Two remarks on elementary theories of groups obtained by free constructions

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Abstract

We give two slight generalizations of results of Poizat about elementary theories of groups obtained by free constructions. The first-one concerns the non-superstability of such groups in most cases, and the second-one concerns the connectedness of most free products of groups.

Recently, first-order theories of free products of groups have been investigated in [JS10] and [Sel10], with to some extent some transfers of arguments from free groups to free products of groups. It is expected that some of this work transfers further to more general classes of groups obtained by free constructions. In this modest and short note we will make slight generalizations of early arguments of Poizat on first-order theories of free products which, so far, did not seem to have been noticed before. Since these generalizations concern the basic model-theory of groups obtained by free constructions, it seems relevant to have them recorded.

In a first series of results we prove that most groups obtained by free constructions do not have a superstable theory. We refer to [OH08] for an approach of the model theory of such groups via actions on trees, but with seemingly weaker results (only the failure of ω-stability). Here, our proofs are mere adpatations of arguments in [Poi83, §7] in the case of free products.

Theorem 1 *Let $P = G *_A H$ be a free product of groups with amalgamated subgroup A, with $A < G$ and $A < H$. Then P has a superstable theory if and only if $A = 1$ and $G \simeq H \simeq \mathbb{Z}_2$.*

In the case of HNN-extensions we also get non-superstability in most cases.

Theorem 2 *Let $G^* = \langle G, t \mid A^t = B \rangle$ be an HNN-extension of a group G, with A or B proper in G. Then the theory of G^* is not superstable.*

We note that we also take the opportunity to make a very basic analysis of generic types in groups as in Theorems 1 and 2, in the event that such groups are stable. We recall that one of the main accomplishements of [Sel10] is a proof of the fact that the free product of two stable groups is still stable. But, conversely, we point out the following question as a possibly difficult one.

Question 3 *Can one have a free product of groups $G * H$ with a stable theory, but with the factor G unstable?*

We merely mention that our basic analysis of generic types in the stable case of Theorems 1 and 2 will not assume the stability of the vertex groups G or H.

In a second type of results, we prove that most free products of groups are connected, i.e., with no proper definable subgroups of finite index. Again this will be a mere adaptation of an argument contained in [Poi83, §7], about the free group on countably many generators, together with a deep elementary equivalence obtained in [Sel10] about free products.

Theorem 4 *Let G and H be two nontrivial groups. Then a group elementarily equivalent to $G * H$ is not connected if and only if $G \simeq H \simeq \mathbb{Z}_2$.*

All groups considered here are considered in the pure group langage, but we will point out when our results remain valid in the case of expansions of groups, especially around non-superstability. We thank Bruno Poizat for his patient explainations of his argument for non-superstability in [Poi83, §7].

1 Non-superstability

We say that a free product of groups $G *_A H$ with amalgamated subgroup A is *non-trivial* when $A < G$ and $A < H$. We say that it is of *dihedral type* when $A = 1$ and $G \simeq H \simeq \mathbb{Z}_2$.

As in stable group theory, we say that a subset X of a group is left-generic in the group when finitely many left-translates of X cover the ambient group.

Lemma 5 *Let $P = G *_A H$ be a non-trivial free product of groups with amalgamated subgroup A, and suppose P not of dihedral type. For any integer $d > 1$ let $Y_d = (G \cup H)^P \cup \{x^d \ : \ x \in P\}$, and for every integer $n \geq 1$ let B_n denote the ball of elements of P of length at most n.*

(1) For every integer $n \geq 1$, there exists α_n in P such that, for every element x in B_n, $x\alpha_n$ is not in Y_d.

(2) For every integer $n \geq 1$, $B_n Y_d$ is a proper subset of P. In particular Y_d is not left-generic in P.

Proof:

(1). We first claim that we can find an element a in one of the factors but not in the amalgam, and elements b and c in the other factor and still not in the amalgam, with $c \neq b$ or $c \neq b^{-1}$. If A is not trivial, then each factor contains elements b and c not in A with $b \neq c$, so our claim follows in this case. If A is trivial, this follows easily from the assumption that P is not of dihedral type.

For every $n \geq 1$, let now $\alpha_n = (ab)^{n+4}(acac^{-1})^{3n+3}$. Now one sees that for every element $x \in B_n$, the element $x\alpha_n$ has a normal form of the form $\gamma(ab)^4(acac^{-1})^{3n+3}$ where γ is an element of length at most $3n$. Contemplating this normal form one sees, as in [Poi83, §7], that $x\alpha_n$ is not conjugated in one of the factors G or H, and that $x\alpha_n$ is not a square, and actually not a d-th power.

(2). The first claim shows that, for every element x of length at most n, $x\alpha_n$ is in $G \setminus Y_d$. In particular, α_n is in $x^{-1}(G \setminus Y_d)$, and α_n is not in $x^{-1}Y_d$. Since x and x^{-1} have the same length, this shows that α_n is not in B_nY_d, and thus B_nY_d is a proper subset of P.

As the length of finitely many elements of P is uniformely bounded, it follows in particular that P cannot be covered by finitely many left-translates of Y_d, and thus Y_d not left-generic in P. □

Recall from [Jal06] that a subset X of a group G is *left-generous* when X^G is left-generic in G. We obtain in particular the following corollary of Lemma 5 (with a proof more general and more direct than the one given in [Pil08, Lemma 2.12]).

Corollary 6 *Let $P = G *_A H$ be a non-trivial free product of groups with amalgamated subgroup A, and suppose P not of dihedral type. Then $(G \cup H)$ is not left-generous in P.*

Since the two generating subgroups of a dihedral group form a left-generous subset, the failure of left-generosity of $(G \cup H)$ in P is actually equivalent to the fact that P is not of dihedral type. Of course, Lemma 5 and Corollary 6 could be proved similarly with the obvious notions of right-genericity, instead of left-generosity. When P is stable in Lemma 5 we get the following.

Corollary 7 *Assume P has a stable theory in Lemma 5. If g is generic over P, then, for every integer $d > 1$, g is not a d-th power, and the unique d-th root of g^d.*

PROOF:

Any element g in $P \setminus Y_d$ is not a d-th power. Furthermore, any such element has a centralizer $C_P(g)$ which is cyclic infinite, and $C_P(g)$ is the centralizer of each of its non-trivial elements. In particular any d-th root of g^d is in $C_P(g)$, and it must coincide with g. This shows that $P \setminus Y_d$ is contained in the definable subset Z_d of elements of P which are not a d-th power and the unique d-th root of their d-th power.

Hence $G \setminus Z_d \subseteq Y_d$ and Lemma 5 implies that $G \setminus Z_d$ is not left-generic. Assuming the stability of P, we get then that, for every $d > 1$, the formula defining Z_d is in all generic types. □

Proof of Theorem 1: If P is of dihedral type, then it is definable in the theory of the infinite cyclic group, and thus it is superstable.

Assume now P has a superstable theory, but is not of dihedral type. By stability, Corollary 7 shows that if g is generic over P, then it is not a d-th root and the unique d-th root of its d-th power (for any $d > 1$). We get thus that g^d is not generic, and that the generic element g is algebraic over the non-generic element g^d. Applying this (just for some fixed $d > 1$), we get a contradiction to the weak regularity in the superstable case as in [Poi83, §7 and p.346]. □

We now pass to HNN-extensions. We say that an HNN-extension $G^* = \langle G, t \mid A^t = B\rangle$ is of *automorphism type* when $A = B = G$.

Lemma 8 *Let $G^* = \langle G, t \mid A^t = B\rangle$ be an HNN-extension, and suppose G^* not of automorphism type. For any integer $d > 1$ let $Y_d = G^{G^*} \cup \{x^d \ : \ x \in G^*\}$, and for every integer $n \geq 1$ let B_n denote the ball of elements of G^* of length at most n.*

(1) For every integer $n \geq 1$, there exists α_n in G^* such that, for every element x in B_n, $x\alpha_n$ is not in Y_d.

(2) For every integer $n \geq 1$, B_nY_d is a proper subset of G^*. In particular Y_d is not left-generic in G^*.

PROOF:
Since G^* is not of automorphism type we can, interchanging A and B if necessary, assume that $B < G^*$. Let now a be any element of $G \setminus B$. Since we are using the convention $g^h = h^{-1}gh$ for conjugates, we get that tat^{-1} is not in the preimage A of B by the underlying automorphism from A to B. Hence there are no cancellations in the word $(at)^{n+4}(atat^{-1})^{3n+3}$. Now one can argue as in Lemma 5, with a playing the same role as a there, and with $t = b = c$ in the proof of Lemma 5. Notice that $t \neq t^{-1}$ here. □

With Lemma 8 we get analogs of Corollaries 6 and 7 in the same way.

Corollary 9 *Let $G^* = \langle G, t \mid A^t = B\rangle$ be an HNN-extension, and suppose it is not of automorphism type. Then G is not left-generous in G^*.*

Corollary 10 *Assume G^* has a stable theory in Lemma 8. If g is generic over G^*, then, for every integer $d > 1$, g is not a d-th power, and the unique d-th root of g^d.*

Proof of Theorem 2: With Corollary 10 we may argue exactly as in the proof of Theorem 1. □

We note that Lemma 5 and its two corollaries, as well as their analogs for HNN-extensions, do not depend on the fact of considering pure group structures: they remain true if the groups considered are expanded by some extra structure definable in some extra language. Similarly, the proofs of non-superstability in Theorems 1 and 2 are not sensitive to structure expanding the group structure. Hence we actually get the following.

Theorem 11 *Let G be any expansion of a non-trivial free product with amalgamation not of dihedral type, or of an HNN-extension not of automorphism type. Then G is not superstable.*

Conversely, of course, there might be expansions of the dihedral group which are not superstable. The question of the superstability of HNN-extensions of automorphism type may depend on the pair consisting of a group with an automorphism considered. For instance, the HNN-extension $\langle \mathbb{Z}, t \mid [t, \mathbb{Z}] \rangle \simeq \mathbb{Z} \times \mathbb{Z}$ is definable in $\mathbb{Z}$ and has a superstable theory.

2 Connectedness

In this section we consider the connectedness of free products of groups without amalgamation. The following argument stems directly from [Poi83, Lemme 6] (see also [Poi93] for related arguments).

Proposition 12 *Let A be a group, $F_\omega = \langle e_i \mid i < \omega \rangle$ the free group on countably many generators e_i, and $G = A * F_\omega$ the free product without amalgamation of A and F_ω.*

(1) If X is a left-generic definable subset of G, then X contains a cofinite subset of the set $\{e_i \mid i < \omega\}$.

(2) G is connected, i.e., with no proper definable subgroup of finite index.

(3) If G is stable, the sequence $(e_i)_{i<\omega}$ is a Morley sequence of the unique generic type p_0 of G over $\emptyset$.

Proof:

(1). Suppose $G = g_1 X \cup \cdots \cup g_k X$ for finitely many elements g_s in G. Let $\{e_1, \cdots, e_r\}$ consists of the set of all the generators of F_ω involved in the parameters needed to define X, together with all the generators of F_ω such that $g_s \in A * \langle e_1, ..., e_r \mid \rangle$ for each s. It suffices to show that e_i is in X for each $i > r$. Since e_i is in $g_s X$ for some s, $g_s^{-1} e_i$ is in X. But since $g_s^{-1} e_i$ is free over $A * \langle e_1, ..., e_r \mid \rangle$, there is an automorphism of $A * \langle e_1, ..., e_r \mid \rangle * \langle e_i \rangle$ fixing $A * \langle e_1, ..., e_r \mid \rangle$ pointwise and sending $g_s^{-1} e_i$ to e_i. This automorphism extends to an automorphism of $A * F_\omega$ and must stabilize X setwise. In particular we get that e_i is in X.

(2). By the preceding, two left-generic definable subsets necessarily have a non-empty intersection, and in particular G cannot have a proper definable subgroup of finite index.

(3). As in [Pil08, Corollary 2.7]. □

With one of the results of [Sel10] we get the following

Proof of Theorem 4: Since elementary equivalence preserves connectedness, we may consider directly $G * H$.

If G and H are cyclic of order 2, then $G * H$ is dihedral and in particular not connected. If $G * H$ is not dihedral, then it is elementarily equivalent to $G * H * F_\omega$ by [Sel10, Theorem 7.2], which is connected by Proposition 12(2); since elementary equivalence preserves connectedness, $G * H$ is connected. □

We note that in the proof of Theorem 4 we only used the elementary equivalence $G * H \equiv G * H * F_\omega$ when $G * H$ is not of dihedral type. It is actually expected that a reworking of [Sel10] "over parameters" would imply the elementary embedding $G * H \preceq G * H * F_1$ in this case (and thus elementary embeddings $G * H \preceq G * H * F_\kappa \preceq G * H * F_{\kappa'}$ for all cardinals $\kappa \leq \kappa'$). In [OH11, Proposition 8.8] an elementary embedding of this type is used in a proof of the connectedness of non-cyclic torsion-free hyperbolic groups, but one could argue without such an elementary embedding as is done here.

Since free products tend to have proper subgroups of finite index, it seems difficult to characterize which expansions of a free product not of dihedral type are still connected. In the pure group langage, we can obtain some realizations of the unique generic type over a non-dihedral free product with the following.

Corollary 13 *Let A be a group, $F_\omega = \langle e_i \mid i < \omega \rangle$, and suppose that $G = A * F_\omega$ is stable. Then any primitive element of F_ω is a realization of the generic type of G over $\emptyset$.*

Proof:
By Proposition 12. □

In particular, for $G * H$ a non-trivial free product not of dihedral type, primitive elements of F_ω realize the generic type in the elementary equivalent group $G * H * F_\omega$. The full characterization of the set of realizations of the generic type, as in [Pil09] in the free group case, seems to depend on the nature of the factors; it is even unclear whether the generic type is realized in the standard model $G * H$. Most probably, one can prove that the generic type is not isolated, as in [Skl11] in the free group case.

Theorem 4 proves that all non-trivial free products of groups are connected, with the single exception of the dihedral case. We believe that such groups are actually definably simple, which would follow from the following more general conjecture.

Conjecture 14 *Let $G * H$ be a free product of two groups. Then any definable subgroup of $G * H$ is of one of the following type:*

- *the ful group,*
- *conjugated in one of the factors G or H,*
- *cyclique infinite and elliptic, or*
- *dihedral (just in case one of the factors contains an element of order 2).*

Most probably, one way to prove Conjecture 14 could be obtained by a direct generalization to free products of groups of the Bestvina-Feighn notion of a *negligeable set* in free groups, and by using the quantifier elimination for definable subsets of $G * H$ from [Sel10]. We refer to [KM11] for a proof in the free group case.

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