for applications to physics.
We managed to simplify the proof with respect to the previous article [HJ07], considerably so in the long time case which was quite intricate before and does not require any special treatment here.
Finally our new result is almost quantitative. For large enough N, it actually tells how close, in Wasserstein distance, the empirical measure is from the limit. In fact if one does not use random initial conditions (and hence we do not need the large deviation result mentioned above), Theorem 3 gives a precise rate of convergence, which is of course quite useful from the point of view of numerical analysis.
Unfortunately, the condition on the kernel F is still the same and does not allow to treat Coulombian interactions. There are some physical reasons for this condition; for instance if $F = -\nabla\phi$ then it guarantees that the potential is bounded. We refer to [BHJ10] for some ideas in how to go beyond this threshold.

The result with cut-off. The result with cut-off presented here is in one sense slighty weaker than the previously known result [GV89], since we just miss the critical case $\alpha = d-1$. But it has also some major advantages, especially if we are not interested by numerical simulation. First of all, it is valid for random initial configurations with propagation of chaos and not only for well distributed initial positions and velocities (on a mesh). Secondly, for α larger but close to one it is valid for smaller cut-offs, much smaller than average (and also minimal) distance between particles.
The result is stated for forces depending on N and satisfying the following condition

$$(S_m^\alpha) \qquad \begin{array}{ll} i) & F \text{ satisfy a } (S^\alpha)-\text{condition} \\ ii) & \forall\, |x| \geq N^{-m},\ F_N(x) = F(x) \\ iii) & \forall\, |x| \leq N^{-m},\ |F_N(x)| \leq N^{-m\alpha}, \end{array} \tag{1.7}$$

which essentially means that the interaction kernel is regularized at scales lower than N^{-m}.
Note that in fact we would not need any estimate on the gradient of F_N for very small x. The result would still be true if F_N only converges to F for large enough x, with an error satisfying $\|F_N - F\|_1 \leq N^{-1/2d}$. The proof could be adapted to that case, but for simplicity we choose this presentation. We also point out that one would like to take m as large as possible if we want to be close to the dynamics without cut-off.

Theorem 2. *Assume that $d \geq 3$, $\gamma \in (0,1)$ and that F_N satisfies a (S_m^α)-condition for some $1 \leq \alpha < d-1$ with a cut-off order m such that*

$$m < m^* := \frac{\gamma}{2d} \min\left(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha} \right).$$

Choose any initial condition $f^0 \in L^\infty$ with compact support and total mass one for the Vlasov equation (1.2). For each $N \in \mathbb{N}^*$, look at the particles system (1.1) with initial positions $(X_i, V_i)_{n \leq N}$ chosen randomly according to the probability $(f^0)^{\otimes N}$. Then for all T there exists positive constants $C_0(f, F)$, $C_1(\gamma, m, f, F)$ and $C_2(f)$ such that for N large enough ($\ln N \gtrsim T$)

$$\mathbb{P}\left(\exists t \in [0,T],\ W_1(\mu_N(t), f(t)) \geq \frac{4e^{C_0 t}}{N^{\gamma/2d}} \right) \leq C_2 N^\gamma\, e^{-C_1 N^\lambda},$$

where $\lambda = \min\left(1-\gamma, \frac{d-\alpha}{2d}\right)$ and $f(t)$ is the unique strong solution of the Vlasov equation. (1.2) with initial condition f^0.

In dimension $d = 3$, the minimal cut-off is given by the order of $m^* = \frac{\gamma}{6}\min((\alpha-1)^{-1}, 5\alpha^{-1})$. As γ can be chosen very close to one, for α larger but close to one, the previous bound tells us that we can choose cut-off of order almost $N^{-5/6}$, i.e. much smaller than the likely minimal inter-particles distance in position space (of order $N^{-2/3}$, see the third section).
With such a small cut-off, one could hope that it is almost never used when we calculate the interaction forces between particles. Only a negligible number of particles will become so close to each others during the time T. This suggests that there should be some way to extend the result of convergence without cut-off at least to some $\alpha > 1$.
Unfortunately, we do not know how to make rigorous the previous probabilistic argument on the close encounters. First it is highly difficult to translate for particles system that are highly correlated. To state it properly we need infinite bounds on the 2 particles marginal. But obtaining such a bound for singular interaction seems difficult. Moreover, it remains

to neglect the influence of particles that have had a close encounters (its trajectory after a encounter is not well controlled) on the other particles.
Let us also mention that astro-physicists doing gravitational simulations ($\alpha = d-1$) with tree codes usually use small cut-off parameters, lower than $N^{-1/d}$ by some order. See [Deh00] for a physical oriented discussion about the optimal length of this parameter.

A short sketch of the proof. As mentioned above, the Vlasov equation (1.2) is satisfied by the empirical distribution μ_N of the interacting particle system provided that $F(0)$ is set to 0. Hence the problem of convergence can be reformulated into a problem of stability of the empirical measures - seen initially as a measure valued perturbation of the smooth profile f^0 - around the solution $f(t)$ of the Vlasov equation.
Our proof uses 3 ingredients to obtain this stability

- Show that with large probability the empirical measure at the initial time $\mu_N^z(0)$ is very close to f^0 and that the particles are not too badly distributed in the phase space.
- Compare the solution $f(t)$ to (1.2) with f^0 as initial data to the solution f_N to (1.2) with a regularization of $\mu_N^z(0)$ as initial data ((Dirac masses are replaced by "blobs").
- Control the distance (in some appropriate Monge-Kantorovitch-Wasserstein distance) between $\mu_N^z(t)$ and $f_N(t)$.

The first two steps are not overly complicated because they rely on previous known results (the second is a standard stability result for Vlasov-Poisson for example) and rather simple probabilistic estimates. The difficulties are hence concentrated in the third step.
This uses a deterministic result that has to be more precise than the one in [HJ07]. First of all it has to allow for a much smaller minimal distance in phase space between the particles at the initial time (and thus as well at later times). Notice that in some average sense the particles are still reasonably well distributed: For instance, the regularization $f_N(0)$ of $\mu_N^z(0)$ at a small scale is bounded in L^∞.
The other complication comes from that we want an explicit bound on the distance between $f_N(t)$ and $\mu_N^z(t)$ in the W_∞ distance as we will need it to control the distribution of particles at time t. This is necessary to control the difference between the force terms and hence the evolution of the distance between $f_N(t)$ and $\mu_N^z(t)$.
We remark that the use of the infinite MKW distance is important. We were not able to perform it with other MKW distance of order $p < +\infty$. It may seem strange to propagate a stronger norm for a problem with low regularity but in fact it turns out to be the only MKW distance with which we can handle a localized singularity in the force and Dirac masses in the distribution.
In essence controlling the distance between $f_N(t)$ and $\mu_N^z(t)$ requires to prove that the difference between the force terms acting on f_N and on $\mu_N^z(t)$ is bounded by this distance plus a small correction; then one may conclude by a Gronwall-like argument. There are several major complications in the proof.
First of all one has to deal with large oscillations in the discrete force term. This is due to particles that may come very close one to another in physical space or even collide, while remaining at a reasonable distance in phase space. The force term may become very large in that case but only for a very short time as the particles do not remain close. To solve this problem, we have to average those oscillations over a short time interval.
Then when one compares the force terms it is necessary to distinguish between 3 domains of interacting particles:

- Interaction between particles far enough in the physical space and that remain far enough over the short time interval where we average. This is the simplest case as one does not see the discrete nature of the problem at that level. The estimates need to be adapted to the distance used here but are otherwise very similar in spirit to the continuous problem or other previous works for mean field limits.

- Interaction between particles close enough in the physical space, far enough in the phase space over the short time interval where we average. Here we start to see the discrete level of the problem and in fact we cannot compare anymore the force term for the continuous Vlasov equation with the one for the particles' system. Instead we just show that both are small. This is simple for the continuous term as the size of the domain is itself quite small. For the discrete force term, it is considerably more complicated. First of all one has to bound the average over the short time interval of the interaction term between any two particles in that situation. Roughly speaking if those have a relative velocity of order v one expects the interaction to behave like
$$\int_t^{t+\tau} \frac{ds}{|\delta + sv|^\alpha},$$
where τ is the size of the average in time and δ is the minimum distance between the two particles over $[t,\ t+\tau]$. This is where the condition $\alpha < 1$ first comes into play as it allows to bound the previous integral independently of δ, provided that v is large enough. To conclude this part it is finally necessary to sum the previous bound over all particles in the corresponding domain. In [HJ07], we could directly bound the number of particles there; here as we do not have the same lower bound on the minimal distance in phase space, we have to use the distance between f_N and μ_N^z (which is why we need the W_∞ distance).

- Interaction between particles close enough in phase space over the short time interval where we average. In [HJ07] this case was relatively simple as the diameter of the corresponding domain in phase space was of the same order as the lower bound on the minimal distance (still in phase space) that we were propagating. Here this lower bound is much smaller (of the order of the square of the diameter). This is where the main technical improvement lies with respect to [HJ07].

Organization of the paper. In the next section, we introduce the notations, and state the deterministic results on which the propagation of chaos relies. In the third section, we explain how to obtain the propagation of chaos from the deterministic results. The fourth section is devoted to the proof of the two deterministic theorems.

2 Notations and other important theorems

2.1 Notations and useful results

We first need to introduce some notations and to define different quantities in order to state the result.

- **Empirical distribution μ_N and minimal inter-particle distance d_N**

Given a configuration $Z = (X_i, V_I)_{i\le N}$ of the particles in the phase space $\mathbb{R}^{2dN}$, the associated empirical distribution is the measure
$$\mu_N^z = \frac{1}{N}\sum \delta_{X_i,V_i}.$$

An important remark is that if $(X_i(t,),V_i(t))_{i\leq N}$ is a solution of the system of ODE (1.1), then the measure $\mu_N^Z(t)$ is a solution of the Vlasov equation (1.2), provided that the interaction force satisfies $F(0)=0$. This condition is necessary to avoid self-interaction of Dirac masses. It means that the interaction force is defined everywhere, but discontinuous and has a singularity at 0. In that conditions, the previously known results [BH77], [NW80] cannot be applied.
For every empirical measure, we define the minimal distance d_N^Z between particles in phase-space:

$$d_N^Z(\mu_N) = \min_{i\neq j}(|X_i-X_j|+|V_i-V_j|). \tag{2.1}$$ eq:dmin

This is a non physical quantity, but it is crucial to control the possible concentrations of particles and we will need to bound that quantity from below.
In the following we often omit the Z superscript Before going on, in order to keep "simple" notations.

- **Infinite MKW distance**

First, we use many times the Monge-Kantorovitch-Wasserstein distance of order one and infinite. The order one distance, denoted by W_1, is classical and we refer to the very clear book of Villani for definition and properties [Vil03]. The second one denoted W_∞ is not widely used, so we recall its definition :

def:Winf **Definition 1.** *For two probability measures μ and ν on X, a polish space, with $\Pi(\mu,\nu)$ the set of transference plan from μ to ν:*

$$W_\infty(\mu,\nu) = \inf\{\lambda - \mathit{esssup}\,|x-y|\ |\lambda\in\Pi\}.$$

In one of the few works on the subject [CDPJ08b] Champion, De Pascale and Juutineen prove that if μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$, then at least one optimal transference plane is given by a optimal transport map. In other words there exists a measurable map $T:\mathrm{Supp}\mu\to X$ such that $(Id,T)_\#\mu\in\Pi$ (it implies in particular that $T_\#\mu=\nu$) and

$$W_\infty(\mu,\nu) = \mu - \mathrm{esssup}_x\,|Tx-x|.$$

Although that is not mandatory (we could also work with optimal transference planes), we will use this result that will greatly simplify the proof.
Optimal transport is useful to compare the discrete sum of the N particles dynamics to the integrals of the continuous Vlasov system. For instance, if f is a continuous distribution and μ_N an empirical distribution we may rewrite the interaction force of μ_N using a transport map $T=(T_x,T_v)$ of f onto μ_N

$$\frac{1}{N}\sum_{i\neq j}F(X_i^0-X_j^0) = \int F(X_i^0-T_x(y,w))f(y,w)\,dydw.$$

Note that in the equality above, the function F is singular at $x=0$. Using infinite MKW distance, the singularity is still localized "in a ball" after the transport. The term under the integral in the right-hand-side has no singularity out of a ball of radius $W_\infty(f,\nu_N)$ in x. Others MKV distance of order $p<+\infty$ destroys that simple localization after the transport, which is why it seems more difficult to use them.

- **The scale ε.** We also introduce a scale

$$\varepsilon(N) = N^{-\gamma/2d}, \tag{2.2}$$ eps

for some $\gamma \in (0,1)$ to be fixed later but close enough from 1. Remark that this scale is larger than the average distance between a particle and its closest neighbor, which is of order $N^{-1/2d}$. We shall do a wide use of that scale in the sequel, and will often define quantities directly in term of ε rather than N. For instance, the cut-off order m used in the (S^α_m)-condition may be rewritten in term of ε, with $\bar m := \frac{2d}{\gamma} m$.

$$(S^\alpha_m) \qquad \begin{array}{ll} i) & F \text{ satisfy a } (S^\alpha)-\text{condition} \\ ii) & \forall\, |x| \geq \varepsilon^{\bar m},\ F_N(x) = F(x) \\ iii) & \forall\, |x| \leq \varepsilon^{\bar m},\ |F_N(x)| \leq \varepsilon^{-\bar m \alpha}, \end{array} \tag{2.3}$$ eq:Ckappa'

- **The solution f_N of Vlasov equation with blob initial condition.**

Now we defined a smoothing of μ_N at the scale $\varepsilon(N)$. For this, we choose a kernel $\phi : \mathbb{R}^{2d} \to \mathbb{R}$ with compact support in $[-\frac12,\frac12]^{2d}$ and total mass one, and denote $\phi_\varepsilon(\cdot) = \varepsilon^{-2d}\phi(\cdot/\varepsilon)$. The precise choice of ϕ is not very relevant, and the simplest one is maybe $\phi = \mathbf{1}_{[-\frac12,\frac12]^{2d}}$. We use this to smooth μ_N and define

$$f_N^0 = \mu_N^0 * \phi_{\varepsilon(N)}, \tag{2.4}$$ eq:deffN

and denote by $f_N(t,x,v)$ the solution to the Vlasov Eq. (eq:vlasov 1.2) for the initial condition f_N^0. The interest of f_N is that we may assume that it belongs to L^∞ (see the deviations estimates of the Proposition prop:largedev 5 in the Appendix). It allows to use standard stability estimates to control its W_1 distance to another solution of the Vlasov equation (See Loeper result Loep06 [Loe06]).

2.2 Statement of the deterministic result without cut-off

As mentioned in the introduction, the dynamic is entirely deterministic. In theorem thm:prob 1 the randomness comes only from the choice of the starting initial data. Precisely, the probability on the initial conditions is used to ensure that some conditions on minimal inter-particle distances and MKV distances are satisfied with large probability. But, once that conditions are fulfilled, we are able to propagate them with deterministic estimates. The following theorem shows that the particles system may be approximated by the solution of the Vlasov equation with the "blob" distribution f_N^0 as initial conditions, provided that two conditions on the minimal inter-particle distance $d_N(0)$ and the infinite norm of f_N^0 are satisfied.

thm:deter **Theorem 3.** *Assume that $d \geq 2$ and that the interaction force F satisfies a (S^α) condition, for some $\alpha < 1$ and let $0 < \gamma < 1$. Assume also that the initial empirical distribution μ_N^0 of the particles and its ε-enlargement f_N^0 satisfy :*

i) $d_N^0 := d_N(\mu_N(0)) \geq \varepsilon^{1+r} = N^{-\gamma(1+r)/2d}$ for some $r \in (1, r^*)$ where $r^* := \frac{d-1}{1+\alpha}$,

ii) $\|f_N^0\|_\infty \leq C_\infty$, a constant independent of N,

iii) For some $R > 0$, $\forall N \in \mathbb{N}$, $\mathrm{Supp}\mu_N^0 \subset B(0,R)$, the ball of radius R and center 0 of R^{2d}.

Then for any $T > 0$, there exists two constants $C_0(R, C_\infty, F, T)$ and $C_1(R, C_\infty, F, \gamma, r, T)$ such that for $N \geq e^{C_1 T}$ the following estimate is true

$$\forall t \in [0,\ T], \quad W_\infty(\mu_N(t), f_N(t)) \leq \frac{e^{C_0 t}}{N^{\gamma/2d}}. \tag{2.5}$$ eq:thm1

Remarks. This is a inequality of the type $W_\infty(t) \le W_\infty(0)e^{Ct}$, where the value of $W_\infty(0)$ has been bounded by $N^{-\gamma/2d}$. But that last bound is true since f_N^0 is a blob approximation of μ_N^0, with blob contained in balls of radius $N^{-\gamma/2d}$ around the Dirac of μ_N^0.
The previous theorem is valid in dimension 2. But unfortunately, its conditions are not generic in that case if the initial conditions are choosen independantly. This is why we cannot conclude to propagation of chaos for $d = 2$.

2.3 Statement of the deterministic result with cut-off

As in the case without cut-off, the probabilistic result 2 relies on a deterministic result, much simpler with cut-off since it does not need any control on the minimal inter-particles distance. The result is the following

thm:cutoff **Theorem 4.** *Assume that $d \ge 2$ and that the interaction force $F = F_N$ satisfies a (S_m^α), for some $1 < \alpha < d-1$, with a cut-off order satisfying*

$$m < m^* := \frac{\gamma}{2d} \min\left(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha} \right).$$

Assume also that the initial empirical distribution of the particles μ_N^0 and its ε enlargement f_N satisfy :

i) $\|f_N^0\|_\infty \le C_\infty$, a constant independent of N,

ii) For some $R > 0$, $\forall N \in \mathbb{N}$, $\operatorname{Supp} \mu_N^0 \subset B(0,R)$, the ball of radius R and center 0 of R^{2d}.

Then for any $T > 0$, there exists two constants $C_0(R, C_\infty, F, T)$ and $C_1(R, C_\infty, F, \gamma, r, T)$ such that for $N \ge e^{C_1 T}$ the following estimate is true

$$\forall t \in [0,\ T], \quad W_\infty(\mu_N(t), f_N(t)) \le \frac{e^{C_0 t}}{N^{\gamma/2d}}. \tag{2.6}$$ eq:thm3

Theorem 4 result has also an interest for numerical simulation because one obvious way to fulfill the hypothesis on the infinite norm of f_N^0 is is to put particles initially on a mesh (with a grid length of $N^{-1/2d}$ in $\mathbb{R}^{2d}$). In that case, the result is even valid with $\gamma = 1$.

3 From deterministic results (Theorem 3 and 4) to propagation of chaos.

The assumptions made in Theorem 3 may seem a little bit strange, but they are in some sense generic, when the initial positions and speed are choosen with the law $(f^0)^{\otimes N}$. Therefore, to prove Theorem 1 form Theorem 3, we need to

- Obtain a bound on the W_1 distance between $f(t)$ and $f_N(t)$, which are two solutions of the Vlasov equation.

- Estimate the probability that empirical measure chosen with the law $(f^0)^{\otimes N}$, do not satisfy the conditions $i)$ and $ii)$ of the deterministic theorem 3, and are far away from f^0 in W_1 distance (the last conditions is important for the previous point on the distance between f and f_N).

For these two points, we will use known results detailed in the next two sections. After that, a good choice of the parameter γ and r will allow us to conclude the proof.

3.1 Stability around solution of the Vlasov equation.

The following result is proved in [Loe06] for $\alpha = d-1$, but its proof may be adapted to our less singular case (The adaptation is done in [Hau09] in the Vortex case)

Proposition 1 (From Loeper)**.** *If f_1 and f_2 are two solutions of Vlasov Poisson equations with different kernel K_1 and K_2 both satisfying a (S^α)-condition, with $\alpha < d-1$, then*

$$\frac{d}{dt} W_1(f_1(t), f_2(t)) \le C \max(\|\rho_1\|_\infty, \|\rho_2\|_\infty)\, W_1(f_1(t), f_2(t)) + C\|\rho_1\|_\infty \|K_1 - K_2\|_1$$

The bound on the density may be obtained in our case with the argument of Pfaffelmoser for solution with compact support [Pfa92]. It is even simpler for $\alpha < 1$ as it is explained in the appendix of [HJ07].

Using that theorem in the case without cut-off ($K_1 = K_2 = F$), with $\alpha < 1$ (for $d \ge 3$) and a $\|f_N^0\|$ compactly supported, we obtain that there exists a constant C_0 depending on F, an uniform bounds on the infinite norms of the f_N and the size of their supports (denoted C_∞ and R in Theorem 3), such that

$$W_1(f(t), f_N(t)) \le e^{C_0 t} W_1(f^0, f_N^0) \le e^{C_0 t}\left(W_1(f^0, \mu_N^0) + N^{-\gamma/2d}\right), \tag{3.1}$$

.

3.2 Estimates in probability on the initial distribution.

Deviations on the infinite norm of the smoothed empirical distribution f_N. The precise result we need is given by the proposition 5 in the Appendix. It tells us that if the approximating kernel is $\phi = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]^{2d}}$, then

$$\mathbb{P}\left(\|f_N^Z\|_\infty \ge 2^{1+2d}\|f\|_\infty \right) \le C_2 N^\gamma e^{-C_1 N^{1-\gamma}}.$$

with $C_2 = (2R^0+2)^d$, R^0 the size of the support of f, and $C_1 = (2\ln 2 - 1)\, 2^n \|f\|_\infty$.

We would like to mention that we were first aware of the possibility of getting such estimates in a paper of Bolley, Guillin and Villani [BGV07], where the authors obtain quantitative concentration inequality for $\|f^N - f\|_\infty$ in infinite norm under the additionnal assumption that f^0 and ϕ are Lipschitz. Unfortunately, they cannot be used in our setting because they would require a too large smoothing paramater. Gao obtain in [Gao03] precise large deviations estimates for $\|f^N - f\|_\infty$, but as usual with large deviations estimates, they are only asymptotics and therefore less convenient for our problem, which is why we reworked them here.

Deviations for the minimal inter-particle distance. It may be proved with simple arguments that the scale η_m is almost surely larger than $N^{-1/d}$ when $f^0 \in L^\infty$. A precise result is stated in the Proposition below, proved in [Hau09]:

Proposition 2. *There exists a constant c_{2d} depending only on the dimension such that if $f^0 \in L^\infty(\mathbb{R}^{2d})$, then*

$$\mathbb{P}\left(d_N(Z) \ge \frac{l}{N^{1/d}} \right) \ge e^{-c_{2d}\|f^0\|_\infty l^d}.$$

We point out that this is not a large deviation result (the inequalities are in the wrong direction). It is that condition that prevents us from obtaining a "large deviation" type result in Theorem 1 (contrarily to the cut-off case of Theorem 2). In fact, the only bound it provides on the "bad" set is

$$\mathbb{P}\left(d_N(Z) \le \frac{l}{N^{1/d}}\right) \le 1 - e^{-c_{2d}\|f^0\|_\infty l^d} \le c_{2d}\|f^0\|_\infty l^d.$$

With the notation of Theorem 3 it comes that if $s = \gamma\frac{1+r}{2} - 1 > 0$ then

$$\mathbb{P}\left(d_N(Z) \le \varepsilon^{1+r}\right) = \mathbb{P}\left(d_N(Z) \le \frac{N^{-s/d}}{N^{1/d}}\right) \le c_{2d}\|f^0\|_\infty N^{-s}. \tag{3.2}$$

Deviations for the W_1 MKW distance. Peyre has obtained in [Pey] the following result

Proposition 3 (Peyre)**.** *Assume that f^0 is a compactly supported measure on $\mathbb{R}^{2d}$. If $d \ge 2$, and the empirical measures μ_N^0 are chosen according to the law $(f^0)^{\otimes N}$, then there exists an explicit constant L_d, depending only on the size of the support, such that*

$$\mathbb{P}\left(W_1(\mu_N^0, f^0) \ge \frac{L}{N^{1/(2d)}}\right) \le e^{L_d - LN^{\frac{d-1}{2d}}}. \tag{3.3}$$

It is of particular interest to us when $L = N^{\frac{1-\gamma}{2d}}$, in which case it maybe rewritten

$$\mathbb{P}\left(W_1(\mu_N^0, f^0) \ge \varepsilon\right) \le Ce^{-N^{\frac{d-\gamma}{2d}}}, \quad \text{with} \quad C = e^{L_d}. \tag{3.4}$$

3.3 Conclusion

Now take the assumptions of Theorem 1. It means that we assume that F satisfies a (S^α) condition for $\alpha < 1$ and $f^0 \in L^\infty$ for $d > 3$. We chose

$$\gamma \in \left(\frac{2+2\alpha}{d+\alpha}, 1\right), \text{ and } \quad r \in \left(\frac{2}{\gamma} - 1, r^* = \frac{d-1}{1+\alpha}\right),$$

the condition on γ ensuring that the second interval is non empty. We also define

$$s := \gamma\,\frac{1+r}{2} - 1 > 0, \quad \lambda = \min\left(1-\gamma, \frac{d-\gamma}{2d}\right)$$

Denote by ω_1, ω_2 the sets of initial conditions s.t. respectively (i), and (ii) (with the constant $C_\infty = 2^{1+2d}\|f^0\|_\infty$) of Theorem 3 hold and ω_3 s.t. $W_1(\mu_N, f^0) \le \frac{1}{N^{\gamma/(2d)}}$.

$$\omega_1 := \{Z \text{ s.t. } d_N(Z) \ge \varepsilon^{1+r}\}, \quad \omega_2 := \{Z \text{ s.t. } \|f_N^0\|_\infty \le 2^{1+2d}\|f^0\|_\infty\}$$
$$\omega_3 := \{Z \text{ s.t. } W_1(\mu_N^0, f^0) \le \varepsilon\}$$

By the results stated in the previous section, one knows that

$$\mathbb{P}(\omega_1^c) \le C\,N^{-s}, \quad \mathbb{P}(\omega_2^c) \le CN^\gamma e^{-C_1 N^{1-\gamma}}, \quad \mathbb{P}(\omega_3^c) \le \ Ce^{-N^{\frac{d-\gamma}{2d}}}.$$

Denote $\omega = \omega_1 \cap \omega_2 \cap \omega_3$. Hence $|\omega^c| \le |\omega_1^c| + |\omega_2^c| + |\omega_3^c|$ and for N large enough

$$\mathbb{P}(\omega^c) \le C\,N^{-s} + C\,N^\gamma\,e^{-\min(1,C_1)\,N^{-\lambda}} \le C\,N^{-s} \tag{3.5}$$

If the initial conditions belong to ω then one may apply Theorem thm:deter 3 and get on $[0,\ T]$

$$W_1(f_N, \mu_N) \le W_\infty(f_N, \mu_N) \le \frac{e^{C_0 t}}{N^{\gamma/(2d)}}.$$

Now apply the stability around solution of Vlasov equation given by (eq:Loeper 3.1) and get

$$W_1(f, f_N) \le W_1(f^0, f_N^0)\, e^{C_0 t} \le \frac{2}{N^{\gamma/(2d)}}\, e^{C_0 t}.$$

The factor 2 comes from the fact that $W_1(f^0, f_N^0) \le W_1(f^0, \mu_N^0) + W_1(\mu_N^0, f_N^0)$. We conclude that

$$W_1(f, \mu_N) \le \frac{3}{N^{\gamma/2d}} e^{C_0 t},$$

which proves that

$$\mathbb{P}(\omega) \le \mathbb{P}\left(\forall t \in [0, T],\ W_1(f, f_N) \le \frac{3e^{C_0 t}}{N^{\gamma/d}} \right).$$

The bound boundomega 3.5 then gives Theorem thm:prob 1.

3.4 From Theorem thm:cutoff 4 to Theorem thm:probcutoff 2

In the cut-off case, one can derive Theorem thm:probcutoff 2 from Theorem thm:cutoff 4 in the same manner. As we do not use the minimal distance in that case, the proof is simpler in the case $\alpha < d-1$ and we get a stronger convergence result. The only difference is that we shall use the Theorem Loeper 1 with $K_1 = F$ and $K_2 = F_N$, so that an error term appears. But that error term is bounded by

$$C\|\rho_f\|_\infty \|F - F_N\|_1 \le C\varepsilon^{d-\alpha} \le CW_\infty(t)$$

for any t so that the proof is unchanged. In fact, with the same λ, we obtain since the set ω_1 is now useless that

$$\mathbb{P}\left(\exists t \in [0, T],\ W_1(\mu_N(t), f(t)) \ge \frac{4e^{C_0 t}}{N^{\gamma/(2d)}} \right) \le CN^\gamma e^{-C_1 N^\lambda}.$$

4 Proof of Theorem thm:deter 3 and thm:cutoff 4

4.1 Definition of the transport

We try now to compare the the dynamics of μ_N and f_N, two distributions which have a compact support. For that, we choose an optimal transport $T^0 (= T_N^0)$ from f_N^0 to μ_N^0 for the infinite MKW distance (See the remark after Definition def:Winf 1). The existence of such a transport is ensured by ChaDePJuu08 [CDPJ08b]. T^0 is defined on the support of f_N^0, which is included in $\{|z| \le R^0\}$ (the size of the support). Since f_N^0 is an ε-enlargement of μ_N^0, it is clear that $W_\infty(f_N^0, \mu_N^0) \le \varepsilon$.
We also denote by $Z^f = (X^f, V^f)$ the smooth flow associated to f_N and by $Z^N = (X^N, V^N)$ the flow of the N particles system (with the convention $Z(t,s)$ transport from time s to time t). A simple way to get a transport of $f_N(t)$ on $\mu_N(t)$ is to transport along the flows the map T^0, i.e. to define

$$T^t = Z^N(t,0) \circ T^0 \circ Z^f(0,t), \qquad \text{and} \quad T^t = (T_x^t, T_v^t)$$

We use the following notation, for a test-"particle" of the continuous system at the position $z_t = (x_t, v_t)$ at time t, $z_s = (x_s, v_s)$ will be its position at time s for $s \in [t-\tau, t]$. Precisely

$$z_s = Z^f(s, t, z_t)$$

Since f_N is the solution of a transport equation, we have $f_N(t, z_t) = f_N(s, z_s)$. And since the vector-field of that transport equation is divergence free

$$\int \Phi(z)\, f_N(s,z)\, dz = \int \Phi(Z^f(s,t,z))\, f_N(t,z)\, dz = \int \Phi(z_s)\, f_N(t,z_t)\, dz_t.$$

Finally let us remark that the f_N are solutions to the (continuous) Vlasov equations with an initial L^∞ norm and support that are uniformly bounded in N. Therefore this remains true uniformly in N for any finite time. In particular there exists a constant C independent of N such that for any $t \in [0,\ T]$

$$\begin{aligned}
&\|f_N(t,.,.)\|_\infty \le C, \ \ \|f_N(t,.,.)\|_{L^1} = 1, \\
&|E|_\infty(t) := \|E(t,\cdot)\|_\infty \le \sup_x \int |F(x-y)|\, f_N(t,y,w)\, dy\, dw \le C \\
&|\nabla E(t,x)| \le \int |\nabla F(x-y)|\, f_N(t,y,w)\, dy\, dw| \le C \\
&\text{supp}\, f_N(t,.,.) \in B(0,R(t)), \quad R(t) \le C,
\end{aligned} \tag{4.1}$$

as of course $R(t) \le R^0 + \int_0^t \|E(s,.)\|_\infty\, ds$. This is always true for $\alpha < 1$. In dimension $d \le 3$ it remains true for $\alpha < d-1$ and even $\alpha = d-1$. In fact, all that estimates where central in the work of see Pfaffelmöser [Pfa92] about existence and uniqueness of compactly supported solution of Vlasov-Poisson equation (See also [Hor93] for a result with improved bounds). The proofs can be adapted to our simpler cases (See the Appendix of [HJ07] for the case $\alpha < 1$).
In dimension $d > 3$ and for attractive forces with $1 < \alpha < d-1$, there can be a blow-up in finite time (for α larger than a critical value depending on the dimension). In that case, we simply restrict ourselves to a time interval on which this does not occur.

In what follows, the final time T is fixed and independent of N. For simplicity, C will denote a generic universal constant, which may actually depend on T, the size of the initial support, the infinite norms of the f_N... But those constants are always independent of N as in (4.1).

4.2 The quantities to control

We will not be able to control the infinite norm of the field (and its derivative) created by the empirical distribution, but only a small temporal average of this norm. For this, we introduce in the case without cut-off a small time step $\tau = \varepsilon^{r'}$ for some $r' > r$ and close to r (the precise condition will appear later).
In the case with cut-off where r and r' are useless, the time step will by $\tau = \varepsilon$.

Before going on, we define some important notations.

- **The MKW infinite distance between $\mu_N(t)$ and $f(t)$.**

  We of course wish to bound $W_\infty(t) := \sup_{0\le s\le t} W_\infty(\mu_N(s), f_N(s))$ (note that W_∞ is hence automatically non decreasing). For the transport introduced before, one has

$$W_\infty(t) \le \sup_{s\le t} \sup_{(x_s,v_s)\in \text{supp}\, f_N(s,.,.)} |T^s(x_s,v_s) - (x_s,v_s)|.$$

In fact, we will provide bound for the quantity of the right hand side. Our result maybe stated for that quantity, rather than the infinite MKW distance. It is a little stronger, since it means that rearrangement in the transport are not necessary to keep the MKW distance bounded. The transport chosen at time $t = 0$ is preserved during the time.

- **The support of μ_N**

  We shall also need a uniform control on the support in position and velocity of the empirical distributions :
  $$R^N(t) = \max_i |(X_i(t),\ V_i(t))|.$$

- **The infinite norm $|E^N|_\infty$ of the time averaged discrete force field.**

  We also define the average of the discrete force field over small time intervals of length τ (the dependence on t is implicit)
  $$|E_N|_\infty = \sup_i \frac{1}{\tau} \int_{t-\tau}^{t} |E_N(X_i(s))|\, ds.$$

- **The infinite norm $|\nabla^N E|_\infty$ of the time averaged discrete derivative of the force field.**

  We also define a version of the infinite norm of its averaged derivative
  $$|\nabla^N E|_\infty = \sup_{i \neq j} \frac{1}{\tau} \int_{t-\tau}^{t} \frac{|E_N(X_i(s)) - E_N(X_j(s))|\, ds}{|X_i(s) - X_j(s)| + \varepsilon^{(1+r')}}\, ds.$$

  For both E_N and $\nabla^N E$, we use the convention that when the interval of integration contain 0 (for $t < \tau$), the integrand is null on the left side. The control on that term is useless in the cut-off case.

- **The minimal distance in phase space d_N**

  which has already be defined by the equation (2.1) in the Section 2.

- **Two useful integrals $I_\alpha(t, z_t)$ and $J_{\alpha+1}(t, z_t)$**

  Finally the technical computations involve
  $$I_\alpha(t, \bar{z}_t, z_t) = \frac{1}{\tau} \int_{t-\tau}^{t} |F(T^s_x(\bar{z}_s) - T^s_x(z_s)) - F(\bar{x}_s - x_s)|\, ds,$$
  which controls the difference of the two force fields at two point related by the "optimal" transport. Defining a second kernel as
  $$K_\varepsilon = \min\left(\frac{1}{|x|^{1+\alpha}},\ \frac{1}{\varepsilon^{1+r'}\, |x|^{\alpha}} \right),$$
  we define a second useful quantity
  $$\begin{aligned} J_{\alpha+1}(t, \bar{z}_t, z_t) &= \frac{1}{\tau} \int_{t-\tau}^{t} K_\varepsilon(|T^s_x(\bar{z}_s) - T^s_x(z_s)|\, ds \\ &= \frac{1}{\tau} \int_{t-\tau}^{t} K_\varepsilon(|X_i(s) - X_j(s)|\, ds, \end{aligned}$$
  if i and j is the indices such that $Z_i(t) = T^t(\bar{z}_t)$ and $Z_j(t) = T^t(z_t)$. $J_{\alpha+1}$ will be useful to control the discrete derivative of the field, and is thus useless in the cut-off case.

All previous quantities are relatively easily bounded by I_α and $J_{\alpha+1}$. Those last two will not be bounded by direct calculation on the discrete system, but we will compare them to similar ones for the continuous system, paying for that in terms of the distance between $\mu_N(t)$ and $f(t)$. That strategy is interesting because the integrals are easier to manipulate than the discrete sums.
We summarize the first easy bounds in the following

Proposition 4. *Under the assumptions of Theorem 3, one has for some constant C uniform in N*

$$\begin{aligned}
&(i) \quad R_N(t) \le W_\infty(t) + R(t) \le W_\infty(t) + C,\\
&(ii) \quad W_\infty(t) \le W_\infty(t-\tau) + C\,\tau \sup_{\bar z_t} \int_{|z_t|\le R(t)} I_\alpha(t,\bar z_t, z_t)\,dz_t,\\
&(iii) \quad |\nabla^N E|_\infty \le C \sup_{\bar z_t} \int_{|z_t|\le R(t)} J_{\alpha+1}(t,\bar z_t, z_t)\,dz_t.\\
&(iv) \quad d_N(t) + \varepsilon^{1+r'} \ge [d_N(t-\tau) + \varepsilon^{1+r'}] e^{-\tau(1+|\nabla^N E|_\infty(t))}.
\end{aligned}$$

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Note that the control on $R_N(t)$ is simple enough that it will actually be used implicitly in the rest many times, and that the $iv)$ is a simple consequence of the $iii)$. In fact, in that proposition the crucial estimates are the $ii)$ and $iii)$.
Remark also that in the case of very singular interaction force ($\alpha \ge 1$) with cut-off - in short (S^α_m) conditions - the control on minimal distance d_N and therefore the control on $|\nabla^N E|_\infty$ are useless, so that the only interesting inequality is the second one.

4.3 Proof of Prop. 4

Let us start with (i). Simply write

$$R^N(t) = \sup_{z_t \in \operatorname{supp} f_N(t,\cdot)} |T^t(z_t)| \le \sup_{z_t \in \operatorname{supp} f_N(t,\cdot)} |T^t(z_t) - z_t| + \sup_{z_t \in \operatorname{supp} f_N(t,\cdot)} |z_t|,$$

So indeed by (4.1)

$$R^N(t) \le W_\infty(t) + R(t) \le W_\infty(t) + C.$$

As for (ii), simply differentiate in time W_∞ to find

$$\frac{W_\infty(t) - W_\infty(t-\tau)}{\tau} \le \sup_{\bar z_t} \int \left| \frac{1}{\tau}\int_{t-\tau}^t [F(T^s_x(\bar z_s) - T^s_x(z_s)) - F(\bar x_s - x_s)]\,ds \right| f_N(z_t)\,dz_t.$$

Since f_N is uniformly bounded in L^∞ and compactly supported in $B(0,R(t))$, one gets by the definition of I_α

$$\frac{W_\infty(t) - W_\infty(t-\tau)}{\tau} \le \|f^0\|_\infty \sup_{\bar z_t} \int_{|z_t|\le R(t)} I_\alpha(t,\bar z_t, z_t)\,dz_t,$$

which is exactly (ii).

Concerning $|\nabla^N E|_\infty$ in (iii), noting that

$$\begin{aligned}
\int_{t-\tau}^t \frac{|E_N(X_i(s)) - E_N(X_j(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}} &= \frac{1}{N}\sum_{k\neq i,j} \int_{t-\tau}^t \frac{|F(X_i(s) - X_k(s)) - F(X_j(s) - X_k(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}}\,ds.\\
&\quad + \frac{1}{N}\int_{t-\tau}^t \frac{|F(X_i(s) - X_j(s)) - F(X_j(s) - X_i(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}}\,ds
\end{aligned}$$

By the assumption (eq:Calpha 1.5), one has that

$$|F(x)-F(y)| \leq C\left(\frac{1}{|x|^{\alpha+1}}+\frac{1}{|y|^{\alpha+1}}\right)|x-y|.$$

So

$$\frac{|F(X_i(s)-X_k(s))-F(X_j(s)-X_k(s))|}{|X_i(s)-X_j(s)|+\varepsilon^{1+r'}} \leq \frac{C}{|X_i(s)-X_k(s)|^{1+\alpha}}+\frac{C}{|X_j(s)-X_k(s)|^{1+\alpha}},$$

and that bound is also true for the remaining term where $k=i$ or j, if we delete the undefined term in the sum. One also obviously has, still by (eq:Calpha 1.5)

$$\begin{aligned}\frac{|F(X_i(s)-X_k(s))-F(X_j(s)-X_k(s))|}{|X_i(s)-X_j(s)|+\varepsilon^{1+r'}} \leq & \frac{C}{\varepsilon^{1+r'}|X_i(s)-X_k(s)|^{\alpha}} \\ & +\frac{C}{\varepsilon^{1+r'}|X_j(s)-X_k(s)|^{\alpha}}.\end{aligned}$$

Therefore by the definition of K_ε

$$\frac{|F(X_i(s)-X_k(s))-F(X_j(s)-X_k(s))|}{|X_i(s)-X_j(s)|+\varepsilon^{1+r'}} \leq K_\varepsilon(X_i(s)-X_k(s))+K_\varepsilon(X_j(s)-X_k(s)).$$

Summing up, this implies that

$$\begin{aligned}|\nabla^N E|_\infty \leq C \max_{i\neq j}\Big(&\frac{1}{\tau}\int_{t-\tau}^t \frac{1}{N}\sum_{k\neq i} K_\varepsilon(X_i(s)-X_k(s))\,ds \\ &+\frac{1}{\tau}\int_{t-\tau}^t \frac{1}{N}\sum_{k\neq j} K_\varepsilon(X_j(s)-X_k(s))ds\Big).\end{aligned}$$

Transforming the sum into integral thank to the transport, we get exactly the bound (iii) involving $J_{\alpha+1}$.

Finally for $d_N(t)$, consider any $i\neq j$, then obviously

$$\frac{d}{ds}|(X_i(s)-X_j(s),\ V_i(s)-V_j(s))| \geq -|V_i(s)-V_j(s)|-|E_N(X_i(s))-E_N(X_j(s))|.$$

Simply write

$$|E_N(X_i(s))-E_N(X_j(s))| \leq \frac{|E_N(X_i(s))-E_N(X_j(s))|}{|X_i(s)-X_j(s)|+\varepsilon^{1+r'}}\ (|X_i(s)-X_j(s)|+\varepsilon^{1+r'})$$

to obtain that

$$\begin{aligned}\frac{d}{ds}|(X_i(s)-X_j(s),\ V_i(s)-V_j(s))| \geq -&\left(1+\frac{|E_N(X_i(s))-E_N(X_j(s))|}{|X_i(s)-X_j(s)|+\varepsilon^{1+r'}}\right) \\ &(|(X_i(s)-X_j(s),\ V_i(s)-V_j(s))|+\varepsilon^{1+r'}).\end{aligned}$$

Integrating this inequality and taking the minimum, we get

$$\begin{aligned}d_N(t)+\varepsilon^{1+r'} &\geq (d_N(t-\tau)+\varepsilon^{1+r'})\inf_{i\neq j}\exp\left(-\tau-\int_{t-\tau}^t \frac{|E_N(X_i(s))-E_N(X_j(s))|}{|X_i(s)-X_j(s)|+\varepsilon^{1+r'}}\,ds\right) \\ &\geq [d_N(t-\tau)+\varepsilon^{1+r'}]\exp^{-\tau(1+|\nabla^N E|_\infty(t))}.\end{aligned}$$

4.4 The bound for I_α and $J_{\alpha+1}$

To close the the system of inequality of the Proposition propeasy 4, it remains to bound the two integrals involving I_α and J_α. It is done with the following lemmas

lem:boundI **Lemma 1.** *Assume that F satisfies an (S^α)-condition with $\alpha < 1$, and that τ is small enough such that*

$$C\,\tau\,(1 + |\nabla^N E|_\infty(t))\,(W_\infty(t) + \tau) \le d_N(t). \tag{4.2}$$

cond:lem1

Then one has the following bounds, uniform in $\bar z_t$

$$\int_{|z_t|\le R(t)} I_\alpha(t, \bar z_t, z_t)\,dz_t \le C\,\big[W_\infty(t) + (W_\infty(t)+\tau)^d\tau^{-\alpha} + (W_\infty(t)+\tau)^{2d}(d_N(t))^{-\alpha}\tau^{-\alpha}\big].$$

$$\int_{|z_t\le R(t)} J_{\alpha+1}(t, \bar z_t, z_t)\,dz_t \le C\,\Big(1 + (W_\infty(t)+\tau)^d\varepsilon^{-(1+r')}\,\tau^{-\alpha} + (W_\infty(t)+\tau)^{2d}\varepsilon^{-(1+r')}\,\tau^{-\alpha}\,(d_N(t))^{-\alpha}\Big).$$

boundIK

In the cut-off case where the interaction force satisfy a (S^α_m) condition (we recall that it means that the cut-off is of size $N^{-m} = \varepsilon^{\bar m}$ with $\bar m = \frac{2d}{\gamma} m$), we only need to bound the integral of I_α, with the result

boundIcut **Lemma 2.** *Assume that $1 \le \alpha < d-1$, and that F satisfies a (S^α_m) condition, one as the following bound, uniform in $\bar z_t$*

$$\int_{|z_t|\le R(t)} I_\alpha(t, \bar z_t, z_t)\,dz_t \le C\,\Big(W_\infty(t) + (W_\infty(t)+\tau)^d\tau^{-1}\varepsilon^{\bar m(1-\alpha)} + (W_\infty(t)+\tau)^{2d}\varepsilon^{-\bar m\alpha}\Big). \tag{4.3}$$

boundIKcut

with the convention (if $\alpha = 1$) that $(\varepsilon^{\bar m})^0 = |\ln(\varepsilon^{\bar m})|$ [1].

The proofs with or without cut-off follow the same line and we will prove the above lemmas at the same time. We begin by an explanation of the sketch of the proof, and then perform the technical calculation.

4.4.1 Rough sketch of the proof

The point $\bar z_t = (\bar x_t, \bar v_t)$ is considered fixed through all this subsection (as the integration is carried over $z_t = (x_t, v_t)$). Accordingly we decompose the integration in z_t over several domains. First

$$A_t = \{ z_t \mid |\bar x_t - x_t| \ge 4W_\infty(t) + 2\tau(|\bar v_t - v_t| + \tau|E|_\infty(t)) \}.$$

This set consist of points z_t such that x_s and $T^s_x(z_s)$ are sufficiently far away from $\bar x_s$ on the whole interval $[t-\tau, t]$, so that they will not see the singularity of the force. The bound over this domain will be obtained using traditional estimates for convolutions.
One part of the integral can be estimated easily on A^c_t (the part corresponding to the flow of the regular solution f_N to the Vlasov equation). For the other part it is necessary to decompose further. The next domain is

$$B_t = A^c_t \bigcap \{ z_t \mid |\bar v_t - v_t| \ge 4\,W_\infty(t) + 4\,\tau|E|_\infty|(t) \}.$$

[1] That convention may be justified by the fact that it implies a very simple algebra $(x^{1-\alpha})' \approx x^{-\alpha}$ even if $\alpha = 1$. It allows us to give an unique formula rather than three different cases.

This contains all particles z_t that are close to $\bar{z}_t$ in position (i.e. x_t close to $\bar{x}_t$), but with enough relative velocity not to interact too much. The small average in time will be useful in that part, as the two particles remains close only a small amount of time.
The last part is of course the remainder

$$C_t = (A_t \cup B_t)^c.$$

This is a small set, but where the particles remains close together a relatively long time. Here, we are forced to deal with the corresponding term at the discrete level of the particles. This is the only term which requires the minimal distance in phase space; and the only term for which we need a time step τ small enough as per the assumption in Lemma boundIK 1.

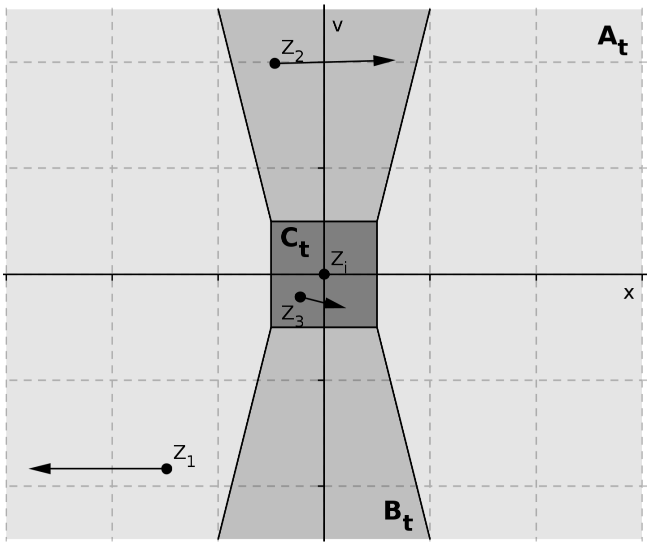

Figure 1: The partition of the phase space.

4.4.2 Step 1: Estimate over A_t

If $z_t \in A_t$, we have for $s \in [t-\tau, t]$

$$\begin{aligned} |\bar{x}_s - x_s| &\geq |\bar{x}_t - x_t| - (t-s)|\bar{v}_t - v_t| - (t-s)^2 |E|_\infty(t) \geq \frac{|\bar{x}_t - x_t|}{2} \quad (4.4) \\ |T_x^s(\bar{z}_s) - T_x^s(z_s)| &\geq |\bar{x}_s - x_s| - 2W_\infty(s) \geq \frac{|\bar{x}_t - x_t|}{2}. \quad (4.5) \end{aligned}$$

For I_α, we use the direct bound for $\bar{z}_t \in A_t$

$$\begin{aligned} |F(T_x^s(\bar{z}_s) - T_x^s(z_s)) - F(\bar{x}_s - x_s)| &\leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} (|T_x^s(\bar{z}_s) - \bar{x}_s| + |T_x^s(z_s) - x_s|) \\ &\leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} W_\infty(s) \leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} W_\infty(t), \end{aligned}$$

and obtain by integration on $[t-\tau, t]$

$$I(t, \bar{z}_t, z_t) \leq \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} W_\infty(t).$$

Then integrating in z_t we may get since $\alpha+1<2\leq d$

$$\begin{aligned}\int_{A_t} I_\alpha(t,\bar z_t,z_t)\,dz_t &\leq C W_\infty(t)\int_{A_t}\frac{dz_t}{|\bar x_t-x_t|^{1+\alpha}}\\ &\leq C\,R(t)^{2d-1-\alpha}\,W_\infty(t)\leq C\,W_\infty(t).\end{aligned} \tag{4.6}$$

At

For $J_{\alpha+1}$, we have used (4.5) on the set A_t the bound

$$|K_\varepsilon(T^s_x(\bar z_t)-T^s_x(z_t))|\leq \frac{C}{|\bar x_t-x_t|^{1+\alpha}}.$$

Integrating with respect to time and z_t we get since $1+\alpha<d$.

$$\begin{aligned}\int_{A_t} J_{\alpha+1}(t,\bar z_t,z_t)\,dz_t &\leq C\int_{A_t}\frac{dz_t}{|\bar x_t-x_t|^{1+\alpha}}\\ &\leq C\,R(t)^{d-1-\alpha}\leq C.\end{aligned} \tag{4.7}$$

At2

For the cut-off case, the estimation on I_α for this step is unchanged.

4.4.3 Step 1' : Estimate over A_t^c for the continuous part of I_α and $J_{\alpha+1}$.

For the remaining term in I_α, we use the rude bound

$$|F(T^s_x(\bar z_s)-T^s_x(z_s))-F(\bar x_s-x_s)|\leq |F(T^s_x(\bar z_s)-T^s_x(z_s))|+|F(\bar x_s-x_s)|.$$

The term involving T^s is complicated and requires the additional decompositions. It will be treated in the next sections. The other term is simply bounded by

$$\begin{aligned}\int_{z_t\in A_t^c}\frac{1}{\tau}\int_{t-\tau}^t |F(\bar x_s-x_s)|ds\;dz_t &\leq \frac{1}{\tau}\int_{t-\tau}^t\int_{z_t\in A_t^c}\frac{C\,dz_t}{|\bar x_s-x_s|^\alpha}\,ds\\ &\leq \frac{1}{\tau}\int_{t-\tau}^t\int_{z_s\in Z^f(s,t,A_t^c)}\frac{C\,dz_s}{|\bar x_s-x_s|^\alpha}\,ds.\end{aligned}$$

From the bound $R(t)$ and $|E|_\infty(t)$ we see that

$$|A_t^c|\leq CR(t)^d(W_\infty(t)+\tau)^d\leq C(W_\infty(t)+\tau)^d,$$

where $|\cdot|$ denote the Lebesgue measure. Since the flow Z_f is measure preserving, the measure of the set $Z^f(s,t,A_t^c)$ satisfies the same bound. This set is also included in $[-R(t),R(t)]^{2d}$ (if R is increasing, a property that we may assume). We use the above lemma which implies that above all the set $Z(s,t,A_t^c)$, the integral reaches is maximum when the set is a cylinder

Lemma 3. *Let $\Omega\subset B(0,R)\subset\mathbb{R}^n$. Let P be a projection from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $m\leq n$. Then for any $a<m$*

$$\int_\Omega \frac{dx}{|Px|^a}\leq C_a R^{a(n-m)/m}\,|\Omega|^{1-a/m}.$$

cylinder

Proof of Lemma 3. We can freely assume that $Px=(x_1,\dots,x_m)$. Now maximize the integral

$$\int_\omega |Px|^{-a}dx$$

over all sets $\omega \subset \mathbb{R}^n$ satisfying $\omega \subset B(0,R)$ and $|\omega| = |\Omega|$. It is clear that the maximum is obtained by concentrating as much as possible ω near $Px = 0$, *i.e.* with a cylinder of the form $B_m(0,r) \times B_{n-m}(0,R)$ where B_k denotes the k-dimensional ball. Since $|\omega| = |\Omega|$ we have $r = |\Omega|^{1/m} R^{1-n/m}$. The integral can now be computed explicitly and gives the lemma. $\square$

Applying the lemma, we get

$$\int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^{t} |F(\bar{x}_s - x_s)|\, dz_t ds \le C(W_\infty(t) + \tau)^{d-\alpha}. \tag{4.8}$$

Ic

That term do not appear in Lemma 1 since it is strictly smaller than the bound of the remaining term (involving T), as we shall see in the next section.
The same bound is valid for $J_{\alpha+1}$ since $\alpha + 1 < d$.
For the cut-off case, the same bound is valid for I_α since $\alpha \le d-1 < d$ (The cut-off cannot in fact help to provide a better bound for this term), and we do not need the estimate on $J_{\alpha+1}$.

At this point, the remaining term to bound in I_α is only

$$\int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^{t} |F(T_x^s(\bar{z}_s) - T_x^s(z_s))|\, ds \le C \int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^{t} \frac{dz_t}{|T_x^s(\bar{z}_s) - T_x^s(z_s)|^\alpha}\, ds. \tag{4.9}$$

eq:remain

For $J_{\alpha+1}$ one may bound the integral of the continuous part on A_t^c in a similar manner. As $K_\varepsilon \le \frac{1}{\varepsilon^{1+r'}|x|^\alpha}$, the remainder can be controlled by (4.9)

$$\int_{A_t^c} J_{\alpha+1}(t, \bar{z}_t, z_t)\, dz_t \le C\,(W_\infty(t) + \tau)^{d-\alpha} + \frac{C}{\varepsilon^{1+r'}} \int_{z_t \in A_t^c} \frac{1}{\tau} \int_{t-\tau}^{t} \frac{dz_t}{|T_x^s(\bar{z}_s) - T_x^s(z_s)|^\alpha}\, ds. \tag{4.10}$$

remainderJ

Therefore in the next sections we focus on giving a bound for (4.9).

4.4.4 Step 2: Estimate over B_t

We recall the definition of B_t

$$B_t = \left\{ z_t \ \text{ s.t. } \begin{array}{l} |\bar{x}_t - x_t| \le 4W_\infty(t) + 2\tau(|\bar{v}_t - v_t| + \tau|E|_\infty(t)) \\ |\bar{v}_t - v_t| \ge 4\,W_\infty(t) + 4\,\tau|E|_\infty|(t) \end{array} \right\}.$$

If $z_t \in B_t$, we have, as for A_t, for $s \in [t-\tau, t]$

$$|\bar{v}_s - v_s - \bar{v}_t + v_t| \ \le \ 2\tau|E_\infty|(t) \le \frac{|\bar{v}_t - v_t|}{2}, \tag{4.11}$$

$$|T_v^s(\bar{z}_s) - T_v^s(z_s) - \bar{v}_t + v_t| \ \le \ |\bar{v}_s - v_s - \bar{v}_t + v_t| + 2W_\infty(s)| \le \frac{|\bar{v}_t - v_t|}{2}. \tag{4.12}$$

This means that the particles involved are close to each others (in the positions variables), but with a sufficiently large relative velocity, so that they do not interact a lot on the interval $[t-\tau, t]$.
First we introduce a notation for the term of (4.9)

$$\int_{z_t \in B_t} I_{bc}(t, \bar{z}_t, z_t)\, dz_t, \qquad \text{with } I_{bc}(t, \bar{z}_t, z_t) = I_{bc}(t,i,j) := \frac{1}{\tau} \int_{t-\tau}^{t} \frac{1}{|T_x^s(\bar{z}_s) - T_x^s(z_s)|^\alpha}\, ds, \tag{4.13}$$

eq:Ibc

where (i,j) are s.t. $T_x^s(\bar{z}_s) = X_i(s)$, $T_x^s(z_s) = X_j(s)$.

For $z_t \in B_t$, define for $s \in [t-\tau,\ t]$

$$\phi(s) = (T^s_x(\bar z_s) - T^s_x(z_s)) \cdot \frac{\bar v_t - v_t}{|\bar v_t - v_t|}.$$

Note that $|\phi(s)| \leq |T^s_x(\bar z_s) - T^s_x(z_s)|$ and that

$$\begin{aligned}\phi'(s) &= (T^s_v(\bar z_s) - T^s_v(z_s)) \cdot \frac{\bar v_t - v_t}{|\bar v_t - v_t|} \\ &= |\bar v_t - v_t| + (T^s_v(\bar z_s) - T^s_v(z_s) - (\bar v_t - v_t)) \cdot \frac{\bar v_t - v_t}{|\bar v_t - v_t|} \geq \frac{|\bar v_t - v_t|}{2},\end{aligned}$$

where we have used (eq:CBF 4.11). Therefore if ϕ attains its minimum on $[t-\tau,\ t]$ at s_0, then

$$|T^s_x(\bar z_s) - T^s_x(z_s)| \geq |t - s_0| \frac{|\bar v_t - v_t|}{2}. \tag{4.14}$$ eq:dispB

Using this directly gives, as $\alpha < 1$

$$|I_{bc}(t,\bar z_t, z_t)| \leq \frac{C}{\tau} |\bar v_t - v_t|^{-\alpha} \int_{t-\tau}^t \frac{ds}{|s-s_0|^\alpha} \leq C\,\tau^{-\alpha}\,|\bar v_t - v_t|^{-\alpha}. \tag{4.15}$$ vtrick

Now integrating

$$\begin{aligned}\int_{z_t\in B_t} |I_{bc}(t,\bar z_t, z_t)|\,dz_t &\leq C\,\tau^{-\alpha} \int_{A_t^c} \frac{dz_t}{|\bar v_t - v_t|^{-\alpha}} \\ &\leq C\,\tau^{-\alpha}\,(W_\infty(t)+\tau)^d\,(R(t))^{d-\alpha},\end{aligned}$$

by using the fact that $B_t \subset B(0, C[W_\infty(t)+\tau]) \times B(0, R(t))$. In conclusion

$$\int_{z_t\in B_t} |I_{bc}(t,\bar z_t, z_t)|\,dz_t \leq C\,\tau^{-\alpha}\,(W_\infty(t)+\tau)^d. \tag{4.16}$$ Bt

With the cut-off where $\alpha > 1$, the reasoning follows the same line up to the bound (vtrick 4.15) which relies on the hypothesis $\alpha < 1$. (vtrick 4.15) is replaced by

$$\begin{aligned}|I_{bc}(t,\bar z_t, z_t)| &\leq \frac{C}{\tau} \int_{t-\tau}^t \frac{ds}{(|s-s_0||\bar v_t - v_t| + 4\varepsilon^{\bar m})^\alpha} \\ &\leq \frac{C}{\tau} \int_0^{\frac{\tau}{2|\bar v_t - v_t|}} \frac{ds}{(s+4\varepsilon^{\bar m})^\alpha} \leq \frac{C\varepsilon^{\bar m(1-\alpha)}}{\tau}.\end{aligned}$$

When $\alpha = 1$, the previous calculation gives

$$\begin{aligned}|I_{bc}(t,\bar z_t, z_t)| &\leq \frac{C}{\tau} \ln\left(1 + \frac{\tau}{2|\bar v_t - v_t|\varepsilon^{\bar m}}\right) \leq \frac{C}{\tau} \ln\left(1+\varepsilon^{-\bar m}\right) \\ &\leq \frac{C\ln(\varepsilon^{-\bar m})}{\tau} = \frac{C(\varepsilon^{-\bar m})^0}{\tau}\end{aligned}$$

In the first line, we used the bound $\frac{\tau}{|\bar v_t - v_t|} \leq \frac{1}{|E|_\infty(t))}$ which is implied by the definition of B_t and then we assumed that $|E|_\infty(t) \geq 1$. In the second line, we use the convention $(\varepsilon^{\bar m})^0 = |\ln(\varepsilon^{\bar m})|$.
In both cases, integrating that bound (which do not depend on v_t) over B_t, we get the estimate

$$\int_{z_t\in B_t} |I_{bc}(t,\bar z_t, z_t)|\,dz_t \leq C\,(W_\infty(t)+\tau)^d\,\tau^{-1}\,\varepsilon^{\bar m(1-\alpha)}\,. \tag{4.17}$$ Btcutoff

4.4.5 Step 3: Estimate over C_t

We recall the definition of C_t

$$C_t = \left\{ z_t \;\text{ s.t. } \begin{array}{l} |\bar x_t - x_t| \le 4W_\infty(t) + 2\tau(|\bar v_t - v_t| + \tau |E|_\infty(t)) \\ |\bar v_t - v_t| \le 4\,W_\infty(t) + 4\,\tau |E|_\infty|(t) \end{array} \right\}.$$

First remark that $C_t \subset \{|z_t| \le C(W_\infty(t)+\tau)\}$, so that its volume is bounded by $C(W_\infty(t)+\tau)^{2d}$. From the previous steps, it only remains to bound

$$\int_{z_t\in C_t} I_{bc}(t,\bar z_t, z_t)\, dz_t.$$

We begin by the cut-off case, which is the simpler one. In that case, one simply bounded $I_{bc} \le C\,\varepsilon^{-\bar m\alpha}$ which implies

$$\int_{z_t\in C_t} I_{bc}(t,\bar z_t, z_t)\, dz_t \le C(W_\infty(t)+\tau)^{2d}\varepsilon^{-\bar m\alpha}. \tag{4.18}$$ Ctcutoff

It remains the case without cut-off. We denote $\tilde C_t = \{j \mid \exists z_t \in C_t,\ s.t.\ Z_j(t) = T^t(z_t)\}$, and transform the integral on C_t in a discrete sum

$$\int_{z_t\in C_t} I_{bc}(t,\bar z_t, z_t)\, dz_t = \sum_{j\in\tilde C_t} a_{ij} I_{Nc}(t,i,j) \quad \text{ with } I_{Nc}(t,i,j) = \frac{1}{\tau}\int_{t-\tau}^t \frac{dz_t}{|X_i(s)-X_j(s)|^\alpha}\, ds,$$

where i is the number of the particle associated to $\bar z_t$ ($T^t(\bar z_t) = Z_i(t)$) and

$$a_{ij} = |\{z_t \in C_t,\ T^t(z_t) = Z_j(t)\}|, \quad \text{so that } \sum_{j\in\tilde C_t} a_{ij} = |C_t|.$$

To bound I_{Nc} over $\tilde C_t$, we do another decomposition in j. Define

$$\begin{aligned} JX_t &= \left\{ j\in\tilde C_t\,,\ |X_j(t)-X_i(t)| \ge \frac{d_N(t)}{2} \right\}, \\ JV_t &= \left\{ j\in\tilde C_t\,,\ |X_j(t)-X_i(t)| \le |V_j(t)-V_i(t)| \text{ and } |V_j(t)-V_i(t)| \ge \frac{d_N(t)}{2} \right\}. \end{aligned}$$

By the definition of the minimal distance in phase space $d_N(t)$, one has that $\tilde C_t = JX_t \cup JV_t$. Since

$$|T^t(z_t) - z_t| \le W_\infty(t),$$

one has by the definition of $\tilde C_t$ and C_t that for all $j\in\tilde C_t$, $|Z_j(t)-Z_i(t)| \le C\,(W_\infty(t)+\tau)$.

Let us start with the bound over JX_t. If $j\in JX_t$, one has that

$$|X_j(s)-X_i(s)| \ge |X_j(t)-X_i(t)| - \int_s^t |V_j(u)-V_i(u)|\, du.$$

On the other hand, for $u\in[s,t]$,

$$|V_j(u)-V_i(u)| \le 2W_\infty(t) + |\bar v_u - v_u| \le 2(W_\infty(t)+\tau|E|_\infty) + |\bar v_t - v_t| \le C(W_\infty(t)+\tau).$$

Therefore assuming that with that constant C

$$C\,\tau(W_\infty(t)+\tau) \le d_N(t)/4, \tag{4.19}$$ assJX

we have that for any $s \in [t-\tau,\ t]$, $|X_j(s)-X_i(s)| \geq d_N(t)/4$. Consequently for any $j \in JX_t$

$$I_{Nc}(t,i,j) \leq C\,[d_N(t)]^{-\alpha}. \tag{4.20}$$ boundJX

For $j \in JV_t$, we write

$$|(V_j(s)-V_i(s))-(V_j(t)-V_i(t))| \leq \int_s^t |E_N(X_j(u))-E_N(X_i(u))|\,du.$$

Note that

$$\begin{aligned}|X_j(s)-X_i(s)| &\leq |X_j(t)-X_i(t)| + \int_s^t |V_j(u)-V_i(u)|\,du\\ &\leq C(W_\infty(t)+\tau) + 2\int_s^t (W_\infty(u)+R(u))\,du\\ &\leq C(W_\infty(t)+\tau).\end{aligned} \tag{4.21}$$ xclose

Hence we get for $s \in [t-\tau, t]$

$$\int_s^t |E_N(X_j(u))-E_N(X_i(u))|\,du \leq C\,\tau\,|\nabla^N E|_\infty\,(W_\infty(t)+\tau+\varepsilon^{1+r'}).$$

Note that the constant C still does not depend on $\tau = \varepsilon^{1+r'}$. Therefore provided that with the previous constant C

$$C\,\tau\,|\nabla^N E|_\infty\,(W_\infty(t)+\tau) \leq d_N(t)/4, \tag{4.22}$$ assJV

one has that

$$|V_j(s)-V_i(s)-(V_j(t)-V_i(t))| \leq d_N(t)/4 \quad \text{and also} \quad |V_i(s)-V_j(s)| \geq \frac{d_N(t)}{4}.$$

As in the step for B_t (See equation (4.14)) this implies the dispersion estimate $|X_j(s)-X_i(s)| \geq |s-s_0|\,d_N(t)/4$ for some $s_0 \in [t-\tau,\ t]$. As a consequence for $j \in JV_t$,

$$I_{Nc}(t,i,j) \leq \frac{C}{\tau}\,(d_N(t))^{-\alpha} \int_{t-\tau}^t \frac{ds}{|s-s_0|^\alpha} \leq C\,\tau^{-\alpha}\,(d_N(t))^{-\alpha}. \tag{4.23}$$ boundJV

Summing (4.20) and (4.23), one gets

$$\sum_{j\in\tilde{C}_t} a_{ij} I_{Nc}(t,i,j) \leq C\,|C_t|\ \left((d_N(t))^{-\alpha} + \tau^{-\alpha}\ (d_N(t))^{-\alpha}\right).$$

Coming back to I_{bc}, using the bound on the volume of $|C_t|$ and keeping only the largest term of the sum

$$\int_{C_t} I_{bc}(t,\bar{z}_t,z_t)\,dz_t \leq C\,(W_\infty(t)+\tau)^{2d}\tau^{-\alpha}\ (d_N(t))^{-\alpha}. \tag{4.24}$$ Ct

4.4.6 Conclusion of the proof of Lemmas 1, 2

Assumptions (4.19) and (4.22) are ensured by the hypothesis of the lemma. Summing up (4.6) for I_α or (4.7) for $J_{\alpha+1}$, with (4.8), (4.16) and (4.24), we indeed find the conclusion of the first lemma.

In the S_m^α case, no assumption is needed, and summing up the bounds (4.6), (4.8), (4.17), (4.18), we obtain the second lemma.

4.5 Conclusion of the proof of Theorem 3 (without cut-off)

In this subsection, in order to make the argument clearer, we number explicitly the constants. Let us summarize the important information of Prop. 4 and Lemma 1. Let us also rescale the interested quantities s.t. all may be of order 1

$$\varepsilon\, \tilde W_\infty(t) = W_\infty(t), \quad \varepsilon^{1+r}\, \tilde d_N(t) = d_N(t).$$

We also assume that $\tilde W_\infty$ and $|\nabla^N E|_\infty$ are non decreasing functions of t, and that $\tilde d_N$ is a non increasing function of t. In fact the bound proved before are also valid for $\sup_{s\le t} \tilde W_\infty$, $\sup_{s\le t} |\nabla^N E|_\infty$, and $\inf_{s\le t} \tilde d_N(s)$. With that convention $\tilde W_\infty(t) \ge \tilde W_\infty(0) \ge 1$. By assumption (i) in Theorem 3, also note that $\tilde d_N(0) \ge 1$.
Recalling $\tau = \varepsilon^{r'}$ (with $r' > r > 1$), the condition of Lemma 1 after rescaling reads

$$C_1\, \varepsilon^{r'-r}\, (1 + |\nabla^N E|_\infty(t))\, \tilde W_\infty(t) \le \tilde d_N(t), \quad \forall\, t \in [0,\ t_0]. \tag{4.25}$$

In Lemma 1, we proved that there exists some constants C_0 and C_1 independent of N (and hence ε), such that if (4.25) is satisfied, then for any $t \in [0,\ t_0]$

$$\begin{aligned} \tilde W_\infty(t) \quad &\le \tilde W_\infty(t-\tau) + C_0\, \varepsilon^{r'} \left(\tilde W_\infty(t) + \varepsilon^{\lambda_1}\, \tilde W^d_\infty(t) + \varepsilon^{\lambda_2}\, \tilde W^{2d}_\infty(t)\, \tilde d_N^{-\alpha}(t) \right), \\ |\nabla^N E|_\infty(t) \quad &\le C_2 \left(1 + \varepsilon^{\lambda_3}\, \tilde W^d_\infty(t) + \varepsilon^{\lambda_4}\, \tilde W^{2d}_\infty(t)\, \tilde d_N^{-\alpha}(t)) \right) \\ \tilde d_N(t) + \varepsilon^{r'-r} \quad &\ge [\tilde d_N(t-\tau) + \varepsilon^{r'-r}] e^{-\tau(1+|\nabla^N E|_\infty(t))}, \end{aligned}$$

where ε appear four times with four different exponents $\lambda_i, i = 1, \dots, 4$ defined by

$$\begin{aligned} \lambda_1 &= d - 1 - \alpha\, r', & \lambda_2 &= 2d - 1 - \alpha(1 + r' + r), \\ \lambda_3 &= d - 1 - r' - \alpha\, r', & \lambda_4 &= 2d - 1 - r' - \alpha(1 + r' + r). \end{aligned}$$

To propagate uniform bounds as $\varepsilon \to 0$ and $N \to \infty$, we need all λ_i to be positive. As $r, r' > 0$, it is clear that $\lambda_1 > \lambda_3$ and $\lambda_2 > \lambda_4$. Thus we need only check $\lambda_3 > 0$ and $\lambda_4 > 0$. As $r' > r$, it is sufficient to have

$$r' < \frac{d-1}{1+\alpha}, \quad \text{and} \quad r' < \frac{2d-1-\alpha}{1+2\,\alpha}.$$

Note that a straightforward calculation shows that

$$\frac{d-1}{1+\alpha} - \frac{2d-1-\alpha}{1+2\,\alpha} = \frac{\alpha^2 - d}{(1+\alpha)(1+2\alpha)} < 0,$$

so that the first inequality is the stronger one. Thanks to the condition given in Theorem 3, $r < r^* := \frac{d-1}{1+\alpha}$, so that if we choose any $r' \in (r, r^*)$, the corresponding λ_i are all positive. We fix a r' as above and denote $\lambda = \min_i(\lambda_i)$. Then by a rough estimate,

$$\begin{aligned} \tilde W_\infty(t) &\le \tilde W_\infty(t-\tau) + C_0\, \tau \left(\tilde W_\infty(t) + 2\, \varepsilon^{\lambda}\, \tilde W^{2d}_\infty(t)\, d_N^{-\alpha}(t) \right), \\ |\nabla^N E|_\infty(t) &\le C_2 \left(1 + 2\, \varepsilon^{\lambda}\, \tilde W^{2d}_\infty(t)\, \tilde d_N^{-\alpha}(t) \right), \\ \tilde d_N(t) &\ge [\, \tilde d_N(0) + \varepsilon^{r'-r}] e^{-t(1+|\nabla^N E|_\infty(t))} - \varepsilon^{r'-r}. \end{aligned} \tag{4.26}$$

If one has (4.25) and

$$2\, \varepsilon^{\lambda}\, \tilde W^{2d}_\infty(t)\, \tilde d_N^{-\alpha}(t) \le 1, \tag{4.27}$$

then using $W_\infty \geq 1$, we get $\tilde W_\infty(t) \leq \tilde W_\infty(t-\tau) + 2C_0\tau \tilde W_\infty(t)$ so that

$$\begin{aligned} \tilde W_\infty(t) &\leq \tilde W_\infty(t-\tau)(1-2C_0\tau)^{-1}, \\ |\nabla^N E|_\infty(t) &\leq 2\,C_2, \\ \tilde d_N(t) &\geq\ e^{-(1+2C_2)\,t} - \varepsilon^{r'-r}. \end{aligned} \tag{4.28}$$

The last inequality implies $\tilde d_N(t) \geq \frac12 e^{-(1+2C_2)\,t}$ if $2\varepsilon^{r-r'}e^{(1+2C_2)\,T} < 1$. That condition is fulfilled for ε small enough, i.e. N large enough ($\ln N \leq T$).
The first inequality in (4.28), iterated gives $W_\infty(t) \leq W_\infty(0)(1-2C_0\tau)^{-\frac{t}{\tau}}$. If $C_0\tau \leq \frac14$, then we can use $-\ln(1-x) \geq 2x$ for $x \in [0, \frac12]$, and get

$$\tilde W_\infty(t) \leq \tilde W_\infty(0) e^{4C_0 t}$$

To summarize

$$\begin{aligned} \tilde W_\infty(t) &\leq e^{4C_0 t}, \\ |\nabla^N E|_\infty(t) &\leq 2\,C_2, \\ \tilde d_N(t) &\geq \frac12\, e^{-(1+2C_2)\,t}. \end{aligned} \tag{4.29}$$

At the discrete level of the particles, the dynamics is continuous in time, at least for initial conditions not leading to collisions. That set is of full measure for $\alpha < 1$, and $d \geq 2$ (See [Hau04]). So as long as (4.25) and (4.27) are satisfied at $t = 0$, there exists a maximal time $t_0 \in]0,\ T]$ (possibly $t_0 = T$) such that they are satisfied on $[0,\ t_0]$.
We show that for N large enough, *i.e.* ε small enough, then one necessarily has $t_0 = T$. Then we will have (4.29) on $[0,\ T]$ which is the desired result. This is simple enough. By contradiction if $t_0 < T$ then

$$C_1\,\varepsilon^{(r-r')}\,(1 + |\nabla^N E|_\infty(t_0))\,\tilde W_\infty(t_0) = \tilde d_N(t_0), \quad \text{or } 4\ \varepsilon^\lambda\,\tilde W_\infty^{2d}(t_0)\,\tilde d_N^{-\alpha}(t_0) = 1.$$

Until t_0, (4.29) holds. Therefore

$$\varepsilon^\lambda\,\tilde W_\infty^{2d}(t_0)\,\tilde d_N^{-\alpha}(t_0) \leq \varepsilon^\lambda\,2^\alpha\,e^{(\alpha + (4d+2\alpha)\max(C_0,C_2))\,t_0} < 1,$$

for ε small enough with respect to T and the C_i. This is the same for (4.25),

$$C_1\,\varepsilon^{(r-r')}\,(1 + |\nabla^N E|_\infty(t_0))\,\tilde W_\infty(t_0)\tilde d_N^{-1}(t_0) \leq 2\varepsilon^{(r-r')}C_1(1+2C_2)e^{(1+6\max(C_0,C_2))t_0} < 1.$$

Hence we obtain a contradiction and prove Theorem 3.

4.6 Conclusion of the proof of Theorem 4 (cut-off case)

In the cut-off case, using Lemma 2 together with the inequality *ii*) of the Proposition 4, we may obtain

$$W_\infty(t) \leq W_\infty(t-\tau) + C_0 W_\infty(t)\left[1 + (W_\infty(t)+\tau)^{d-1}\tau^{-1}\varepsilon^{\bar m(1-\alpha)} + (W_\infty(t)+\tau)^{2d-1}\varepsilon^{-\bar m\alpha}\right].$$

We again rescale the quantity $W_\infty(t) = \varepsilon \tilde W_\infty(t)$ and replace $\tilde W_\infty(t)$ by $\tilde W_\infty(t) + 1$. Choosing in that case $\tau = \varepsilon$, it comes for $1 \leq \alpha < d-1$,

$$\tilde W_\infty(t) \leq \tilde W_\infty(t-\tau) + C_0\tilde W_\infty(t)\left[1 + \varepsilon^{d-2-\bar m(\alpha-1)}\,\tilde W_\infty^{d-1}(t) + \varepsilon^{2d-1-\bar m\alpha}\,\tilde W_\infty^{2d-1}(t)\right].$$

As in the previous section, we will get a good bound provided that the power of ε appearing in parenthesis are positive. The two conditions read

$$\bar{m} < \bar{m}^* := \min\left(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha}\right).$$

In that case, for N large enough (with respect to e^{Ct}), we get a control of the type

$$\frac{d}{dt}\tilde{W}_\infty(t) \leq 4C_0 \tilde{W}_\infty(t),$$

(but discrete in time) which gives the desired result.

Remark 2. *In the cut-off case (and also in the case without cut-off), it seems important to be able to say that the initial configurations Z we choose have a total energy close from the one of f^0. Because, if the empirical distribution μ_N^Z is close form f^0, but has a different total energy, we would not expect that they do not remains close a very long time.*
Fortunately, such a result is true and under the assumptions of Theorem 3 and 4, the total energy of the empirical distributions are close from the total energy of f^0.
Unfortunately, we do not have a simple proof of this fact. But, it can be done using the argument for the proof of the deterministic theorems. First, the difference between the kinetic energy is easily controlled because our solutions are compactly supported and that there is no singularity there. Next, performing calculations very similar to the ones done in the proofs, we can control the difference between a small average in time of the potential energies, on the small interval of time $[0,\tau]$. Then, we control the average of the total energy, which is constant.

A Appendix : Large deviation on the infinite norm of f_N.

Proposition 5. *Assume that ρ is a probability on $\mathbb{R}^n$ with support included in $[-R^0, R^0]^n$ and and bounded density $f(x)\,dx$. Let ϕ be a bounded cut-off function, with support in $[-\frac{L}{2}, \frac{L}{2}]^n$ and total mass one, and define the usual $\phi_\varepsilon := \frac{1}{\varepsilon^n}\phi(\frac{\cdot}{\varepsilon})$. For any configuration $Z_N = (Z_i)_{i\leq N}$ we define*

$$f_N^Z := \mu_N^Z * \phi_\varepsilon(N).$$

If $\varepsilon(N) = N^{\frac{\gamma}{n}}$ and the Z_N are distributed according to $f^{\otimes N}$, then we have the explicit "large deviations" bound with $c_\phi = (2L)^n\|\phi\|_\infty$ and $c_0 = (2R^0+2)^d L^{-n}$.

$$\mathbb{P}\left(\|f_N^Z\|_\infty \geq \beta c_\phi \|f\|_\infty\right) \leq c_0 N^\gamma e^{-(\beta\ln\beta-\beta+1)(2L)^n\|f\|_\infty N^{1-\gamma}}. \tag{A.1}$$

In particular, for $\phi = \mathbf{1}_{[-1/2,1/2]^n}$ and $\beta = 2$, we get

$$\mathbb{P}\left(\|f_N^Z\|_\infty \geq 2^{1+n}\|f\|_\infty\right) \leq (2R^0+2)^d N^\gamma e^{-(2\ln 2-1)2^n\|f\|_\infty N^{1-\gamma}}. \tag{A.2}$$

Proof. For any $Z \in \mathbb{R}^{nN}$ and $z \in \mathbb{R}^n$, we have

$$\begin{aligned}
f_N^Z(z) &= \frac{1}{N}\sum_{i=1}^N \phi_\varepsilon(z - Z_i) = \frac{1}{N\,\varepsilon^n}\sum_{i=1}^N \phi\left(\frac{z-Z_i}{\varepsilon}\right) \\
&\leq \frac{\|\phi\|_\infty}{N\,\varepsilon^n}\#\{i \text{ s.t. } |z-Z_i|_\infty \leq \tfrac{L\varepsilon}{2}\} \\
\|f_N^Z\|_\infty &\leq \frac{\|\phi\|_\infty}{N\,\varepsilon^n}\sup_{z\in\mathbb{R}^n}\#\{i \text{ s.t. } |z-Z_i|_\infty \leq \tfrac{L\varepsilon}{2}\},
\end{aligned}$$

where # stands for the cardinal (of a finite set). It remains to bound the supremum on all the cardinals. The first step will be to replace the sup on all the $z \in \mathbb{R}^n$ by a supremum on a finite number of point. For this, we cover $[-R^0, R^0]^n$ by M cubes C_k of size $L\varepsilon$, centered at the points $(c_k)_{k \le M}$. The number M of square needed depends on N via ε, and is bounded by

$$M \le \left[\frac{2(R^0+1)}{L\varepsilon} \right]^n.$$

Next, for any $z \in \mathbb{R}^d$, there exists a $k \le M$ such that $|z - c_k| \le \frac{L\varepsilon}{2}$. This implies that

$$\sup_{z \in \mathbb{R}^n} \#\{i \text{ s.t. } |z - Z_i|_\infty \le \tfrac{L\varepsilon}{2}\} \le \sup_{k \le M} \#\{i \text{ s.t. } |c_k - Z_i|_\infty \le L\varepsilon\}$$

Now we denote by $H^N_k := \#\{i \text{ s.t. } |c_k - Z_i|_\infty \le L\varepsilon\}$. H^N_k follows a binomial law $B(N, p_k)$ with $p_k = \int_{2C_k} f(z)\, dz$, where $2C_k$ denotes the square with center c_k, but size $2L\varepsilon$. Remark that

$$p_k \le \bar{p} := (2L\varepsilon)^n \|f\|_\infty.$$

For any λ, the exponential moments of H^N_k are therefore given and bounded by

$$\begin{aligned} \mathbb{E}(e^{\lambda H^N_k}) &= \left[1 + (e^\lambda - 1)p_k\right]^N \\ &\le \left[1 + (e^\lambda - 1)(2L\varepsilon)^n \|f\|_\infty\right]^N \\ &\le e^{(e^\lambda - 1)N(2L\varepsilon)^n \|f\|_\infty}. \end{aligned}$$

Now for the supremum of the H^N_k

$$\begin{aligned} \mathbb{E}(e^{\lambda \sup_k H^N_k}) &\le \mathbb{E}(e^{\lambda H^N_1}) + \cdots + (e^{\lambda H^N_M}) \\ &\le M e^{(e^\lambda - 1)N(2L\varepsilon)^n \|f\|_\infty} \\ &\le \left[\frac{2(R^0+1)}{L\varepsilon} \right]^n e^{(e^\lambda - 1)N(2L\varepsilon)^n \|f\|_\infty} \end{aligned}$$

Using finally Chebyshev's inequality, we get for any $\beta > 0$

$$\begin{aligned} \mathbb{P}\left(\|f^Z_N\|_\infty \ge \beta(2L)^n \|\phi\|_\infty \|f\|_\infty\right) &\le \mathbb{P}\left(\sup_k H^N_k \ge \beta \|f\|_\infty N(2L\varepsilon)^n\right) \\ &\le \mathbb{E}(e^{\lambda \sup_k H^N_k}) e^{-\lambda\beta \|f\|_\infty N(2L\varepsilon)^n} \\ &\le \left[\frac{2(R^0+1)}{L\varepsilon} \right]^n e^{\left(e^\lambda - 1 - \lambda\beta\right) N(2L\varepsilon)^n \|f\|_\infty}. \end{aligned}$$

For $\beta > 1$, the optimal λ is $\ln \beta$ and we get with $c_\phi = (2L)^n \|\phi\|_\infty$

$$\mathbb{P}\left(\|f^Z_N\|_\infty \ge \beta c_\phi \|f\|_\infty\right) \le \left[\frac{2(R^0+1)}{L\varepsilon} \right]^n e^{-(\beta \ln\beta - \beta + 1)N(2L\varepsilon)^n \|f\|_\infty}.$$

With the scaling $\varepsilon(N) = N^{-\frac{\gamma}{n}}$, we get

$$\mathbb{P}\left(\|f^Z_N\|_\infty \ge \beta c_\phi \|f\|_\infty\right) \le c_f N^\gamma e^{-(\beta \ln\beta - \beta + 1)(2L)^n \|f\|_\infty N^{1-\gamma}}.$$

Remark finally that the choice of scale $\varepsilon(N) = (\ln N) N^{-\frac{1}{n}}$ is also sufficient to get a probability vanishing faster than any inverse power. $\square$

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