

Remarks on global controllability for the shallow-water system with two control forces

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Abstract

In this paper we deal with the compressible Navier-Stokes equations with a friction term in one dimension on an interval. We study the exact controllability properties of this equation with general initial condition when the boundary control is acting at both endpoints of the interval. Inspired by the work of Guerrero and Imanuvilov in [19] on the viscous Burger equation, we prove by choosing irrotational data and using the notion of effective velocity developed in [22, 23] that the exact global controllability result does not hold for any time T .

1 Introduction

We consider the viscous shallow water model with friction term. This model is also called by the french community the Saint-Venant equations and is generally used in oceanography. Indeed it allows to model vertically averaged flows in terms of the horizontal mean velocity field u and the depth variation h . In the rotating framework, the model is described by the following system:

$$\begin{cases} \frac{\partial}{\partial t} h + \partial_x(hu) = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ \frac{\partial}{\partial t}(hu) + \partial_x(h|u|^2) - \partial_x(\mu h \partial_x u) + \frac{\partial_x h}{Fr^2} + rhu = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ u(t, 0) = v_1(t), \quad u(t, 1) = v_2(t), & t \in (0, T), \end{cases} \quad (1.1)$$

$Fr > 0$ denotes the Froude number and we set $r = \frac{1}{\mu Fr^2}$. System (1.1) is supplemented with initial conditions

$$h|_{t=0} = h_0, \quad u|_{t=0} = u_0. \quad (1.2)$$

This model is derived from the compressible Navier-Stokes equations and the turbulent regime ($r \geq 0$) is obtained from the friction condition on the bottom, see [28]. μ is the viscosity coefficient and verifies $\mu > 0$.

Here, $T > 0$ is a given final time and $v_1(t)$ and $v_2(t)$ are control functions which are acting over our system at both endpoints of the segment $(0, 1)$. We are now going to

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assume that the controls v_1 and v_2 belong to the space $H^{3/4}(0, T)$ and the initial data (h_0, u_0) are in $H^2(0, 1) \times H^1(0, 1)$ with $h_0 \geq c > 0$. Furthermore we suppose the following compatibility assumption:

$$v_1(0) = u_0(0) \quad \text{and} \quad v_2(0) = u_0(1). \quad (1.3)$$

Under these assumptions, it is classical to see that there exists a solution (h, u) of system (1.1) which belongs for a $T > 0$ to the space $X_T = L^\infty(0, T; H^2(0, 1)) \times (L^2(0, T; H^2(0, 1)) \cap H^1(0, T; L^2(0, 1)))$ and a constant $C > 0$ such that:

$$\|(h, u)\|_{X_T} \leq C(\|u_0\|_{H^1(0,1)} + \|h_0\|_{H^2(0,1)} + \|v_1\|_{H^{3/4}(0,T)} + \|v_2\|_{H^{3/4}(0,T)}), \quad (1.4)$$

(see, for instance, [27]). We would like to mention that we can obtain global solution provided that we choose $u_0 = -\mu \partial_x \ln h_0$ (we shall detail more precisely in the sequel this point which will play a crucial role in your results, we refer also to [23]).

In the sequel we are interesting in proving some negative results about the global exact controllability of the system (1.1). Before giving some elements of the proof we would like to recall the notion of global exact controllability property for system (1.1).

Definition 1.1 *The system (1.1) will be said global exact controllable if for any $(h_0, u_0) \in (H^2(0, 1) \times H^1(0, 1))$ with $h_0 \geq c > 0$, and for any $(h_1, u_1) \in (H^2(0, 1) \times H^1(0, 1))$ with $h_1 \geq c_1 > 0$, for any time $T > 0$ there exist controls $v_1 \in H^{3/4}(0, T)$ and $v_2 \in H^{3/4}(0, T)$ such that the corresponding solution of (1.1) satisfies $u(T, x) = u_1(x)$ in $(0, 1)$ and $h(T, x) = h_1(x)$ in $(0, 1)$.*

Remark 1 *In our previous definition, we impose only two controls on the system (1.1) which is natural inasmuch as the result of existence of strong solution for the system (1.1) do not admit boundary condition on the height h .*

Let us recall what is known about the controllability of the compressible Navier-Stokes equations, as far as we know there exists very few results. Indeed one of the main difficulty consists in the non controllability of the linear system associated to (1.1) around the stable equilibrium $(1, 0)$. This has been observed indirectly in [29] and more recently in [11]. By developing a moment method in one dimension, we can observe that the operator associated to the linear system of (1.1) admits accumulation point on the eigenvalues, this is due in particular to the behavior of the height which generates a damping effect. However in [11], we check that the linearized system is approximate controllable. To overcome this difficulty, it would be natural to use the return method developed by Coron (see [4, 7, 8]), however the linearized system around the trajectory of [4] remains non controllable. It is one of the main reason why the problem of exact controllability for the compressible Navier-Stokes equations is so difficult. However very recently in a very interesting work Ervedoza, Glass, Guerrero and Puel in [12] have obtained the local exact controllability for the 1-D compressible Navier-Stokes equation by using Carleman estimates taking into account the flow associated to the transport equation on the height. The study of the exact local incompressible Navier-Stokes equations has been investigated by many authors, we would like in particular recall the results of Fernández-Cara et al in [14, 15]. Global exact controllability in two dimension has been obtained by Coron and Fursikov for a manifold without boundary by combining some result of global approximate

control obtained via the return method and some results of local exact controllability via Carleman estimates.

The goal of this paper is to show that the compressible shallow water system with friction term is not globally exactly controllable and not even globally approximately controllable. To do this, we have constructed special solutions of the system (1.1) which are not controllable (they are also called *quasi solution* in [?] when there is no friction term). Here compared with the compressible Navier-Stokes equation we also take into account the friction term νu which is physically justified to model the friction condition on the bottom of the ocean (see [28]). Naturally on a mathematical point of view, the term νu does not add any difficulties for obtaining global weak solutions or global strong solutions with small initial data, however in our study this term shall turn out to be essential in order to cancel out the coupling between the height h and the velocity u . Initially in [22] we obtain the existence of global strong solutions for the Korteweg system with friction term with large initial data (at least for the irrotational part of the initial data) in dimension $N \geq 2$. Indeed coupled with the pressure term, the friction term shall introduce a damping effect on a new unknown called *effective velocity* and introduced in [21]. Roughly speaking this friction term allows to cancel out the coupling between the height h and the velocity u (we can check in particular that $(h, -\mu \nabla \ln h)$ is a particular solution of the system where h verifies a heat equation), it is one of the main difficulties in order to obtain the existence of global strong solutions for the compressible Navier-Stokes equation (see [20], in this case the coupling is between the velocity and the density). Indeed in this last case, it is difficult to obtain a damping effect on the density and the coupling between the velocity and the pressure terms impose a smallness condition on the initial density.

This last result was extended for the shallow water system without friction in [23], in this paper we introduce the notion of quasi-solution $(h_1, -\mu \nabla \ln h_1)$ (where h_1 verifies a heat equation) which is a particular solution of the shallow-water system when the pressure is considered null. We consider then the pressure term as a perturbation by taking profit of the regularizing effect on h_1 which induces a smallness condition on the pressure in high frequencies.

In this paper we are going to verify that $(h, -\mu \partial_x \ln h)$ is a particular solution of (1.1). Via the Cole-Hopf formula we verify that the global controllability problem of (1.1) comes to deal in reality with the viscous Burgers equation studied in particular by Guerrero and Imanuvilov in [19]. They prove that this equation is not globally exactly controllable by using some comparison argument on the heat equation. We shall follow in our proof the argument of [19] which shall be an angular stone in our result.

Before giving our main result, let us briefly mention some previous works which deal with the exact controllability for the viscous and non viscous Burgers equation in order to understand better this equation.

Exact controllability for the non viscous Burgers equations

In [25], Horsin shows that by using one boundary control, the Burgers equation can be driven from the null initial condition to a constant final state M in a time $T \geq 1/M$. To do this, he uses the so-called return method introduced by Coron (see [8] for the case of the global approximate control of the Navier-Stokes equation with slip boundary condition). More general results are obtained in [1] when we consider general scalar

non-linear conservation laws.

Exact controllability for the viscous Burger equations

With one boundary control

First, in [10], the author proves that the approximate controllability to some target states does not arise for the solutions of the viscous Burgers equation with one boundary control. Indeed he proves the existence of some invariant, more precisely that there exists a constant $C > 0$ such that, for every $T > 0$ one has:

$$u(t, x) \leq \frac{C_0}{1-x} \quad \forall (t, x) \in (0, T) \times (0, 1).$$

Next, in [13], the authors prove that the global null controllability does not hold with one control force by using in a crucial way the comparison principle.

With two boundary controls

In [19], Guerrero and Imanuvilov prove that the viscous Burger equation is not exact global controllable even with two boundary controls. To do this they combine the Cole-Hopf transformation and some comparison principle for the heat equation.

In [18], Glass and Guerrero show the uniform exact controllability for the viscous Burger equation when the viscosity is small enough. In [26], Imanuvilov and Puel obtain the global exact controllability for a type of Burger equations. To finish we would like to mention the works of Chapouly in [3] who proves the global exact controllability of the viscous Burger equation when we consider three controls: two are the boundary values, one is the right member of the equation and is constant with respect to the space variable. This result is to relate with the paper of Coron [8] inasmuch as the right control replace in some sense the pressure term of the Navier-Stokes equations. In some sense in [3], Chapouly via the source control forces the system to be incompressible as for Navier-Stokes.

In the present paper, we have two main objectives. One concerning the global exact null controllability for small time and the other one concerning the global exact controllability for any time $T > 0$. Both results are of negative nature.

As long as the first one is concerned, we prove that there exists a final time T and an initial condition u_0 such that the solution of (1.1) is far away from zero. That is to say, the global null controllability for the compressible Navier-Stokes equation with friction and with two control forces does not hold. The precise result is given in the following theorem:

Theorem 1.1 *There exists $T > 0$ and $u_0 = -\mu \partial_x \ln h_0 \in H^1(0, 1)$ with $h_0 \geq c > 0$ such that, for any control functions $v_1 \in H^{3/4}(0, T)$ and $v_2 \in H^{3/4}(0, T)$ satisfying the compatibility conditions (1.3), the associated solution $(h, u) \in X_T$ to (1.1) satisfies:*

$$\|u(T, \cdot)\|_{H^1(0,1)} \geq C > 0, \quad (1.5)$$

for some positive constant depending on T and h_0 .

The second main result is a negative exact controllability result:

Theorem 1.2 *For any $T > 0$, there exists an initial condition $u_0 - \mu \partial_x \ln h_0 \in H^1(0, 1)$ and a target function $u_1 = -\mu \partial_x \ln h_1 \in H^1(0, 1)$ such that, for any $v_1 \in H^{3/4}(0, T)$ and $v_2 \in H^{3/4}(0, T)$ satisfying (1.3), the associated solution $(h, u) \in X_T$ to (1.1) satisfies:*

$$\|u(T, \cdot) - u_1(\cdot)\|_{H^1(0,1)} \geq C > 0, \quad (1.6)$$

for some positive constant C depending on T , h_0 and h_1 .

Remark 2 *Let us observe that we have global existence of solution for the system (1.1) via the fact that $(h, -\mu \partial_x \ln h)$ is a particular solution.*

Remark 3 *Let us remark that in [8], J-M Coron proves the global approximate controllability of the Navier-Stokes equations with slip boundary condition by using the return method (in [9], the authors obtain the global exact controllability). In [19], Guerrero and Imanuvilov prove that the viscous Burger equation in one dimension is not global exactly controllable. We show that it is also the case for the compressible Navier-Stokes equations with friction. This is due in particular to the fact that the flow is compressible, indeed in our case we are working with a specific solution of the form $u = -\mu \partial_x \ln h$ which is naturally not incompressible but irrotational. In particular it gives an idea why in the case of slip boundary condition we can hope global controllability. Indeed by imposing a Neuman condition on the height which verifies an heat equation in the context of the quasi solution, we verify that $u = -\mu \partial_x \ln h$ is such that $u \cdot n = 0$.*

Remark 4 *We recall that all global controllability properties obtained up to now for the incompressible NavierStokes system come essentially from the same property for the Euler equation (see the works of Coron [7, 17]). In the case of the compressible Navier-Stokes equation we use the properties of the viscous Burger equations and not these of the non viscous Burger equations as for incompressible Navier-Stokes equations. On the other side the structure of quasi solution is purely non linear.*

Remark 5 *In a forthcoming paper we are going to prove the same type of results for the shallow water system without friction by using the same type of argument than in [23] which corresponds to neglect in high frequencies the pressure term.*

In order to prove these results, in section 2 we first show the equivalence of the controllability problem for the system (1.1) and some controllability problem for a one-dimensional linear heat equation with positive boundary controls. To do this, we combine a Cole-Hopf transformation and the introduction of particular solution introduced in [21]. Finally, in the section 3 we prove both theorems 1.1 and 1.2 by using a method developed by Guerrero and Imanuvilov in [19].

2 Simplification of the problem

In this section, we show that with particular initial data (it means irrotational initial data on the velocity which depends on h_0) the exact controllability properties for the system (1.1) (see definition 1.1) can be reduced to some controllability assumptions for

the heat equation. In the sequel $T > 0$ is a fixed final time. Let us rewrite the system (1.1):

$$\begin{cases} \frac{\partial}{\partial t} h + \partial_x(hu) = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ \frac{\partial}{\partial t} u + u\partial_x u - \mu\partial_{xx}u + \frac{\partial_x \ln h}{Fr^2} + ru = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ u(t, 0) = v_1(t), \quad u(t, 1) = v_2(t), \quad t \in (0, T), \\ u(0, x) = u_0(x), \quad h(0, x) = h_0(x). \end{cases} \quad (2.7)$$

Next we remark that the controllability problem of definition 1.1 implies the controllability problem for a semilinear parabolic system (2.8) with time-dependent controls. We are writing the system (2.8) as follows:

$$\begin{cases} \frac{\partial}{\partial t} h - \mu\partial_{xx}h = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ \frac{\partial}{\partial t} w - \mu\partial_{xx}w - \mu|\partial_x w|^2 = v_5'(t) + v_3(t), & (t, x) \in Q = (0, T) \times (0, 1) \\ w(t, 0) = v_5(t), \quad w(t, 1) = v_4(t), \quad t \in (0, T), \\ h(t, 0) = e^{v_5(t)}, \quad h(t, 1) = e^{v_4(t)}, \quad t \in (0, T), \\ w(0, x) = \ln h_0(x), \quad h(0, x) = h_0(x). \end{cases} \quad (2.8)$$

Lemma 1 *The system (1.1) is global exact controllable implies that there exists a solution to the following controllability problem:*

For any $w_0, w_1 \in H^2(0, 1)$ and $h_0 = e^{w_0}$, $h_1 = e^{w_1}$ with $w_0(0) = w_1(0) = 0$, there exists control $v_3 \in L^2(0, T)$ and $v_4, v_5 \in H^1(0, T)$ and a solution of (2.8) such that $w(T, x) = w_1(x)$ in $(0, 1)$ and $h(T, x) = h_1(x)$ with $w \in Y_T = L^2(0, T; H^3(0, 1)) \cap H^1(0, T; H^1(0, 1))$ and $h = e^w$.

Proof: It suffices to set:

$$\ln h(t, x) = w(t, x) = -\frac{1}{\mu} \int_0^x u(t, y) dy + v_5(t), \quad \forall (t, x) \in Q,$$

and with:

$$\begin{aligned} v_3(t) &= -\frac{v_1^2(t)}{\mu} + u_x(t, 0), \quad v_4(t) = -\frac{1}{\mu} \int_0^1 u(t, y) dy + v_5(t), \quad w_0(x) = -\frac{1}{\mu} \int_0^x u_0(y) dy, \\ w_0(x) &= -\frac{1}{\mu} \int_0^x u_0(y) dy + v_5(0), \quad w_1(x) = -\frac{1}{\mu} \int_0^x u_1(y) dy + v_5(1). \end{aligned}$$

From (1.4) and $v_1 \in H^{\frac{3}{4}}(0, T)$, we obtain that $v_3 \in L^2(0, T)$ and $v_4, v_5 \in H^1(0, T)$. ■

Let us now prove that the previous controllability result is equivalent to a controllability problem for the system (2.9) with two time-dependent controls, one of bilinear nature (multiplying the state function) and the other one acting at $x = 1$. We are writing the

system (2.9):

$$\begin{cases} \frac{\partial}{\partial t}h - \mu\partial_{xx}h = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ \frac{\partial}{\partial t}h - \mu\partial_{xx}h - (v_5'(t) + v_3(t))h = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ h(t, 0) = e^{v_5(t)}, \quad h(t, 1) = e^{v_4(t)}, & t \in (0, T), \\ h(0, x) = h_0(x). \end{cases} \quad (2.9)$$

Lemma 2 *The system (2.9) is global exact controllable implies that there exists a solution to the following controllability problem:*

For any $h_0, h_1 \in H^2(0, 1)$ with $0 < c \leq h_0 < M$, $0 < c_1 \leq h_1 < M_1$, there exists control $v_4, v_5 \in H^1(0, T)$ and a solution of (2.9) such that $h(T, x) = h_1(x)$ in $(0, 1)$ with $h \in Y_T = L^2(0, T; H^3(0, 1)) \cap H^1(0, T; H^1(0, 1))$.

Proof: By multiplying (2.8) by h , we check that h verifies the second equation of (2.9). In order to ensure the compatibility between the first and the second equation of (2.8), we assume that:

$$v_5(t) = - \int_0^t v_3(s)ds. \quad (2.10)$$

To summarize we have proved that if the system (1.1) is global exact controllable, it implies that the following heat equation:

$$\begin{cases} \frac{\partial}{\partial t}h - \mu\partial_{xx}h = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ h(t, 0) = v_6(t), \quad h(t, 1) = v_7(t), & t \in (0, T), \\ h(0, x) = h_0(x). \end{cases} \quad (2.11)$$

is global exact controllable with:

$h_0, h_1 \in H^2(0, 1)$ with $0 < c \leq h_0 < M$, $0 < c_1 \leq h_1 < M_1$ and with $v_6 = e^{v_5(t)}$, $v_7 = e^{v_4(t)} \in H^1(0, T)$ strictly positive where:

$$\begin{aligned} h(t, x) &= \exp\left(-\frac{1}{\mu} \int_0^x u(t, y)dy - \int_0^t v_3(s)ds\right), \\ v_3(t) &= -\frac{v_1^2(t)}{\mu} + u_x(t, 0), \end{aligned}$$

and:

$$h_1(x) = K \exp\left(-\frac{1}{\mu} \int_0^x u(t, y)dy\right),$$

with $K = - \int_0^T v_3(s)ds$.

3 Proofs of theorem 1.1 and 1.2

In this section, we will prove both theorems 1.1 and 1.2. We start with giving technical results for the dual system of the heat equations that we shall use for proving theorem 1.1 and 1.2:

Proposition 3.1 *Let $0 < \xi_0 < \xi_1 < \xi_2 < 1$. Then, for each $\theta > 0$ there exists a time $T = T(\theta) > 0$ such that the solution of the backwards heat equation:*

$$\begin{cases} -U_t - U_{xx} = 0, & (t, x) \in (0, T^*) \times (0, 1), \\ U(t, 0) = U(t, 1) = 0, & t \in (0, T^*), \\ U(T^*, x) = \delta_{\xi_0} - \theta\delta_{\xi_1} + \delta_{\xi_2}, & x \in (0, 1) \end{cases} \quad (3.12)$$

satisfies

$$U_x(t, 0) > 0 \text{ and } U_x(t, 1) < 0 \quad \forall t \in (0, T^*).$$

Here δ_x is the Dirac measure at x .

The following proposition is a generalization of the previous one which was proved in [2]:

Proposition 3.2 *Let $T > 0$ and $0 < \xi_0 < \xi_1 < \xi_2 < 1$. Then there exists $\theta > 0$ such that the solution of the backwards heat equation:*

$$\begin{cases} -U_t - U_{xx} = 0, & (t, x) \in (0, T^*) \times (0, 1), \\ U(t, 0) = U(t, 1) = 0, & t \in (0, T^*), \\ U(T^*, x) = \delta_{\xi_0} - \theta\delta_{\xi_1} + \delta_{\xi_2}, & x \in (0, 1) \end{cases} \quad (3.13)$$

satisfies

$$U_x(t, 0) > 0 \text{ and } U_x(t, 1) < 0 \quad \forall t \in (0, T^*).$$

3.1 Proof of theorem 1.1 via the method of Guerrero et Imanuvilov ([19])

We have verified that the global exact controllability of (1.1) implies the exact controllability problem of the heat equation:

$$\begin{cases} \frac{\partial}{\partial t} h - \mu \partial_{xx} h = 0, & (t, x) \in Q = (0, T) \times (0, 1) \\ z(t, 0) = v_7(t), \quad z(t, 1) = v_8(t), & t \in (0, T), \\ h(0, x) = h_0(x). \end{cases} \quad (3.14)$$

for any $h_0, h_1 \in H^2(0, 1)$ with $h_0(x), h_1(x) > 0$ in $(0, 1)$ and $h_0(0) = 1$ and $h_1(0) = K > 0$, and with two controls $v_7(t), v_8(t) \in H^1(0, T)$ which are strictly positive in $[0, T]$ such that the solution of (2.9) satisfies $h(T, x) = h_1(x)$ in $(0, 1)$.

We are going to show the theorem 1.1 by contradiction. Thus, we assume that for any $T > 0$ and any $h_0 \in H^2(0, 1)$ with $h_0 > 0$ and $h_0(0) = 1$, there exists a constant $K > 0$ and two controls $0 < v_1 \in H^1(0, T)$ and $0 < v_2 \in H^1(0, T)$ such that the solution of (3.14) satisfies $h(T, x) = K$ in $(0, 1)$.

Multiplying the equation of (3.14) by U given by proposition 3.1 (for $\theta \geq 2$) and integrating over $(0, T) \times (0, 1)$ with $T = T^*$, we obtain that:

$$\int_0^T (U_x(t, 0)\tilde{v}_1(t) - U_x(t, 1)\tilde{v}_2(t))dt + K(2 - \theta) - \int_0^1 U(0, x)h_0(x)dx = 0 \quad (3.15)$$

for any $h_0 \in H^2(0, 1)$. By using the facts that the normal derivative of U is negative and $\theta \geq 2$, we observe that the two first terms of (3.15) are non-positive. Our goal now consists in choosing an initial condition h_0 with $h_0(0) = 1$ such that:

$$-\int_0^1 U(0, x)h_0(x)dx < 0,$$

in order to obtain a contradiction with the equality (3.15) and the fact that the two first terms of this equality are non positive.

Using the fact that the normal derivative of U is negative and U verifies homogeneous Dirichlet boundary conditions, we observe by Taylor formula that there exists $\delta > 0$ such that:

$$U(0, x) \geq \delta x, \forall x \in (0, \delta) \text{ and } U(0, x) \geq \delta(1 - x), \forall x \in (1 - \delta, 1).$$

On the other hand since U satisfies a heat equation (3.13), we obtain an instantaneously regularizing effect on U in Sobolev norms which depends on some power of $\frac{1}{T^\alpha}$ with $\alpha > 1$, more precisely there exists a positive constant $C^*(T^*, \theta)$ such that:

$$\|U(0, \cdot)\|_{L^2(0,1)} \leq C^*. \quad (3.16)$$

We are going to follow the choice of h_0 as in [19], let $h_0 = h_0(x) \in (0, 1)$ for all $x \in (0, 1)$ such that

$$h_0(x) = \frac{4C^*}{\delta^3} \forall x \in \left(\frac{\delta}{4}, \frac{3\delta}{4}\right) U\left(1 - \frac{3\delta}{4}, 1 - \frac{\delta}{4}\right),$$

we have by using the fact that h_0 is positive:

$$\begin{aligned} -\int_0^1 U(0, x)h_0(x)dx &\leq -\int_{\frac{\delta}{4}}^{\frac{3\delta}{4}} \frac{4C^*}{\delta^3} U(0, x)dx - \int_{1-\frac{3\delta}{4}}^{1-\frac{\delta}{4}} \frac{4C^*}{\delta^3} U(0, x)dx - \int_0^{1-\delta} U(0, x)h_0(x)dx \\ &\leq -C^* + (1 - 2\delta)C^* = -2\delta C^* < 0. \end{aligned}$$

This achieves the proof. ■

3.2 Proof of theorem 1.2

In this paragraph, we prove the theorem 1.2. The proof follows the same lines than in [19] and than in the previous section. That why we are just pointing out the differences and explaining how to deal with them.

We are interested in showing that for any $h_0, h_1 \in H^2(0, 1)$ with $h_0(x), h_1(x) > 0$ in $(0, 1)$ and $h_0(0) = 1, h_1(0) = K$, there exists two controls $v_7(t), v_8(t) \in H^1(0, T)$ which are strictly positive in $[0, T]$ such that the solution of (2.9) satisfies $h(T, x) = h_1(x)$ in $(0, 1)$. As in the previous section, we multiply the system (3.14) by U checking the proposition 3.2, we have then:

$$\int_0^{T^*} (U_x(t, 0)\tilde{v}_1(t) - U_x(t, 1)\tilde{v}_2(t))dt + (h_1(\xi_0) - \theta h_1(\xi_1) + h_1(\xi_2)) = 0 \quad (3.17)$$

for all $0 < h_0 \in H^2(0, 1)$ and $0 < h_1 \in H^2(0, 1)$.

The first term of (3.17) is non positive, we are then using the same argument of contradiction than in the previous proof and in particular showing that:

$$-\int_0^1 U(0, x)h_0(x)dx < 0,$$

for a suitable choice on h_0 . We proceed exactly as in the previous section with the same h_0 and furthermore we take h_1 such that:

$$h_1(\xi_0) - \theta h_1(\xi_1) + h_1(\xi_2) < 0.$$

This is a contradiction with identity (3.17) which concludes the proof. ■

Acknowledgements: The authors would like to thank Jean-Michel Coron for many helpful and fruitful discussions.

References

- [1] F. Ancona, A. Marson, On the attainable set for scalar nonlinear conservation laws with boundary control, *SIAM J. Control Optim.* 36 (1) (1998) 290312.
- [2] M.I. Belishev, On approximating properties of solutions of the heat equation, in: *Control Theory of Partial Differential Equations*, in: *Lecture Notes in Pure and Appl. Math.*, vol. 242, Chapman and Hall/CRC, Boca Raton, FL, 2005, pp. 4350.
- [3] M. Chapouly, Global controllability of nonviscous and viscous Burgers equations. *SIAM J. Control Optim.*, Volume 48, Issue 3 (2009), pp. 1567-1599.
- [4] J.-M. Coron, Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations, *ESAIM: Control, Optimisation and Calculus of Variations*, 8, 2002, p. 513-554.
- [5] J.-M. Coron, On the controllability of the 2-D incompressible perfect fluids, *J. Math. Pures et Appliquées*, 75 (2), 1996, p. 155-188.
- [6] J.-M. Coron, On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions, *ESAIM Control Optim. Calc. Var.* 1 (1995/96) 3575.
- [7] J.-M. Coron, Some open problems on the control of nonlinear partial differential equations, in: H. Berestycki, M. Bertsch, B. Peletier, L. Vron (Eds.), *Perspectives in Nonlinear Partial Differential Equations: In Honor of Ham Brezis*, in: *Contemporary Mathematics*, American Mathematical Society, Providence, RI, in press.
- [8] J.-M. Coron, Global asymptotic stabilization for controllable systems without drift, *Math. Control Signals Systems* 5 (3) (1992) 285312.
- [9] J.-M. Coron et A. Fursikov, Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary, *Russian Journal of Mathematical Physics*, 4 (4), 1996, p. 429-448.
- [10] J.I. Daz, Obstruction and some approximate controllability results for the Burgers equation and related problems, in: *Control of Partial Differential Equations and Applications*, in: *Lecture Notes in Pure and Appl. Math.*, vol. 174, Dekker, New York, 1995, pp. 6376.

- [11] A. Drici and B. Haspot, Some remarks on the exact controllability of the compressible Navier-Stokes equations, in preparation.
- [12] S. Ervedoza, O. Glass, S. Guerrero and J-P Puel. Local exact controllability for the 1-D compressible Navier-Stokes equation. *preprint*.
- [13] E. Fernández-Cara, S. Guerrero, On the controllability of Burgers system, C. R. Acad. Sci. Paris, Ser. I 341 (2005) 229-232.
- [14] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov and J-P Puel, Local exact controllability of the Navier-Stokes system, *J. Math. Pures. Appl.* 83 (2004), 1501-1542.
- [15] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov and J-P Puel, Some controllability results for the N dimensional Navier-Stokes and Boussinesq systems with $N - 1$ scalar controls, *SIAM J. Control optim.* 45 (2006), 146-173.
- [16] A. Fursikov, O.Yu. Imanuvilov, On controllability of certain systems simulating a fluid flow, in: Flow Control, Minneapolis, MN, 1992, in: IMA Vol. Math. Appl., vol. 68, Springer, New York, 1995, pp. 149-184.
- [17] O. Glass, Exact boundary controllability of 3-D Euler equation, *ESAIM Control Optim. Calc. Var.* 5 (2000) 144.
- [18] O. Glass and S. Guerrero, Uniform controllability of a transport equation in zero diffusion-dispersion limit, *Mathematical Models and Methods in Applied Sciences*, Vol 19, No. 9 (2009) 1567-1601.
- [19] Sergio Guerrero and Oleg Imanuvilov, Remarks on global controllability for the Burgers equation with two control forces, *Annales de l'Institut Henri Poincaré Analyse Non Linéaire* 24 (2007), no. 6, 897–906.
- [20] B. Haspot, Existence of global strong solutions in critical spaces for barotropic viscous fluids, *Archive for Rational Mechanics and Analysis*, Volume 202, Issue 2 (2011), Page 427-460.
- [21] B. Haspot, New entropy for Korteweg's system, existence of global weak solution and Prodi-Serrin theorem, *preprint* (arxiv March 2011).
- [22] B. Haspot, Existence of strong global solutions for the Korteweg system with large initial data, *preprint*.
- [23] B. Haspot, Global existence of strong solution for shallow water system with large initial data on the irrotational part, *preprint* arxiv January 2012.
- [24] B. Haspot and E. Zuazua, Local exact controllability for the Korteweg system, in preparation.
- [25] T. Horsin, On the controllability of the Burgers equation, *ESAIM Control Optim. Calc. Var.* 3 (1998) 839-855.
- [26] O. Yu. Imanuvilov and J-P Puel, On global controllability of 2D Burgers equations, *Discrete and Continuous Dynamical Systems*, Vol 23, 1/2, (2009) 299-313.

- [27] J.-L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. I, Translated from the French by P. Kenneth. Die Grundlehren der Mathematischen Wissenschaften, Band 181, Springer-Verlag, New YorkHeidelberg, 1972.
- [28] J. Pedlosky, Geophysical Fluid Dynamics, Berlin Heidelberg-New York, *Springer Verlag*: 1987.
- [29] L. Rosier and P. Rouchon, On the exact controllability of a wave equation with structural damping, in *Proceedings of the 13th IFAC Workshop on Control Applications of Optimisation*, Cachan 2006.