Martin representation, Relative Fatou Theorem and Hardy spaces for fractional Laplacian with a gradient perturbation

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7 février 2012

Abstract

Let $L = \Delta^{\alpha/2} + b \cdot \nabla$ with $\alpha \in (1,2)$. We prove several properties of singular L-harmonic functions on a $\mathcal{C}^{1,1}$ bounded open set D : the Martin representation, the Relative Fatou Theorem and the representation theorem for Hardy spaces. [1]

1. This research was partially supported by ANR-09-BLAN-0084-01 and grants MNiSW N N201 397137, MNiSW N N201 422539.

Keywords: gradient perturbation, fractional Laplacian, Ornstein-Uhlenbeck stable process, Martin representation, Relative Fatou Theorem, Hardy spaces

MSC: 60J50, 60J75, 60J45, 42B30, 31B25, 30H10

1 Introduction and Preliminaries

Stable stochastic processes with gradient perturbations on $\mathbb{R}^d$, $d \geq 2$, i.e. with the infinitesimal generator

$$L = \Delta^{\alpha/2} + b \cdot \nabla, \tag{1}$$

where $\alpha \in (0, 2)$, constitute an important class of jump processes, intensely studied in recent years. Their most celebrated case are the Ornstein-Uhlenbeck stable processes with $b(x) = \lambda x$, $\lambda \in \mathbb{R}$. They have important physical and financial applications and form a part of Lévy-driven Ornstein-Uhlenbeck processes, cf. [10, 33]. The motivations of this paper were to:

(i) establish the theory of the Martin representation for singular L-harmonic non-negative functions
(ii) study boundary limit properties of L-harmonic functions and to obtain a Relative Fatou Theorem for them
(iii) develop the theory of Hardy spaces of L-harmonic functions.

All these topics are fundamental for the knowledge of L-harmonic functions. The topics (i), (ii) and (iii) are well developed for fractional Laplacians. The Martin representation was established in this case in [3], [32] and [8]. The Relative Fatou Theorem was proved for α-harmonic functions on Lipschitz domains in [30], see also the survey [4, Chapter 3] and [22]. Also some important variants of stable processes such as relativistic and truncated stable processes were studied from the point of view of topics (i) and (ii), see [4, Section 3.4], [23] and [25]. Nevertheless, the methods of these extensions do not apply to the operator L of the form (1). The theory of Hardy spaces for α-harmonic functions was developed in [31] and recently in [5, 29]. It was never discussed for any versions of stable processes. Let us notice that all our results are also true for Ornstein-Uhlenbeck stable processes.

On the other hand, in the case of diffusion operators, Hardy spaces and Fatou-type theorems for harmonic functions were widely studied in the literature, see [15, 34, 35, 2, 37, 14, 21, 26, 1] for the results on the classical Laplacian $\Delta = \sum_{i=1}^d \partial^2/\partial x_i^2$, and [36, 28] for the Laplacian with various perturbations.

Let us mention that the methods of this article give also interesting new results for operators different from L. In the case of Laplacians with gradient perturbations, i.e. $\alpha = 2$, we get new perturbation formulas for the Green function and the Martin and Poisson kernels (Section 3.4). For relativistic α-stable processes we have a representation theorem for Hardy spaces, see Remark 2.

The potential theory of stable stochastic processes with gradient perturbations was started in the Ornstein–Uhlenbeck case by T. Jakubowski [19, 20]. Next, in the general context of gradient perturbations and $\alpha > 1$, with a function b from the Kato class $\mathcal{K}_d^{\alpha-1}$ it was developed by K. Bogdan and T. Jakubowski [6, 7]. This work is a natural continuation of the research presented in [7].

In particular, we send the reader to [7] for the definitions of the fractional Laplacian, $\mathcal{C}^{1,1}$ domains, Green functions and Poisson kernels, with respect to both operators $\Delta^{\alpha/2}$ and L. The definitions of α-harmonic, regular α-harmonic and singular α-harmonic functions can be found e.g. in the monography [4, page 61] and are analogous for L-harmonic functions.

Throughout this paper, like in [7], we suppose $1 < \alpha < 2$, unless stated otherwise. We consider an open set D of class $\mathcal{C}^{1,1}$ and a vector field $b \in \mathcal{K}_d^{\alpha-1}$ on $\mathbb{R}^d$ i.e.

$$\lim_{\epsilon\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-z|<\epsilon} |b(z)||x-z|^{\alpha-1-d} dz = 0.$$

The potential theory objects related to the operator L defined in (1) will be denoted with a tilde $\tilde{}$, while those related to the operator $\Delta^{\alpha/2}$ will be denoted without it. In particular $\tilde{G}_D$ is the Green function of L on D and G_D is the Green function of $\Delta^{\alpha/2}$ on D. We fix throughout this paper a point $x_0 \in D$ and define the Martin kernel of D by

$$M_D(x,Q) = \lim_{y\to Q} \frac{G_D(x,y)}{G_D(x_0,y)}, \quad x \in D, Q \in \partial D.$$

The L-Martin kernel is defined by

$$\tilde{M}_D(x,Q) = \lim_{y\to Q} \frac{\tilde{G}_D(x,y)}{\tilde{G}_D(x_0,y)}, \quad x \in D, Q \in \partial D$$

and we show in Section 3 its existence.

The starting point of the research contained in this paper are the following mutual estimates of Green functions and Poisson kernels for L and $\Delta^{\alpha/2}$ (see [7, Theorem 1 and (72)]).

Comparability Theorem. *There exists a constant $C = C(\alpha, b, D)$ such that for all $x, y \in D$ and $z \in (\overline{D})^c$,*

$$C^{-1} G_D(x,y) \le \tilde{G}_D(x,y) \le C G_D(x,y), \tag{2}$$

$$C^{-1} P_D(x,z) \le \tilde{P}_D(x,z) \le C P_D(x,z). \tag{3}$$

One of the main elements of the proof of (2) is the following perturbation formula, that will be also very useful in our present work.

Perturbation formula for Green functions. *[7, Lemma 12] Let $x, y \in \mathbb{R}^d, x \neq y$. We have*

$$\tilde{G}_D(x,y) = G_D(x,y) + \int_D \tilde{G}_D(x,z) b(z) \cdot \nabla_z G_D(z,y) dz. \tag{4}$$

We start our paper by proving in Section 2.1 a generalization of the Comparability Theorem: according to Lemma 1, the constant C in the estimates (2) may be chosen the same for sets D_r sufficiently close to D. The same phenomenon holds also for the Poisson kernels $\tilde{P}_D(x,y)$ and $P_D(x,y)$. In Section 2 we also prove the α-harmonicity of the first partial derivatives of the Green function $G_D(\cdot, z)$ and a uniform integrability result, that will be needed in proving the main results of the paper, contained in Sections 3 and 4.

In Section 3 we develop the Martin theory of L-harmonic functions. We prove the existence of the L-Martin kernel which is L-harmonic (Theorems 8 and 12). Next we obtain the Martin representation of singular L-harmonic non-negative functions on D, see Theorem 13.

The formula (4) allows us to prove very useful perturbation formulas for Martin kernels (17), Poisson kernels (20) and singular α-harmonic functions (31). In Section 3.4, (4) and (17) are proved in the diffusion case $\alpha = 2$. Also a perturbation formula (36) for the L-Poisson kernel is derived.

Section 4 is devoted to important fine boundary properties of singular L-harmonic-functions. First we prove the Relative Fatou Theorem (Theorem 23). We provide a direct proof of it based on the perturbation formula for singular α-harmonic functions (31). An alternative proof based on comparability of L-harmonic and α-harmonic functions(Corollary 15) is also given.

Next we establish the representation theorem of Hardy spaces of L-harmonic functions. The results on the L-Hardy spaces constitute a stable counterpart for the results on the diffusions with a gradient perturbation proved by T. Luks in [28]. By the same methods one can prove the representation theorem of Hardy spaces of relativistic α-harmonic functions (Remark 2).

2 Preparatory results

In this section we prove some results, interesting independently, that will be useful in proving the main results of the paper, coming in the next sections.

2.1 Uniform comparability of Green functions and Poisson kernels

In what follows, $\mathbb{R}^d$ denotes the Euclidean space of dimension $d \geq 2$, dy stands for the Lebesgue measure on $\mathbb{R}^d$. Without further mention we will only consider Borelian sets, measures and functions in $\mathbb{R}^d$. By $x \cdot y$ we denote the Euclidean scalar product of $x, y \in \mathbb{R}^d$. Writing $f \approx g$ we mean that there is a constant $C > 0$ such that $C^{-1}g \leq f \leq Cg$. As usual, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. We let $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$. For $U \subset \mathbb{R}^d$ we denote

$$\delta_U(x) = \operatorname{dist}(x, U^c)\,,$$

the distance to the complement of U.

In what follows D is a bounded $C^{1,1}$ domain.

For $r \geq 0$ define

$$D_r = \{x \in D : \delta_D(x) > r\}\,.$$

When r is sufficiently small, then D_r is also a $C^{1,1}$ domain, see [31, Lemma 5].

In the sequel we will often use the estimates of the Green function ([27], [11], see also [18]) of a $C^{1,1}$ domain

$$G_D(y, z) \approx |y - z|^{\alpha - d} \left(\frac{\delta_D(y)^{\alpha/2} \delta_D(z)^{\alpha/2}}{|y - z|^{\alpha}} \wedge 1 \right), \tag{5}$$

and of the Martin kernel ([12])

$$M_D(x, Q) \approx \frac{\delta_D(x)^{\alpha/2}}{|x - Q|^d}. \tag{6}$$

Moreover, in the stable case, the estimates (5) are uniform when we consider the sets D_r sufficiently close to D, i.e. there exist constants $c, r_0 > 0$ depending only on D and α such that for all $r \in [0, r_0]$ and $x, y \in D_r$ we have

$$c^{-1}|y-z|^{\alpha-d}\left(\frac{\delta_{D_r}(y)^{\alpha/2}\delta_{D_r}(z)^{\alpha/2}}{|y-z|^{\alpha}}\wedge 1\right) \leq G_{D_r}(y,z) \leq c|y-z|^{\alpha-d}\left(\frac{\delta_{D_r}(y)^{\alpha/2}\delta_{D_r}(z)^{\alpha/2}}{|y-z|^{\alpha}}\wedge 1\right). \tag{7}$$

We will now show analogous uniformity of constants for the processes with a gradient perturbation.

Lemma 1. *(i) There exist constants $c, r_0 > 0$ depending only on D and α such that for all $r \in [0, r_0]$ and $x, y \in D_r$ we have*

$$c^{-1}G_{D_r}(x,y) \leq \tilde{G}_{D_r}(x,y) \leq cG_{D_r}(x,y).$$

(ii)There exist constants $C, r_0 > 0$ depending only on D and α such that for all $r \in [0, r_0]$, $x \in D_r$ and $y \in D_r^c$ we have

$$C^{-1}P_{D_r}(x,y) \leq \tilde{P}_{D_r}(x,y) \leq CP_{D_r}(x,y).$$

Proof. In order to show (i) we follow the proof of the Theorem 1 in [7]. We analyse below its crucial points.

1. *Comparison of Green functions $\tilde{G}_S(x,y)$ and $G_S(x,y)$ for "small" sets S*, [7, Lemma 13], based on estimates from [7, Lemma 11]. Thanks to property (7), we see that the comparison of Green functions $\tilde{G}_{S_r}(x,y)$ and $G_{S_r}(x,y)$ for small sets S holds with a common constant c, when $r \in [0, r_0]$.

2. *Harnack inequalities for L and the Boundary Harnack Principle*, [7, Lemmas 15, 16]. Thanks to 1., we get them uniformly with respect to $r \in [0, r_0]$.

3. Now the proof of (i) for any $C^{1,1}$ domain D is the same as in Section 5 of [7].

The part (ii) is implied by (i), applying the Ikeda-Watanabe formula for the Poisson kernel $\tilde{P}_D$, see [7, Lemma 6 and (39)]. Recall that the Lévy system for the process $\tilde{X}_t$ is given by the Lévy measure of the α-stable process X_t. □

2.2 Derivatives of the Poisson kernel and of the Green function for $\Delta^{\alpha/2}$

In this section we prove useful gradient estimates for the Poisson kernel of α-stable processes, $0 < \alpha < 2$. Next, for $1 < \alpha < 2$, we study the α-harmonicity of the Green function.

Consider a ball $B = B(\xi_0, r) \subset \overline{B} \subset D$ and let P_B be the Poisson kernel of B

$$P_B(x,y) = C_\alpha^d \left[\frac{r^2 - |x-\xi_0|^2}{|y-\xi_0|^2 - r^2} \right]^{\alpha/2} |x-y|^{-d}, \quad x \in B, y \in \bar{B}^c \tag{8}$$

and equal to 0 elsewhere. By [9, Lemma 3.1],

$$|\nabla_x P_B(x,y)| \le (d+\alpha) \frac{P_B(x,y)}{r - |x-\xi_0|}, \qquad x \in B, y \in (\overline{B})^c. \tag{9}$$

We will now show analogous estimate for arbitrary open sets.

Lemma 2. *Suppose $0 < \alpha < 2$. Let U be an arbitrary open set in $\mathbb{R}^d$ and let $P_U(x,y)$ be the Poisson kernel of $\Delta^{\alpha/2}$ for U. Then we have*

$$|\nabla_x P_U(x,y)| \le (\alpha + d) \frac{P_U(x,y)}{\delta_U(x)}, \quad x \in U, y \in (\overline{U})^c. \tag{10}$$

Proof. For $x \in U$ denote $B_x = B(x, \delta_U(x))$. In view of [8, (29)] we have

$$P_U(x,y) = P_{B_x}(x,y) + \int_{B_x^c} P_{B_x}(x,z) P_U(z,y) dz.$$

By (9) and bounded convergence we have

$$|\nabla_x P_U(x,y)| \le |\nabla_x P_{B_x}(x,y)| + |\nabla_x \int_{B_x^c} P_{B_x}(x,z) P_U(z,y) dz|$$

$$\le (d+\alpha) \frac{P_{B_x}(x,y)}{\delta_U(x)} + \int_{B_x^c} |\nabla_x P_{B_x}(x,z)| P_U(z,y) dz \le (d+\alpha) \frac{P_U(x,y)}{\delta_U(x)}.$$

□

From (9) and the dominated convergence theorem it follows, that if f is α-harmonic in D then

$$\frac{\partial}{\partial x_i} f(x) = \int_{B^c} \frac{\partial}{\partial x_i} P_B(x,y) f(y)\, dy, \qquad i = 1, \ldots, d. \tag{11}$$

The esitmate (9) and (11) gives ([9, Lemma 3.2])

Lemma 3. *Let U be an arbitrary open set in $\mathbb{R}^d$ and let $\alpha \in (0,2)$. For every nonnegative function u on $\mathbb{R}^d$ which is α-harmonic in U, we have*

$$|\nabla u(x)| \le d \frac{u(x)}{\delta_U(x)}, \quad x \in U. \tag{12}$$

Since $G_U(\cdot, y)$ is α-harmonic in $U \setminus \{y\}$, for every $y \in U$ we obtain

$$|\nabla_x G_U(x,y)| \le d \frac{G_U(x,y)}{\delta_U(x) \wedge |x-y|}, \quad x, y \in U,\ x \ne y. \tag{13}$$

In the next proposition we will prove that the derivatives of G_D are also α-harmonic.

Proposition 4. *Suppose that $\alpha \in (1,2)$.*

(a) Fix $z \in D$. Then the partial derivative $\frac{\partial}{\partial x_j} G_D(x,z)$ is singular α-harmonic in x on $D \setminus \{z\}$.

(b) Fix $x \in D$. Then the partial derivative $\frac{\partial}{\partial x_j} G_D(x,z)$ is regular α-harmonic in z on $D \setminus \overline{B(x,r)}$ for every $r > 0$.

Proof. (a) Let $x \in D \setminus \{z\}$. Consider a ball $B = B(\xi_0, r) \subset \bar{B} \subset D \setminus \{z\}$ and let P_B be the Poisson kernel of B (see (8). The function $x \to G_D(x,z)$ is α-harmonic on $D \setminus \{z\}$ containing $\bar{B}$. There exists $s_0 > 0$ such that for $s \in \mathbb{R}$ satisfying $0 < |s| < s_0$

$$\begin{aligned} G_D(x+se_j, z) &= \int_{B(\xi_0+se_j,r)^c} P_{B(\xi_0+se_j,r)}(x+se_j, y) G_D(y,z) dy \\ &= \int_{B(\xi_0+se_j,r)^c} P_{B(\xi_0,r)}(x, y-se_j) G_D(y,z) dy \\ &= \int_{B(\xi_0,r)^c \setminus (z+\mathbb{R}e_j)} P_{B(\xi_0,r)}(x,y) G_D(y+se_j, z) dy \end{aligned}$$

and consequently,

$$\frac{G_D(x+se_j,z)-G_D(x,z)}{s}=\int_{B^c\setminus(z+\mathbb{R}e_j)}P_B(x,y)\frac{G_D(y+se_j,z)-G_D(y,z)}{s}dy. \tag{14}$$

Now we take limit when $s\to 0$ in the formula (14). In order to justify the passage with the limit under the integral on the right-hand side of (14) we use (5) and (13).

Let $0<s<s_0$ be fixed. Choose $\delta>0$ such that $B(z,\delta)\subset D_{2s_0}$ and $B(z,2\delta)\cap B(\xi_0,r)=\emptyset$. For $y\in D\cap(B(\xi_0,r)^c\setminus(z+\mathbb{R}e_j))$ we consider the following cases.

1. $y\in D_{2s}$ and $|y-z|\geq\delta$. By the gradient estimates and (5) we have

$$\left|\frac{G_D(y+se_j,z)-G_D(y,z)}{s}\right|\leq\frac{1}{s}\int_0^s|\nabla G_D(y+te_j,z)|dt$$
$$\leq\frac{C}{s}\int_0^s\delta_D(y+te_j)^{\alpha/2-1}dt\leq C(\delta_D(y)-s)^{\alpha/2-1}\leq C(\delta_D(y)/2)^{\alpha/2-1}.$$

2. $y\in D\setminus D_{2s}$, so $|y-z|\geq\delta$. Then $\delta_D(y)\leq 2s$ and we get

$$\frac{G_D(y+se_j,z)}{s}\leq C\frac{\delta_D(y+se_j)^{\alpha/2}}{s}\leq C\frac{(\delta_D(y)+s)^{\alpha/2}}{s}$$
$$\leq C3^{\alpha/2}s^{\alpha/2-1}\leq C\delta_D(y)^{\alpha/2-1},$$

   and similarly $\frac{G_D(y,z)}{s}\leq C\delta_D(y)^{\alpha/2-1}$.

3. $y\in B(z,\delta)$ and $|y-z|>2s$. We have

$$\left|\frac{G_D(y+se_j,z)-G_D(y,z)}{s}\right|\leq\frac{1}{s}\int_0^s|\nabla G_D(y+te_j,z)|dt$$
$$\leq\frac{C}{s}\int_0^s|y+te_j-z|^{\alpha-1-d}dt\leq C(|y-z|-s)^{\alpha-1-d}\leq C|y-z|^{\alpha-1-d}.$$

4. $y\in B(z,\delta)$ and $|y-z|\leq 2s$. Then

$$\frac{G_D(y+se_j,z)}{s}\leq C|y+se_j-z|^{\alpha-d}s^{-1}\leq C|y+se_j-z|^{\alpha-d}|y-z|^{-1}$$
$$\leq C|y+se_j-z|^{\alpha-1-d}+C|y-z|^{\alpha-1-d},$$

   and $\frac{G_D(y,z)}{s}\leq C|y-z|^{\alpha-1-d}$.

We remark that in all cases the constants do not depend on $s \in (0, s_0)$ and $y \in D \cap (B(\xi_0, r)^c \setminus (z + \mathbb{R}e_j))$. The argument is identical for $s \in (-s_0, 0)$. We finally obtain

$$\left| \frac{G_D(y + se_j, z) - G_D(y, z)}{s} \right| \leq C(\delta_D(y)^{\alpha/2-1} + |y + se_j - z|^{\alpha-1-d} + |y - z|^{\alpha-1-d}),$$

and the last term is uniformly in s integrable against $P_{B(\xi_0,r)}(x, y)dy$ when $1 < \alpha < 2$. It now follows from (14) that

$$\frac{\partial}{\partial x_j} G_D(x, z) = \int_{B^c} P_B(x, y) \frac{\partial}{\partial y_j} G_D(y, z)dy,$$

so the function $x \to \frac{\partial}{\partial x_j} G_D(x, z)$, $x \in D$, is singular α-harmonic on $D \setminus \{z\}$.

(b) It suffices to consider the case $\overline{B(x, r)} \subset D$. We use the fact that the function $G_D(x, z)$ is regular α-harmonic in z on $V = D \setminus \overline{B(x, r)}$, see e.g. [3, p. 231]. Thus, for $z \in V = D \setminus \overline{B(x, r)}$

$$G_D(x, z) = \int_{V^c} P_V(z, w)G_D(x, w)dw = \int_{B(x,r)} P_V(z, w)G_D(x, w)dw.$$

It follows that

$$\begin{aligned} \frac{G_D(x + se_j, z) - G_D(x, z)}{s} &= \\ = \int_{B(x,r)\setminus(x+\mathbb{R}e_j)} & P_V(z, w) \frac{G_D(x + se_j, w) - G_D(x, w)}{s} dw. \end{aligned} \tag{15}$$

Let $0 < |s| < r/2$ and $w \in B(x, r) \setminus (x + \mathbb{R}e_j)$. We estimate the quotient $\frac{G_D(x+se_j,w)-G_D(x,w)}{s}$, similarly to the proof of the part (a)3. and 4., separately for $|w - x| > 2|s|$ and $|w - x| \leq 2|s|$. We obtain

$$\left| \frac{G_D(x + se_j, w) - G_D(x, w)}{s} \right| \leq C(|x + se_j - w|^{\alpha-1-d} + |x - w|^{\alpha-1-d}).$$

The last term is uniformly in s integrable against $P_V(z, w)dw$ on $B(x, r)$. By (15) we conclude that

$$\frac{\partial}{\partial x_j} G_D(x, z) = \int_{V^c} P_V(z, w) \frac{\partial}{\partial x_j} G_D(x, w)dw.$$

□

Remark 1. *It would be interesting to discuss the question of α-harmonicity of $\frac{\partial f}{\partial x_j}$ for any non-negative function f singular α-harmonic on D. Already the example $f(x) = M_B(x,Q)$, $Q \in \partial B$ and B is a ball, shows that the α-harmonicity of $\frac{\partial f}{\partial x_j}$ must be redefined when $\frac{\partial f}{\partial x_j}$ is a distribution.*

2.3 A uniform integrability result

One of the important results of [7] is

Lemma 5. *$G_D(y,w)/[\delta(w) \wedge |y-w|]$ is uniformly in y integrable against $|b(w)|dw$.*

In the next lemma we will show a similar property for the family of functions $G_{D_{2^{-n}}}(x,w)M_D(w,Q)\delta_{D_{2^{-n}}}(w)^{-1}$.

Lemma 6. *Let $x \in D$ be fixed. There exists $N = N(D,x) \in \mathbb{N}$ such that the functions*

$$G_{D_{2^{-n}}}(x,w)M_D(w,Q)\delta_{D_{2^{-n}}}(w)^{-1}$$

are uniformly in $Q \in \partial D$ and $n > N$ integrable against $|b(w)|dw$.

Proof. In view of the properties of D and of the estimates of $G_{D_{2^{-n}}}(x,w)$ and $M_D(w,Q)$, we can choose $N = N(D,x) \in \mathbb{N}$ sufficiently large, such that for all $n > N$ we have

$$\frac{G_{D_{2^{-n}}}(x,w)M_D(w,Q)}{\delta_{D_{2^{-n}}}(w)} \le \frac{c\mathbf{1}_{D_{2^{-n}}}(w)\delta_D(w)^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w-Q|^d|w-x|^{d-\alpha}}$$

$$\le \tilde{c}\mathbf{1}_{D_{2^{-n}}}(w)\left(|w-x|^{\alpha-d} + \frac{\delta_D(w)^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w-Q|^d}\right),$$

where c and $\tilde{c}$ depends only on D, α and x. The first term in the parentheses is integrable against $|b(w)|dw$ independently of Q, n, so we only need to consider the second one. For $w \in D_{2^{-N}}$ and $Q \in \partial D$ we have

$$\frac{\delta_D(w)^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w-Q|^d} \le \operatorname{diam}(D)^{\alpha/2}2^{(N+1)(d+1-\alpha/2)},$$

and for $w \in D_{2^{-n}} \setminus D_{2^{-N}}$, $Q \in \partial D$ we get

$$\frac{\delta_D(w)^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w-Q|^d} = \frac{(\delta_{D_{2^{-n}}}(w) + 2^{-n})^{\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w-Q|^d}$$

$$\leq 2^{\alpha/2}\left(|w-Q|^{\alpha-d-1}+\frac{2^{-n\alpha/2}}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w-Q|^d}\right).$$

Since $\mathbf{1}_{D_{2^{-n}}}(w)|w-Q|^{\alpha-d-1}$ is uniformly in Q,n integrable against $|b(w)|dw$, we can restrict our attention to the function

$$H_n(w,Q)=\frac{2^{-n\alpha/2}\mathbf{1}_{D_{2^{-n}}}(w)}{\delta_{D_{2^{-n}}}(w)^{1-\alpha/2}|w-Q|^d}$$

Let $R>N$, $R\in\mathbb{N}$. For $k,m,n\in\mathbb{N}$, $k\geq R$, $m\geq N$, $n>N$ we define

$$W^n_{k,m}(Q,R)=\left\{w\in D_{2^{-n}}:\frac{1}{2^{k+1}}<\delta_{D_{2^{-n}}}(w)\leq\frac{1}{2^k},\quad \frac{1}{2^{m+1}}<|w-Q|\leq\frac{1}{2^m}\right\},$$

$$W^n_k(Q,R)=\left\{w\in D_{2^{-n}}:\frac{1}{2^{k+1}}<\delta_{D_{2^{-n}}}(w)\leq\frac{1}{2^k},\quad |w-Q|>\frac{1}{2^N}\right\}.$$

We note that $W^n_{k,m}(Q,R)=\emptyset$ for $k<m$ or $m\geq n$. $W^n_{k,m}(Q,R)$ can be covered by $c_1(2^{k-m})^{d-1}$ balls of radii 2^{-k}, where $c_1=c_1(D)$. For $r>0$ denote

$$K_r=\sup_{z\in\mathbb{R}^d}\int_{B(z,r)}|b(w)||z-w|^{\alpha-d-1}dw.$$

Then $K_r\to 0$ as $r\downarrow 0$. We have

$$\int_{W^n_{k,m}(Q,R)}H_n(w,Q)|b(w)|dw$$

$$\leq (2^{k+1})^{1-\alpha/2}(2^{m+1})^d 2^{-n\alpha/2}c_1(2^{k-m})^{d-1}\sup_{z\in D}\int_{B(z,2^{-k})}|b(w)|dw$$
$$\leq (2^{k+1})^{1-\alpha/2}(2^{m+1})^d 2^{-n\alpha/2}c_1(2^{k-m})^{d-1}(2^k)^{\alpha-d-1}K_{2^{-k}}$$
$$\leq c_2K_{2^{-R}}(2^k)^{\alpha/2-1}2^m2^{-n\alpha/2},$$

where $c_2=c_2(D,b,\alpha)$. Furthermore, $W^n_k(Q,R)$ can be covered by $c_3(2^k)^{d-1}$ balls of radii 2^{-k}, where $c_3=c_3(D)$, and thus

$$\int_{W^n_k(Q,R)}H_n(w,Q)|b(w)|dw$$

$$\leq (2^{k+1})^{1-\alpha/2}2^{Nd}2^{-n\alpha/2}c_3(2^k)^{d-1}\sup_{z\in D}\int_{B(z,2^{-k})}|b(w)|dw$$

$$\leq (2^{k+1})^{1-\alpha/2} 2^{Nd} 2^{-n\alpha/2} c_3 (2^k)^{d-1} (2^k)^{\alpha-d-1} K_{2^{-k}}$$
$$\leq c_4 K_{2^{-R}} (2^k)^{\alpha/2-1} 2^{-n\alpha/2},$$

where $c_4 = c_4(D, b, \alpha)$. Let $A_R^n = \{w \in D_{2^{-n}} : \delta_{D_{2^{-n}}}(w) \leq 2^{-R}\}$. Then

$$A_R^n = \sum_{k=R}^{\infty} W_k^n(Q, R) + \sum_{m=N}^{n-1} \sum_{k=R\vee m}^{\infty} W_{k,m}^n(Q, R),$$

and we obtain

$$\int_{A_R^n} H_n(w, Q)|b(w)|dw$$
$$\leq c_4 K_{2^{-R}} 2^{-n\alpha/2} \sum_{k=R}^{\infty} (2^k)^{\alpha/2-1} + \sum_{m=N}^{n-1} \sum_{k=R\vee m}^{\infty} c_2 K_{2^{-R}} (2^k)^{\alpha/2-1} 2^m 2^{-n\alpha/2}$$
$$\leq c_5 K_{2^{-R}} \left(2^{-n\alpha/2} (2^R)^{\alpha/2-1} + \sum_{m=N}^{n-1} (2^{n-m})^{-\alpha/2} \right) \leq c_6 K_{2^{-R}},$$

where $c_6 = c_6(D, b, \alpha)$. For $w \in D_{2^{-n}} \setminus A_R^n$ we have

$$H_n(w, Q) < 2^{-n\alpha/2} (2^R)^{1-\alpha/2} \left(\frac{1}{2^R} + \frac{1}{2^n} \right)^{-d} < 4^{dR},$$

so $B_R^n(Q) := \{w : H_n(w, Q) > 4^{dR}\} \subset A_R^n$ for all $Q \in \partial D$ and $n > N$. Therefore

$$\lim_{R\to\infty} \sup_{Q\in\partial D, n>N} \int_{B_R^n(Q)} H_n(w, Q)|b(w)|dw \leq \lim_{R\to\infty} c_6 K_{2^{-R}} = 0.$$

□

3 Martin kernel and Martin representation

3.1 Existence and Perturbation formula for the L-Martin kernel

Lemma 7. *For all $x \in D$ and $Q \in \partial D$ we have*

$$\lim_{y\to Q} \frac{\nabla_x G_D(x, y)}{G_D(x_0, y)} = \nabla_x M_D(x, Q).$$

Proof. Let $z \in D, Q \in \partial D$ and choose $r > 0$ such that $\overline{B(z,r)} \subset D$ and $B(z,r) \cap B(Q,r) = \emptyset$. Since $G_D(\cdot, y)$ is α-harmonic in $B(z,r)$ for $y \in B(Q,r) \cap D$, by (11), we have

$$\begin{aligned}\frac{\nabla_x G_D(x,y)}{G_D(x_0,y)} &= \nabla_x \int_{B(z,r)^c} P_{B(z,r)}(x,w)\frac{G_D(w,y)}{G_D(x_0,y)}dw \\ &= \int_{B(z,r)^c} \nabla_x P_{B(z,r)}(x,w)\frac{G_D(w,y)}{G_D(x_0,y)}dw, \qquad x \in B(z,r).\end{aligned}$$

Furthermore, by (9) and (5),

$$|\nabla_x P_{B(z,r)}(x,w)|\frac{G_D(w,y)}{G_D(x_0,y)} \le C\frac{P_{B(z,r)}(x,w)}{r-|x-z|}\frac{G_D(w,y)}{\delta_D(y)^{\alpha/2}}.$$

We now use the estimate [7, (25)] and by considering the cases $\delta_D(w) > |w-y|$ and $\delta_D(w) \le |w-y|$ we get $\frac{G_D(w,y)}{\delta_D(y)^{\alpha/2}} \le C|w-y|^{\alpha/2-d}$. Hence the last term is uniformly in $y \in B(Q,r/2) \cap D$ integrable against dw, and thus

$$\lim_{y\to Q}\int_{B(z,r)^c} \nabla_x P_{B(z,r)}(x,w)\frac{G_D(w,y)}{G_D(x_0,y)}dw$$

$$= \int_{B(z,r)^c} \nabla_x P_{B(z,r)}(x,w)M_D(w,Q)dw = \nabla_x M_D(x,Q).$$

The last equality follows from (11) and the α-harmonicity of the Martin kernel.

Alternative proof. Lemma 7 can be also justified by using the classical theorem of the derivation of function sequences. As

$$\lim_{y\to Q}\frac{G_D(z,y)}{G_D(x_0,y)} = M_D(z,Q),$$

we need to show that the functions $\dfrac{\nabla G_D(z,y)}{G_D(x_0,y)}$ converge locally uniformly w.r. to z when $y \to Q$. We will show that they satisfy the Cauchy condition locally uniformly in z, i.e. for any $z_0 \in D$ and $\epsilon > 0$ there exist $\eta > 0$ and $\delta > 0$ such that

$$\sup_{z\in B(z_0,\eta)}\ \sup_{y,y'\in B(Q,\delta)\cap D}\left|\frac{\nabla G_D(z,y)}{G_D(x_0,y)} - \frac{\nabla G_D(z,y')}{G_D(x_0,y')}\right| < \epsilon. \tag{16}$$

By Proposition 4(b), the function $\nabla_z G_D(z,y)$ is regular α-harmonic in y on $D \setminus B(z, r_0)$, for any z in a neighborhood of z_0 contained in $D \setminus B(Q, \epsilon_0)$ and a sufficiently small r_0. Then (16) follows from a local version of BHP, see [3, Lemma 3]. □

Theorem 8. *Let $x \in D$ and $Q \in \partial D$. Let $M_D(x,Q)$ be the Martin kernel of D for $\Delta^{\alpha/2}$. Denote*

$$l_D(x,Q) = M_D(x,Q) + \int_D \tilde{G}_D(x,z) b(z) \cdot \nabla_z M_D(z,Q) dz$$

The function $l_D(x,Q)$ is well defined for $x \in D$ and $Q \in \partial D$ and $l(x,Q) > 0$. Moreover the following limit exists and equals:

$$\lim_{y \to Q} \frac{\tilde{G}_D(x,y)}{\tilde{G}_D(x_0,y)} = \frac{l_D(x,Q)}{l_D(x_0,Q)}.$$

Thus the Martin kernel of D for $L = \Delta^{\alpha/2} + b \cdot \nabla$ exists and equals

$$\tilde{M}_D(x,Q) = \frac{1}{l_D(x_0,Q)} \left[M_D(x,Q) + \int_D \tilde{G}_D(x,z) b(z) \cdot \nabla_z M_D(z,Q) dz \right]. \tag{17}$$

Proof. We divide the perturbation formula (4) for the Green function $\tilde{G}_D(x,y)$ by $G_D(x_0,y)$ and let $y \to Q$.

The exchange of $\lim_{y \to Q}$ and $\int_D$ is justified by Lemma 11 of [7], see the formula (49) in its proof. Note that by the Boundary Harnack Principle, $G_D(x_0,y) \approx G_D(x,y)$ when $y \in B(Q, \epsilon_0)$, a sufficiently small ball around Q. We also use the estimates (5), (13) and (2).

The exchange of $\lim_{y \to Q}$ and ∇_z is justified by Lemma 7. Finally

$$\lim_{y \to Q} \frac{\tilde{G}_D(x,y)}{G_D(x_0,y)} = M_D(x,Q) + \int_D \tilde{G}_D(x,z) b(z) \cdot \nabla M_D(z,Q) = l_D(x,Q).$$

The strict positivity of the function $l_D(x,Q)$ follows from (2), which implies that there exists $a > 0$ such that

$$l_D(x,Q) \geq a M_D(x,Q) > 0. \tag{18}$$

Now we consider the quotient

$$\frac{\tilde{G}_D(x,y)}{\tilde{G}_D(x_0,y)} = \frac{\tilde{G}_D(x,y)}{G_D(x_0,y)} \frac{G_D(x_0,y)}{\tilde{G}_D(x_0,y)} \to \frac{l_D(x,Q)}{l_D(x_0,Q)},$$

when $y \to Q$. □

Directly from the definition of $\tilde{M}_D(x,Q)$ and (2) we obtain the following corollary.

Corollary 9. *There is a constant c such that for all $x \in D$ and $Q \in \partial D$,*

$$c^{-1} M_D(x,Q) \le \tilde{M}_D(x,Q) \le c M_D(x,Q). \tag{19}$$

3.2 Properties of the L-Martin kernel

Lemma 10. *Consider a $\mathcal{C}^{1,1}$ domain $U \subset \overline{U} \subset D$.*
(i) **(Perturbation formula for the Poisson kernel)** *For all $x \in U, z \in (\overline{U})^c$*

$$\tilde{P}_U(x,z) = P_U(x,z) + \int_U \tilde{G}_U(x,w) b(w) \cdot \nabla P_U(w,z) dw. \tag{20}$$

(ii) Let $Q \in \partial D$. We have the following expression for the L-Poisson integral of the Martin kernel M_D:

$$\begin{aligned} \tilde{P}_U M_D(x,Q) &:= \int_{U^c} \tilde{P}_U(x,y) M_D(y,Q) dy \\ &= M_D(x,Q) + \int_U \tilde{G}_U(x,z) b(z) \cdot \nabla M_D(z,Q) dz, \quad x \in U. \end{aligned} \tag{21}$$

Proof. In the following we apply the Ikeda-Watanabe formula for the Poisson kernels $\tilde{P}_U$ and P_U. By (4) and Fubini's theorem, for any $x \in U$ and $z \in U^c$,

$$\begin{aligned} \tilde{P}_U(x,z) &= \int_U \mathcal{A}_{d,\alpha} \frac{\tilde{G}_U(x,y)}{|z-y|^{d+\alpha}} dy \\ &= \int_U \frac{\mathcal{A}_{d,\alpha}}{|z-y|^{d+\alpha}} \left[G_U(x,y) + \int_U \tilde{G}_U(x,w) b(w) \cdot \nabla G_U(w,y) dw \right] dy \\ &= P_U(x,z) + \int_U \tilde{G}_U(x,w) b(w) \cdot \nabla P_U(w,z) dw. \end{aligned}$$

For the necessary exchanges of order of integration and derivation in the last formula, we apply (13), (2), Lemma 5 and bounded convergence theorem. In order to prove (ii), we use (i) and insert the formula (20) in $\int_{U^c} \tilde{P}_U(x,y) M_D(y,Q) dy$. We obtain

$$\int_{U^c} \tilde{P}_U(x,y) M_D(y,Q) dy = M_D(x,Q) + \int_U \tilde{G}_U(x,z) b(z) \cdot \nabla M_D(z,Q) dz$$

In the last equality the use of Fubini theorem and the exchange of $\int$ and ∇ are justified by (10), (6), Lemma 5 and bounded convergence. □

Lemma 11. *The Martin kernel $\tilde{M}_D(\cdot,\cdot)$ is jointly continuous on $D \times \partial D$.*

Proof. By Theorem 8 and the continuity of $M_D(\cdot,\cdot)$, it suffices to show the joint continuity on $D \times \partial D$ of the function

$$f(x,Q) = \int_D \tilde{G}_D(x,z)b(z) \cdot \nabla M_D(z,Q)dz.$$

Let $z \in D$. By (11) and the α-harmonicity of $M_D(\cdot,Q)$, for $r > 0$ sufficiently small, we have

$$\nabla M_D(z,Q) = \nabla \int_{B(z,r)^c} P_{B(z,r)}(z,w)M_D(w,Q)dy$$

$$= \int_{B(z,r)^c} \nabla P_{B(z,r)}(z,w)M_D(w,Q)dw.$$

From (9) and (6) it follows, that $\nabla P_{B(z,r)}(z,w)M_D(w,Q)$ is uniformly in Q integrable against dw. This implies that $\nabla M_D(z,\cdot)$ is continuous on ∂D for every $z \in D$. Let now $x \in D$ and choose $r > 0$ such that $\overline{B(x,r)} \subset D$. By (2), (5), (12) and (6), for all $y \in B(x,r)$, $z \in D$ and $Q \in \partial D$, we have

$$\tilde{G}_D(y,z)|\nabla M_D(z,Q)| \le \frac{C\delta_D(z)^{\alpha-1}}{|y-z|^{d-\alpha}|z-Q|^d} \le \frac{C}{|y-z|^{d-\alpha}|z-Q|^{d+1-\alpha}}. \quad (22)$$

Hence, $\tilde{G}_D(y,z)|\nabla M_D(z,Q)|$ is uniformly in $y \in B(x,r)$ and $Q \in \partial D$ integrable against $|b(z)|dz$, which gives the continuity of $f(\cdot,\cdot)$. □

Theorem 12. *For every $Q \in \partial D$ the Martin kernel $\tilde{M}(x,Q)$ is a singular L-harmonic function of x on D.*

Proof. Fix $Q \in \partial D$. By (17),

$$l_D(x_0,Q)\tilde{M}_D(x,Q) = M_D(x,Q) + \int_D \tilde{G}_D(x,z)b(z)\cdot\nabla M_D(z,Q)dz, \qquad x \in D.$$

Thus, we need to show that the following function is singular L-harmonic on D:

$$l(x) = M_D(x,Q) + \int_D \tilde{G}_D(x,z)b(z) \cdot \nabla M_D(z,Q)dz.$$

Define

$$l_n(x) = M_D(x,Q) + \int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x,z)b(z) \cdot \nabla M_D(z,Q)dz.$$

We have $\tilde{G}_{D_{1/n}}(x,z) \nearrow \tilde{G}_D(x,z)$ as $n \to \infty$ and, by (22),

$$\int_D \tilde{G}_D(x,z)|b(z)||\nabla M_D(z,Q)|dz < \infty.$$

By the Lebesgue dominated convergence theorem we get $\lim_n l_n(x) = l(x)$. By Lemma 10

$$l_n(x) = \int_{D^c_{1/n}} \tilde{P}_{D_{1/n}}(x,y) M_D(y,Q)dy,$$

so the functions $l_n(x)$ are L-harmonic on $D_{1/n}$. Fix open $B \subset\subset D$ and let $x \in B$. For n big enough we have $B \subset\subset D_{1/n}$ and hence,

$$l(x) = \lim_n l_n(x) = \lim_n \int_{B^c} \tilde{P}_B(x,y) l_n(y)dy.$$

Also, by Lemma 1 we get $l_n(y) \leq cM_D(y,Q)$, where $c > 0$ depends only on D and α. Furthermore, $\tilde{P}_B(x,y) \leq CP_B(x,y)$, where C depends only on B and α. Hence,

$$\int_{B^c} \tilde{P}_B(x,y) M_D(y,Q)dy \leq CM_D(x,Q),$$

and the Lebesgue dominated convergence theorem gives the desired conclusion. □

Alternative proof of Theorem 12. First consider a $\mathcal{C}^{1,1}$ domain $U = D_r$. We note that

$$\tilde{G}_D(x,w) = \tilde{G}_U(x,w) + \int_{U^c} \tilde{P}_U(x,z)\tilde{G}_D(z,w)dz. \tag{23}$$

By (21), (20), (23) and Fubini's theorem

$$\begin{aligned}
\tilde{P}_U l_D(x,Q) &= \int_{U^c} \tilde{P}_U(x,z) l_D(z,Q)dz \\
&= \int_{U^c} \tilde{P}_U(x,z) M_D(z,Q)dz \\
&+ \int_{U^c} \tilde{P}_U(x,z) \int_D \tilde{G}_D(z,w) b(w)\cdot \nabla M_D(w,Q)dwdz \\
&= M_D(x,Q) + \int_{U^c}\int_U \tilde{G}_U(x,w)b(w)\cdot\nabla P_U(w,z)dwM_D(z,Q)dz \\
&+ \int_D \left[\int_{U^c} \tilde{P}_U(x,z)\tilde{G}_D(z,w)dz\right] b(w)\cdot\nabla M_D(w,Q)dw \\
&= M_D(x,Q) + \int_U \tilde{G}_U(x,w)b(w)\cdot\nabla M_D(w,Q)dw \\
&+ \int_D [\tilde{G}_D(x,w) - \tilde{G}_U(x,w)] b(w)\cdot\nabla M_D(w,Q)dw \\
&= M_D(x,Q) + \int_D \tilde{G}_D(x,w)b(w)\cdot\nabla M_D(w,Q)dw = l_D(x,Q).
\end{aligned}$$

Thus the function $l_D(x,Q)$ is regular L-harmonic on each set $U = D_r$ for r sufficiently small. By the strong Markov property, it has the mean value property on each open set $U \subset \bar{U} \subset D$. □

3.3 L-Martin representation

Theorem 13. *For every nonnegative finite measure ν on ∂D the function u given by*

$$u(x) = \int_{\partial D} \tilde{M}_D(x,Q)d\nu(Q), \tag{24}$$

is singular L-harmonic on D. Conversely, if u is nonnegative singular L-harmonic on D, then there exists a unique nonnegative finite measure ν on ∂D verifying (24).

Proof. The L-harmonicity of the Martin integral (24) and the uniqueness of the representation follow from Theorem 12, Lemma 11, (6), (19) and Fubini theorem, in the same way as in the case of the Martin representation for α-harmonic functions in [3, proof of Theorem 1]. We will now focus on the existence part. By L-harmonicity of u and by (20) we have for each n

$$
\begin{aligned}
u(x) &= \int_{D_{1/n}^c} \tilde{P}_{D_{1/n}}(x,y)u(y)dy = \\
&= \int_{D_{1/n}^c} u(y) \left[P_{D_{1/n}}(x,y) + \int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x,w)b(w)\cdot\nabla P_{D_{1/n}}(w,y)dw \right] dy.
\end{aligned}
$$

Denote

$$u_n^*(x) = \int_{D_{1/n}^c} P_{D_{1/n}}(x,y)u(y)dy.$$

By (10), [7, (72)] and Lemma 5 we have

$$\int_{D_{1/n}^c}\int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x,w)|b(w)||\nabla P_{D_{1/n}}(w,y)|u(y)dwdy$$

$$\leq C\int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x,w)|b(w)|\frac{u(w)}{\delta_{D_{1/n}}(w)}dw < \infty,$$

where $C = C(\alpha, b, D_{1/n}) > 0$. Hence, by Fubini theorem

$$u(x) = u_n^*(x) + \int_{D_{1/n}} \tilde{G}_{D_{1/n}}(x,w)b(w)\cdot\int_{D_{1/n}^c} \nabla P_{D_{1/n}}(w,y)u(y)dydw.$$

The function u_n^* is α-harmonic on $D_{1/n}$, so it is differentiable. In order to justify the exchange of $\int$ and ∇ in the last integral we fix $w \in D_{1/n}$. Then by (10) and [7, (72)], for $\varepsilon > 0$ sufficiently small and all $w' \in B(w,\varepsilon)$ and $y \in D_{1/n}^c$ we have

$$|\nabla_{w'} P_{D_{1/n}}(w',y)u(y)| \leq C\frac{u(y)}{\delta_{D_{1/n}}(y)^{\alpha/2}},$$

where $C = C(\alpha, b, D_{1/n}, \varepsilon) > 0$. Since the last term is integrable on $D_{1/n}^c$, by the dominated convergence we obtain

$$u(x) = u_n^*(x) + \int_D \tilde{G}_{D_{1/n}}(x,w)b(w)\cdot\nabla u_n^*(w)dw. \tag{25}$$

We now study the sequence $u_n^*(x)$ in the same way as K. Bogdan [3] in the proof of the existence part of the $\Delta^{\alpha/2}-$Martin representation, with the difference that in our case the function u under the integral defining u_n^* is not α-harmonic.

Like in [3, (2.27)] we have

$$u_n^*(x) = \int_{D_{1/n}^c} P_{D_{1/n}}(x,y)u(y)dy = \int_{D_{1/n}^c}\int_{D_{1/n}} u(y)\mathcal{A}_{d,\alpha}\frac{G_{D_{1/n}}(x,\xi)}{|\xi-y|^{d+\alpha}}d\xi dy$$

Set $\mu_n(d\xi) = \mathcal{A}_{d,\alpha}G_{D_{1/n}}(x_0,\xi)\int_{D_{1/n}}\frac{u(y)}{|\xi-y|^{d+\alpha}}dyd\xi$. Lemma 1 implies that

$$\mu_n(\mathbb{R}^d) = \int_{D_{1/n}^c} P_{D_{1/n}}(x_0,y)u(y)dy \le C\int_{D_{1/n}^c}\tilde{P}_{D_{1/n}}(x_0,y)u(y)dy = cu(x_0) < \infty$$

(recall that if u was α-harmonic, then $\mu_n(\mathbb{R}^d) = u(x_0)$). We obtain

$$u_n^*(x) = \int_{D_{1/n}}\frac{G_{D_{1/n}}(x,\xi)}{G_{D_{1/n}}(x_0,\xi)}\mu_n(d\xi).$$

The only other property of the function u intervening in the proof of the existence part of the $\Delta^{\alpha/2}-$Martin representation in [3] is

$$\lim_n \int_{D_{1/n}^c} u(y)dy = 0$$

and it also holds in our case: the L-harmonic function u is integrable on $D_{1/n}^c$ for every n. The sequence (μ_n) of simultaneously bounded finite measures with support contained in $\bar{D}$ is tight. We choose a subsequence μ_{n_k} converging to a finite (perhaps zero) measure μ. This choice is common for all x. Without loss of generality, we may suppose that (n_k) is a subsequence of (2^{-n}). The limit measure μ satisfies

$$\mathrm{supp}(\mu) \subset \partial D.$$

Exactly as in the proof of the existence part of the $\Delta^{\alpha/2}-$Martin representation in [3], we deduce that for all $x \in D$ the limit

$$\lim_k u_{n_k}^*(x) = u^*(x)$$

exists and

$$u^*(x) = \int_{\partial D} M_D(x,Q)d\mu(Q). \tag{26}$$

Furthermore, in view of (11), for $x \in D_{1/n}$ and $r > 0$ sufficiently small we have

$$\nabla u_n^*(x) = \nabla \int_{B(x,r)^c} P_{B(x,r)}(x,y) u_n^*(y) dy = \int_{B(x,r)^c} \nabla P_{B(x,r)}(x,y) u_n^*(y) dy.$$

By Lemma 1 and (9) we have

$$|\nabla P_{B(x,r)}(x,y) u_n^*(y)| \leq C \frac{P_{B(x,r)}(x,y)}{r - |x|} u(y),$$

and by the dominated convergence we get $\nabla u_{n_k}^*(x) \to \nabla u^*(x)$ as $k \to \infty$. We also have $\tilde{G}_{D_{1/n}}(x,w) \nearrow \tilde{G}_D(x,w)$. In order to justify the passage with the limit under the integral sign in (25) with n_k instead of n we observe that the functions $\tilde{G}_{D_{1/n_k}}(x,w) b(w) \cdot \nabla u_{n_k}^*(w)$ are uniformly integrable on D. Clearly, by Lemma 1 we have $c^{-1} u_n^*(w) \leq u(w) \leq c u_n^*(w)$, where c does not depend on n, thus $u_n^*(w) \leq c u^*(w)$. By the gradient estimates we get

$$\tilde{G}_{D_{1/n}}(x,w) |b(w)| |\nabla u_n^*(w)| \leq \tilde{G}_{D_{1/n}}(x,w) |b(w)| \frac{u^*(w)}{\delta_{D_{1/n}}(w)},$$

and the uniform integrability follows from (26), Lemma 1 and Lemma 6. Therefore

$$u(x) = u^*(x) + \int_D \tilde{G}_D(x,w) b(w) \cdot \nabla u^*(w) dw, \tag{27}$$

which, using (26), becomes

$$\begin{aligned} u(x) &= \qquad (28) \\ &= \int_{\partial D} M_D(x,Q) d\mu(Q) + \int_D \tilde{G}_D(x,w) b(w) \cdot \nabla \int_{\partial D} M_D(w,Q) d\mu(Q) dw. \end{aligned}$$

By the gradient estimates and dominated convergence we also get

$$\nabla \int_{\partial D} M_D(w,Q) d\mu(Q) = \int_{\partial D} \nabla M_D(w,Q) d\mu(Q), \quad w \in D.$$

Define a measure ν on ∂D by $\nu(dQ) = l_D(x_0,Q) d\mu(Q)$. As the function $Q \to l_D(x_0,Q)$ is continuous positive, the measure ν is finite positive on ∂D.

Using Fubini theorem in (28) and the perturbation formula for $\tilde{M}_D$ from Theorem 8, we obtain

$$u(x) = \int_{\partial D} \tilde{M}_D(x, Q) d\nu(Q).$$

□

Corollary 14. *(***Perturbation formula for singular** L**-harmonic functions***) Let $v(x) \geq 0$ be a singular L-harmonic function on D with the Martin representation*

$$v(x) = \int_{\partial D} \tilde{M}_D(x, Q) d\nu(Q), \quad x \in D. \tag{29}$$

Define a singular α-harmonic function v^* on D by

$$v^*(x) = \int_{\partial D} M_D(x, Q) \frac{d\nu(Q)}{l(x_0, Q)}, \quad x \in D \tag{30}$$

Then the following formula holds

$$v(x) = v^*(x) + \int_D \tilde{G}_D(x, w) b(w) \cdot \nabla v^*(w) dw. \tag{31}$$

Proof. Observe that by (18) there exists $\delta > 0$ such that

$$l_D(x_0, Q) > \delta > 0$$

for all $Q \in \partial D$. Thus the measure $d\mu(Q) = \frac{d\nu(Q)}{l_D(x_0,Q)}$ is finite and the function v^* is well defined. By the unicity of the Martin representation and the formula (26), the function v^* defined by (30) is the same as the function v^* defined by a limit procedure and associated to v in the proof of the Theorem 13. Hence the formula (27) holds for v and v^*. It is equivalent to (31). □

Corollary 15. *Let $v(x) \geq 0$ be a singular L-harmonic function on D. The functions v and v^* are comparable: there exists $c > 0$ such that for all $x \in D$*

$$c^{-1} v^*(x) \leq v(x) \leq c v^*(x). \tag{32}$$

Proof. We use the Martin representations (29), (30), the Corollary 9 and the fact that $l_D(x_0, Q) > \delta > 0$ for all $Q \in \partial D$. □

3.4 Perturbation formulas in the diffusion case

In the present article we exploit the perturbation formulas in the case of the singular operator $L = \Delta^{\alpha/2} + b \cdot \nabla$, $1 < \alpha < 2$. In this short chapter we make a parenthesis and briefly discuss the case $\alpha = 2$ and $d \geq 3$, corresponding to the diffusion operator

$$L = \frac{1}{2}\Delta + b \cdot \nabla$$

on $\mathbb{R}^d, d \geq 3$. The potential theory for such diffusion generators was studied by Cranston and Zhao[13], and more recently by Ifra and Riahi[17], Kim and Song[24] and Luks[28]. Our methods allow to enrich this theory by some new perturbation formulas.

We suppose that $b \in \mathcal{K}_d^1$. Recall that Cranston and Zhao[13] worked under this condition and a complementary second condition $|b|^2 \in \mathcal{K}_{d-1}^1$; Kim and Song[24] supressed the condition on $|b|^2$ and considered signed measures in the place of b.

Proposition 16. *Let $L = \frac{1}{2}\Delta + b \cdot \nabla$ with $b \in \mathcal{K}_d^1$. Then the following perturbation formula for the L-Green function $\tilde{G}_D$ holds if $x, y \in \mathbb{R}^d, x \neq y$.*

$$\tilde{G}_D(x,y) = G_D(x,y) + \int_D \tilde{G}_D(x,z) b(z) \cdot \nabla_z G_D(z,y) dz. \tag{33}$$

Proof. Note that by [24, Theorem 6.2], we have the estimate

$$\tilde{G}_D(x,y) \leq C|x-y|^{2-d}, \quad x, y \in \mathbb{R}^d. \tag{34}$$

The proof of the Proposition is the same as the proof of [7, Lemma 12] in the case $1 < \alpha < 2$, with (34) replacing [7, Lemma 7]. □

Let us mention that a perturbation formula for the L-Green function was proposed in [17], but under a restrictive assumption of boundedness of the Kato norm $\|b\|$ of b. A simpler direct proof of the estimate (34) without using the precise estimates [24, Theorem 6.2] should be available.

Next we obtain a perturbation formula for the Martin kernel of Laplacians with a gradient perturbation.

Proposition 17. *Let $L = \frac{1}{2}\Delta + b \cdot \nabla$ with $b \in \mathcal{K}^1_d$. Then the following perturbation formula for the L-Martin kernel $\tilde{M}_D$ holds if $x \in D$ and $Q \in \partial D$.*

$$\tilde{M}_D(x,Q) = \frac{1}{l_D(x_0,Q)}\left[M_D(x,Q) + \int_D \tilde{G}_D(x,z)b(z)\cdot\nabla_z M_D(z,Q)dz\right] \tag{35}$$

where $l_D(x_0,Q)$ is a continuous function on ∂D, equal

$$l_D(x_0,Q) = M_D(x_0,Q) + \int_D \tilde{G}_D(x_0,z)b(z)\cdot\nabla_z M_D(z,Q)dz > 0.$$

Proof. We follow the proof of the Theorem 8 in the case $\alpha = 2$. □

The next perturbation formula concerns the L-Poisson kernel $\tilde{P}_D(x,Q)$.

Proposition 18. *Let $L = \frac{1}{2}\Delta + b \cdot \nabla$ with $b \in \mathcal{K}^1_d$. Then the following perturbation formula for the L-Poisson kernel $\tilde{P}_D$ holds if $x \in D$ and $Q \in \partial D$.*

$$\tilde{P}_D(x,Q) = P_D(x,Q) + \int_D \tilde{G}_D(x,z)b(z)\cdot\nabla_z P_D(z,Q)dz. \tag{36}$$

Proof. Observe that by the formula (33) the function $\tilde{G}_D$ has the same differentiability properties as the function G_D. In particular the inner normal derivative $\frac{\partial\tilde{G}_D}{\partial n}(x,Q)$ exists for $x \in D$ and $Q \in \partial D$. It is known (see [17, page 173]) and possible to prove by the Green formula that

$$\tilde{P}_D(x,Q) = \frac{\partial\tilde{G}_D}{\partial n}(x,Q).$$

The formula (36) then follows by differentiating of the formula (33) in the direction of the inner normal unit vector n. We omit the technical details. □

Let us finish this section by some remarks. The formula $\tilde{P}_D(x,Q) = \frac{\partial\tilde{G}_D}{\partial n}(x,Q)$ implies, like in the Laplacian case, that the L-Martin and the L-Poisson kernels are related by the formula

$$\tilde{M}_D(x,Q) = \frac{\tilde{P}_D(x,Q)}{\tilde{P}_D(x_0,Q)}. \tag{37}$$

On the other hand, if we insert the formula $M_D(x,Q) = \frac{P_D(x,Q)}{P_D(x_0,Q)}$ into (36), we obtain using (35)

$$\tilde{P}_D(x,Q) = P_D(x_0,Q)l_D(x_0,Q)\tilde{M}_D(x,Q).$$

Evaluating the last equation at x_0 we obtain a formula for the function $l_D(x_0, Q)$ intervening in the perturbation formula (35)

$$l_D(x_0, Q) = \frac{\tilde{P}_D(x_0, Q)}{P_D(x_0, Q)}$$

and another proof of the formula (37).

4 Boundary properties of L-harmonic functions

We prove in this section two important boundary properties of L-harmonic functions: the Relative Fatou Theorem and the representation theorem of Hardy spaces. As in the preceding sections, we consider a nonempty bounded $\mathcal{C}^{1,1}$ domain D.

4.1 Relative Fatou Theorem

Recall the Relative Fatou Theorem in the α-stable case. It was proved in [30] for Lipschitz sets D.

Theorem 19. *Let g and h be two non-negative singular α-harmonic functions on D, with Martin representations*

$$g(x) = \int_{\partial D} M_D(x, Q) d\mu^{(g)}(Q), \quad h(x) = \int_{\partial D} M_D(x, Q) d\mu^{(h)}(Q), \quad x \in D.$$

Then, for $\mu^{(h)}$-almost all $Q \in \partial D$,

$$\lim_{x \to Q} \frac{g(x)}{h(x)} = f(x)$$

where f is the density of the absolute continuous part of $\mu^{(g)}$ in the decomposition $\mu^{(g)} = f d\mu^{(h)} + \mu^{(g)}_{sing}$ with respect to the measure $\mu^{(h)}$, and $x \to Q$ non-tangentially.

Our objective in this section is to prove an analogous limit property for non-negative singular L-harmonic functions u and v on D.

If we denote the integral part of the perturbation formula (31) by

$$I_{v^*}(x) = \int_D \tilde{G}_D(x, w) b(w) \cdot \nabla v^*(w) dw$$

then we have

$$u = u^* + I_{u^*}, \quad v = v^* + I_{v^*}$$

where u^* and v^* are singular α-harmonic non-negative functions. We write

$$\frac{u(x)}{v(x)} = \frac{u^*(x)}{v^*(x)} \frac{1 + \frac{I_{u^*}(x)}{u^*(x)}}{1 + \frac{I_{v^*}(x)}{v^*(x)}} \tag{38}$$

The limit boundary behaviors of the quotients $\frac{u(x)}{v(x)}$ and $\frac{u^*(x)}{v^*(x)}$ will be related if we control the limit behavior of the quotients $\frac{I_{u^*}(x)}{u^*(x)}$ and $\frac{I_{v^*}(x)}{v^*(x)}$. Thus we start with discussing the properties of the quotient $\frac{I_h(x)}{h(x)}$ for a singular α-harmonic non-negative function h.

Lemma 20. *Let the Martin representation $h(x) = \int_{\partial D} M(x, Q) d\mu^{(h)}(Q)$ for some non-negative finite measure μ on ∂D. Then, if $Q \notin \mathrm{supp}(\mu^{(h)})$*

$$\lim_{x \to Q} h(x) = 0$$

and if $Q \in \mathrm{supp}(\mu^{(h)})$ and $x \to Q$ non-tangentially

$$\lim_{x \to Q} h(x) = +\infty.$$

Proof. The limit in the case $Q \notin \mathrm{supp}(\mu^{(h)})$ follows easily from the Martin representation of h and the Lebesgue theorem. In the case $Q \in \mathrm{supp}(\mu^{(h)})$ we use the following result of Wu [37].

Let f be a Δ-harmonic function on D, corresponding via the Martin representation to a finite measure $\mu = \mu^{(h)}$ on ∂D. If $Q \in \mathrm{supp}\mu$, then

$$\liminf_{x \to Q} f(x) > 0,$$

provided $x \to Q$ non-tangentially. We have, on D of class $\mathcal{C}^{1,1}$

$$\begin{aligned} f(x) &= \int_{\partial D} P_D^{\Delta}(x, y)\mu(dy) \leq c \int_{\partial D} \frac{\delta_D(x)}{|x-y|^d} \mu(dy) \\ &= c\delta_D(x)^{1-\frac{\alpha}{2}} \int_{\partial D} \frac{\delta_D(x)^{\frac{\alpha}{2}}}{|x-y|^d} \mu(dy) \leq C\delta_D(x)^{1-\frac{\alpha}{2}} \int_{\partial D} M_D(x, y)\mu(dy) \end{aligned}$$

Consequently

$$h(x) \geq \frac{1}{C} \frac{f(x)}{\delta_D(x)^{1-\frac{\alpha}{2}}}$$

and the second part of the Lemma follows.

□

Lemma 21. *The quotient* $\frac{I_h(x)}{h(x)}$ *is bounded. More exactly, there exists* $c > 0$ *such that*

$$c^{-1} \leq 1 + \frac{I_h(x)}{h(x)} \leq c. \tag{39}$$

Proof. Observe that by Corollary 15 and the formula (31), the quotient $\frac{I_{v^*}(x)}{v^*(x)}$ is bounded. More exactly, there exists $c > 0$ such that

$$c^{-1} \leq 1 + \frac{I_{v^*}(x)}{v^*(x)} \leq c.$$

As the function $l_D(x_0, Q)$ is bounded, any singular α-harmonic non-negative function h is of the form v^* for a singular L-harmonic non-negative function v. □

By (2), if we denote

$$J_h(x) = \int_D G_D(x, w) b(w) \cdot \nabla h(w) dw$$

then

$$I_h(x) \sim J_h(x)$$

In particular, by Lemma 21, the quotient $J_h(x)/h(x)$ is bounded. We prove a much stronger property of this quotient in the following lemma.

Lemma 22. *Let* h *be a non-negative singular* α*-harmonic function on* D*, with the Martin representation* $h(x) = \int_{\partial D} M_D(x, Q) d\mu^{(h)}(Q)$ *for a finite measure* $\mu^{(h)}$ *on* ∂D*. Then, when* $Q \in \operatorname{supp}(\mu^{(h)})$ *and* $x \to Q$ *non-tangentially, we have*

$$\lim_{x \to Q} \frac{J_h(x)}{h(x)} = 0.$$

Proof. We will show that $\frac{G_D(x,w)h(w)}{h(x)\delta_D(w)}$ is uniformly integrable in $x \in D$ against the measure $|b(w)|dw$. Let $\varepsilon > 0$. Since $J_h(x)/h(x)$ is bounded it suffices to show that there is $\delta > 0$ such that

$$\int_F \frac{G_D(x,w)h(w)}{h(x)\delta_D(w)}|b(w)|dw \le \varepsilon, \tag{40}$$

provided $\lambda(F) < \delta$. Here, λ denotes the Lebesgue measure on $\mathbb{R}^d$. First, we note that

$$\begin{aligned}
&\int_F \frac{G_D(x,w)h(w)}{h(x)\delta_D(w)}|b(w)|dw \\
&= \int_F \int_{\partial D} \frac{G_D(x,w)M_D(w,Q)}{h(x)\delta_D(w)} d\mu^{(h)}(Q)|b(w)|dw \\
&= \int_{\partial D} \frac{M_D(x,Q)}{h(x)} \left(\int_F \frac{G_D(x,w)M_D(w,Q)}{M_D(x,Q)\delta_D(w)}|b(w)|dw \right) d\mu^{(h)}(Q). \qquad (41)
\end{aligned}$$

The function $\frac{G_D(x,w)G_D(w,y)}{G_D(x,y)\delta(w)}$ is uniformly integrable in $x, y \in D$ against $|b(w)|dw$ (see the proof of [7, Lemma 11]). Hence, there exists $\delta > 0$ such that for $\lambda(F) < \delta$,

$$\int_F \frac{G_D(x,w)G_D(w,y)}{G_D(x,y)\delta_D(w)}|b(w)|dw < \varepsilon, \qquad x, y \in D,$$

and consequently

$$\begin{aligned}
&\int_F \frac{G_D(x,w)M_D(w,Q)}{M_D(x,Q)\delta_D(w)}|b(w)|dw = \int_F \lim_{D\ni y\to Q} \frac{G_D(x,w)G_D(w,y)}{G_D(x,y)\delta_D(w)}|b(w)|dw \\
&= \lim_{D\ni y\to Q} \int_F \frac{G_D(x,w)G_D(w,y)}{G_D(x,y)\delta_D(w)}|b(w)|dw \le \varepsilon.
\end{aligned}$$

Now, (40) follows from (41) and Martin representation of h. For $Q \in \mathrm{supp}\mu^{(h)}$, $\lim_{D\ni x\to Q} h(x) = \infty$ from the Lemma 20. Hence, by uniform integrability,

$$\begin{aligned}
\lim_{D\ni x\to Q} \frac{|J_h(x)|}{h(x)} &\le c \lim_{D\ni x\to Q} \int_D \frac{G_D(x,w)h(w)}{h(x)\delta_D(w)}|b(w)|dw \\
&= c \int_D \lim_{D\ni x\to Q} \frac{G_D(x,w)h(w)}{h(x)\delta_D(w)}|b(w)|dw = 0.
\end{aligned}$$

□

Now, we return to the Relative Fatou Theorem for L-harmonic functions. Let u and v be two non-negative singular L-harmonic functions on D. By Theorem 13, they have a Martin representation

$$u(x) = \int_{\partial D} \tilde{M}_D(x,Q)d\mu(Q), \quad v(x) = \int_{\partial D} \tilde{M}_D(x,Q)d\nu(Q), \quad x \in D$$

where μ and ν are two Borel finite measures concentrated on ∂D.

We decompose the measure μ into its absolutely continuous and singular parts with respect to the measure ν

$$d\mu = f\ d\nu + d\mu_{sing}$$

with a non-negative function $f \in L^1(\nu)$ and $\nu(\mathrm{supp}(\mu_{sing})) = 0$.

Theorem 23. *(***Relative Fatou Theorem***) For ν-almost every point $Q \in \partial D$ we have*

$$\lim_{x \to Q} \frac{u(x)}{v(x)} = f(Q) \tag{42}$$

when $x \to Q$ non-tangentially.

Proof. We will use the Relative Fatou Theorem for the singular α-harmonic functions u^* and v^* defined according to (30).

Let $Q \in \mathrm{supp}(\nu) \setminus \mathrm{supp}(\mu)$. Then, if $x \to Q$, $v^*(x) \to \infty$ and $u^*(x) \to 0$, so $\lim_{x\to Q} \frac{u^*(x)}{v^*(x)} = 0$. The formulas (39) and (38) imply that in this case

$$\lim_{x \to Q} \frac{u(x)}{v(x)} = 0.$$

Let us consider the case $Q \in \mathrm{supp}(\nu) \cap \mathrm{supp}(\mu)$. As

$$\frac{d\mu(Q)}{l_D(x_0,Q)} = f\ \frac{d\nu(Q)}{l_D(x_0,Q)} + \frac{d\mu_{sing}(Q)}{l_D(x_0,Q)},$$

the Relative Fatou Theorem for the singular α-harmonic functions u^* and v^* says that for ν-almost every point $Q \in \partial D$

$$\lim_{x \to Q} \frac{u^*(x)}{v^*(x)} = f(Q)$$

when $x \to Q$ non-tangentially. The formula (42) then follows by the formula (38) and the Lemma 22. $\square$

Alternative proof. It is possible to prove the Theorem 23 in a different way. The idea of another proof is to use the Corollary 15 and to follow the basic steps of the proof of the Relative Fatou Theorem for α-harmonic functions of K. Michalik and M. Ryznar[30], as explained in the survey [4, Section 3.3].
Step 1. *The estimates of the L-Martin kernel* $\tilde{M}_D$ are the same as in [4, Lemma 3.6, Corollary 3.7 and Lemma 3.8] thanks to the Corollary 9.
Step 2. *The limit behaviour of an L-harmonic function.* Similarly as for singular $\alpha-$harmonic functions(see Lemma 20), when the domain D is of class $\mathcal{C}_{1,1}$, we have, thanks to the Corollary 15

$$\lim_{x\to Q} v(x) = +\infty$$

for ν-almost all Q, provided the limit is nontangential.
Step 3. *Nontangential Maximal Estimate for L-harmonic functions.* Using [4, Lemma 3.10], the Corollary 15 and the fact that $l(x_0, Q) > \delta > 0$ for all $Q \in \partial D$, we obtain that for any $x \in D$, $Q \in \partial D$ such that $|x - Q| \leq t\delta_D(x)$ with a $t > 0$ there exists a constant $C = C(t, Q)$ such that

$$\frac{u(x)}{v(x)} \leq C \sup_{r>0} \frac{\mu(B(Q,r))}{\nu(B(Q,r))}.$$

Step 4. *Showing that* $\frac{u(x)}{v(x)} \to Q$ *for* ν*-almost all* Q. This final step of the proof, explained in [4, page 70], is common for singular α-harmonic and L-harmonic functions.

4.2 L-Hardy spaces

We recall first the analytic construction of Hardy spaces of singular α-harmonic functions discussed in [31]. Let $\mathcal{M}(\partial D)$ denote the family of all finite signed Borel measures on ∂D and for $\mu \in \mathcal{M}(\partial D)$ let $\|\mu\|$ denote the total variation norm of μ. Choose a constant $r_0 > 0$ such that the $C^{1,1}$ characteristics of D_r, $0 < r < r_0$ may be fixed independently of r (see [31, Lemma 5] for details). In particular we may assume that the sets D_r verify the ball condition with the same radius r_0. For $0 < r < r_0$ let $\pi_r \colon \partial D \mapsto \partial D_r$ be the orthogonal projection, i.e., $\pi_r(x)$ means the closest point to x on ∂D_r. Let σ be the $(d-1)$-dimensional Hausdorff surface measure on ∂D and set

$$N(x) = M_D[\sigma](x) = \int_{\partial D} M_D(x, y)\sigma(dy).$$

The Hardy space $H^p_\alpha(D)$, $p \geq 1$, is defined as the family of all singular α-harmonic functions u on D such that

$$\|u\|_{H^p_\alpha} := \sup_{0<r<r_0} \left(\int_{\partial D} \left| \frac{u(\pi_r(x))}{N(\pi_r(x))} \right|^p \sigma(dx) \right)^{1/p} < \infty.$$

The following representation theorem is a consequence of [31, Theorem 2 and Theorem 3].

Theorem 24. *Let u be singular α-harmonic on D. Then*

1. *$u \in H^1_\alpha(D)$ if and only if $u = M_D[\mu]$ for some $\mu \in \mathcal{M}(\partial D)$. Furthermore, μ is unique and there exists a positive constant $C = C(\alpha, D)$ such that*
$$C^{-1}\|\mu\| \leq \|u\|_{H^1_\alpha} \leq C\|\mu\|.$$
2. *$u \in H^p_\alpha(D)$ for a given $p > 1$ if and only if $u = M_D[f]$ for some function $f \in L^p(\partial D, \sigma)$. Furthermore, f is unique and there exists a positive constant $C = C(\alpha, D)$ such that*
$$C^{-1}\|f\|_p \leq \|u\|_{H^p_\alpha} \leq C\|f\|_p.$$

We point out that Theorem 24 holds for all $\alpha \in (0, 2)$. We will now prove analogous version of the representation theorem for singular L-harmonic functions on D with $\alpha \in (1, 2)$. Set $\tilde{N}(x) = \tilde{M}_D[\sigma](x)$. We define the Hardy space $\tilde{H}^p_\alpha(D)$, $p \geq 1$, as the family of all singular L-harmonic functions u on D verifying

$$\|u\|_{\tilde{H}^p_\alpha} := \sup_{0<r<r_0} \left(\int_{\partial D} \left| \frac{u(\pi_r(x))}{\tilde{N}(\pi_r(x))} \right|^p \sigma(dx) \right)^{1/p} < \infty.$$

We note that, since ∂D is compact, $\tilde{H}^p_\alpha(D) \subset \tilde{H}^1_\alpha(D)$ for all $p > 1$. Furthermore, in view of Corollary 9, $\tilde{N} \approx N$, and hence there exists a constant $c = c(\alpha, b, D) > 0$ such that for every Borel function u on D we have

$$c^{-1}\|u\|_{H^p_\alpha} \leq \|u\|_{\tilde{H}^p_\alpha} \leq c\|u\|_{H^p_\alpha}. \tag{43}$$

Theorem 25. *Let u be singular L-harmonic on D. Then*

1. *$u \in \tilde{H}^1_\alpha(D)$ if and only if $u = \tilde{M}_D[\mu]$ for some $\mu \in \mathcal{M}(\partial D)$. Furthermore, μ is unique and there exists a positive constant $C = C(\alpha, b, D)$ such that*
$$C^{-1}\|\mu\| \leq \|u\|_{\tilde{H}^1_\alpha} \leq C\|\mu\|.$$

2. *$u \in \tilde{H}^p_\alpha(D)$ for a given $p > 1$ if and only if $u = \tilde{M}_D[f]$ for some functions $f \in L^p(\partial D, \sigma)$. Furthermore, f is unique and there exists a positive constant $C = C(\alpha, b, D)$ such that*

$$C^{-1}\|f\|_p \le \|u\|_{\tilde{H}^p_\alpha} \le C\|f\|_p.$$

Proof. Suppose first that $u = \tilde{M}_D[\mu]$ for some $\mu \in \mathcal{M}(\partial D)$. Then by the Hahn decomposition $\mu = \mu^+ - \mu^-$ and Corollary 14 we get the perturbation formula

$$u(x) = u^*(x) + \int_D \tilde{G}_D(x, w)b(w) \cdot \nabla u^*(w)dw,$$

where $u^*(x) = \int_{\partial D} M_D(x, Q)/l_D(x_0, Q)d\mu(Q)$. Furthermore, as in Corollary 15 we obtain $u \approx u^*$. Since $l_D(x_0, \cdot)$ is bounded and separated from 0, it follows from Theorem 24 that $u^* \in H^1_\alpha(D)$ and $C^{-1}\|\mu\| \le \|u^*\|_{H^1_\alpha} \le C\|\mu\|$. By (43) we then get $C^{-1}\|\mu\| \le \|u\|_{\tilde{H}^1_\alpha} \le C\|\mu\|$ and hence $u \in \tilde{H}^1_\alpha(D)$.

Conversely, suppose that $u \in \tilde{H}^1_\alpha(D)$. Define

$$u_r(x) = \int_{D \setminus D_r} \tilde{P}_{D_r}(x, y)|u(y)|dy.$$

By the strong Markov property, u_r increases as $r \downarrow 0$ and we may define $v = \lim_{r\downarrow 0} u_r$. By the monotone convergence and the Harnack inequality (see [7, Lemma 15]) we obtain that either v is finite and L-harmonic on D or $v \equiv \infty$ on D. In order to show that $v < \infty$ we follow [31, proof of Theorem 3]. Fix $x \in D$ and let $0 < s < r < r_0$. By (6), Corollary 9 and the estimates of the classical Poisson kernel of the Laplacian (see the proof of Lemma 20) we have $\tilde{N}(\pi_s(y)) \le Cs^{\alpha/2-1}$, $y \in \partial D$, where $C = C(\alpha, b, D)$. Furthermore, $\delta_{D_r}(\pi_s(y)) = r - s$, $\delta_{D_r}(x) \le \delta_D(x)$ and $|x - \pi_s(y)| \ge \delta_{D_r}(x) \ge 1/2\delta_D(x)$ for sufficiently small r. For such r, by Lemma 1 and by the estimates of the Poisson kernel for $\Delta^{\alpha/2}$ with a constant common for all D_r ([31, (15)]) we obtain

$$\tilde{P}_{D_r}(x, \pi_s(y)) \le \frac{C\delta_{D_r}(x)^{\alpha/2}}{\delta_{D_r}(\pi_s(y))^{\alpha/2}(1 + \delta_{D_r}(\pi_s(y)))^{\alpha/2}|x - \pi_s(y)|^d} \le \frac{C\delta_D(x)^{\alpha/2-d}}{(r - s)^{\alpha/2}},$$

where $C = C(\alpha, b, D)$. Therefore

$$u_r(x) \le C \int_0^r \int_{\partial D} \tilde{P}_{D_r}(x, \pi_s(y))\tilde{N}(\pi_s(y)) \left| \frac{u(\pi_s(y))}{\tilde{N}(\pi_s(y))} \right| \sigma(dy)ds$$

$$\leq C\delta_D(x)^{\alpha/2-d}\int_0^r \frac{s^{\alpha/2-1}}{(r-s)^{\alpha/2}}\int_{\partial D}\left|\frac{u(\pi_s(y))}{\tilde{N}(\pi_s(y))}\right|\sigma(dy)ds$$

$$\leq C\delta_D(x)^{\alpha/2-d}\|u\|_{\tilde{H}^1_\alpha}\int_0^r \frac{s^{\alpha/2-1}}{(r-s)^{\alpha/2}}ds \leq C\delta_D(x)^{\alpha/2-d}\|u\|_{\tilde{H}^1_\alpha}.$$

Hence, v is finite, singular L-harmonic and non-negative on D. By Theorem 13, $v = \tilde{M}_D[\nu]$ for some non-negative measure $\nu \in \mathcal{M}(\partial D)$. Since $|u| \leq u_r \leq v$, the function $v-u$ is also singular L-harmonic and non-negative on D, so $v-u = \tilde{M}_D[\mu]$ for some non-negative $\mu \in \mathcal{M}(\partial D)$. Therefore

$$u = v-(v-u) = \tilde{M}_D[\nu] - \tilde{M}_D[\mu] = \tilde{M}_D[\nu-\mu].$$

This gives the first part of the theorem.

To prove the second part fix $p > 1$. If $u = \tilde{M}_D[f]$ for some $f \in L^p(\partial D, \sigma)$, then $u^* = \tilde{M}_D[f/l_D(x_0,\cdot)]$ and as in the proof of the first part, using Theorem 24 and (43) we conclude that $C^{-1}\|f\|_p \leq \|u\|_{\tilde{H}^p_\alpha} \leq C\|f\|_p$ and $u \in \tilde{H}^p_\alpha(D)$. Conversely, if $u \in \tilde{H}^p_\alpha(D)$, then $u \in \tilde{H}^1_\alpha(D)$ and by the first part, $u = \tilde{M}_D[\mu]$ for some $\mu \in \mathcal{M}(\partial D)$. Hence, $u^* = M_D[\mu/l_D(x_0,\cdot)]$. Since $u \approx u^*$, by (43) we get $u^* \in H^p_\alpha(D)$ and Theorem 24 implies that $\mu(dx) = f(x)\sigma(dx)$ for some $f \in L^p(\partial D, \sigma)$. This ends the proof. □

In the proof of the Theorem 25 we used strongly the interplay between a L-harmonic function u and its α-harmonic counterpart u^*. Using the unicity of the Martin representations for L and $\Delta^{\alpha/2}$ we obtain the following Corollary.

Corollary 26. *Let $p \geq 1$. The Hardy spaces $H^p_\alpha(D)$ and $\tilde{H}^p_\alpha(D)$ are topologically isomorphic. More exactly, the mapping*

$$u(x) \to \tilde{u}(x) = u(x) + \int_D \tilde{G}_D(x,w)b(w)\cdot\nabla u(w)dw$$

is a bijection between $H^p_\alpha(D)$ and $\tilde{H}^p_\alpha(D)$ and

$$\|u\|_{H^p_\alpha} \approx \|\tilde{u}\|_{\tilde{H}^p_\alpha}.$$

Proof. The corollary follows from Theorem 24, Theorem 25, Corollary 14 and its generalization for signed measures given in the proof of Theorem 25. □

In view of Corollary 26 we can say that the space $\tilde{H}^p_\alpha(D)$ is equal to a perturbed space $H^p_\alpha(D)$.

Remark 2. *By similar methods as to prove the Theorem 25, we can obtain the representation theorem for Hardy spaces of relativistic α-harmonic functions.*

We observe that, according to [16], stable and relativistic Green functions and Poisson kernels are comparable on bounded $\mathcal{C}^{1,1}$ domains. Consequently, the Martin kernels are also comparable. A careful reading of the proofs of all these estimates allows one to see that common constants may be chosen for sets D_r sufficiently close to D.

The Martin representation of singular relativistic α-stable non-negative functions was proved in [23].

Acknowledgements

We thank Krzysztof Bogdan, Jacek Małecki and Michał Ryznar for discussions about this paper.

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