WASSERSTEIN DECAY OF ONE DIMENSIONAL JUMP-DIFFUSIONS

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ABSTRACT. We are interested by a one dimensional Markov process which moves following a diffusion for some random time and then jumps. It can represent some natural phenomena like size of cell or data transmission over the Internet. The paper begin with some results about Lipschitz contraction of semigroup. Our approach is connected with the notion of curvature introduced by Ollivier and Joulin. Our main results for jump-diffusions are quantitative estimates in Wasserstein distance, when the jump times depend of the space motion. We use different techniques which are a particular Feynmann-Kac interpretation and a non coalescent coupling. Several examples and applications are developed, including explicit formulas for the equilibrium, application to branching measure-valued processes and integrals of compound Poisson process with respect to a Brownian motion.

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1. INTRODUCTION AND MAIN RESULT

1.1. **Introduction and related work.** We are interested by a process that models a fragment which remains stable and moves for some random time and then jumps. It represents some natural phenomena that can be observed at a great variety of scales. To give just few examples, let us simply mention the studies of stellar fragments in astrophysics, fractures and earthquakes in geophysics, degradation of large polymer chains in chemistry, size of cell in biology, fragmentation of a hard drive in computer science. More precisely, This process $X = (X_t)_{t\geq 0}$ has E, a subinterval of $\mathbb{R}$, as state space and its infinitesimal generator is given, for any smooth enough function $f : E \mapsto \mathbb{R}$, by

$$\forall x \geq 0,\ \mathcal{L}f(x) = \sigma^2(x)f''(x) + g(x)f'(x) + r(x)\left(\int_0^1 f\left(F(x,\theta)\right) - f(x)d\theta\right) \tag{1}$$

where σ, r, g and $x \mapsto F(x,\cdot)$ are C^∞, $\theta \mapsto F(\cdot,\theta)$ is measurable, r and σ are non negative. Betweens the jumps, this process evolves like a diffusion $(Y_t)_{t\geq 0}$ which satisfies the following stochastic differential equation :

$$dY_s = g(Y_s)ds + \sqrt{2}\sigma(Y_s)dB_s \tag{2}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Then, at a random time T which verifies

$$\int_0^T r(Y_s)ds \sim \text{Exp}(1) \Leftrightarrow \mathbb{P}\left(T > t \mid Y_s,\ s \leq t\right) = \exp\left(-\int_0^t r(Y_s)ds\right)$$

it jumps following F. Indeed $X_T = F(X_{T-},\Theta) = F(Y_{T-},\Theta)$ where Θ is a uniform variable on $[0,1]$. Then, the process evolves like the diffusion and we repeat these steps again and again. This process is an hybrid process [5] and when $\sigma = 0$ it is a piecewise deterministic Markov process (PDMP) [14].

Some subclass of this process was studied. For instance, the articles [3, 10, 15, 17, 18, 26, 25, 32] study a version which represent the modeling of the famous Transmission Control Protocol (TCP) used for data transmission over the Internet. And in [1, 2, 13, 24, 31], some others examples represent the growth of some biological content of the cell (nutriments, parasites,...) which is shared randomly in the daughter cells when the cell divide. There is again a lot of application, like reliability theory and queuing [14, 23, 25]. Some theorems about the ergodicity and others properties are established in these papers. But to find an explicit bound for the speed of convergence is an interesting question which it remains to establish.

Our aim in this paper is to get quantitative estimates for the convergence to equilibrium of X. For some particular coefficient, it is ergodic and admits a unique invariant law. We can find conditions in [23], when the jumps are additive, and in [1, proposition 5.1], when the jumps are multiplicative. These two results are proved via Lyapunov techniques. Nevertheless, this process is, in general, irreversible since time reversed sample paths are not sample paths and it has infinite support. This makes Lyapunov techniques less efficient for the derivation of quantitative exponential ergodicity. Furthermore, another main difficulties is that entropy methods fails. In general, the process do not verify a Poincaré or log-Sobolev inequality.

Using different techniques (gradient estimate via Feynmann-Kac formula and non coalescent coupling), we arrive to prove different bounds in Wasserstein metric. These bounds are optimal in term of curvature of semigroup. The proofs are inspirated by [9, 10, 27] which gives similar result for others models or for particular cases. Recently, to find quantitative bounds, Harnack inequalities, functional inequalities or Wasserstein curvature, for jumps processes, has spurred an enormous amount of research [3, 4, 9, 12, 16, 17, 22, 23, 37].

The remainder is explained after the main results.

1.2. **Lipschitz contraction and Wassertsein curvature.** In this section, we present briefly some basic definition about Lipschitz contraction of Markov process. The results presented here hold in a very general setting. Some of these are new, even if there is some close theorem, and some others are not new, but we hope our proof is simpler. Let (E, d) be a Polish space, we denote by $\mathcal{P}_d(E)$ the set of probability measures μ on E such that $\int_{x\in E} d(x, x_0)\mu(dx) < \infty$ for some (or equivalently for all) $x_0 \in E$. We recall that the Wasserstein distance between two probability measures $\mu_1, \mu_2 \in \mathcal{P}_d(E)$ is defined by

$$\mathcal{W}_d(\mu_1, \mu_2) = \inf_{\nu\in \mathrm{Marg}(\mu_1,\mu_2)} \iint_{E^2} d(x, y)\nu(dx, dy), \tag{3}$$

where $\mathrm{Marg}(\mu_1, \mu_2)$ is the set of probability measures on E^2 such that the marginal distributions are μ_1 and μ_2, respectively. This infimum is attained [34]. The Kantorovich-Rubinstein duality [34, Theorem 5.10] gives that we also have the following representation:

$$\mathcal{W}_d(\mu_1, \mu_2) = \sup_{g\in \mathrm{Lip}_1(d)} \int_E g\, d\mu_1 - \int_E g\, d\mu_2, \tag{4}$$

where $\mathrm{Lip}(d)$ is the set of Lipschitz function g with respect to the distance d, i.e.

$$\|g\|_{\mathrm{Lip}(d)} := \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|g(x) - g(y)|}{d(x, y)} < \infty,$$

and $\mathrm{Lip}_1(d) = \{g \in \mathrm{Lip}(d) \mid \|g\|_{\mathrm{Lip}(d)} \leq 1\}$. If the semigroup is smooth enough, we have $\delta_x P_t \in \mathcal{P}_d(\mathbb{R}_+)$, for all $x \in E$ and $t \geq 0$. Hence, the semigroup is well defined on $\mathrm{Lip}(d)$. The Wasserstein curvature of $(X_t)_{t\geq 0}$ with respect to a given distance d is the optimal (largest) constant ρ in the following contraction inequality:

$$\|P_t\|_{\mathrm{Lip}(d)\to\mathrm{Lip}(d)} \leq e^{-\rho t}, \quad t \geq 0. \tag{5}$$

Here $\|P_t\|_{\mathrm{Lip}(d)\to\mathrm{Lip}(d)}$ denotes the supremum of $\|P_t f\|_{\mathrm{Lip}(d)}$ when f runs over $\mathrm{Lip}_1(d)$. It is actually equivalent to the property that

$$\mathcal{W}_d(\mu_1 P_t, \mu_2 P_t) \leq e^{-\rho t}\, \mathcal{W}_d(\mu_1, \mu_2), \quad \mu_1, \mu_2 \in \mathcal{P}_d(\mathbb{R}_+), \quad t \geq 0.$$

This notion of curvature was introduced by Joulin [22] and Ollivier [29] and is connected to the notion of Ricci curvature of Riemannian manifold. This definition of curvature is relatively close to the notion of Wasserstein spectral gap (see for instance, [19]) : A semigroup possesses a Wasserstein spectral gap if there exist $\lambda > 0$ and $C < 0$ such that

$$\mathcal{W}_d\left(\delta_x P_t, \delta_y P_t\right) \leq C e^{-\lambda t} d(x, y).$$

The presence or not of the constant C may be important for concentration inequalities. If the optimal constant is positive, and the process has a stationary probability measure π then the semigroup converges exponentially fast in Wasserstein distance $\mathcal{W}_d$ to the stationary distribution (it is enough to take $\mu_2 = \pi$). We can notice that an exponential decay is possible even though the Wasserstein curvature is null:

Theorem 1.1 (Exponential decay when $\rho = 0$)**.** *Let X be a (general) Markov process such that its Wasserstein curvature defined at (5) is null (or non-negative). If there exists $t_0 > 0$ such that*

$$K_{t_0} = \sup_{x_0, y_0 \in E} \frac{\mathcal{W}_d(\delta_{x_0} P_{t_0}, \delta_{y_0} P_{t_0})}{d(x_0, y_0)} < 1.$$

Then there exists $\kappa > 0$, such that for all $x_0, y_0 \in E$,

$$\forall t \geq t_0, \mathcal{W}_d(\delta_{x_0} P_t, \delta_{y_0} P_t) \leq e^{-t\kappa} d(x_0, y_0).$$

Furthermore, κ can be chosen as

$$\kappa = -\frac{\ln(K_{t_0})}{t_0}.$$

The Wasserstein curvature is a local characteristic. It corresponds to the worst exponential decay. In this paper we use two different distances. When $d(x, y) = |x - y|$ for all $x, y \in E$, $\mathcal{W}_d$ is the standard Wasserstein distance (also called Kantorovich distance). So, in this case, we denote it by $\mathcal{W}$ and $d = |\cdot|$. When $E \subset \mathbb{R}$ and F_1, F_2 are the cumulative functions of μ_1 and μ_2, we have

$$\mathcal{W}(\mu_1, \mu_2) = \int_E |F_1(x) - F_2(x)| dx = \|F_1 - F_2\|_{L^1}.$$

The convergence in Wasserstein distance is equivalently to the convergence in law and the convergence of the first moment.
When $d(x, y) = \mathbf{1}_{\{x=y\}}$ for all $x, y \in E$, $\mathcal{W}_d$ is the standard total variation distance. So, in this case, we denote it by d_{TV}. The equation (4) becomes

$$d_{\text{TV}}(\mu_1, \mu_2) = \frac{1}{2} \sup_{\|f\|_\infty \leq 1} \int_E f d\mu_1 - \int_E f d\mu_2.$$

If μ_1, μ_2 have density n_1, n_2 with respect to the Lebesgue measure, we have

$$d_{\text{TV}}(\mu_1, \mu_2) = \frac{1}{2} \int_E |n_1(x) - n_2(x)| dx = \frac{1}{2} \|n_1 - n_2\|_{L^1}.$$

We finish this section with two results. The first one gives the existence of an invariant distribution and the second one describes the link between Wasserstein decay and L^2-spectral gap.

Theorem 1.2 (Existence and uniqueness of the stationary distribution)**.** *Let $(P_t)_{t \geq 0}$ be a Markov semigroup, if its Wasserstein curvature ρ is positive then there is a unique invariant measure π. Furthermore,*

$$\forall \nu \in \mathcal{P}_d(E), \mathcal{W}_d(\mu P_t, \pi) \leq e^{-\rho t} \mathcal{W}_d(\nu, \pi)$$

We can compare this theorem (and its proof) with [11, Theorem 5.23].

Theorem 1.3 (Wassertsein contraction implies L^2-spectral gap for reversible semigroup)**.** *Let $(P_t)_{t \geq 0}$ be a semigroup of a Markov process. Assume:*

- *Its Wasserstein curvature, ρ, is positive.*
- *The semigroup is reversible.*
- *$Lip_1(d) \cap L^\infty(\pi) \cap L^2(\pi)$ is dense in $L^2(\pi)$.*

Then there is a L^2-spectral gap. Indeed,

$$Var_\pi(P_t f) \leq e^{-2\rho t} Var_\pi(f).$$

Or equivalently,

$$\left\| P_t f - \int f d\pi \right\|_{L^2(\pi)} \leq e^{-\rho t} \left\| f - \int f d\pi \right\|_{L^2(\pi)}.$$

We can see, in the proof, the importance of the reversibility. Nevertheless, for the most part of our examples, this condition is never verified. This theorem is just the continuous time adaptation of [19, Proposition 2.8]. We can compare it (and its proof) with [11, Theorem 5.23] and [36, Theorem 2.1 (2)] for continuous time version and [29, Proposition 30]. Notice that the two last theorems are also true when there is a Wasserstein spectral gap instead a positive curvature. Let us end this section by advertising on a parallel and independent work [33] which give some related results.

1.3. **Main results : Quantitative bounds of jump-diffusions.** In this section, we first recall some properties of the process that we study, then we state our main results. Let $P_t f(x) = E[f(X_t)|X_0 = x]$ be the semigroup generated by (1). By the Itô-Dynkin, it means that

$$P_t f(x) = f(x) + \int_0^t P_s \mathcal{L} f(x) ds = f(x) + \int_0^t \mathcal{L} P_s f(x) ds \tag{6}$$

for all f smooth enough (see [5] or [14] for the domain of the generator). We can describe the evolution of X in terms of stochastic differential equations (SDE). Let $(B_s)_{s\geq 0}$ be a standard Brownian motion, and let $Q(ds, du, d\theta)$ be a Poisson point measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times [0,1]$, of intensity $ds\ du\ d\theta$, independent from the Brownian motion. Here, $ds, du, d\theta$ are the Lebesgue measures on $\mathbb{R}_+, \mathbb{R}_+$ and $[0,1]$. Then we have,

$$\begin{aligned} X_t = X_0 &+ \int_0^t g(X_s) ds + \int_0^t \sqrt{2}\sigma(X_s) dB_s \\ &+ \int_0^t \int_{E\ \times [0,1]} \mathbf{1}_{\{u \leq r(X_{s-})\}} \left(F(X_{s-}, \theta) - X_{s-}\right) Q(ds, du, d\theta). \end{aligned} \tag{7}$$

We are interested by the long time behavior and not by the existence. Hence, in all the paper, the coefficient are supposed to be smooth. Now, we are able to express our main results.

Theorem 1.4 (Wasserstein curvature for jump-diffusion which conserves the monotonicity)**.** *Assume*

$$\int_0^1 \partial_x F(x,\theta) d\theta \neq 0 \text{ and } r'(x) \int_0^1 x - F(x,\theta) d\theta \leq 0.$$

If

$$\inf_{x\geq 0} \left(-g'(x) + r(x) \left(1 - \int_0^1 \partial_x F(x,\theta) d\theta\right) - r'(x) \int_0^1 x - F(x,\theta) d\theta \right) = \rho > 0.$$

Then the contraction inequality (5) *is satisfied with the optimal constant* ρ. *Indeed, we have*

$$\mathcal{W}(\mu P_t, \nu P_t) \leq e^{-\rho t} \mathcal{W}(\mu, \nu)$$

for all measure μ *and* ν *which have a first moment.*

In section **??**, we will see that, under this condition, the process is stochastically monotone. Indeed, the semigroup preserves the monotonicity. It is easy to see that the conserve is true. In particular this theorem can be used when

- r is decreasing.
- $F(x,\theta) < x$, for almost all $\theta \in [0,1]$.

- $x \mapsto F(x, \theta)$ is increasing.

For instance, the jumps can be multiplicative (indeed $F(x, \theta) = \varphi(\theta)x$, where $\varphi \leq 1$), additive (indeed $F(x, \theta) = x - \varphi(\theta)$, where $\varphi \geq 0$), or a combination of these two types. We can, of course, also suppose that r is increasing, and, for almost all $\theta \in [0, 1]$, $F(x, \theta) > x$.

In the expression of ρ, we can see the interplay between the drift parameter (indeed g) and the jump mechanism (indeed r and F). We also see that the curvature do not depend to the diffusive term σ. We also give an other criterion to find the Wasserstein curvature, which is just a generalisation of the result of [10, theorem 2.3]:

Theorem 1.5 (Wasserstein curvature when r is affine)**.** *If the following assumptions hold*

- $E \subset \mathbb{R}_+$.
- $F(x, \theta) < x$*, for almost all* $\theta \in [0, 1]$.
- $x \mapsto F(x, \theta)$ *is increasing.*
- $\partial_x F(x, \theta) \leq 1$.
- $r(x) = ax + b$*, for all* $x \geq 0$*, where* $a, b \geq 0$

Then the contraction inequality (5) *verifies*

$$\rho \geq -\sup_{z \in E} g'(z) + b\left(1 - \sup_{z \in E} \int_0^1 \partial_x F(z, \theta) d\theta\right).$$

This bound looks like the bound of theorem 1.4 (because $E \subset \mathbb{R}_+$ and $F(x, \theta) < x$ implies $\inf_{x \in E} x - F(x, \theta) = 0$). It will be interesting to prove that this bound is optimal and right for all r. These two theorems give interesting estimates in Wasserstein distance. It is also interesting to prove some bounds in total variation distance, but it is more difficult. For instance, for the TCP windows size process, between the jumps the trajectories are parallel, thus it seems impossible to intersect it (see section 4). A first approach is given in [3].

It will be interesting to find the same type of result when we add a discrete component to our process. It will be generated by

$$Lf(x, n) = \sigma(x, n) f''(x) + g(x, n) f'(x) + r(x, n)\left(\int_0^1 f\left(F(x, n, \theta)\right) d\theta - f(x, n)\right).$$

In this case, $X = (Z, N)$, and Z may be continuous (see [16], for instance). It is also interesting to investigate the multidimensional case. The proof of theorem 1.4, is not adaptable in these two cases.

The papers is organised as follow: The next section is devoted to the proof of theorem about Lipschitz contraction. The proof of theorem 1.4 and theorem 1.5 are in section 3. Many examples and applications are given in the last section.

2. General properties about Lipschitz contraction

In this section we prove the theorems of section 1.2

2.1. **Exponential decay when $\rho = 0$ (proof of theorem 1.1).** Let X be a Markov process such that its Wasserstein curvature defined at (5) is null (or non-negative). Assume there exists $t_0 > 0$ such that

$$K_{t_0} = \sup_{x_0, y_0 \in E} \frac{\mathcal{W}_d(\delta_{x_0} P_{t_0}, \delta_{y_0} P_{t_0})}{d(x_0, y_0)} < 1.$$

We will prove that there exists $\kappa > 0$, such that for all $x_0, y_0 \in E$,

$$\forall t \geq t_0, \mathcal{W}_d(\delta_{x_0} P_t, \delta_{y_0} P_t) \leq e^{-t\kappa} d(x_0, y_0).$$

Let ω be defined, for all $x \neq y$, by

$$\omega(t,x,y) = \frac{\mathcal{W}_d\left(\mathcal{L}(X_t \mid X_0 = x), \mathcal{L}(X_t \mid X_0 = y)\right)}{|x-y|},$$

and

$$\bar{\omega}(t) = \sup_{x,y \in E} \omega(t,x,y).$$

Let (Y, Z) be any coupling of the law of X. Indeed $Y \overset{d}{=} Z \overset{d}{=} X$ but Y and Z are not necessary independent. The Markov property gives

$$\begin{aligned}
\omega(t,y,z)d(y,z) &= \sup_{f \in \mathrm{Lip}_1} \mathbb{E}\left[f(Y_{t+s}) \mid Y_0 = y\right] - \mathbb{E}\left[f(Z_{t+s}) \mid Z_0 = z\right] \\
&= \sup_{f \in \mathrm{Lip}_1} \mathbb{E}\left[\mathbb{E}\left[f(\tilde{Y}_t) \mid \tilde{Y}_0 = Y_s\right] - \mathbb{E}\left[f(\tilde{Z}_t) \mid \tilde{Z}_0 = Z_s\right] \mid Y_0 = y, Z_0 = z\right] \\
&\leq \mathbb{E}\left[\, \bar{\omega}(t) d(Y_s, Z_s) \mid Y_0 = y, Z_0 = z\right],
\end{aligned}$$

where $(\tilde{Y}, \tilde{Z})$ is a coupling of the law of X. Taking the infimum on all coupling we deduce,

$$\omega(t+s,x,y) \leq \bar{\omega}(t)\omega(s,x,y) \;\Rightarrow\; \bar{\omega}(t+s) \leq \bar{\omega}(t)\bar{\omega}(s).$$

Now, by the hypothesis, the curvature is non-negative, thus

$$\forall t > 0,\ \bar{\omega}(t) \leq 1.$$

Hence, $\bar{\omega}$ is decreasing. But, by the assumptions, there exists $t_0 > 0$ such that $\bar{\omega}(t_0) < 1$, and so

$$\forall t \geq t_0,\ \bar{\omega}(t) < 1.$$

Finally, for all $t \geq t_0$, there exists $n \in \mathbb{N}$ such that $t \geq n t_0$, and then

$$\begin{aligned}
\mathcal{W}_d(\delta_x P_t, \delta_y P_t) &\leq \bar{\omega}(t)\, \mathcal{W}_d(\delta_x, \delta_y) \leq \bar{\omega}(n t_0)\, \mathcal{W}_d(\delta_x, \delta_y) \\
&\leq \bar{\omega}(t_0)^n\, \mathcal{W}_d(\delta_x, \delta_y) \leq \exp\left(t\, \frac{\ln(\bar{\omega}(t_0))}{t_0}\right)\, \mathcal{W}_d(\delta_x, \delta_y).
\end{aligned}$$

2.2. **Existence criterion of the invariant distribution (proof of theorem 1.2).** Let X be a Markov process and $(P_t)_{t \geq 0}$ its semigroup. First, let us assume that

- $\mathcal{W}_d(\mu P_t, \nu P_t) \leq e^{-\rho t} \mathcal{W}_d(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}_d(E)$.
- $C = \sup_{t \geq 0} \mathcal{W}_d(\delta_x, \delta_x P_t) < +\infty$ for some $x \in E$.

We will prove that gives the existence of an invariant measure. The proof which follows is simple and comes from a discussion with D.Villemonais. He has developed the same type of criterion, for the existence of Quasi-stationary distribution, with the help of P. Del moral in a work still to be completed (see [35, chapter 5]). Consider the sequence $(\delta_x P_t)_{t \geq 0}$, it verifies

$$\begin{aligned}
\forall t, s \geq 0,\ \mathcal{W}_d(\delta_x P_{t+s}, \delta_x P_t) &\leq e^{-\rho s} \mathcal{W}_d(\delta_x P_s, \delta_x) \\
&\leq e^{-\rho s} C.
\end{aligned}$$

Furthermore, the space $(\mathcal{P}_d(E), \mathcal{W}_d)$ is known to be complete [11, theorem 5.4] thus the sequence $(\partial_x P_t)_{t\geq 0}$ converges to a measure $\pi \in \mathcal{P}_d(E)$. Then

$$\lim_{t\to+\infty} \mathcal{W}_d(\eta P_{t+s}, \eta P_t) = \mathcal{W}_d(\pi P_s, \pi) = 0.$$

Thus π is invariant, and the Lipschitz contraction gives the uniqueness and the exponential decay. Now, it rest to prove that $\sup_{t\geq 0} \mathcal{W}_d(\delta_x, \delta_x P_t) < +\infty$ for some $x \in E$. We have

$$\sup_{\|f\|_{\mathrm{Lip}_1}\leq 1} d(P_t f(x), P_t f(y)) \leq e^{-\rho t} d(x, y).$$

Hence P_t remains stable the space of Lipschitz function. As we have $\mathcal{W}_d(\delta_x, \delta_x P_t) = P_t f_x(x)$, where $f_x(y) = d(x, y)$, the mapping $x \mapsto \mathcal{W}_d(\delta_x, \delta_x P_t)$ is continuous and for all $x \in E$ and $\varepsilon > 0$ we have

$$\sup_{t\in[0,\varepsilon+1]} \mathcal{W}_d(\delta_x, \delta_x P_t) < +\infty. \tag{8}$$

And we have for all $k \in \mathbb{N}^*$ and $t \in [\varepsilon, \varepsilon + 1]$,

$$\begin{aligned}
\mathcal{W}_d(\delta_x, \delta_x P_{kt}) &\leq \sum_{j=1}^{k-1} \mathcal{W}_d(\delta_x P_{jt}, \delta_x P_{(j+1)t}) \\
&\leq \sum_{j=1}^{k-1} e^{-j\rho t} \mathcal{W}_d(\delta_x, \delta_x P_t) \\
&\leq \mathcal{W}_d(\delta_x, \delta_x P_t) \frac{1}{1 - e^{-\varepsilon\rho}}
\end{aligned}$$

The last inequalities and (8) gives the needed boundedness.

2.3. **Lipschitz contraction and L^2-Spectral Gap (proof of theorem 1.3).** The proof of this result is a continuous-time adaptation to [19, Proposition 2.8]. Let f be a non-negative, Lipschitz and bounded function such that $\int_E f d\pi = 1$. Using the reversibility, we have,

$$\begin{aligned}
\mathrm{Var}_\pi(P_t f) &= \int_E f P_{2t} f d\pi - \left(\int_E f d\pi\right)^2 \\
&\leq \|f\|_{\mathrm{Lip}(d)} \mathcal{W}_d\left(P_{2t} f d\pi, \pi\right)
\end{aligned}$$

where $P_{2t} f d\pi$ is the measure which verifies, for all smooth φ

$$\begin{aligned}
\int_E \varphi P_{2t} f d\pi &= \int_E \int_E \varphi(x) f(y) P_{2t}(x, dy) \pi(dx) \\
&= \int_E \int_E \varphi(x) f(y) P_{2t}(y, dx) \pi(dy).
\end{aligned}$$

Because π is reversible. Hence,

$$\begin{aligned}
\mathrm{Var}_\pi(P_t f) &\leq \|f\|_{\mathrm{Lip}(d)} \mathcal{W}_d\left((f\pi) P_{2t}, \pi\right) \\
&\leq C_f e^{-2\rho t}.
\end{aligned} \tag{9}$$

By translation/dilatation, the last inequality holds for all Lipschitz and bounded f, and by density, for all $f \in L^2(\pi)$. Now, let f be a measurable function such that,

$$\int_E f(x)\pi(dx) = 0 \text{ and } \int_E f(x)^2\pi(dx) = 1.$$

Applying spectral theorem, Jensen inequality and (9), we find

$$\begin{aligned}\mathrm{Var}_\pi(P_t f) &= \int P_t f^2 d\pi = \int_E \int_0^\infty e^{-\lambda t} dE_\lambda(f) d\pi \\ &\leq \left(\int_E \int_0^\infty e^{-\lambda(t+s)} dE_\lambda(f) d\pi \right)^{\frac{t}{t+s}} \\ &\leq C_f^{\frac{t}{t+s}} e^{-\rho t}\end{aligned}$$

Taking the limit $s \to +\infty$, we conclude the proof.

3. Exponential decay for jump-diffusions

In this section, we prove the announced results about jumps-diffusion and add some remark or corollary.

3.1. **Gradient estimate (proof of theorem 1.4).** In this section we follow the same approach to [9]. Using a Feynmann-Kac semigroup, we estimate the derivative of our semigroup. So, we begin to prove, for any smooth enough function f,

$$\forall x \geq 0,\ (P_t f)'(x) = \mathbb{E}\left[f'(Y_t) e^{-\int_0^t V(Y_s)ds} \right]. \tag{10}$$

Where Y is a Markov process generated by

$$\begin{aligned}\mathcal{L}_S f(x) &= \sigma^2(x) f''(x) + \left(2\sigma(x)\sigma'(x) + g(x)\right) f'(x) \\ &\quad + r(x) \int_0^1 \partial_x F(x,\theta) d\theta \left(\frac{1}{\int_0^1 \partial_x F(x,\theta)d\theta} \int_0^1 \partial_x F(x,\theta) f(F(x,\theta)) d\theta - f(x) \right) \\ &\quad - r'(x) \int_0^1 x - F(x,\theta) d\theta \left(\frac{1}{\int_0^1 x - F(x,\theta)d\theta} \int_0^1 \int_{F(x,\theta)}^x f(u) du \, d\theta - f(x) \right).\end{aligned}$$

Lemma 3.1 (Gradient estimate via Feynmann-Kac formula)**.** *Assume*

$$\int_0^1 \partial_x F(x,\theta) d\theta \neq 0 \text{ and } r'(x) \int_0^1 x - F(x,\theta) d\theta \leq 0.$$

If

$$\forall x \in E,\ \mathbb{E}\left[\exp\left(-\int_0^t V(Y_s) ds \right) \mid Y_0 = x \right] < +\infty.$$

where

$$V(x) = -g'(x) + r(x) \left(1 - \int_0^1 \partial_x F(x,\theta) d\theta \right) - r'(x) \int_0^1 x - F(x,\theta) d\theta.$$

Then (10) *holds.*

Proof. First, we have $(\mathcal{L}f)' = (\mathcal{L}_\mathcal{S} - V)\, f'$. Then, by the Itô-Dynkin formula (6),

$$\begin{aligned}(P_t f)'(x) &= f'(x) + \int_0^t (\mathcal{L} P_s f)'(x) ds \\ &= f'(x) + \int_0^t (\mathcal{L}_\mathcal{S} - V)\,(P_s f)'(x) ds\end{aligned}$$

Thus $t \mapsto (P_t f)'$ verifies this partial differential equation: $\partial_t u = (\mathcal{L}_\mathcal{S} - V)\, u$. But, it is known that the following semigroup verify this equation too:

$$S_t f(x) = \mathbb{E}\left[f(Y_t) e^{-\int_0^t V(Y_s) ds} \right]$$

where Y is generated by $\mathcal{L}_S$. As there is a unique solution, we get $(P_t f)' = S_t f'$. □

Corollary 3.2 (Propagation of monotonicity)**.** *Under the same assumption, if f is non-increasing then $P_t f$ is also non-increasing.*

This corollary implies that, for all $x < y$, there exists a coupling (X^x, X^y), such that the marginals are generated by $\mathcal{L}$, which start from (x, y), and

$$\forall t \geq 0,\; X_t^x \leq X_t^y \quad \text{a.s..}$$

It is why this type of formula is available only for a non general class of r.

proof of theorem 1.4. Let f be a Lipschitz and smooth function. For any $x, y \geq 0$, we have by the intertwining identity (10),

$$\begin{aligned}|P_t f(x) - P_t f(y)| &\leq \sup_{z \geq 0} |(P_t f)'(z)| |x - y| \\ &\leq \sup_{z \geq 0} \mathbb{E}\left[|f'(Y_t)| e^{-\int_0^t V(Y_s) ds} | Y_0 = z \right] |x - y| \\ &\leq \sup_{z \geq 0} \mathbb{E}\left[e^{-\int_0^t V(Y_s) ds} | Y_0 = z \right] |x - y|.\end{aligned}$$

so that dividing by $|x - y|$ and taking suprema entail the inequality:

$$\|P_t\|_{Lip \to Lip} \;\leq\; \sup_{z \geq 0} \mathbb{E}\left[\exp\left(- \int_0^t V(Y_s)\, ds \right) \;|\; Y_0 = z \right].$$

Finally, the right-hand-side of the latter inequality is nothing but $\|P_t(id)\|_{Lip}$, showing that the supremum over Lip is attained for the function $id : x \mapsto x$. It achieves the proof. □

Remark 3.3 ($h-$transform and first eigenvalue)**.** *Assume that $\mathcal{L}_\mathcal{S} - V$ have a first eigenvalue such that its eigenvector ψ is positive. Using an $h-$transform with $h = e^{-\lambda t}\psi$, we get for any Lipschitz function f,*

$$\mathbb{E}\left[f'(Y_t) e^{-\int_0^t V(Y_s) ds} \right] = e^{-\lambda t} \psi(x) \mathbb{E}\left[\frac{f'(Z_t)}{\psi(Z_t)} \right] \leq e^{-\lambda t} \mathbb{E}\left[\frac{\psi(Z_0)}{\psi(Z_t)} \right].$$

Then, if ψ is smooth enough, the Wasserstein spectral gap is λ. See section **??** *for an application.*

Theorem 3.4 (Exponential decay when the curvature is null)**.** *Under the assumption of the previous lemma and if Y is irreducible, $V \geq 0$, and there exist $\mathcal{O} = (a, b)$ and $\varepsilon > 0$ such that $\mathcal{O}^c$ is compact and*

$$\forall x \in \mathcal{O}, V(x) \geq \varepsilon.$$

Then there exist $t_0 \geq 0$ and $\kappa > 0$ such that

$$\forall t \geq t_0,\ \mathcal{W}(\mu P_t, \nu P_t) \leq e^{-\kappa t}\, \mathcal{W}(\mu, \nu)$$

for any $\mu, \nu \in \mathcal{P}_{|.|}(\mathbb{R}_+)$.

a and b may be infinite if there is not in the same time. Notice that, if there exist $x \in E$, such that $V(x) = 0$ then the curvature is null.

Proof. The proof is adapted to [27, section 5.1]. Let

$$D(t, x) = \mathbb{E}_x \left[\exp\left(- \int_0^t V(Y_s) ds \right) \right]$$

and $\bar{D}(t) = \sup_{x \geq 0} D(t, x)$. It is easy to see that for all $x \in E$, $D(\cdot, x)$ and $\bar{D}$ are decreasing. Furthermore, $D(t, x) = 1$ if and only if, starting from x, $Y_s \in \mathcal{O}^c$ almost surely, for almost all $s \leq t$. Then, as Y is irreducible, we have

$$\forall x \geq 0,\ \forall t > 0, D(t, x) < 1. \tag{11}$$

If $x \in \mathcal{O}$, then it is also right for $t = 0$. We begin to prove that there exist t_0 such that

$$\forall t \geq t_0,\ \bar{D}(t) < 1.$$

Then, we will prove the assertion of the theorem. For $x \in \mathcal{O}$, set $\tau = \inf\{t \geq 0 \mid Y_t \in \mathcal{O}^c\}$. We have,

$$\begin{aligned} D(t, x) &= \mathbb{E}_x \left[\mathbf{1}_{\tau < t} \exp\left(- \int_0^t V(Y_s) ds \right) \right] + \mathbb{E}_x \left[\mathbf{1}_{\tau \geq t} \exp\left(- \int_0^t V(Y_s) ds \right) \right] \\ &\leq \mathbb{E}_x \left[\mathbf{1}_{\tau < t} \exp\left(- \int_0^t V(Y_s) ds \right) \right] + e^{-\varepsilon t} \end{aligned}$$

And

$$\begin{aligned} \mathbb{E}_x \left[\mathbf{1}_{\tau < t} \exp\left(- \int_0^t V(Y_s) ds \right) \right] \leq&\ \mathbb{E}_x \left[\mathbf{1}_{\tau < t} e^{-\varepsilon t} \mathbb{E}_x \left[\exp\left(- \int_\tau^t V(Y_s) ds \right) \mid \mathcal{F}_\tau \right] \right] \\ \leq&\ \mathbb{E}_x \left[\mathbf{1}_{\tau < t} e^{-\varepsilon t} \max_{c \in \{a,b\}} (D(t - \tau, c)) \right] \\ \leq&\ \mathbb{E}_x \left[\mathbf{1}_{\tau < t/2} e^{-\varepsilon t} \max_{c \in \{a,b\}} (D(t - \tau, c)) \right] \\ &+ \mathbb{E}_x \left[\mathbf{1}_{t/2 \leq \tau < t} e^{-\varepsilon t} \max_{c \in \{a,b\}} (D(t - \tau, c)) \right] \\ \leq&\ \max_{c \in \{a,b\}} (D(t - \tau, c)) + e^{-\varepsilon t/2} \end{aligned}$$

where $\mathcal{F}$ is the filtration associates to Y. Thus,

$$\sup_{x\in\mathcal{O}} D(t,x) \leq \max_{c\in\{a,b\}} (D(t-\tau,c)) + e^{-\varepsilon t} + e^{-\varepsilon t/2}.$$

Then we deduce that $\limsup_{t\to+\infty} \sup_{x\in\mathcal{O}} D(t,x) < 1$ and then the existence of t_0 such that for all $t \geq t_0$,

$$\sup_{x\in\mathcal{O}} D(t,x) < 1.$$

When $x \in \mathcal{O}^c$, we see that he Feynmann-Kac semigroup is continuous because Y verifies the Feller property, then, the supremum of $\sup_{x\in\mathcal{O}^c} D(t,x)$ is attained. And finally, by (11), we have $\bar{D}(t) < 1$ for all $t \geq t_0$. Now, we can use theorem 1.1 to conclude. We can also use the argument to [27]. That is, the Markov property ensures that

$$\begin{aligned} D(t+s,x) &= \mathbb{E}_x\left[\exp\left(-\int_0^t V(Y_u)du\right)\mathbb{E}_{Y_t}\left[\exp\left(-\int_0^t V(Y_u)du\right)\right]\right] \\ &\leq \mathbb{E}_x\left[\exp\left(-\int_0^t V(Y_u)du\right)\bar{D}(s)\right] = D(t,x)\bar{D}(s) \\ &\leq \bar{D}(t)\bar{D}(s) \end{aligned}$$

and then $\bar{D}(t+s) \leq \bar{D}(t)\bar{D}(s)$. For all $t > t_0$, there exists $n \in \mathbb{N}$ such that $t \geq nt_0$. Hence, we have

$$\begin{aligned} \mathcal{W}(\mu P_t, \nu P_t) &\leq \bar{D}(t)\,\mathcal{W}(\mu,\nu) \leq \bar{D}(nt_0)\,\mathcal{W}(\mu,\nu) \\ &\leq \bar{D}(t_0)^n\,\mathcal{W}(\mu,\nu) \leq \exp\left(t\,\frac{\ln(\bar{D}(t_0))}{t_0}\right)\,\mathcal{W}(\mu,\nu). \end{aligned}$$

□

3.2. **Contraction by coupling (Proof of theorem 1.5).** As in [10], we make a coupling which favour the simultaneous jumps. Because, by the properties of F, during a simultaneous jump, the distance is decreasing. More precisely, this couple is generated by

$$\begin{aligned} Gf(x,y) &= \sigma(x)\partial_{xx}f(x,y) + \sigma(y)\partial_{yy}f(x,y) + g(x)\partial_x f(x,y) + g(y)\partial_y f(x,y) \\ &\quad + r(y)\int_0^1 f\left(F(x,\theta),F(y,\theta)\right)d\theta \\ &\quad + (r(x)-r(y))\int_0^1 f\left(F(x,\theta),y\right)d\theta - r(x)f(x,y) \end{aligned}$$

if $x \geq y$, and

$$\begin{aligned} Gf(x,y) &= \sigma(x)\partial_{xx}f(x,y) + \sigma(y)\partial_{yy}f(x,y) + g(x)\partial_x f(x,y) + g(y)\partial_y f(x,y) \\ &\quad + r(x)\int_0^1 f\left(F(x,\theta),F(y,\theta)\right)d\theta \\ &\quad + (r(y)-r(x))\int_0^1 f\left(x,F(y,\theta)\right)d\theta - r(y)f(x,y) \end{aligned}$$

in the other case. The first line describe the space motion between the jumps and the second line the jump mechanism. The dynamics of the couple of components is as follows:

(1) After an "appropriate" time which depends of the two trajectories (or only one in some cases). The upper coordinate jumps.
(2) Simultaneously, the other one "tosses an appropriate coin" whose probability of success depends on the positions on the two components to decide whether or not it jumps too.
(3) In the case of joint jumps, both components use the same uniform variable.
(4) Then, we repeat these three first steps again and again...

Let α be defined by $\alpha_{(x,y)}(t) = \mathbb{E}[|X_t - Y_t|]$, where (X, Y) is generated by G and starting from (x, y). It verifies $\alpha'_{(x,y)}(0) = Gf(x, y)$ where $f : (u, v) \mapsto |u - v|$. Assume $x > y$, we have,

$$
\begin{aligned}
\alpha'_{(x,y)}(0) &= g(x) - g(y) + (ay+b)\int_0^1 (F(x,\theta) - F(y,\theta))d\theta \\
&+ a(x-y)\int_0^1 |F(x,\theta) - y|\, d\theta - (ax+b)(x-y) \\
&\leq (x-y)[\sup_{z\in E} g'(z) + (ay+b)\sup_{z\in E}\int_0^1 \partial_x F(z,\theta)d\theta \\
&+ a\int_0^1 |F(x,\theta) - y|\, d\theta - (ax+b)] \\
&\leq (x-y)[\sup_{z\geq 0} g'(z) + b\left(\sup_{z\in E}\int_0^1 \partial_x F(z,\theta)d\theta - 1\right) \\
&+ a\int_0^1 \left(F(x,\theta) - x + y(1 - \sup_{z\in E}\int_0^1 \partial_x F(z,\theta)d\theta)\right)\mathbf{1}_{F(x,\theta)\geq y} \\
&+ \left((y-x) + y(1 - \sup_{z\in E}\int_0^1 \partial_x F(z,\theta)d\theta)\right)\mathbf{1}_{F(x,\theta)<y} d\theta] \\
&\leq (x-y)\left[\sup_{z\geq 0} g'(z) + b\left(\sup_{z\in E}\int_0^1 \partial_x F(z,\theta)d\theta - 1\right)\right]
\end{aligned}
$$

Then, by the Markov property,

$$
\begin{aligned}
\alpha'_{(x,y)}(t) &\leq \left(\sup_{z\geq 0} g'(z) + b\left(\sup_{z\in E}\int_0^1 \partial_x F(z,\theta)d\theta - 1\right)\right)\alpha_{(x,y)}(t) \\
\Rightarrow \alpha_{(x,y)}(t) &\leq \exp\left(\sup g' + b\left(\sup_{z\in E}\int_0^1 \partial_x F(z,\theta)d\theta - 1\right)t\right)|x-y|.
\end{aligned}
$$

The end of the proof is standard.

4. Examples and Applications

We begin by the easier example. This example show the interplay between the growth and the jump rate. After, we consider five examples. The first one represents a process, which grows linearly, and jumps additively. Then, the three next examples are different growth-fragmentation processes. We called growth-fragmentation process, a process generated by

$$\mathcal{L}f(x) = \sigma^2(x)f''(x) + g(x)f'(x) + r(x)\left(\int_0^1 f(hx)\mathcal{H}(dh) - f(x)\right). \tag{12}$$

The state space can be $E = \mathbb{R}$ or $E = \mathbb{R}_+$. And we finish by an example of structured-population.

4.1. **Straightforward consequences.**

4.1.1. *Jumps independent of the current state.* Assume $F(x,\theta) = F(\theta)$ for almost all $\theta \in [0,1]$. It means that, when a jump occurs, the new position of the process do not depend of the previous position. This process forgets its origin just after its first jump. We can find an expression of the invariant distribution in [25]. In this case, the theorem 1.4, is valid when r increases before $\int_0^1 F(\theta)d\theta$ and decreases after. And we have that the curvature ρ verifies

$$\rho \geq \inf_{z\in E} r(z) - \sup_{z\in E} g'(z).$$

4.1.2. *Feller diffusion with multiplicative jumps : The Bansaye-Tran process.* Let us consider the process studied in [2]. It evolves like a Feller diffusion between the jumps. Indeed, it moves following the E.D.S.:

$$\forall t \geq 0,\ dY_t = \mathbf{g}Y_t dt + \sqrt{2\sigma Y_t}dB_t.$$

Where $\mathbf{g}$ are σ are two positive numbers. And, when it jumps from x, this new state is Hx parasite, where $H \sim \mathcal{H}$, and $\mathcal{H}$ is a measure supported in $[0,1]$. This process models the rate of parasite in a cell population. The parasites grow in each cell and, sometimes, the cells divide. These two phenomena do not unwind in the same time scale. The parasite are born and die faster than the cell divide. Hence, the rate of parasite is modelled by a Feller diffusion. This one can be understood as the limit of birth and death process. The jumps models the division of cells. In this setting, we have

Corollary 4.1 (Exponentially decreasing to 0)**.** *If r is decreasing or affine and*

$$\mathbf{g} < \inf_{x\geq 0} r(x) \int_0^1 h\mathcal{H}(dh)$$

then

$$\mathbb{E}[|X_t|] \leq \exp\left(-t\ \left(\inf_{x\geq 0} r(x)\int_0^1 h\mathcal{H}(dh) - \mathbf{g}\right)\right)\mathbb{E}[X_0].$$

Proof. Under theorem 1.4 and theorem 1.5, the Wasserstein curvature is positive. Furthermore, δ_0 is invariant and $\mathcal{W}(\delta_x P_t, \delta_0) = P_t Id(x) = \mathbb{E}[X_t \mid X_0 = x]$. □

The results of [2] are better even if this corollary gives a bound for the L^1-convergence. To compare, for instance, [2, proposition 3.1]) tells us

Theorem 4.2 (Extinction criterion when r is constant)**.** *We have the following duality.*
(i) If $\mathbf{g} \leq -r\int_0^1 \log(h)\mathcal{H}(dh)$, *then* $\mathbb{P}(\exists t > 0,\ Y_t = 0) = 1$.
Moreover if $\mathbf{g} < -r\int_0^1 \log(h)\mathcal{H}(dh)$,

$$\exists\alpha > 0, \forall x_0 \geq 0, \exists c > 0, \quad \mathbb{P}(X_t > 0 \mid X_0 = x_0) \leq ce^{-\alpha t} \qquad (t \geq 0).$$

(ii) If $\mathbf{g} > -r\int_0^1 \log(h)\mathcal{H}(dh)$*, then* $\mathbb{P}(\forall t \geq 0,\ X_t > 0) > 0$.
Furthermore, for every $0 \leq \alpha < \mathbf{g} + r\int \log(h)\mathcal{H}(dh)$*,*

$$\mathbb{P}\left(\lim_{t\to+\infty} e^{-\alpha t} Y_t = \infty\right) = \{\forall t,\ X_t > 0\} \quad \textit{a.s.}$$

When r is constant and $\mathbf{g} < r\kappa$, we have $\mathbf{g} < -r\int_0^1 \log(h)\mathcal{H}(dh)$. Thus X converges almost surely to 0, and

$$\mathbb{E}[|X_t|] \leq e^{-(r\kappa-\mathbf{g})}\mathbb{E}[|X_0|].$$

More precisely, a rapidly calculation (which is given in the more general case hereafter) gives

$$\mathbb{E}[X_t] = e^{(\mathbf{g}-r\kappa)t}\mathbb{E}[X_0].$$

So, if

$$\int_0^1 -\log(h)\mathcal{H}(dh) > \frac{\mathbf{g}}{r} > \kappa = 1 - \int_0^1 h\mathcal{H}(dh)$$

then

$$\lim_{t\to+\infty} X_t = 0 \text{ a.s. but } \lim_{t\to+\infty} \mathbb{E}[X_t] = +\infty.$$

These two convergences explain why our criterion is not applicable and why the convergence is slow in this case.

4.1.3. *Speed of convergence for branching measure-valued processes.* Consider a model of structured population. We observe a content Y of a cell which evolves following the stochastic differential equation (2) and after an exponential time T, the cell dies and is replaced by a random number of offspring K. The content of the mother is shared in each new cells. Indeed, the offspring $j \leq K$ have the content $F(Y_T, \Theta)$. Consider the measure defined, for all $t \geq 0$, by

$$Z_t = \sum_{u\in V_t} \delta_{X_t^u}$$

where X_t^u is the content of the cell u at time t, and V_t the set of individual alive. It was proved in [1] and [13] that

$$\mathbb{E}\left[\int_E f(x)Z_t(ds)\right] = \mathbb{E}\left[f(X_t)\right].$$

Where X is generated by (1), with biased parameter. Hence, we deduce the long time behavior, and the contraction properties of the mean measure. A same formula holds when r is not constant (see [13]). It will be interesting to capture the speed of convergence to Z instead of $\mathbb{E}[Z]$. A first approach is given in [13, theorem 1.2].

4.1.4. *Stress release, repairable system and workload models.* Assume that $E = \mathbb{R}$, and let us define

$$\mathcal{L}_1 f(x) = \mathbf{g}f'(x) + r(x)\int_0^1 f(x - G(x,\theta)) - f(x)d\theta,$$

where $\mathbf{g}$ is a positive number and G is a non-negative and measurable function. The process generated by $\mathcal{L}_1$ was studied in [23]. This class represents a generic model of applied probability and is of importance in earthquake modeling, reliability theory and queuing. We have

Corollary 4.3 (Wasserstein curvature when r is decreasing)**.** *If r is decreasing, $x \mapsto G(x,\cdot)$ is, almost surely, increasing, we have*

$$\rho \geq \inf_{z\in R} \left(r(z) \int_0^1 \partial_x G(x,\theta) d\theta \right) - \sup_{z\in\mathbb{R}} r'(z) \int_0^1 G(x,\theta) d\theta.$$

In particular, if G do not depend to x, then we have

$$\rho = -\sup_{z\in\mathbb{R}} r'(z) \int_0^1 G(\theta) d\theta.$$

This corollary implies a bound to the Wasserstein curvature of the virtual waiting time (also workload) process. Indeed, this process is generated by

$$\mathcal{L}_2 f(x) = -\mathbf{g} \mathbf{1}_{x\geq 0} f'(x) + r(x) \int_0^1 f(x + G(x,\theta)) - f(x) d\theta$$

where $\mathbf{g}$ is a positive number and G is non-negative. If W is generated by $\mathcal{L}_2$, it is a lipschitz transformation of a process generated by $\mathcal{L}_1$. Indeed, we can write

$$\forall t \geq 0, W_t = \max(X_t, 0).$$

where X is generated by $\mathcal{L}_1$ (with an other G).

4.2. **Kolmogorov-Langevin processes.** Let us consider the process which verifies the following S.D.E.:

$$dX_t = \sqrt{2} dB_t - q'(X_s) ds$$

where q is C^∞ and B is a standard Brownian motion. It is already known, under suitable assumption, that this process converges to the Gibbs (or Boltzmann) measure $\pi(du) = e^{-q(u)} du/\mathcal{Z}$ (where $\mathcal{Z}$ is a renormalizing constant). The theorem 1.4 shows that the curvature is equal to $\rho = \inf_{z\in\mathbb{R}} q''(z)$. This result was already known but we can hope an exponential decay to the invariant measure when $\rho < 0$. The case where $\rho = 0$ was studied in [27]. Our main example of interest is

$$q(x) = x^4 - x^2.$$

We begin by a preliminary lemma

Lemma 4.4 (Existence of eigenelements of a certain operator)**.** *Let us define H the closure on L^2 of the operator defined by*

$$Hf = -f''(x) + f \frac{q''}{2}$$

for all smooth enough f. If q is C^∞ and

$$\lim_{|x|\to+\infty} q''(x) = +\infty,$$

there exist a unique positive function $\varphi \in L^2 \cap C^\infty$ and a real number $\lambda > 0$ such that $H\varphi = \lambda\varphi$.

Proof. It is know from [6, theorem 3.1 p. 57] that if

$$\lim_{|x|\to+\infty} \frac{q''(x)}{2} = +\infty$$

then H has a discrete spectrum. Furthermore, there exist a unique positive eigenvector. It corresponds to the smaller eigenvalue $\lambda > 0$ and we denote it by φ (see [6, chapter 2]). The regularity of φ comes from the regularity of q and the uniqueness of the solution of the linear equation. □

Theorem 4.5 (Rates of convergence)**.** *Under the same assumption of the previous lemma, we have for all* $x, y \in \mathbb{R}$*,*

$$\mathcal{W}\left(\delta_x P_t, \delta_y P_t f\right) \leq e^{-\lambda t} \sup_{z\in[x,y]} e^{q(z)/2} \left(C_1 + C_2 \beta^t \frac{1}{\varphi(z)} \right) |x-y|$$

where C_1, C_2 *are two positives constants (*C_1 *is explicit) and* $\beta \in (0,1)$.

Proof. Let $(P_t)_{t\geq 0}$ be the semigroup of X. As can be view in the proof of theorem 1.4, we have, for all f smooth enough,

$$\partial_x P_t f = S_t f'$$

where $(S_t)_{t\geq 0}$ is a Feynamnn Kac semigroup generated by $\tilde{L}$ defined by,

$$\tilde{L} f = f''(x) - q'(x) f'(x) - q'' f(x).$$

Let $(Q_t)_{t\geq 0}$ be defined for all smooth enough f by

$$Q_t(f) = \frac{e^{\lambda t}}{\varphi} e^{-q/2} S_t(f \varphi e^{q/2})$$

We have

$$\begin{aligned}
\partial_t Q_t(f) &= \lambda \frac{e^{\lambda t}}{\varphi} e^{-q/2} S_t(f\varphi e^{q/2}) + \frac{e^{\lambda t}}{\varphi} e^{-q/2} S_t(\tilde{L}(f\varphi e^{q/2})) \\
&= \frac{e^{\lambda t}}{\varphi} e^{-q/2} S_t(f e^{q/2} H\varphi) + \frac{e^{\lambda t}}{\varphi} e^{-q/2} S_t(-e^{q/2} H(f\varphi)) \\
&= \frac{e^{\lambda t}}{\varphi} e^{-q/2} S_t(e^{q/2}\varphi G f) \\
&= Q_t(Gf).
\end{aligned}$$

Where

$$Gf = f'' + 2\frac{\varphi'}{\varphi} f'.$$

This relation can be understood as

$$\partial_x P_t f(x) = e^{-\lambda t} e^{q(x)/2} \varphi(x) \mathbb{E}\left[\frac{f'(Y_t)}{\varphi(Y_t)} e^{-q(Y_t)/2} \mid Y_0 = x \right]$$

where Y is a Kolmogorov-Langevin process generated by G and starting from x. Hence,

$$|\partial_x P_t f(x)| \leq e^{-\inf_{z\in\mathbb{R}} q(z)/2} e^{-\lambda t} e^{q(x)/2} \varphi(x) Q_t\left(\frac{1}{\varphi}\right)(x).$$

But

$$G\frac{1}{\varphi} = \frac{-\varphi''}{\varphi^2} = \frac{\lambda - q''}{\varphi}.$$

So, by [28, theorem 6.1] with $V = 1/\varphi$, we have

$$\left| Q_t\left(\frac{1}{\varphi}\right)(x) - \tilde{\pi}\left(\frac{1}{\varphi}\right) \right| \leq B\beta^t \left(1 + \frac{1}{\varphi(x)}\right)$$

for all $x \in \mathbb{R}$, where $\beta \in (0,1)$, $B \geq 0$ and $\tilde{\pi}(dx) = e^{\int^x 2\varphi'/\varphi} dx/\mathcal{Z} = \varphi(x)^2 dx/\mathcal{Z}$ ($\mathcal{Z}$ is a renormalizing constant). Finally,

$$|\partial_x P_t f(x)| \leq e^{-\inf_{z\in\mathbb{R}} q(z)/2} e^{-\lambda t} e^{q(x)/2} \varphi(x) \left(\tilde{\pi} \left(\frac{1}{\varphi} \right) + B\beta^t \left(1 + \frac{1}{\varphi(x)} \right) \right).$$

□

Example 4.6 (Ornstein-Uhlenbeck)**.** *If $q(x) = \mu x^2/2$ then the theorem is not available. But we see that $\varphi = 1$ is an eigenvector of $\mathcal{H}$ with respect to the eigenvalue $\mu/2$. It is the smaller eigenvalue such that the eigenvector is positive. With a same computation to the previous theorem, we can prove*

$$\mathcal{W}(\delta_x P_t, \delta_y P_t) \leq e^{-\mu t/2} |x - y|.$$

We can compare this result with the one of theorem 1.4:

$$\mathcal{W}(\delta_x P_t, \delta_y P_t) \leq e^{-\mu t} |x - y|.$$

In fact, it easy to see that it is an equality.

This example prove the non-optimality to the previous theorem. But as can be viewed by the following example, this techniques can be useful for more complicated example.

Example 4.7 (An example where q is not convex)**.** *Let us consider*

$$q(x) = x^4 - x^2.$$

Hence $q''(x)/2 = 6x^2 - 1$ and $H\varphi(x) = \lambda\varphi(x)$, when

$$\lambda = \sqrt{6} - 1 \text{ and } \varphi(x) = \exp\left(-x^2\sqrt{6}/2\right).$$

So, as π, the invariant measure, integrates $1/\varphi$ we have,

$$\mathcal{W}(\mu P_t, \pi) \leq C_{\mu,\pi} e^{-(\sqrt{6}-1)t}$$

where μ is a measure which integrate $1/\varphi$ and $C_{\mu,\pi}$ a constant which depend to μ and π.

4.3. **TCP Windows size process.**

4.3.1. *An ideal model for the TCP windows size.* Now, we study some model which represent the TCP congestion. The first one was developed in [30]. Before a jump, the process evolves as

$$\partial_t X_t = \frac{\mathbf{g}}{X_t^m}.$$

The jumps arise following a Poisson process of intensity r with points $(T_k)_{k\geq 0}$. And the jumps are multiplicative, indeed, $X_{T_k} = \mathbf{h} X_{T_k -}$. The parameters verify

$$0 < \mathbf{h} < 1, \ m > -1, \ \mathbf{g} > 0 \text{ and } r > 0.$$

In [30], the proof of the ergodicity and an explicit formula for the invariant measure is proved:

Theorem 4.8 (Convergence to equilibrium)**.** *For any initial condition, X converges in law to a measure π which has a density defined by*

$$x \mapsto \frac{rx^m}{\mathbf{g}} \sum_{k\geq 0} a_k(\mathbf{h}) \exp\left(-\frac{\mathbf{h}^{-k(m+1)} r}{\mathbf{g}(m+1)} x^{m+1} \right)$$

where

$$a_k(\mathbf{h}) = \frac{(-1)^k \mathbf{h}^{\frac{k(k-1)}{2}(m+1)}}{\prod_{j=1}^{+\infty}(1-\mathbf{h}^{j(m+1)}) \prod_{j=1}^{k}(1-\mathbf{h}^{j(m+1)})}.$$

By theorem 1.4, if $m \geq 0$, the Wasserstein curvature is $\rho = r(1-\mathbf{h})$.

Corollary 4.9 (Wasserstein exponential ergodicity)**.** *Assume* $m \geq 0$ *, for any probability measure* $\mu \in \mathcal{P}_{|\cdot|}(\mathbb{R}_+)$ *(indeed, which has a first moment), we have*

$$\mathcal{W}(\mu P_t, \pi) \leq e^{-r(1-\mathbf{h})t} \mathcal{W}(\mu, \pi).$$

4.3.2. *A more realistic model.* Now, we consider that the jumps depend of the position and the growth is linear. Let $(X_t)_{t\geq 0}$ be the Markov process generated by

$$\forall x \geq 0,\ \mathcal{L}f(x) = f'(x) + r(x)\left(\int_0^1 f(hx)\mathcal{H}(dh) - f(x)\right). \tag{13}$$

To fix an idea, $r(x) = x$ is an interesting value [3, 15, 18], but, the case where r is constant, is often used for its simplicity [26, 30]. The invariant measure is known when $r(x) = \mathbf{r}x^\alpha$(it is explain in the following). We have:

Corollary 4.10 (Wasserstein curvature)**.** *If* r *is decreasing, we have that* ρ*, defined at* (5)*, verifies*

$$\rho = \left(1 - \int_0^1 h\mathcal{H}(dh)\right) \inf_{x\geq 0}\left(r(x) - xr'(x)\right).$$

If otherwise, $r(x) = ax + b$*, we have*

$$\rho \geq \left(1 - \int_0^1 h\mathcal{H}(dh)\right) b = \left(1 - \int_0^1 h\mathcal{H}(dh)\right) \inf_{x\geq 0}\left(r(x) - xr'(x)\right).$$

.

In particular, when $r(x) = x$, we have $\rho \geq 0$. When the process is near than 0, the jump rate is null. Hence the process evolves like $t \mapsto X_0 + t$ which have a null curvature. So we see that $\rho = 0$. A recent work [3] prove that, nevertheless the curvature is null, this process converges exponentially rapidly to its invariant distribution.

4.3.3. *The embedded chain.* We always consider a process generated by (13). Let $(\hat{X}_n)_{n\geq 0}$ be the embedded chain. It is defined by,

$$\hat{X}_n = X_{T_n} \text{ where } T_n = \inf\{t \geq T_{n-1} \mid X_{t+} \neq X_{t-}\} \text{ for } n \geq 1 \text{ and } T_0 = 0. \tag{14}$$

This Markov chain is easier to study than the continuous time process. For instance, for $r(x) = ax^\alpha$, it is easy to see that

$$R(\hat{X}_{T_{n+1}}) = R(H_n(R^{-1}(E_n + R(\hat{X}_{T_n})))) = H_n^{\alpha+1}(E_n + R(\hat{X}_{T_n}))$$

where R is the antiderivative of r. Then, it is ergodic and its limit $\hat{X}_\infty$ verifies

$$R(\hat{X}_{T_\infty}) \overset{d}{=} H_n^{\alpha+1}(E_n + R(\hat{X}_\infty)).$$

Now, using [18, proposition 5], we deduce that $\hat{X}_\infty$ have a density given by

$$x \mapsto \frac{1}{\prod_{n\geq 1}(1-\mathbf{h}^{(\alpha+1)n})} \sum_{n\geq 0} \prod_{k=1}^{n} \frac{\mathbf{h}^{-(\alpha+1)(n+1)}}{1-\mathbf{h}^{-(\alpha+1)k}} a x^\alpha e^{-\mathbf{h}^{-(\alpha+1)(n+1)} a(\alpha+1)^{-1} x^{(\alpha+1)}}.$$

Now, applying [14, theorem 34.31], we can deduce the invariant law of the continuous time process. This result generalises [10, Theorem 2.1] but it is already known via others techniques [32]. For this Markov chain, we arrive to bounded all Wasserstein distance. Indeed, recall that for every $p \geq 1$, the $\mathcal{W}^{(p)}$ Wasserstein distance between two laws μ_1 and μ_2 on E with finite p^{th} moment is defined by

$$\mathcal{W}^{(p)}(\mu_1, \mu_2) = \inf_{\{X\sim\mu_1, Y\sim\mu_2\}} \left(\mathbb{E}\left[|X-Y|^p\right]\right)^{1/p}$$

where the infimum runs over all coupling of μ_1 and μ_2.

Theorem 4.11 (Wasserstein exponential ergodicity for the embedded chain)**.** *Assume that $\mathcal{L}(X_0)$ and $\mathcal{L}(Y_0)$ have finite p^{th} moment for some real $p \geq 1$ and r is increasing. Let $\hat{X}$ and $\hat{Y}$ be the embedded chains of X and Y. Then, for any $n \geq 0$, with a random variable $H \sim \mathcal{H}$,*

$$\mathcal{W}^{(p)}(\mathcal{L}(\hat{X}_n), \mathcal{L}(\hat{Y}_n)) \leq \mathbb{E}(H^p)^{n/p} \mathcal{W}_p(\mathcal{L}(X_0), \mathcal{L}(Y_0)).$$

In particular, if $\hat{\pi}$ is the invariant law of $\hat{X}$ then

$$\mathcal{W}^{(p)}(\mathcal{L}(\hat{X}_n), \hat{\pi}) \leq \mathbb{E}(H^p)^{n/p} \mathcal{W}_p(\mathcal{L}(X_0), \hat{\pi}).$$

Proof. It is sufficient to provide a good coupling. Let $x \geq 0$ and $y \geq 0$ be two non-negative real numbers, and let $(E_n)_{n\geq 1}$ and $(H_n)_{n\geq 1}$ be two independent sequences of i.i.d. random variables with respective laws the exponential law of unit mean and the law $\mathcal{H}$. Let $\hat{X}$ and $\hat{Y}$ be the discrete time Markov chains on $[0, \infty)$ defined by

$$\hat{X}_0 = x \quad \text{and} \quad \hat{X}_{n+1} = H_{n+1} R^{-1}(R(\hat{X}_n) + E_{n+1}) \quad \text{for any } n \geq 0$$
$$\hat{Y}_0 = y \quad \text{and} \quad \hat{Y}_{n+1} = H_{n+1} R^{-1}(R(\hat{Y}_n) + E_{n+1}) \quad \text{for any } n \geq 0.$$

the law of $\hat{X}$ (respectively $\hat{Y}$) is the law of the embedded chain of a process generated by L and starting from x (respectively y). Now, let a be a non-negative number, if $\varphi_a : x \mapsto R^{-1}(a + R(x))$ then

$$\varphi_a'(x) = \frac{r(x)}{r(a+x)} \Rightarrow |\varphi_a(x) - \varphi_a(y)| \leq |x-y|. \tag{15}$$

And we get

$$\forall p \geq 1,\ \mathbb{E}[|\hat{X}_{n+1} - \hat{Y}_{n+1}|^p] = \mathbb{E}[H_{n+1}^p |\varphi_{E_{n+1}}(\hat{X}_n) - \varphi_{E_{n+1}}(\hat{Y}_n)|^p]$$
$$\leq \mathbb{E}[H_{n+1}^p |\hat{X}_n - \hat{Y}_n|^p] = \mathbb{E}[H_{n+1}^p]\mathbb{E}[|\hat{X}_n - \hat{Y}_n|^p]$$

A straightforward recurrence leads to

$$\mathbb{E}[|\hat{X}_n - \hat{Y}_n|^p] \leq \mathbb{E}[H_1^p]^n |x-y|^p.$$

This gives the desired inequality when the initial laws are Dirac masses. The general case follows by integrating this inequality with respect to couplings of the initial laws. □

The fact that, between the jumps, the growth is linear is fundamental in this proof. It is why, we do not generalise it. So, we arrive to prove a convergence criterion for the continuous time process when r decreases and for the embedded chain when r increases. We are not able to associate these two approaches.

4.4. **Integral of Lévy processes with respect to a Brownian motion.** Before to talk about integral of Lévy process, we give an example of growth-fragmentation processes, where some calculates are explicit.

4.4.1. *Generator which conserves the polynomial functions.* Consider the process generated by (12) with the following parameters:

$$g(x) = g_0 + g_1 x, \ \sigma(x) = \sigma_0 + \sigma_1 x + \sigma_2 x^2.$$

And r is constant. In this case, the generator conserves the polynomial functions and their degree. We then deduce,

Theorem 4.12 (Moments estimates)**.** *Let $t \geq 0$ and $x \geq 0$, we have*

$$\mathbb{E}[X_t \mid X_0 = x] = \left(x + \frac{g_0}{g_1 - r\kappa_1} \right) e^{(g_1 - r\kappa)t} - \frac{g_0}{g_1 - r\kappa}$$

and

$$\forall n \geq 2, \ \mathbb{E}[X_t^n \mid X_0 = x] = A_0^{(n)} + \sum_{k=1}^{n} A_k^{(n)} e^{(\sigma_2 k(k-1) + g_1 k + r\kappa_k)t}$$

where $\kappa_n = 1 - \int h^n \mathcal{H}(dh)$ and for all $k \in \{1, .., n-1\}$ we have

$$A_0^{(n)} = \frac{-1}{\sigma_2 n(n-1) + g_1 n - r\kappa_n} (\sigma_0 n(n-1) A_0^{(n-2)} + (\sigma_1 n(n-1) + g_0 n) A_0^{(n-1)}).$$

$$A_k^{(n)} = \frac{1}{1 - (\sigma_2 n(n-1) + g_1 n - r\kappa_n)} (\sigma_0 n(n-1) A_k^{(n-2)} + (\sigma_1 n(n-1) + g_0 n) A_k^{(n-1)}).$$

$$A_{n-1}^{(n)} = \frac{1}{1 - (\sigma_2 n(n-1) + g_1 n - r\kappa_n)} ((\sigma_1 n(n-1) + g_0 n) A_{n-1}^{(n-1)}).$$

$$A_n^{(n)} = x - \sum_{k=0}^{n-1} A_k^{(n)}.$$

Proof. Let $t \geq 0, \ n \in \mathbb{N}$ and $\alpha_n(t) = \mathbb{E}[X_t^n]$, we get,

$$\alpha_0(t) = 1, \ \alpha_1'(t) = g_0 + (g_1 - r\kappa_1)\alpha_1(t)$$

and,

$$\alpha_n'(t) = \sigma_0 n(n-1)\alpha_{n-2}(t) + (\sigma_1 n(n-1) + g_0 n)\alpha_{n-1}(t) + (\sigma_2 n(n-1) + g_1 n - r\kappa_n)\alpha_n(t).$$

Then the result follows by a straightforward reccurence. □

Remark 4.13 (Convergence to the equilibrium)**.** *This result gives the convergence of the moments when $g_1 - r\kappa < 0$. In this case, the n^{th} moment converges to $A_0^{(n)}$ which is easily calculable. When this sequence verifies the Carleman criterion, that is*

$$\sum_{n \geq 0} (A_0^{(n)})^{-1/2n} = +\infty$$

We have the convergence, in law, of X to a measure with moments defined by the sequence A_0.

Example 4.14 (A first link with the Lévy process)**.** *The If $\sigma_1 = \sigma_2 = 0$ and $g_0 = 0$, then X is the exponential of a Levy process L. The condition $g_1 - r\kappa < 0$ implies that X converges to 0 almost surely. The Wasserstein decay of the theorem 1.4 gives a speed of convergence in L^1-norm.*

This example is near to the example of theorem 4.2. In the following, we give an other link with the Lévy processes and an other example which have a different behaviour.

4.4.2. *Integral of Lévy processes with respect to a Brownian.* Heuristically, if you take the logarithm of a growth-fragmentation process, you transform the multiplicative jumps on additive jumps. Then, as the jump times are Poissonian, we can obtain a continuous process by renormalising our process with a Lévy process. Formally, let (L_t) be the Lévy process defined by,

$$L_t = -\int_{\mathbb{R}_+ \times [0,1]} \mathbf{1}_{\{u \leq r\}} \ln(h) Q(ds, du, dh) = -\ln \left(\prod_{j=1}^{\mathcal{N}_t} H_j \right)$$

and let $\bar{X}$ the continuous process defined by $\bar{X}_t = X_t e^{L_t}$. We get,

Lemma 4.15 (Stochastic differential equation for $\bar{X}$ and X)**.** *If X is generated by (12) then, for any non negative number t, we get*

$$\bar{X}_t = X_0 + \int_0^t e^{L_s} g(X_s) ds + \int_0^t e^{L_s} \sigma(X_s) dB_s$$

and,

$$X_t = X_0 e^{-L_t} + \int_0^t e^{-L_s} g(X_{t-s}) ds + \int_0^t e^{-L_s} \sigma(X_{t-s}) dB_s.$$

Proof. All the stochastic integrals that we write are well defined as local martingales. Using Itô's formula with jumps [20, Theorem 5.1, p.67] (see also [21, Theorem 4.57, p.57]) and since $\bar{X}$ is continuous, we get:

$$\begin{aligned}
\bar{X}_t &= X_0 + \int_0^t e^{L_s} \left(g(X_s) ds + \sigma(X_s) dB_s \right) \\
&\quad + \int_0^t X_{s-} e^{L_{s-}} - X_s e^{L_s} \mathbf{1}_{\{u \leq r\}} Q(ds, du, dh) \\
&= X_0 + \int_0^t e^{L_s} \left(g(X_s) ds + \sigma(X_s) dB_s \right).
\end{aligned}$$

Then, we deduce,

$$\begin{aligned}
X_t &= X_0 e^{-L_t} + \int_0^t e^{L_s - L_t} \left(g(X_s) ds + \sigma(X_s) dB_s \right) \\
&= X_0 e^{-L_t} + \int_0^t e^{-L_{t-s}} \left(g(X_s) ds + \sigma(X_s) dB_s \right) \\
&= X_0 e^{-L_t} + \int_0^t e^{-L_s} g(X_{t-s}) ds + \int_0^t e^{-L_s} \sigma(X_{t-s}) dB_s.
\end{aligned}$$

□

This lemma is a generalisation of the relation of [2, lemma 3.2] and [26, section 6]. In [2], it is a preliminary for the proof of theorem 4.2 and in [26], they deduce that when g is constant, and $\sigma = 0$, we get,

$$\lim_{t \to +\infty} X_t = g \int_0^{+\infty} e^{-L_s} ds.$$

The behavior of the right hand side was studied in [7] and [8] for general Lévy processes. We give an other application:

Theorem 4.16 (Long time behavior of Integral of Lévy process with respect to a Brownian motion)**.** *If*

$$Y_t = \int_0^t e^{-L_s} dB_s$$

where $L_t = \sum_{k=1}^{N_t} H_j$ *is a lévy process independent of the Brownian motion* B*. Then it converges to a measure* π *such that*

$$\int x^n \mu(dx) = \frac{n!}{r^{n/2} \prod_{k=1}^{n/2} \kappa_{2k}}$$

for all $n \in 2\mathbb{Z}$ *and*

$$\int x^n \mu(dx) = \frac{n!}{r^{(n+1)/2} \prod_{k=1}^{(n+1)/2} \kappa_{2k-1}}$$

otherwise. Furthermore, for all $t \geq 0$*,*

$$\mathcal{W}\left(\mathcal{L}(Y_t), \pi\right) \leq e^{-\kappa t} \mathcal{W}(\delta_0, \pi),$$

where $\kappa = r\left(1 - \int_0^1 h\mathcal{H}(dh)\right)$.

Proof. In this case, we have that Y is generated by (12), where, $g = 0, \sigma = 1$, r is constant and $X_0 = 0$. The result is an application of the results of the previous sections. □

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