Euler, infinitesimals and limits

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Abstract. This paper examines the Eulerian notion of infinitesimal or evanescent quantity and compares it with the modern notion of limit and non-standard analysis concepts. The Eulerian infinitesimal, when interpreted using the conceptual instruments available to modern mathematics, seems to be a fluid mixture of different elements, a continuous leap from a vague idea of limit to a confused notion of infinitesimal. However, while the modern notions of limit, hyperreal numbers etc. derive from precise definitions or constructions and are developed within non-equivocal contexts, Euler did not define the notion of evanescent quantity, either explicitly or implicitly: it is substantially a 'common notion', directly derived from the experience of physical and geometric quantities, which can diminish until they vanish. Infinitesimals are fictions, useful instruments for dealing with finite quantities, and thus Euler could state that the algorithm of the calculus did not concern differentials but functions. The vague and imprecise Eulerian notions can be reformulated in modern terms only by introducing extraneous elements into Euler's work and changing the logical and conceptual framework of his mathematics.

1. Introduction

In this paper I investigate the Eulerian notion of evanescent or infinitesimal quantity with the aim of contributing to the understanding of the foundations of calculus in the eighteenth century.[1] When interpreted in the light of modern theories such as real analysis or non-standard analysis, the Eulerian notion appears to be an extremely confused mixture of different elements which are sometimes irreducible to each other. Nevertheless, the comparison with modern theories is problematic since the latter use concepts (such as set, real number, structure) which do not exist in eighteenth-century mathematics, which was based on the classic notion of quantity. The modern notions of limit, hyperreal numbers etc. derive from precise definitions or constructions and are inserted in non-equivocal contexts. Euler, however, does not define the notion of evanescent quantity, either explicitly or implicitly: it is substantially a 'common notion', directly derived from the experience of physical and geometric quantities, which can diminish until they vanish. Rather than making use of modern concepts, the Eulerian use of infinitesimals can be better understood if they are thought of as 'fictions', represented by symbols with which one operated in analogy with 'real' quantities. Unlike modern mathematics, Euler dealt with two species of objects, which differ ontologically: actually existing quantities (in the sense that these were conceived of as the abstract but immediate representations of physical objects) and fictitious quantities. Actually existing quantities were expressed in the calculus by means of functions and thus Euler could think that infinitesimal or differential (Euler *de facto* reduced

[1] I have dealt elsewhere with other aspects of the Eulerian foundations of the calculus. in particular the notions of the sum of a series and of a function (see [Ferraro 2000a and 2000b]).

differentials to infinitesimals)[2] were only an instrument for dealing with functions and that the algorithm of the calculus did not transform differentials into differentials but functions into functions.[3]

My point of view is different from that of some recent papers, such as [McKinzie-Tuckey 1997] and [Pourciau 2001]. In this writing the authors recast the early procedures directly in terms of the modern foundation of analysis or interpret the earlier results in terms of modern theory of non-standard analysis and understand the results in the light of this later context. I instead think that the attempt to specify Euler's vague and ambiguous notions by applying modern concepts is only possible if elements are added which essentially alien to it, and thus transforming Eulerian mathematics into something wholly different. Of course, I am not stating that eighteenth-century mathematics had to be investigated without considering modern theories. Modern concepts are essential for understanding eighteenth-century notions and why these led to meaningful results, even when certain procedures, puzzling from the present views, were used. However, eighteenth-century analysis had its own principles, different from those of modern analysis: a unproblematic translation of certain chapters in the history of mathematics into modern terms tacitly assumes that the same logical and conceptual framework guiding work in modern mathematics also guided work in past mathematics.

2. Infinitesimals, infinites, evanescent quantities and infinitely increasing quantities

The starting point underlying Euler's thought is his claim regarding the absolute correctness of the calculus: "geometric rigour escapes even the slightest error".[4] These words echo Newton's and Berkeley's statements: "errors are not to be disregarded in mathematics, no matter how small".[5] Based on this assertion, Berkeley, in his famous *The Analyst*, argued that the calculus, in contrast to other branches of mathematics, was not an exact theory. He did not cast any doubt upon the usefulness of the calculus in solving many problems of physics or geometry; nevertheless, he believed that it did not possess solid foundations. In particular, he criticised the principle of cancellation of higher-order differentials (in other words, the rule according to which d^nx+d^mx was equal to d^nx if $n<m$). In Berkeley's opinion, by writing $d(x^2)=(x+dx)^2-x^2=2xdx+dx^2=2xdx$, firstly one considered $dx\neq 0$, then one let $dx=0$: thus there was a negation of the first hypothesis and an elementary logical principle was violated. The same criticism had been made of the method of first and last reasons: indeed, in the expression $((x+h)^n-x^n)/h$, one first considered a finite

[2] The differential was regarded as an infinitesimal obtained by making the increment ω of a variable quantity tend to zero (see Euler [1755, pp. 5-6 and p.84]). Euler often simply treats differentials and infinitesimals as the same thing (for instance, see [Euler 1755, p.70]).

[3] It is worth clarifying that this article does not seek to investigate the origin of Eulerian notions and the connections of Euler's evanescent quantity with the analogous ideas of Leibniz and with the Newtonian method of first and last reasons.

[4] "[R]igor geometricus etiam a tantillo errore abhorret" [Euler 1755, 6].

[5] "Errores quam minimi in rebus mathematicis non sunt contemnendi" [Newton 1704, 334]. In §.4 of [1734] Berkeley emphasised: "It is said that the minutest errors are not to be neglected in mathematics" and, in §.9, he quoted Newton's Latin words.

increase h≠0, but one then let h=0 and obtained nx^{n-1} [Berkeley 1734, sections 13-18]. Berkeley thought that the calculus achieved correct results only thanks to a compensation of mistakes. This justification was considered by Euler to be insufficient: he observed that by ignoring infinitely small quantities, but not noughts, one could still commit extremely serious errors.[6] Neither could the correctness of the calculus be based on an exhibition of examples in which differential calculus and elementary geometry reached the same results [Euler 1755, 6]. In his writings, Euler asserted the independence of the calculus from geometry as a result of which one could not use geometry to establish a solid basis of the analysis.[7]

The crux of the question lay in knowing what meaning to attribute to the equation a+dx=a. The exactness of mathematics required, according to Euler, that the differential dx should be precisely equal to 0: simply by assuming that dx=0, the outrageous attacks on the calculus would be shown to lack any basis. Those who considered infinitesimal quantities as different from nothing could effectively be accused of ignoring geometric rigour and so the deductive process based on such assumptions were really in doubt.[8]

Nevertheless, it is difficult to base the calculus on differentials understood as zeros: indeed, if we examine the definitions of infinitesimal that Euler offers in the third chapter of *Institutiones calculi differentialis*, it can immediately be seen that they cannot be reduced to the mere statement that an infinitesimal is zero. In [1755] Euler asserts: "There is no doubt that every quantity can be diminished until it vanishes completely and is reduced to nothing. But an infinitely small quantity is simply an evanescent quantity and therefore actually equal to zero."[9] Immediately afterwards, he argues that an infinitesimal can also be defined as a quantity which is less than any assignable quantity (in more modern terms, less than any finite quantity), since a quantity which is less than any assignable quantity is necessarily equal to zero.[10]

The statement that an infinitesimal ω or a differential dx are equal to 0 can be understood in two ways: ω is 'numerically' equal to zero (this justifies the principle of cancellation) or else ω is a variable quantity tending

[6] "[Q]uantumvis enim exigua haec infinita parvae concipiantur, tamen non solum singulis, sed etiam pluribus atque adeo innumerabilibus simul reiiciendis errorem tandem inde enormem resultare posse" [Euler 1755, 6]

[7] Furthermore it must have appeared completely obvious to Euler that calculus could not be based on the use of kinematical concepts (in this matter, he was forced to agree with Berkeley's criticism of Newton [Berkeley 1734, §§. 30-31]).

[8] "Cum autem hoc modo […] principia Calculi differentialis stabiliuntur, omnes obtrectationes, quae contra hunc calculum proferri sunt solitae, sponte corruunt; quae tamen summam vim retinerent, si differentialia seu infinite parva non plane annihilarentur. Pluribus autem, qui Calculi differentialis praecepta tradidere, visum est differentialia a nihilo absoluto secernere peculiaremque ordinem quantitatum infinite parvarum, quae non penitus evanescat, sed quantitatem quandam, quae quidem esset assignabili minor, retineat, constituere; his igitur iure est obiectum rigorem geometricum negligi et conclusiones inde deductas, propterea quod huismodi infinite parva negligerentur, merito esse suspectas" [Euler 1755, 6].

[9] "Nullium […] est dubium, quin omnis eousque diminui queat, quoad penitus evanescat atque in nihilum abeat. Sed quantitas infinite parva nil aliud est nisi quantitas evanescens ideoque revera erit =0." [Euler 1755, 69].

[10] "Consentit quoque ea infinite parvorum definitio, qua dicuntur omni quantitate assignabili minora; si enim quantitas tam fuerit parva, ut omni quantitate assegnabili sit minor, ea certe non poterit non esse nulla; namque nisi esset =0, quantitas assignari posset ipsi aequalis, quod est contra hypothesin." [Euler 1755, 69]

towards zero' (this justifies the existence of different symbols which denote the different way of tending towards zero). From a modern point of view, the difference between the two notions is such that Euler's argument is incomprehensible. Let us imagine, for example, that we subdivide a given quantity *q* into two equal parts and then divide one of the halves again, and so on. We have a quantity *q* that vanishes; according to Euler's definition, it is an infinitesimal. If we denote the act of vanishing of *q* by ω and observe that it can analytically expressed as $q2^{-n}$ or, more simply, as 2^{-n} (by assuming *q*=1); therefore $\omega=2^{-n}$ is an infinitesimal $\omega=2^{-n}=0$.

The meaning of $\omega=2^{-n}=0$ is unclear: it may be a way (a rather infelicitous one) of denoting a limit (therefore $2^{-n}=0$ would mean $\lim_{n\to\infty} 2^{-n}=0$), or it may symbolise a equation valid for a process which occurs a really infinite number of times (therefore $2^{-n}=0$ would mean $st(2^{-n})=0$,[11] where n is an infinite number).[12]

The definition of the infinitesimal as a quantity smaller than any assignable quantity ε is vague and does not clarify the question: this leads to the relation $\omega=2^{-n}<\varepsilon$, but this relation could be understood as $2^{-n}<\varepsilon$, for every ε and for a suitable n depending on ε, or else as $2^{-n}<\varepsilon$, for every ε and for n as a fixed (and therefore infinite) number.

The Eulerian definitions seem to confuse various notions (limit, infinitesimal, value of a variable) which, from a modern viewpoint, are clearly different. It might be thought that this situation is due to an imperfect formulation of the definitions themselves: however, this is an essential characteristic of Eulerian analysis, which is actually composed of an inextricable interplay of elements which today appear as different ones. The following example will help to clarify this point.

In section 114 of the *Introductio in analysin infinitorum* (the first section of chapter V) Euler stated that if a is a number greater than one and ω and ψ are infinitesimals (or rather such small fractions that they are almost equal to nothing), then it follows that

(1) $$a^{\omega}=1+\psi.$$

Euler justifies (1) by making a vague reference to what he had stated in the preceding chapter IV: "Indeed, from the preceding chapter, it was established that if ψ was not an infinitely small number, then neither could ω be an infinitely small number."[13] In reality, in chapter IV, exponential and logarithmic functions had been introduced without making any mention of infinitesimals [Euler 1748, 1:103-105]. He suggested the idea that the difference between a^{z_1} and a^{z_2} of the exponential function a^z might be made equal to a tiny finite quantity,

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[11] Of course, the symbol st (a) denotes the standard part of the hyperreal number a.

[12] It certainly does not mean that 2^{-n} is approximately equal to 0 because this is contrary to a literal and substantial interpretation of Euler (see also section 4). The exactness of mathematics makes it necessary that $a+2^{-n}=a$ when n=infinity, is a precise equation.

[13] "Quia est $a^0=1$, atque crescente exponente ipsius a simul valor potestatis augetur, si quidem *a* est numerus unitate major; sequitur si esponens infinite parum cyphram excedat, potestatem ipsam quoque infinite parum unitatem esse superaturam. Sit ω numerus infinite parvus, seu fractio tam exigua, ut tantum non nihilo sit aequalis, ω erit $a^{\omega}=1+\psi$, existente ψ quoque numero infinite parvo. Ex praecedente enim capite constat, nisi ψ esset numerus infinite parvus, neque ω talem esse posse." [Euler 1748, 1:122].

provided that z_1 and z_2 are taken very close together (in other words, at a tiny finite distance).[14] In the following chapter this idea was expressed by (1), or rather by means of explicit use of the language of infinitesimals, as if there were no difference between any randomly chosen (but finite) quantity and infinitesimal quantities and if one could move from one language to another in an immediate and natural way, without an adequate theoretical construction.

Euler continued by assuming that the infinitesimal ψ is equal to $k\omega$ and that therefore $a^{\omega}=1+k\omega$ and

(2) $$\omega=\log_a(1+k\omega).$$

To prove that k depends on the base a of the logarithm, he put a=10 and $k\omega=1/1000000$ and found that $\omega=0.00000043429$ and k=2.30258 [Euler 1748, 1:122]. This example, according to Euler, was sufficient to show that k is a finite number and that, by changing the base of the logarithm, the result changes. Euler assigned a finite value to the infinitesimal: his reasoning implied that $k\omega$ can assume increasingly small values and that, by reducing the value $k\omega$, one obtained a larger number of exact decimal figures of k=log 10 (the exact value of k can be obtained when ω was evanescent). An infinitesimal cannot be distinguished, for practical purposes, from a sequence of numbers that becomes very small.

In the following paragraph 118 of chapter V, Euler multiplied (2) by *i* and obtains $i\omega= \log_a(1+k\omega)^i$. He observed that the greater the value of *i*, the more $(1+k\omega)^i$ exceeded one: assuming that *i* is an infinite number, the value of $(1+k\omega)^i$ could be any number greater than one [Euler 1748, 1:125]. He used *i* as an infinite number, but the properties of the infinite number *i* derived from the fact that the finite number *i* increased beyond all limits (and so became infinite): it is not easy to separate the infinite number *i* from the process of growth of a finite variable *i*.

Apart from this point, it should be pointed out that the nature of the quantity ω is far from clear: indeed, when *i* is finite, it is necessary to suppose that ω is finite so that$(1+k\omega)^i$ grows beyond all limits; if ω is an infinitesimal (as Euler has assumed from the beginning of section 118), $(1+k\omega)^i$ remains near to one.[15] It is reasonable to suppose that Euler considered ω as a small but finite quantity and then imagined that he could make it vanish (which precisely corresponds to the fact that ω is an evanescent quantity).

By putting $(1+k\omega)^i=1+x$, Euler [1748, I:125-126] obtained $i\omega=\frac{1}{k}(1+x)^{1/i}-\frac{1}{k}$ and $\log_a(1+x)$ $=\frac{i}{k}((1+x)^{1/i}-1)$. For k=1, he had

(3) $$\log(1+x)=i((1+x)^{1/i}-1).$$[16]

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[14] Indeed, the idea underlying Euler's construction is that the value of a^z, when z is an irrational number, could be obtained by considering rational approximations of the irrational number z. He did not take into consideration, for example, the case where $a<0$, because in this case the function proceeded by jumps.

[15] Euler is perfectly aware of this: indeed, immediately afterwards, at p.125 of the *Introductio in analysin infinitorum*, he posed $(1+k\omega)^i=1+x$ and stated that *i* must be ?finite in order to be x a finite quantity.

[16] Euler considered the case k=1 (corresponding to a=e, where e is the base of the natural logarithm), separately at pp.128-132 of [1748].

In the *Institutiones calculi differentialis*, where Euler explicitly formulated the notion of infinitesimal as an evanescent quantity or zero, he wrote (3) by setting 1+x=p and 1/*i*=0:

(4) $$\log p = \frac{p^0 - 1^0}{0}.$$ [17]

Equation (4) was used in [Euler 1755, 122] to prove that d(logp) is equal to dp/p, where p is a function of x (it was the second proof of this rule: the first was a simple consequence of the procedures illustrated in section 6 of this paper). Euler posed ω=0 and wrote (4) in the form logp=(p^ω-1)/ω. He then derived d(logp)=d((p^ω-1)/ω)=$p^{\omega-1}$dp and (ω being equal to 0) d(logp)=dp/p. Euler, therefore, differentiated $\frac{p^0 - 1^0}{0}$ $= \frac{p^\omega - 1}{\omega}$ as if ω=0 was a constant (an infinitesimal number) and not a variable quantity.

Despite the ambiguity of the procedures discussed here, an ambiguity that perfectly corresponds to the ambiguous definition of the infinitesimal, it is well-known that the results obtained are substantially correct and that the same Eulerian procedures can be reformulated in forms that are acceptable from a modern viewpoint by using the theory of limits or non-standard analysis. Nevertheless, such a reformulation leads to complex problems which will be examined in the following sections.

3. The evanescent quantity and the modern notion of limit

Let us first consider an example which is particularly suited, at least apparently, to being translated into the modern language of limits. In *De progressionibus transcendentibus*, Euler derived the formula:

(5) $$\frac{f+(n+1)g}{g^{n+1}}\int_0^1 \frac{g}{f+g}(1-x^{\frac{g}{f+g}})^n\,dx = \frac{n!}{(f+g)(f+2g)...(f+ng)},$$

and observed that if f=1 and g=0 then $\int_0^1 \frac{(1-x^0)^n}{0^n}dx$ =n!. He interpreted 0 as the limit value of a variable quantity z and, by applying the l'Hôpital rule, found that the value of $\frac{1-x^z}{z}$ as z vanishes (cum z evanescint) was $\left(\frac{1-x^z}{z}\right)_{z=0} = \left(\frac{(x^z \log x)dz}{dz}\right)_{z=0} = -\log x$. In this way he obtained n!=$\int_0^1 (-\log x)^n\,dx$, which, historically, was the first integral expression of the gamma (factorial) function Γ(z)= $\int_0^\infty t^{z-1}e^{-t}dt$.

[17] "Sequitur haec eandem regula [d(logp)=dp/p] quoque ex forma $\frac{p^0 - 1^0}{0}$, ad quem superiori libro [*Introductio in analysin infinitorum*] logarithmum ipsius p reduximus" [Euler 1755, 122].

It is completely natural to translate $\left(\frac{1-x^z}{z}\right)_{z=0} = \left(\frac{(x^z \log x)dz}{dz}\right)_{z=0}$ as $\lim_{z\to 0}\frac{1-x^z}{z} = \lim_{z\to 0}(x^z \log x)$; however, such a translation of the language of limits produces some straining in the meaning.[18] From a modern perspective, finding the value of $\frac{1-x^z}{z}$ when z=0 means that:

a) for every fixed value x>0, one considers the function (function in the modern sense of the term, not as an analytical expression) f(z)=$\frac{1-x^z}{z}$ defined for z≠0,

b) the domain of f(z) has a point of accumulation at 0 so that we can attempt to calculate the limit as z→0;

c) the application of l'Hospital's rule, under whose hypotheses our case falls, makes it possible to state that such a limit exists and is equal to -logx;

d) finally, we defined a new function F(z), which will be continuous at the point 0, by setting

$$F(z)=\begin{cases}\frac{1-x^z}{z} & z\neq 0\\ -\log x & z=0.\end{cases}$$

In this procedure we use notions such as limit, value and extension of a function, whose meaning is opportunely and explicitly defined. Indeed, $\lambda=\lim_{x\to c} f(x)$, where f(x) is a function with domain D in R, means:

(D) given any ε>0 there exists a δ>0 such that if x belongs to D and |x-c|<δ then |f(x)-λ|<ε.

By λ=f(c) we intend: λ is the number that the function f associates with the number c. If $\lim_{x\to c} f(x)$ exists and is equal to λ, while the function f(x) is not defined at the point c or f(c)≠λ, we can remove the discontinuity at c by defining the new function $x\to\begin{cases}f(x) & x\in D-\{c\}\\ \lambda & x=c.\end{cases}$

These definitions presuppose knowledge of the notions of set, real numbers, function in the modern sense, continuity, etc.: it is such concepts that enable real analysis to give a clear and rigorous formulation of the intuitive notion of one quantity approaching another. However, Euler did not have the mathematical concept of set, nor the theory of real numbers nor the modern notion of function. He based the calculus on the classic notion of quantity. Quantity was conceived of as that which could be increased or reduced. A distinction was made between two types of quantities: continuous quantities, which were the main subject of the calculus and whose archetype was the segment of a straight line, and discrete quantities, which were modelled on the sequence of the natural numbers 1, 2, 3, 4….[19] A quantity was not a set. Indeed the

[18] A different opinion seems to underlie Pourciau's recent studies about Newton [See Pourciau 1998 and 2001].

[19] In *Institutiones calculi differentialis*, Euler observed that any quantity could be increased *in infinitum* and stated: "In singulis quantitatum speciebus hoc etiam clarius perspecietur. Sic nemo facile reperietur, qui statuerit seriem numerorum naturalium 1,2,3,4,5,6 etc. ita usquam esse determinatam, ut ulterius continuari non possit. Nullus enim datur numerus, ad quem non in super unitas addi sicque numerus sequens maior exhiberi queat, hinc series numerorum naturalium sine fine progreditur neque unquam pervenitur ad numerum maximum, quo maior prorsus non detur. Simili modo linea recta numquam eousque

segment was not understood as a set of points but as a unique object that increased or diminished remaining practically the same. The sequence of natural numbers was also considered as a quantity because it was an ordered sequence and not because it was a numerical set. Moreover, a quantity was an intrinsically variable object: it could be determined or assume a given value. Nevertheless, the calculus referred to indeterminate quantities, subject to possible variations, whether increases or decreases.[20] It is worth observing that while a series of integers constituted a quantity, a single number, precisely because it is single, did not: a number cannot be increased or diminished, if not transformed into another number.

In such a situation, it was impossible to give a precise definition of limit[21] and distinguish between the extension of a function, limit process and assignment of a value to a function: it was only possible to refer to a vague intuition of a process of whereby a quantity A approaches a quantity B. In contrast to other eighteenth century mathematicians (*e.g.*, d'Alembert [Diff] and [Lim]), Euler generally avoided even the use

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produci potest, ut in super ulterius prolongari non posset. Quibus evincitur tam numerus in infinitum augeri quam lineas in infinitum produci posse. Quae cum sint species quantitatum, simul intelligetur omni quantitati, quantumvis sit magna, adhuc dari maiorem hacque denuo maiorem sicque augendo continuo ulterius sine fine, hoc est in infinitum, procedi posse" [Euler 1755, 65].

[20] It is well-known that eighteenth-century mathematicians subdivided quantities into constants and variables. However Euler explained: "Primum [...] hic calculus circa quantitates variabilis versatur: etsi enim omnis quantitas sua natura in infinitum augeri et diminui potest; tamen dum calculus ad certum quoddam institutum dirigitur, aliae quantitates costanter eandem magnitudinem retinere concipiuntur, aliae vero per omnes gradus auctionis ac diminutionis variari: ad quam distinctiomem notandam illae quantitates costantes, hae vero variabiles vocari solent; ita ut *hoc discrimen non tam in rei natura, quam in quaestionis, ad quam calculus refertur, indole sit positum*." [Euler 1755, 3, my emphasis]. As an illustration, Euler considered the trajectory of a bullet. He observed that it was determined by four quantities (the amount of gunpowder, the angle of fire, the range, and the time); however, each of them could be conceived of as a variable or constant according to circumstances. In the *Introductio*, he defined: "Quantitas constans est quantitas determinate, perpetuo eudem valorem servans" [Euler, 1748, 17]. A quantity was variable by itself, however, it could be determinate and fixed (if the problem required this) and the it was considered as a constant.

[21] See the problematic distinction between "limit-achieving" and "limit-avoiding" in [Grattan-Guiness 1969-70, 219-221].

of the term ‘limit’.[22] Furthermore, he never considered the general case of a variable quantity A which goes to any finite quantity B, but only variables that vanish or endlessly increase.[23]

Even though Euler based the calculus on the notions of ‘vanishing’, ‘evanescent quantity’,…, he never offered a definition, not even a vague or imprecise one, of these terms. The idea of an evanescent quantity seems to be more a notion borrowed from the natural world than from mathematical notions. The mental image that may be associated with it is that of a physical entity (such as the quantity of gunpowder, in the initial example of the *Institutiones calculi differentialis*) which we can consider to an increasingly smaller extent or else, still remaining within the field of mathematics, the image of a segment which increasingly diminishes until it becomes a single point and disappears as a segment.

Euler’s use of the evanescent quantity corresponds to Euclid’s use of the ‘common notion’ in the *Elements*: a part is smaller than the whole. This is not an axiom in the modern sense of the term but a general and rather vague principle where use is made of the terms ‘part’, ‘smaller’ and ‘whole’ without their meaning being defined (not even implicitly). In the same way, Euler regarded the fact that a quantity vanishes or increases infinitely as simply part of the idea of quantity and not in need of further explanation or clarification: it is a notion which forms part of the knowledge of every human being.

4. Non-standard analysis and Eulerian infinitesimals

In his [1748], Euler observed that $a^{i\omega} = (1+k\omega)^i = \sum_{r=0}^{\infty}\binom{i}{r}(k\omega)^r$ for any *i*. He assumed x to be a finite number and posed *i*=x/ω, where ω is an infinitesimal. This yielded $a^x = \left(1+\frac{kx}{i}\right)^i = \sum_{r=0}^{\infty}\binom{i}{r}\left(\frac{kx}{i}\right)^r$. For an infinitely large numbers i, $\frac{i-1}{i}$=1, $\frac{i-2}{i}$=1, $\frac{i-3}{i}$=1, …, and therefore the development of a^x is $\sum_{r=0}^{\infty}\frac{1}{r!}(kx)^r$.

In this proof, Euler considers ω and *i* as infinitesimal and infinite numbers respectively; nevertheless, he justifies the relation $\frac{i-1}{i}$=1 by stating that “however much larger the number that we substitute for *i*, the

[22] As far as I am aware, only in the preface to *Institutiones calculi differentialis* [Euler 1755, 7] (see footnote 60) did Euler use ‘limit’ to mean “approaching to a limit”.

[23] Euler seems to believe that every other case of nearing a limit can be included within these two cases. Thus, in order to calculate the value of $\frac{x^{\alpha} - x^{\alpha+m\beta}}{1-x^{\beta}}$ at the point x=1 and $\frac{\pi^2}{6n(n-1)} + \frac{1}{n(n-1)^2} - \frac{(2n-1)\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)}{n^2(n-1)^2}$ for n=1, Euler took x=1-w and n=1+a, respectively (see [Euler 1732-33, 44] and [Fuss 1843, 2:229-231]).

more the value of the fraction (i-1)/i comes closer to unity. Therefore, if i is a larger number than any assignable one, the fraction $\frac{i-1}{i}$ equals the unity."[24]

The equation $\frac{i-1}{i}=1$ was thus obtained by first interpreting i as a finite variable and then as infinite number. It was an approximate equation if i was finite; nevertheless, when i was an infinite number, it needed to be considered as a precise equation, equivalent to the statement that the *infinitesimal was precisely zero*.

The difference between Eulerian infinitesimals and non-standard analysis is therefore clear. Nowadays a hyperreal number a has a standard part b, from which it differs by an infinitesimal (in symbolic form, b=st(a) or $a \approx b$). In particular, one obtains st($\frac{i-1}{i}$)=1, an equation which we can regard as the non-standard version of $\lim_{i\to\infty}\frac{i-1}{i}=1$. According to Euler, however, the form a+dx=a had to be an exact equation, not an approximate one: his objective was precisely to eliminate approximations such as $\frac{i-1}{i}\approx 1$. He refused the idea of basing the calculus on infinitesimals as really existent entities since they implied errors of approximation. For instance, in chapter VII of *Institutionum calculi integralis* [1768-70, 1:183-184], Euler expressed the integral y(x)= $\int_a^x X(x)dx$ in the form

(6) y(x)= $\int_a^x X(x)dx$ =b+a(X(a)+X(a+α)+X(a+2α)+…+X(n+nα)), where x=a+nα.

He observed that (6) could be assumed as the definition of an integral, taking α as an infinitesimal. However, (6) is precise only if one considers infinitesimals as nothing: if, however, infinitesimals are not considered nothing, then (6) offers a useful rule for the approximate calculation of integrals (and, effectively, this formula is included in the chapter of the treatise dedicated to this subject). According to Euler, the concept of the integral as the sum of an infinite number of infinitesimals is similar to the concept of lines as aggregates of infinite points. Both can be admitted (or better tolerated), as long as there is reference to the true principles of the calculus and geometry,[25] thus avoiding every kind of sophism (see [Euler 1768-1770, I:185]. In other words, it is possible to take into account non-null infinitesimals but they constitute only an

[24] "Cum autem i sit numerus infinite magnus, erit $\frac{i-1}{i}=1$; pater enim, quo maior numerus loco i substituatur, eo proprius valorem fractionis $\frac{i-1}{i}$ ad unitatem esse accessurum; hinc si i sit numerus omni assignabili maior, fractio quoque $\frac{i-1}{i}$ ipsam unitatem adaequabit." [Euler 1748, 124].

[25] In the case of integration, the integral is regarded as an antidifferential.

imprecise and approximate version of the notion of infinitesimal, which nevertheless has useful applications.[26]

There are other reasons that make the assimilation of Eulerian calculus into non-standard analysis problematic.[27] Today infinitesimal and infinitely large numbers are used in non-standard analysis as elements of a set *R: they are elements a rich and well-organised algebraic structure which encodes how a sequence approaches a limit. A possible construction of *R is the following. Let m be a finitely additive measure on the set N of the positive integers such that: for all A⊂N, m(A) is 0 or 1; m(A)=0 if A is finite, m(N)=1.[28] Now, consider the equivalence relation ~ on the set **S** of the all the sequence of numbers:

$$\{a_n\}\sim\{b_n\} \text{ iff } m\{n: \{a_n - b_n = 0\}=1$$

The set *R of the hyperreal numbers is defined by *R=S/~. The classes of equivalence of the sequence {0}, {1/n}, $\{1/n^2\}$ are elements of *R and are examples of infinitesimals; the classes of equivalence of the sequence {n}, $\{n^2\}$, $\{n^3\}$ are three examples of infinite numbers.

The intuitive idea of approaching a limit is at the basis of the construction of *R; however, in no case can a demonstration concerning objects of *R refer to such an intuition but only to the axioms and the definitions which enable us to arrive at *R. The same is true for the calculation of the limits in R, which entirely depends on the definition of limit and the construction of the set R.

Viewed from the perspective of the refined concepts of the twenty-first century, the way in which Euler codifies the idea of approaching is naive and over-simplified: an infinite number is simply an short way of indicating a variable that goes to infinity while an infinitesimal is a short way of indicating a variable that goes to zero. The rules that Euler uses upon infinite and infinitesimal numbers constitute an immediate extrapolation of the behaviour of a finite variable *i* tending to ∞ or to 0: what is wholly missing is the complex construction of *R and the assumptions upon which it is based. For instance, in *De progressionibus harmonicis observations*, given the equality $1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{i}=\log(1+i)+\gamma$, where γ is the constant named today after Euler and Mascheroni (on the meaning of this relation, see section 4), Euler derived:

$$\log n = \log\frac{ni+1}{i+1} = \log(ni+1)-\log(i+1) = \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{ni}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{i}\right)$$

$$= \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)-1+\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}\right)-\frac{1}{2}+\left(\frac{1}{2n+1}+\frac{1}{2n+2}+...+\frac{1}{3n}\right)-\frac{1}{3}+...$$

By specialising, he had $\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+...$, $\log 3 = 1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+...,...$ [Euler 1734-35a, 90].

[26] Naturally, if one imagines α as an evanescent quantity (that is, finite and therefore making α tend towards 0), it might be possible to follow a definition of an integral which is acceptable from a Eulerian point of view. However, Euler is not interested in such a possibility and regards the definition of an integral as an antidifferential as satisfactory.

[27] On the difference between the seventeenth and eighteenth-century calculus and non-standard analysis, see [Bos 1974, 81-86].

[28] For a proof of the existence of this measure, see [Lindstrøm 1988, 84-85].

There is no doubt that Euler uses infinitely large numbers but it is difficult to point to what distinguishes the infinite number *i* from a short way of denoting finite sums for which one wants the limit for $i \to \infty$.[29]

Rather than making use of sophisticated twentieth century concepts, the Eulerian use of infinite numbers[30] and infinitesimals can be better understood if they are thought of as 'fictions': infinite numbers and infinitesimals are fictions of variable quantities, represented by symbols with which one operates in analogy with 'real' quantities. Leibniz had also made use, in certain occasions, of the notion of fiction in order to justify infinitesimals; he used it as the basis for using imaginary quantities and other entities introduced in mathematics.[31]

The use of fictions made Eulerian mathematics extremely different from modern mathematics. A fiction presupposes the existence of real objects to which it relates by imitating them. The mathematics of fictions distinguishes between true and false objects; the latter express qualities or characteristics of true ones. Today, infinite numbers and infinitesimals, as elements of *R, are not ontologically different from other elements of *R, which all derive from the construction of *R. According to Euler, an infinitesimal quantity was not an actual quantity in the true sense of the term but a short way of indicating how 'true' quantities vanish. Analogously, in modern mathematics, 0, as an element of the set (0, 1, 2, ….), is a number that exists in the same way as the other numbers 1, 2, 3…; In Euler's opinion, however, 0 was the symbol that represented the absence of quantity; it was not a quantity in the true sense of the term but a fiction that could be treated as a quantity.[32] There was an ontological difference between the fictions and the 'real' objects of Eulerian mathematics, a difference which does not exist in modern mathematics.

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[29] in other words a short way of saying that

$$\log n = \lim_{i\to\infty} \log \frac{ni+1}{i+1} = \lim_{i\to\infty} \{\log(ni+1) - \log(i+1)\} = \lim_{i\to\infty} \left\{ \left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{ni}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{i}\right) \right\}$$

$$= \lim_{i\to\infty} \left\{ \left(1+\frac{1}{2}+\dots+\frac{1}{n}\right) - 1 + \left(\frac{1}{n+1}+\dots+\frac{1}{2n}\right) - \frac{1}{2} + \left(\frac{1}{2n+1}+\dots+\frac{1}{3n}\right) - \frac{1}{3} + \dots + \left(\frac{1}{(i-1)n+1}+\dots+\frac{1}{in}\right) - \frac{1}{i} \right\}.$$

[30] Euler stated: "[E]tiamsi quis neget in mundo numerum infinitum revera existere, tamen in speculationibus mathematicis saepissime occurrunt questiones, ad quas, nisi numerus infinitus admittatur, responderi non posset" [Euler 1755, 69]. In his writings Euler is ready to admit the existence of infinity; nevertheless, it is not strictly necessary. Euler justifies the need to introduce the use of infinity for numbers by observing that the sum of the numbers 1+2+3+4+… is larger than any finite quantity and therefore does not exist among finite quantities: this is the sense in which it is infinite. Given the non-existence of the object 1+2+3+ ….., however, he behaves as though one could pretend that it existed.

[31] Leibnitian notion of fiction has often discussed in the secondary literature. For instance, see [Bos 1974, 54-57], Giorello [1992, 148-151], [Horvàth 1982 and 1986]. Here is …

[32] See (1748, p…), where the zero is placed at the same level as imaginary numbers, with a different degree of reality compared to natural, fractional and irrational numbers. At this point it is clear that when Euler refers to evanescent quantity, in the sense of a quantity that vanishes, we are dealing with something which differs ontologically from the modern idea of tending towards zero.

The extreme vagueness of fictions needs to be underlined. I will look in detail at this question below. For the moment I will give the following example.

In *Institutionum calculi differentialis Sectio III*, [33] Euler (see fig.1) considers the finite increments Pp=Δx, mn=Δy of the variables AP=x and y=PM, then imagines that the point p continually approaches P (or rather, in Eulerian language, Pp becomes infinitely small or reduces to nothing). He states that:

A) when p meets P, Pp will form the differential dx (1862, 336-337).

B) the figure cannot represent infinitesimal quantities so that an appeal is made to the *fictio animi* that Pp does not so much represent itself, so to speak, but an infinitesimal part of itself dx.[34]

C) the element of the infinitesimal curve Mm can be considered as a small straight line: indeed, when the arc Mm is was infinitely small, it coincides with the straight line Mm since the difference between the arc and the segment becomes smaller than any pre-established quantity.

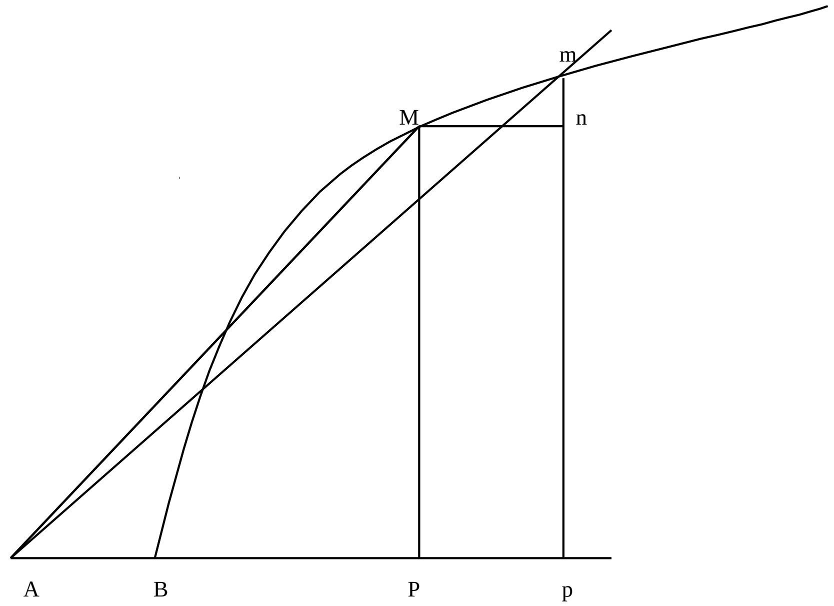

fig.1

In other words, the infinitesimals Pp and Mm are simultaneously considered as a way of indicating a limit (evanescent quantities), null (or pairs of coincidental points to use modern terminology), but also as *elements* of the lines. It follows that the limit P reached by p is indistinguishable from the limit process p→P and both are confused with the infinitesimal trait pP=dx: this infinitesimal part can be treated as a fiction, or rather as if it were a really existent quantity.[35] The precise nature of such a *fictio animi* is not specified however.

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[33] Euler had intended this work to form the third part of the treatise on differential calculus; it was published posthumously in 1862.

[34] Euler refers to fiction of the infinitesimal part of a line as a reference to representation of infinitesimals in a diagram; however, by reasoning in analytical terms, this means that there is a pretence that dx is an infinitesimal part of the variable x.

[35] Euler seems to refer to the Leibnizian tradition, which he attempted to rescue. In Leibniz's opinion, "it was perfectly legitimate to mark a point… with a bidimensional configuration, that of the characteristic triangle, even if it was inassignable" and that "a point can be viewed as *linea evanescens*". [Giorello 1992, 149]) It should also be noted that the method of

5. Further aspects of infinite numbers and infinitesimals

The Eulerian notion of infinite number and infinitesimal contains other interesting aspects apart from those already considered. Consider the following example. In the chapter VII of his *Institutiones calculi differentialis*, Euler obtained

$$S(x)=1^n-2^n+3^n-4^n+\ldots+(-1)^{x+1}x^n=(-1)^{x+1}\left(\frac{1}{2}x^n+\sum_{m=1}^{s}(-1)^{m+1}\binom{n}{2m-1}\frac{(2^{2m}-1)B_{2m}}{2m}x^{n-2m+1}\right)+C$$

where $n\geq 0$, $s=[(n+1)/2]$ is the integral part of $(n+1)/2$, B_n are the Bernoulli numbers[36] and the constant C is determined by the condition $S(0)=0$. This constraint implies that if n is even, then C=0; if n=0, then C=1/2; if n is greater than 0 and is odd, then $C=(-1)^s\frac{(2^{n+1}-1)B_{n+1}}{n+1}$.[37]

Consequently $1^n-2^n+3^n-4^n+\ldots+(-1)^{x+1}x^n$ can be expressed as $C+(-1)^x f(x)$, where C is a constant and f(x) is appropriate function of the index x. Euler thought that one could make $(-1)^x f(x)$ equal to zero for $x=\infty$ since “if x is an infinite number, which is neither even nor odd, these considerations [he intends the alternating sequence of + and – in $+(-1)^x f(x)$] had to end and, therefore, the sum of the ambiguous terms should be rejected. The sum of the series to infinity are now given by the constant term”.[38] In other terms, this means $(-1)^\infty=0$[39] and the sum are

$$1^n-2^n+3^n-4^n+\ldots=C=(-1)^{[n/2]}\frac{(2^{n+1}-1)B_{n+1}}{n+1}. \tag{7}$$

Equation (7) cannot be expressed by means of real analysis or non-standard analysis. Thus the use of such theories to reformulate Eulerian calculus distorts the meaning, on the one hand, transforming fictions into ‘true’ numbers; on the other hand, it only manages to obtain a part of Euler’s results. Reformulating (7) in terms that are acceptable today means using the theory of summability. This is at the very least problematic. Indeed, summability presupposes a conception of mathematics as a set of theories syntactically derived from

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fluxions was understood by Newton as translatable in the terms of infinitely small figures (provided one acted with caution), even when he stated that in the method of fluxions there should be no need to introduce infinitely small figures into geometry: “… volui ostendere, quod in methodo fluxionum non opus sit figures infinite parvas in geometriam introducere. Peragi tamen potest analysis, in figuris quibuscunque, seu infinite parvis, quae figuris evanescentibus fingutur similes … modo caute procedeas [Newton 1704, 337]. On Newton, see [Guicciardini 1999, 32-37 and 160].

[36] There are several definition of Bernoulli numbers B_r. Here I refer to the following: $\frac{t}{e^t-1}=1+\sum_{r=1}^{\infty}(-1)^{[r/2]+1}\frac{B_r}{r!}t^r$ ($|t|<2\pi$, [x] is the integral part of x), $B_0=1$, which is closer to Eulerian use.

[37] Euler’s derivation of this result is illustrated in [Goldstine 1977, 131-135].

[38] [Euler 1755, 384]. Euler’s words remember Leibniz’s in a famous letter to Wolff: with regard to infinite number, “paris imparisque jura confundutur” [Leibniz 1713, 269].

[39] If one work fomally, as Euler did, it is clear that $+(-1)^\infty$ involves $\cos\infty=0$ (being $e^{i\pi}=-1$).

axioms and definitions. The usual notion of a sum of a series, of the limit of a sequence and function, of an improper integral is not considered intrinsic to the actual nature of mathematical objects (and thus the only possible one); on the contrary, it is assumed that different and arbitrary definitions of sum, limit and improper integral can be given. However, Euler thought that alternative theories based on alternative definitions of sum and limit could not exist.

Moreover, nowadays, every object is introduced into mathematics by means of an appropriate definition, although it is arbitrary in principle. For example, if *n* and *a* are positive integers, the power a^n denotes the object a^n =aa..a (n times). The objects a^0, a^1, a^r (r real), a^∞, $a^{-\infty}$ do not come under this definition and are therefore defined separately. In the eighteenth century no mathematician would have stated: 'I define a^0 by letting $a^0=1$'. Instead he would have explained that a^0 is equal to 1 because $a^0=a^{n-n}=a^n:a^n=1$. The objects a^0, a^1, a^r (r real), a^∞, $a^{-\infty}$ were used not because they were mathematically defined objects but as fictions that were useful for reasoning: their meaning could not be arbitrary but derived 'naturally' from the meaning of a^n. Given this context, it is not surprising that Euler thought of using the symbol $(-1)^\infty$ without defining it: if he had behaved differently it would have been anomalous with respect to standard practice. He was definitely aware that a clear piece of intuition in terms of infinitely growing or evanescent quantities corresponded to objects such as 2^∞ or $2^{-\infty}$ while this did not occur with $(-1)^\infty$. Nevertheless, he thought he could provide an *a posteriori* justification for $(-1)^\infty$,[40] in the same way that the equation $a^0=1$ was justified *a posteriori* rather than being defined *a priori*. Such a justification in the case of $(-1)^\infty$, rather than being linked to the fairly confused idea of infinite parity[41], was based on the conscious acceptance of the calculus as algebraic formalism and of the principle of generality in algebra. However, it is not my intention in this article to examine this point in detail (even though I will briefly return to it in the last paragraph). I simply intend to emphasise the point that Euler's use of infinity also includes (7) and that a translation of the vague Eulerian notions in the precise terminology of non-standard analysis eliminates aspects that were regarded as unitary.

A further remarkable aspect of the Eulerian notion of infinite quantity and infinitesimal is the meaning that could be attributed to the equation '0=0' and '∞=∞'. Let us consider the relation

(8) $$\frac{c}{a}+\frac{c}{a+b}+\ldots+\frac{c}{a+(i-1)b}=C+\frac{b}{c}\log(a+ib)$$

where *i* is an infinite number and C is a finite constant.

In [1734-35a, 90] Euler had proved (8) by assuming that if *i* is infinite, then the general term of the sequence $\frac{c}{a}+\frac{c}{a+b}+\ldots+\frac{c}{a+(i-1)b}+\ldots$ is $\Delta s=s_i-s_{i-1}=\frac{c}{a+(i-1)b}$ is infinitely small and that $ds=\frac{c}{a+(i-1)b}di$. By integrating he obtained $s=C+\frac{b}{c}\log(a+ib)$ .He then points out that, if *i* is infinite, log *i* and log (a+ib) are an infinite times smaller than any power of *i*. Justification for the fact that the logarithm is a smaller infinite with respect to i^n, for every n, can be found in (1778). It is based on dell'Hospital's rule, which considers the

[40] It should be noted that not all 18th century mathematicians accepted the idea of assigning a meaning to $(-1)^\infty$. For instance, ….

[41] The fact that this idea was far from fundamental to Euler's thought can be deduced from what he states in (1754-55, section 4).

infinite number *i* as a variable (see Bos(1974, 84-87)]. The relation (8) essentially states that the two quantities behave in the same way when i increases unlimitedly. The following theorem, for example, should be interpreted in this way:

the sum of the series of the reciprocals of prime numbers $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ldots$ is infinitely large but is nevertheless infinitely smaller than the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$, the sum of the first is equal to the logarithm of the sum of the second. Since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ is equal to log ∞, one has $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ldots$ =log (log∞). [Euler 1737, 242-243].

Comparing infinite and infinitesimal quantities meant for Euler assessing their behaviour when they simultaneously increased unlimitedly or vanished. This idea is at the basis of the distinction between arithmetic and geometric equations, by which Euler aims to provide an improved mathematical explanation of the equation between infinitesimals (0=0) and between infinite numbers (∞=∞).

In [1755, 70 and 74] Euler stated that, given two quantities a and b, the equation a=b can be understood in an arithmetic sense (in other words, a=b, if a-b=0) and in a geometric sense (a=b, if a/b=1). The arithmetic equality coincides with the geometric one for finite quantities but the situation is different for infinite numbers and infinitesimals. For infinitesimals, the arithmetic equation, which is always verified, does not imply a geometric one: in Euler's rudimentary symbolism, 0=0 does not imply 0/0=1.[42] For infinite numbers, the geometric equation does not imply an arithmetical one, that is ∞/∞=1 does not imply ∞=∞.

By using this distinction, Euler believed he could justify the principle of cancellation. He observes that a±ndx=a (n being any number) is true in that it is not only verified in an arithmetical sense ((a±ndx)-a=ndx, being ndx=0) but also in a geometric sense. Indeed, one obtains (a±ndx)/a=1 and this means that the infinitesimals vanish before any finite quantity. The situation is analogous for higher order infinitesimals. Euler observes that "the infinitely small quantity dx^2 vanishes before dx", since the quantities $dx+dx^2$ and dx (both evanescent: $dx+dx^2=dx=0$) go to zero in the same way $(dx+dx^2):dx=1+dx=1$. More generally, if m<n, then adx^m+bdx^n was equal to adx^m because the arthimetical and geometrical equlities ($adx^m+bdx^n=dx^m=0$ and $\frac{adx^m+bdx^n}{adx^m}=1+\frac{a}{b}dx^m=1$) were verified.

We could briefly say: if A=B+C, and if C goes to zero before B (in other words, A/(B+C)=1), then A=B. In modern terms, A and B are asymptotically equal[43] or have the same asymptotic behaviour.[44] Nevertheless, while it is fairly clear what is meant by the ratio ψ/ξ between infinitesimal and infinite quantities ψ and ξ, it

[42] In (1755), Euler shows how the laws of elementary arithmetic can be adapted to the case 0=0…Naturally the example does not show us how 0:0 becomes exactly equal to a certain n. This can only be understood by studying the zero as the limit of variable.

[43]

[44] In particular, the idea of the approach of two quantities, which we can see here in operation, rather than the usual definition of limit, makes us think of the following generalised definition:….

not at all clear how it is possible to move from asymptotic equations to algebraic equations, or rather how the quantities ψ□ and ξ, which are asymptotically equal, can be manipulated as though they were equal numbers.[45]

Naturally there is no intention of attributing to Euler modern asymptotic notions[46] or the definition of the Peano derivative: this would be as difficult as stating that Euler uses infinitesimals in the sense of non-standard analysis. I merely noted that Eulerian notions possessed many facets which, with the hindsight of modern knowledge, we can appreciate but which were hazy to Euler: thus the idea of evanescent quantity contained subtleties that lead us to think of the standard notion of limit or of modern infinitesimals or of asymptotic processes or generalised limit concepts. I would not be surprised if traces of other modern notions are found in Eulerian infinitesimals. Nevertheless, one cannot, in my opinion, speak of confusion between different aspects, since Euler could not confuse notions that did not yet exist. Instead, he refers to a primordial idea of approaching (which we can term 'proto-limit') from which later mathematics has derived various modern notions.

6. The calculus as the calculus of functions

In his [1748], Euler introduced functions as autonomous objects in analyis and constructed an organic theory of these objects. Functions became the heart of mathematics and Euler claimed also that functions the subject-matter of the calculus and that differentials were mere tools for dealing with function. In *De usu functionum discontinuarum in Analysi*, Euler stated:

"differential calculus is not concerned with investigating the quantity of differentials, which is nothing, but with defining their mutual ratio, which had a determinate quantity in any case. It certainly investigates not so much the differential dy of the function y but its ratio with the differential dx (p.11-12)".

He subsequently made it clear that the value of the fraction dy/dx gives rise, for any possible case, to a determinate quantity which can be considered a new function of x. Similar statements can be found in [Euler 1755 and 1768]. According to Euler, given the function, for example $y=x^3$ whose differential is $dy=3x^2dx$, the real object of the calculus was the study of the differential coefficient (in the example, $3x^2$) and not of the differential $3x^2dx$. The algorithm of the calculus transformed functions into functions and not differentials into differentials: this represents a rather innovative viewpoint which became part of the general rewriting of analysis as a theory of functions and separated itself from the tradition of Leibniz and Bernouill.

In the preface to (1755) and in the *De usu*, Euler provided examples where the calculation of the differential coefficient seems to look forward to the modern definition of the derivate. In (1755, p.5), he considered the function $y=x^2$ and observed that $y(x+\omega):\omega=(2x+\omega):1$ (being $y(x+\omega)$ the increment of $2x\omega+\omega^2$ of x^2), so that the smaller ω is, the smaller is $(2x\omega+\omega^2):\omega$ becomes close to 2x:1, even though it actually reaches 2x:1 only when the increment has completely vanished.

[45] It should be noted that Euler did not necessarily regard the quantities ψ and ξ as functions but as variable quantities.

[46]

The similarity with the modern notion of derivate is striking, but it must not deceive us. In chapters 3 and 4 of [1755], where the rules of the calculus are formulated,[47] the introduction of differential coefficients was considerably more tortuous. Indeed, Euler tried to include the analysis of infinites organically into the method of finite differences, considering it as a special case of this method "which occurs when the differences, which were previously supposed to be finite, are taken infinitely small." [Euler 1755, 84]. The idea of limit, which had already been expressed with sufficient clarity in the preface to (1755), was thus absorbed by the notion of infinitely small.[48]

In chapter 1, Euler defined the finite differences of any order[49] and had set out the rules of the sum and the product of finite differences [1755, 16-20]. He then calculated the differences of algebraic functions and exponential, logarithmic, trigonometric functions. For example, in [Euler 1755, 28-29], by using the series expansion of the logarithm

(9) $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Euler obtained

(10) $$\Delta y = y^{(1)} - y = \log(x+\omega) - \log x = \log\left(1+\frac{\omega}{x}\right) = \frac{\omega}{x} - \frac{\omega^2}{2x^2} + \frac{\omega^3}{3x^3} - \frac{\omega^4}{4x^4} + \ldots$$

On the basis of the study of these functions (which constituted the universe of the most commonly studied functions), he stated that the difference Δy, for every function y, could be expressed in the form

(11) $$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \ldots$$

Analogously he asserted that higher-order differences could be written in the form:

(12) $$\Delta^2 y = P\omega^2 + Q\omega^3 + R\omega^4 + \ldots,$$

(13) $$\Delta^3 y = P\omega^3 + Q\omega^4 + R\omega^5 + \ldots, \text{ etc.}$$

Differential calculus originated by letting $\omega=dx$ infinitesimal in the (11): in this way Δy also became an infinitesimal and, by ignoring higher order infinitesimals (which vanish before ω) one obtained $\Delta y=P\omega$ or, in a different notation, could be written as dy=Pdx. Euler thought he had proved that the differential dy of a function could be expressed by means of a certain function P of x multiplied by the differential dx and that the relationship between dx and dy was finished, since dy:dx=P:1. [Euler 1755, 86].

If one takes into account the examples given in the preface, he could have defined the differential coefficient as

[47] In the preface to (1755) and in the *De usu*, Euler briefly discusses the nature of calculus, restricting himself to the examples mentioned here.

[48] In his [1972], Bos stated that the definition of the analysis of infinite in chapter of the *Institutiones calculi differentialis* "is rather at variance with" his remark quoted in footnote ??, "a contradiction which shows that his arguments about the infinitely small did not really influence his presentation of the calculus" [Bos 1972, 68-69]. I think that one can see a contradiction in the *Institutiones* only if, unlike Euler, one distinguishes between limits and infinitesimals.

[49] Set $y^{(n)}=y(x+n\omega)$, for a nonnegative integer n, and $y=y^{(0)}$, Euler defined $\Delta y=y^{(1)}-y$, $\Delta y^{(n)}= y^{(n+1)}-y^{(n)}$, $\Delta^m y= \Delta^{m-1}y^{(1)}- \Delta^{m-1}y$, $\Delta^m y^{(n)}= \Delta^{m-1}y^{(n+1)}- \Delta^{m-1}y^{(n)}$, for m>1 and n>0.

$$\frac{\Delta y}{\omega}=\frac{P\omega+Q\omega^2+R\omega^3+S\omega^4+...}{\omega}=P+Q\omega+R\omega^2+S\omega^3+...$$

for ω as an evanescent quantity. However, Euler preferred to give a direct definition of the differential dy of the function y(x) (by means of $\Delta y=P\omega+Q\omega^2+R\omega^3+S\omega^4+....$) and only a indirect definition the differential coefficient.[50] For example, in order to determine the differential of $y=x^n$, he considered

$dy==y^{(1)}-y=(x+dx)^n-x^n=nx^{n-1}dx+\frac{n(n-1)}{1\cdot 2}x^{n-2}dx^2+\ldots$

By neglecting higher order infinitesimals, which vanish before dx, he had $dx^n=nx^{n-1}dx$.

Similarly, since $dy=\log(x+dx)-\log x=\log(1+\frac{dx}{x})$, he derived

(14) $$dy=\frac{dx}{x}-\frac{dx^2}{2x^2}+\frac{dx^3}{3x^3}-\frac{dx^4}{4x^4}+...$$

by applying (9). In formula (14), the terms dx^n/nx^n, for n>1, vanish in comparison with dx/x, and he obtained d(logx)=dy=dx/x [Euler 1755, 122].

It should be observed that this presents a rather peculiar situation. Even though the calculus dealt with finite quantities dy/dx, in the actual construction of the calculus, he considered dy/dx as the real ratio of the differentials dy and dx. Moreover, the symbols dy and dx have a meaning as variable quantities tending to zero or evanescent quantity, but this idea was not developed. Euler used differentials on the basis of the fact that the differential dy of the function y(x) could be considered as the increment of y(x) when x has an infinitesimal increment dx [Euler 1755, 85], namely on the basis of the formula dy=y(x+dx)-y(x). Thus, in the calculation of the differential of x^n, there is no reference to limit process and evanescent quantity, in contrast to the example presented in the preface to *Institutiones* and in *De usu*. Despite these ambiguities, Euler really did place differential coefficients at the centre of attention and he made a systematic attempt to reduce the calculus to a calculus of finite quantities: this is particularly clear in the case of functions with several variables and in the elimination of higher order infinitesimals.

In (1755) the differential of a function V(x,y,z) in the variables x,y,z was introduced by Euler by letting dV=V(x+dx,y+dy,z+dz)-V(x,y,z), without making further mention of finite increments tending to zero. By analysing two examples, he observed that dV can be expressed as dV=pdx+qdy+rdz where p,q,r are functions of x,y,z… He also noted that if y and z were taken as constants then dy=0, dz=0, and dV=pdx. Similarly, if x and y were constants then dx=0, dy=0, and dV=rdz; if x and z were constants, then dx=0, dz=0 and dV=qdy. Consequently, dV is obtained by calculating the differentials of V supposing, on each occasion, that two of the variables are constant [Euler 1755, 144-146].

Euler then demonstrated the theorem of mixed differentials which, in the case of functions with two variables, could be formulated as follows: if dV=Pdx+Qdy then the differential of P for variable y and constant x and the differential of Q for variable x and constant y are equal [Euler 1755, 153-154].[51] Subsequently [Euler 1755, 156-157], he posed dP=rdy (constant x) and dQ=qdx (constant y), and observed that dPdx=rdxdy and dQdx=qdxdy. Since the mixed differentials are equal, he had r=q. Only at this point Euler

[50] Euler's approach seems to be an attempt to fit Newtonian conception in the Leibnizian tradition. For instance, …

[51] Euler first published in [1734-35b]. An hand-written version was published in [Engelsman 1984, 205-213]. This proof is well-known, it is therefore not illustrated here (for instance, see [Engelsman 1984, 128-130]).

decided to introduce a symbolism to indicate the functions r and q in a convenient and unambiguous way. He denoted r by means of the symbol $\left(\frac{dP}{dy}\right)$, which meant *the differential of P for variable y and constant x* (that is, considering P as a function of the single variable y) divided by dy. Similarly $\left(\frac{dP}{dx}\right)$ indicated *the differential of Q for variable x and constant y*.[52] Therefore the condition that linked the finite quantities P and Q in the differential dV=Pdx+Qdy can be expressed as $\left(\frac{dP}{dy}\right)=\left(\frac{dP}{dx}\right)$.

In his treatises, Euler systematically used the coefficients $\left(\frac{dP}{dy}\right)$ and this constituted an essential contribution to the transformation of the calculus with multiple variables into a calculus of finite quantities. Nevertheless, the underlying problems and ambiguities seem evident: to an even greater extent than for functions with one variable, the notion of differential dV appears to be hard to reconcile with the meaning of evanescent quantity and, *de facto*, the symbols $\left(\frac{dP}{dy}\right)$ and $\left(\frac{dP}{dx}\right)$ were introduced as ratios of differentials.

The situation does not differ significantly from that regarding the elimination of higher order differentials. These were regarded as particularly problematic from the origins of the calculus and were, for example, the object of an attack by Nieuwentijdt upon Leibniz. In chapter IV of the *Institutiones*, Euler [1755, 84 and 88] stated that higher order differentials derived from higher order finite differences $\Delta^n y=P\omega^n+Q\omega^{n+1}+R\omega^{n+2}+...$ putting ω=dx in the same way in which first differentials derived from $\Delta y=P\omega+Q\omega^2+R\omega^3+...$ posing ω=dx. For example, as regards the second differentials, the terms $Q\omega^3$, $R\omega^4$... of (12) vanished before $P\omega^2$ and, therefore, $d^2y=Pdx^2$, where dx^2 was the square of dx. Naturally, d^2y was equal to 0 while the ratio between d^2y and dx^2 was finite and equal to P:1 [Euler 1755, 88].

At §.126 of the first part of *Institutiones*, Euler stated that the second differential was simply the differential of the first differential (since the second difference was simply the difference of the first difference) [Euler 1755, 88-89]. Nevertheless, stating that d^2y was simply d(dy) led to the fundamental question as to what is meant by differentiating dy. The question was far from clear. In reality, it involved making something become evanescent when it was already evanescent. It is also worth pointing out that, if it is true that the finite second difference $\Delta^2 y$ is by definition equal to $\Delta(\Delta y)$, it does not appear at all obvious that the second differential, defined by means of (12), must coincide with the differential of dy.

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[52] "Brevitas, gratia autem hoc autem capite quantitates r et q ita commode denotari solent, ut r indicetur per $\left(\frac{dP}{dy}\right)$, qua scriptura designatur P ita differentiari, ut sola y tanquam variabilis tractetur atque differentiale istud per dy dividatur; sic enim prodibit quantitas finita r. Simili modo significabit $\left(\frac{dP}{dx}\right)$ quantitatem finitam q, quia hac ratione indicatur functionem Q sola x posita variabili differentiari tumque differentiale per dx dividi habere." [Euler 1755, 157].

Furthermore, given a function y=y(x), whose first and second differentials are dy=pdx and $d^2y=qdx^2$, the problem arises of establishing the connection between p and q. Having used (12) to introduce second differentials, there is no *a priori* assurance that there exists a simple relationship between p and q. In order to determine such a relationship, Euler observed that one could set dp=qdx (since all the differentials of functions possessed this form) and that ndp=nqdx, where n represented any constant quantity.[53] If one then posed n=dx (therefore dx is constant), one obtained pdx=qdx. Remembering that dy=pdx and dp=qdx, one obtained $d^2y=d(pdx)=qdx^2$: the second differential of y had a finite relationship with dx^2, which coincided with the differential coefficient of p [Euler 1755, 89].[54]

In this proof demonstration, Euler assumed that dx was a constant (non null) and that the second differences and the higher order differences were null. In order to justify this assertion, he assumed that the variable quantity x received equal increments, or rather that the sequence of values x, $x^{(1)}=x+dx$, $x^{(2)}=x+2dx$, …, $x^{(n)}=x+ndx$ was assigned to the variable x.[55] In this way, however, d^2x was constantly equal to zero (in Euler's terminology, the higher order differentials d^2x, d^3x, d^4x, … were *per se* equal to zero): they vanished in a different sense from dx and the intuitive meaning of a differential understood as an evanescent variable disappeared. Euler therefore attempted to provide a different interpretation of the differential according to d^2x. If one applied (12) to the function y=x, then P=Q=R=…=0, that is $d^2y=d^2x=0dx^2+0dx^3+…$ and P=Q=R=…=0: it was possible to state that the second differential d^2x vanished before dx^2 and all the higher powers of dx [Euler 1755, 88]. Therefore, d^2x was both zero *per se* and a relative zero (namely, d^2x was actually equal to zero and was a quantity that vanished before dx^2, d^3x, d^4x,…) A similar interpretation can be given for d^3x, d^4x, … .

The aim of these complex and occasionally rather unclear considerations is to prove that the second differential coefficient is obtained by differentiating the first coefficient: it is natural to ask ourselves why Euler did not directly define the second differential as $d^2y=qdx^2$, where q=dy/dx. Yet even in the preface to *Institutiones*, Euler seemed to suggest precisely this definition for d^2y. Indeed, he had observed that since infinitesimals were equal to zero, the higher order differentials were never considered *per se* but in relation to each other. More precisely, given a function y=f(x), whose differential coefficient is a certain function p, the second differentials were obtained by considering the ratio of increment of the function p with other increments and the symbols of the differentials serve only to give a convenient representation of certain finite quantities [Euler 1755, 8].

Instead, in chapter IV of the *Institutiones*, Euler made use of the complex construction described above to introduce higher order differential coefficients: he seemed to be worried about the confrontation with the

[53] Euler justifies this step by appealing finite differences; however, its extension to differentials is not a source of further difficulties.

[54] Naturally, the reasoning can be repeated; if dq=rdx, then $d^3y=dq=rdx^3$, if dr=tdx, then $d^4y=dr=tdx^4$,…: therefore the higher order differentials of y can be calculated one after another by differentiating p,q,r,t, etc.

[55] It is possible to think the sequence $x^{(n)}=x+ndx$ as a sequence $x^{(n)}=x+n\omega$, where the increment ω is initially finite and subsequently made evanescent; dx=constant means that the increment diminishes in the same way for all the points $x^{(n)}=x+ndx$. Whatever value is assigned to dx tending uniformly to zero, we will always have d(dx) constantly = 0.

Leibnizian tradition and to demonstrate how Leibniz' and Bernoulli's differential calculus was capable of being translated into the new calculus of functions and their differential coefficients.

The Leibnizian calculus was not based on functions but on curves analytically expressed by an equation f(x,y)=0. The independent variable was not chosen *a priori* and therefore it was not established *a priori* that dx was a constant. The notion of function is based rather on a clear distinction between dependent and independent variable: the entire Eulerian construction, beginning with the formulae of finite differences, is made using the hypothesis that x is the independent variable.[56] Euler emphasised that the increments of x are taken as constants (that is, x is the independent variable and $d^2x=0$) and that he is studying the variability of functions in relation to x. In the case that constant increments were not assumed, the second differentials have a more complicated expression.

Indeed, if dx was not constant, then the differential of p(x)dx was derived by considering the increment dx of x and d^2x of dx. Put p(x+dx)=p(x)+q(x)dx, he had

$d^2y=p(x+dx)\cdot(dx+d(dx))-pdx=(p+qdx)\cdot(dx+d^2x)-pdx=pdx+pdx^2+qdx^2+qdxdx^2-pdx= pdx^2+qdx^2+qdxdx^2$.

Since $qdxdx^2$ vanished before qdx^2, he obtained $d^2y=pdx^2+qdx^2$ [Euler 1755, 88-90].

In chapter IV Euler restricted himself to observing that nothing can be said with certainty about second differentials if dx is not assumed to be constant: there is no explanation about how d^2x can be reduced in this case to the theory of evanescent quantities. He dealt with the question more thoroughly in chapter VIII, where he applied the rules of differentiation to functions such as $V=(dx^2+dy^2)^{1/2}$ and V=ydy/dx by considering dy and dx as variables. For instance, he found that the differential of $V=(dx^2+dy^2)^{1/2}$ was $dV=\frac{dxd^2x+dxd^2y}{\sqrt{dx^2+dy^2}}$

and that the differential of V=ydy/dx was $dV=dx+\frac{yd^2x}{dy}-\frac{ydxd^2y}{dy^2}$ [Euler 1755, 167-168].

Euler [1755, 168] observed that the formulae containing higher order differentials have a vague meaning since they do not possess any determined value *per se* but assume values which vary according to which differential is taken as constant. For example, given the expression $\frac{yd^2x+xd^2y}{dxdy}$, if one considered dx as a constant then the differential was $\frac{xd^2y}{dxdy}$, while if dy was a constant, then one obtained a different result $\frac{yd^2x}{dxdy}$.

[56] In chapter 1, Euler had already noted that, as regards finite differences, the first difference $\Delta y=y(x_1)-y(x)=y(x+\omega)-y(x)$ was not influenced by the sequence x_i, while the second differences changed according to the nature of x_n. Indeed, in general it is $\Delta^2y=\Delta^{(1)}y-\Delta y=y(x_2)-2y(x_1)+y(x)$, and, for $x_n=x+n\omega$, $\Delta^2y=\Delta^{(1)}y-\Delta y=y(x+2\omega)-y(x+\omega)-y(x+\omega)+y(x)=y(x+2\omega)-2y(x+\omega)+y(x)$.

If one did not consider a sequence in arithmetic progression, the result was different. These remarks are also valid for differentials. The question of choice of the progression of variables and of the indeterminacy of higher order differentials in the eighteenth-century calculus is treated in detail by Bos (see, in particular [Bos 1974, 25-31]).

Furthermore, Euler believed that higher order differentials did not have an effective use in analysis [Euler 1755, 263]: they can be eliminated due to the differential coefficients p,q,r…, (defined by the relations dy=pdx, dp=qdx, dq=rdx,...); namely, putting $d^2y=qdx^2$, $d^3y=dq=rdx^3$,…. For instance, Euler stated that $\sqrt{dx^2+dy^2}$ as a constant was often found in the applications of the calculus. Set dy=pdx and dp=qdx, he obtained $dx\sqrt{1+p^2}$=constant, $d^2x\sqrt{1+p^2}+\frac{pqdx^2}{\sqrt{1+p^2}}=0$ and $d^2x=-\frac{pqdx^2}{1+p^2}$. He also derived $d^2y=qdx=qdx^2+pd^2x=qd^2x-\frac{p^2qdx^2}{1+p^2}=\frac{qdx^2}{1+p^2}$. Similarly one could derive d^3x, d^4x,…, d^3y, d^4y,… [Euler 1755, 178].[57]

At this point, a fundamental clarification must be made in order to understand Eulerian calculus. Differentiation is not a pointwise defined operation;[58] in other words, the differential is not defined at a specific point x_0 of the domain of the function. Euler never considers limitations to the domain of variables: such a concept is completely absent from his mathematics and he only considers general or abstract quantities (see, for instance, (1755, 15 and 83). Differentiation was regarded as a global operation that involved the entire analytical expression y(x) and was aimed at determining another analytical expression p(x). The statement that the calculus transformed functions into functions should be understood in the sense that calculus transformed analytical expressions (functions in the eighteenth century use of the term) into analytical expressions. Since this article is principally concerned with offering an analysis of the notion of differential, I will not provide a detailed examination of this aspect, which is an integral part of the eighteenth century concept of function (see [Ferraro 2000a]), and instead will merely offer two examples.

The first example concerns finite differences. Given a finite difference $\Delta y=z(x,\omega)$, Euler posed the inverse problem of finding the quantity that generates $z=\Delta y$ (he used the symbol $y=\Sigma z$ and termed the quantity y the sum of z). He stated that the sum y of $z=\Delta y$ is not unique but is of the type $y=\Sigma x+C$, where C is an arbitrary constant. For instance, if $\Delta y=aw$ then $\Sigma a\omega=ay+C$. Hence (ω being a constant quantity), one derives that the sum of $\Delta y=\omega^2$ is $\Sigma\omega^2=\omega y+C$, and that the sum of $\Delta y=\omega^3$ is $\Sigma\omega^3=\omega^2y+C$. However it should be noted that if $y=\Sigma z$ is the sum of $z=\Delta y$, the most general sum is not $y=\Sigma x+C$ but $y=\Sigma x+f(\omega)$, where $f(\omega)$ is a periodic function that assumes the same value at the points $x+\omega$, $x+2\omega$, $x+3\omega$,…, such as $C\cos\frac{2\pi x}{\omega}$, for fixed x and ω [Euler 1755, 32].[59] Assuming that the sum is of the type $y=\Sigma x+C$, with constant C, means that Euler does not reason on a fixed set of points { $x+\omega$, $x+2\omega$, $x+3\omega$,…} but upon the general or abstract quantities ω and x (for any ω); namely his procedure holds for the variable ω and x, not for the specific value that ω and x assume.

Another example concerns the differential of the logarithm function. In [1755], once he had established that d(logx)=dx/x, Euler also did not hesitate to apply this formula to the case where the variable is

[57] On Euler's program to eliminate higher-order differentials from analysis, see [Bos 1984, 66-77].

[58] Today a derivative is locally defined in the sense that, given a point x_0 of the domain of a function f(x), the derivative is defined at the point x_0.

[59] The fact that Euler could make such considerations can be deduced from his work (1750-51), in particular 465-467.

imaginary, without making any distinction between real and imaginary variable. Thus he found the differential $dy = \frac{dx}{\sqrt{1-x^2}}$ of the function y $y = \frac{1}{\sqrt{-1}} \log\left(x\sqrt{-1} + \sqrt{1-x^2} \right)$ and asserted that the differential of an imaginary quantity could be real [Euler 1755, 124]. More explicitly, in [Euler 1749] he had stated: "Car, comme ce calcul roule sur des quantités variables, c'est-à-dire sur des quantités considérées en général, s'il n'etoit pas vrai généralment qu'il tût d.lx=dx/x, quelque quantité qu'on donne à x, soit positive ou negative, ou même imaginaire, on ne pourrait jamais se servir de cette régle, la verité du calcul differential étant fondée sur la généralité des régles qu'il renferme."[60]

The only, and in any case partial, exception to the approach to the calculus as an algorithm transforming analytical expressions into analytical expressions concerns the treatment of equations of partial derivates. Without going into detail, it can be stated that, in order to resolve this type of equation, Euler also considered functions which did not have an analytical expression which he called discontinuous functions. He denoted the differential of a discontinuous function f:y by the symbol dyf':y [Euler 1768-70, III:69]), he took the geometric meaning of a function into account and stated that if f(x) represented a curve then f'(x) was the slope of the tangent whereas, if f(x) was interpreted as an area then the differential coefficient f'(x) was a curve. There was no adequate development of this idea: Euler never actually dealt with such functions but merely stated that it was possible to define the differential coefficient and that these formed part of the solution to the equations of partial derivates.[61]

7. Conclusion

This article has attempted to highlight some aspects of the Eulerian foundations of the calculus. It has been observed that Eulerian calculus is not based on the notion of the set nor that of real number but on that of quantity. Only natural numbers and rational numbers were considered numbers in their own right. Irrational numbers, zero and imaginary numbers were ways of dealing with quantities; they were useful fictions for mathematics. The notions of evanescent quantity and indefinitely increasing quantity, when interpreted using the conceptual instruments available to modern mathematics, seem to be an ambiguous mixture of different elements, a continuous leap from a vague idea of limit to a confused notion of infinitesimal. In reality, Euler does not confuse the modern notions of limit and the modern concept of infinitesimal: he simply did not possess such notions, but merely a primordial idea (directly derived from the physical world) of bringing a variable quantity close to a certain limit. The application of modern concepts to Euler's vague and ambiguous notions results in transforming Eulerian mathematics into something different.

It has also been said that Euler, in principle, conceived the calculus as the calculus of finite quantities, having as an object not the differentials dy,dx,… but the differential coefficients dy/dx; nevertheless, the first order differentials not only serve to introduce differential coefficients but, as fictions, can be used per se and play an important role in the calculus. Moreover, the differential coefficient p(x) of a function f(x) was not

[60] "For, as this calculus concerns variable quantities, that is quantities considered in general, if it were not generally true that d(logx)=dx/x, whatever value we give to x, either positive, negative or even imaginary, we would never able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains" [15, 143-144].

[61] On this subject, see [Truesdell 1960] and [Ferraro 2000a].

punctually defined, as in the case of the modern derivate (that is, given the function f(x), subject to a specific condition, to a number x_0 belonging to an interval I is associated with an appropriately defined value $p(x_0)$ but globally transformed analytical expressions into analytical expressions.

Euler's successors[62] developed certain concepts of his foundation of the calculus and rejected others. For example, Lagrange rejected the use of evanescent quantities and infinitely large quantities; however, he accepted the attempt to give a systematic translation of the calculus of differentials into a calculus of finite quantities as well as the use of rules applied globally to analytical expressions and not locally to functions in the modern sense. Shortly afterwards, Cauchy rejected this latter aspect of Eulerian theory, but accepted the fundamental idea that the algorithm of the calculus consisted in determining the differential coefficient. Cauchy's use of the idea of limit was explicit but the use of infinitesimals and infinite numbers persisted provided that they were effectively capable of being reduced to limits (accordingly a case such as $(-1)^{\infty}$ was excluded).

[62] In this article, I have not dealt in detail with the interpretations given to the calculation of the Eulerian zeros in the eighteenth century. I will simply mention the fact that Lagrange considered the foundations of the calculus of Euler and d'Alembert as analogous and contrasted both with the Leibnizians.

References

Jacques Dutka, (1984-85) The Early History of the Hypergeometric Function, *Archive for History of Exact Sciences* **31** (1984-85) 15-34.

Euler, L. (*Opera omnia*) *Leonhardi Euleri Opera omnia*. series I: Opera mathematica, Bern, 1911-1975.

(1730-31a) De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt, *Opera omnia* (1) 14: 1-24.

(1730-31b) De summatione innumerabilium progressionum, *Opera omnia* (1) 14:25-41.

(1732-33) Methodus generalis summandi progressiones, *Opera omnia* (1) 14: 42-72.

(1734-35a) De summis serierum reciprocarum, *Opera omnia* (1) 14:73-86.

(1734-35b) De progressionibus harmonicis observationes, *Opera omnia* (1) 14 87-100.

(1736a) Methodus universalis serierum convergentium summas quam proxime inveniendi, *Opera omnia* (1) 14: 101-107.

(1736b) Inventio summae cuiusque seriei ex dato termino generali, *Opera omnia*, (1), 14, 108-123.

(1736c) Methodus universalis series summandi ulterius promota, *Opera omnia* (1), 14, 124-137.

(1737a) De fractionibus continuis dissertatio, *Opera omnia* (1) 14, 197-215.

(1737b) Variae observationes circa series infinitas, *Opera omnia* (1) 14, 216-244.

(1737c) De variis modis circuli quadraturam numeris proxime exprimendi, *Opera omnia* (1) 14, 245-259.

(1739a) Consideratio progressionis cuiusdam ad circuli quadraturam inveniendam idoneae, *Opera omnia* (1), 14, 350-365.

(1739b) De Productis ex infinitis factoribus ortis, *Opera omnia* (1) 14: 260-290.

(1739d) Methodus facilis computandi angulorum sinus ac tangentes quam naturales quam artificiales, *Opera omnia* (1) 14, 366-406.

(1739e) De fractionibus continuis observationes, *Opera omnia* (1) 14, 291-349.

(1740a) De seriebus quibusdam considerationes, *Opera omnia* (1) 14, 407-462.

(1743a) De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera in qua eaedem summationes ex fonte maxime diverso derivantur; *Opera omnia* (1) 14, 138-159.

(1743b) Démonstration de la somme de cette suite $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\text{etc}$., *Opera omnia* (1) 14, 176-186.

(1748) *Introductio in analysin infinitorum*; *Opera omnia*, (1), 8-9.

(1750-51a) De serierum determinatione seu nova methodus inveniendi terminos generales serierum, *Opera omnia*, (1), 14, 463-515.

(1750-51b) Consideratio quarumdam serierum quae singularibus proprietatibus sunt praeditae, *Opera omnia*, (1), 14, 516-541.

(1754-55a) Subsidium calculi sinuum, *Opera omnia*, (1) 14: 542-582.

(1754-55b) De seriebus divergentibus, *Opera omnia*, (1) 14: 583-617.

(1755) *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*; *Opera omnia*, (1), 10.

(1774) Summatio progressionum sin. φ^λ+ sin. $2\varphi^\lambda$+ sin. $3\varphi^\lambda$+…+ sin. $n\varphi^\lambda$, cos. φ^λ+ cos. $2\varphi^\lambda$+cos. $3\varphi^\lambda$+…+ cos. $n\varphi^\lambda$; *Opera omnia*, (1), 15:168-190.

(1797) Delucidationes in capita postrema calculi mei differentialis de functionibus inexplicabilibus; *Opera omnia* (1), 16_1, 1-33.

Ferraro, G. (1998) Some Aspects of Euler's Theory of Series. Inexplicable functions and the Euler-Maclaurin summation formula, *Historia mathematica,* 1998, 25, 290-317.

(1999) The first modern definition of the sum of a divergent series. An aspect of the rise of the 20th century mathematics, *Archive for History of Exact Sciences, to appear.*

[A] Some Aspects of 18th century Infinitesimal Analysis. Functions, Functional relations and the Law of Continuity in Euler, *unpubblished manuscript*.

Fraser, C. (1985) J.L. Lagrange's Changing Approach to the Fondations of the Calculus of Variations, *Archive of History of Exact Science*, 32 (1985), 151-191.

(1989) The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century, *Archive for history of exact sciences*, 39 (1989), 317-335.

Gauss

[1813] Disquisitiones generales circa seriem infinitam $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+2)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)}x^3+etc.$, *Werke*, Bd.3, 124-162.

Publication List

Giovanni Ferraro, The integral as an anti-differential. An aspect of Euler's attempt to transform the calculus into an algebraic calculus, *Quaderns d'història de l'enginyeria,* 9 (2008), 25-58.

Giovanni Ferraro, Manuali di geometria elementare nella Napoli preunitaria (1806-1860), *History of Education & Children's Literature*, 3 (2008), 103-139.

Giovanni Ferraro, D'Alembert visto da Eulero, *Bollettino di Storia delle Scienze Matematiche*, 28 (2008), 257-275.

Giovanni Ferraro, Convergence and formal manipulation in the theory of series from 1730 to 1815, *Historia Mathematica,* 34 (2007), 62-88.

Giovanni Ferraro, The foundational aspects of Gauss's work on the hypergeometric, factorial and digamma functions, *Archive for History of Exact Sciences* 61 (2007), 457-518.

Giovanni Ferraro, Functions, series and integration of differential equations, *Oberwolfach Reports*, 1, (2005), 2729-2794.

Giovanni Ferraro, Differentials and differential coefficients in the Eulerian foundations of the calculus, *Historia Mathematica,* 31 (2004), 34-61.

Giovanni Ferraro, M.Panza) Developing into Series and Returning from Series. A Note on the Foundation 18th Century Analysis, *Historia mathematica*, 30 (2003), 17-46.

Giovanni Ferraro, Convergence and formal manipulation of series in the first decades of the eighteenth century, *Annals of Science,* 59 (2002), 179-199.
Giovanni Ferraro, Analytical symbols and geometrical figures in Eighteenth Century Calculus, *Studies in History and Philosophy of Science Part A, 32 (2001), 535-555.*
Giovanni Ferraro, Functions, Functional Relations and the Laws of Continuity in Euler, *Historia mathematica*, 27 (2000), 107-132.
Giovanni Ferraro, The value of an infinite sum. Some Observations on the Eulerian Theory of Series, *Sciences et Techniques en Perspective*, 4 (2000), 73-113.
Giovanni Ferraro, True and Fictitious Quantities in Leibniz's Theory of Series, *Studia Leibnitiana*, 32 (2000), 43-67.
Giovanni Ferraro, The first modern definition of the sum of a divergent series. An aspect of the rise of the 20th century mathematics, *Archive for History of Exact Sciences*, 54 (1999), 101-135.
Giovanni Ferraro, Rigore e dimostrazione in Matematica alla metà del Settecento, *Physis*, (2) 36 (1999), 137-163.
Giovanni Ferraro, Some Aspects of Euler's Theory of series. Inexplicable functions and the Euler-Maclaurin summation formula, *Historia mathematica*, 25 (1998), 290-317.
Giovanni Ferraro, Su alcuni scritti inediti del matematico Ernesto Cesàro (1859-1906), *Annali Dist.*, (15) 2000, 17-31.
Giovanni Ferraro, Continuità e derivabilità in Catalan. Alcune osservazioni sui fondamenti dell'analisi prima di Weiestrass e Dedekind, *Annuario 1999/00*, I.T.C.G, 59-65.
Giovanni Ferraro, È necessario definire i numeri reali? Brevi note su 'Continuità e numeri irrazionali' di Dedekind, *Progetto Alice*, 1 (2000), 415-423.
Giovanni Ferraro, Sperimentazioni didattiche in matematica; *Annuario 1982/83 - 1995/96*, I. Sereni, Afragola, 1996, 99-101.
Giovanni Ferraro, L'insegnamento della Geometria a Napoli nell'Ottocento e i suoi influssi sulle scuole del Regno d'Italia, *Annali Dist.*, 10 (1995), 66-82.
Giovanni Ferraro, P. De Lucia, F. Palladino), Alcuni tratti della matematica napoletana da prima a dopo la repubblica partenopea del 1799, *Rendiconto dell'Accademia di Scienze Matematiche e Fisiche*, 62 (1995), 225-274.
Giovanni Ferraro, F.Palladino), Sui manoscritti di Nicolò Fergola, *Bollettino di Storia delle Scienze Matematiche*, 13 (1993), 147-197.
Giovanni Ferraro, F.Palladino), Contributo alla conoscenza del matematico Giulio Carlo de' Toschi di Fagnano (con lettere a C. Galiani e C. Grandi), *Archivio Storico per le Province Napoletane*, 110 (1992), 153-181.
Giovanni Ferraro, Pure and Mixed Mathematics in the Work of Leonhard Euler in *Computational Mathematics: Theory, Methods and Applications*, a c. di Peter G. Chareton, Nova Science Publishers, Hauppauge, New York, 2011, 35-61.

Giovanni Ferraro, Mathematics and Natural Philosophy in Euler's Investigation of Saturn's Perturbations, in *First International Meeting on Cultural Astronomy*, Napoli, Loffredo editore, 2010, 125-157.

Giovanni Ferraro, Dimostrazioni matematiche e conoscenza scientifica in Alessandro Piccolomini, in *Saggi di Letteratura architettonica da Vitruvio a Winckelmann*, a c. di H. Burns, F. P. Di Teodoro e G. Bacci, vol. III, Firenze, Olschki, 2010, 215-233.

Giovanni Ferraro, Baldi, le matematiche, l'architettura in *Saggi di Letteratura architettonica da Vitruvio a Winckelmann*, a c. di F.P. Di Teodoro, vol. I, Firenze, Olschki 2009, 207-220.

Giovanni Ferraro, Tra filosofia naturale e matematica: il paradosso della rota Aristotelis in Cardano, de Guevara e Galileo, in *Saggi di Letteratura architettonica da Vitruvio a Winckelmann*, a cura di L. Bertolini, vol. II, Firenze, Olschki, 2009, 121-138.

Giovanni Ferraro, Euler's treatises on infinitesimal analysis*: Introductio in analysin infinitorum, Institutiones calculi differentialis, Institutionum calculi integralis*, in *Euler Reconsidered. Tercentenary Essays*, a c. di R. Baker, Heber City, UT, Kendrick Press, 2007, 39-101.

Giovanni Ferraro, *The rise and development of the theory of series up to the early 1820s*, New York, Springer, Sources and Studies in the History of Mathematics and Physical Sciences, 2008.

Giovanni Ferraro, *Bernardino Baldi e il recupero del pensiero tecnico-scientifico dell'antichità*, Alessandria, Edizioni dell'Orso, 2008.

Giovanni Ferraro, *L'evoluzione della matematica. Alcuni momenti critici.* Napoli, Ernesto Ummarino Editore, 2007.

Giovanni Ferraro F.Palladino, *Il Calcolo sublime di Eulero e Lagrange esposto col metodo sintetico nel progetto di Nicolò Fergola*, Napoli, Istituto Italiano per gli Studi Filosofici, Seminari di Scienze, Edizioni La Città del Sole, 1995.