

Small values of the Euler function and the Riemann hypothesis

Jean-Louis Nicolas *

3 février 2012

À André Schinzel pour son 75ème anniversaire, en très amical hommage.

Abstract

Let φ be Euler's function, γ be Euler's constant and N_k be the product of the first k primes. In this article, we consider the function $c(n) = (n/\varphi(n) - e^\gamma \log \log n) \sqrt{\log n}$. Under Riemann's hypothesis, it is proved that $c(N_k)$ is bounded and explicit bounds are given while, if Riemann's hypothesis fails, $c(N_k)$ is not bounded above or below.

Keywords : Euler's function, Riemann hypothesis, Explicit formula.

2010 Mathematics Subject Classification : 11N37, 11M26, 11N56.

1 Introduction

Let φ be the Euler function. In 1903, it was proved by E. Landau (cf. [5, §59] and [4, Theorem 328]) that

$$\limsup_{n \rightarrow \infty} \frac{n}{\varphi(n) \log \log n} = e^\gamma = 1.7810724179 \dots$$

where $\gamma = 0.5772156649 \dots$ is Euler's constant.

In 1962, J. B. Rosser and L. Schoenfeld proved (cf. [9, Theorem 15])

$$(1.1) \quad \frac{n}{\varphi(n)} \leq e^\gamma \log \log n + \frac{2.51}{\log \log n}$$

for $n \geq 3$ and asked if there exists an infinite number of n such that $n/\varphi(n) > e^\gamma \log \log n$. In [6], (cf. also [7]), I answer this question in the affirmative. Soon after, A. Schinzel told me that he had worked unsuccessfully on this question, which made me very proud to have solved it.

*Research partially supported by CNRS, Institut Camille Jordan, UMR 5208.

For $k \geq 1$, p_k denotes the k -th prime and

$$N_k = 2 \cdot 3 \cdot 5 \cdots p_k$$

the primorial number of order k . In [6], it is proved that the Riemann hypothesis (for short RH) is equivalent to

$$\forall k \geq 1, \quad \frac{N_k}{\varphi(N_k)} > e^\gamma \log \log N_k.$$

The aim of the present paper is to make more precise the results of [6] by estimating the quantity

$$(1.2) \quad c(n) = \left(\frac{n}{\varphi(n)} - e^\gamma \log \log n \right) \sqrt{\log n}.$$

Let us denote by ρ a generic root of the Riemann ζ function satisfying $0 < \Re \rho < 1$. Under RH, $1 - \rho = \bar{\rho}$. It is convenient to define (cf. [2, p. 159])

$$(1.3) \quad \beta = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma - \log \pi - 2 \log 2 = 0.0461914179 \dots$$

We shall prove

Theorem 1.1 *Under the Riemann hypothesis (RH) we have*

$$(1.4) \quad \limsup_{n \rightarrow \infty} c(n) = e^\gamma(2 + \beta) = 3.6444150964 \dots$$

$$(1.5) \quad \forall n \geq N_{120569} = 2 \cdot 3 \cdot \dots \cdot 1591883, \quad c(n) < e^\gamma(2 + \beta).$$

$$(1.6) \quad \forall n \geq 2, \quad c(n) \leq c(N_{66}) = c(2 \cdot 3 \cdot \dots \cdot 317) = 4.0628356921 \dots$$

$$(1.7) \quad \forall k \geq 1, \quad c(N_k) \geq c(N_1) = c(2) = 2.2085892614 \dots$$

We keep the notation of [6]. For a real $x \geq 2$, the usual Chebichev's functions are denoted by

$$(1.8) \quad \theta(x) = \sum_{p \leq x} \log p \quad \text{and} \quad \psi(x) = \sum_{p^m \leq x} \log p.$$

We set

$$(1.9) \quad f(x) = e^\gamma \log \theta(x) \prod_{p \leq x} (1 - 1/p).$$

Mertens's formula yields $\lim_{x \rightarrow \infty} f(x) = 1$. In [6, Th. 3 (c)] it is shown that, if RH fails, there exists b , $0 < b < 1/2$, such that

$$(1.10) \quad \log f(x) = \Omega_{\pm}(x^{-b}).$$

For $p_k \leq x < p_{k+1}$, we have $f(x) = e^\gamma \log \log(N_k) \frac{\varphi(N_k)}{N_k}$. When $k \rightarrow \infty$, by observing that the Taylor development in neighborhood of 1 yields $\log f(p_k) \sim f(p_k) - 1$, we get

$$\log f(p_k) \sim f(p_k) - 1 = \frac{\varphi(N_k)}{N_k} \frac{c(N_k)}{\sqrt{\log N_k}} \sim \frac{e^{-\gamma}}{\log \log N_k} \frac{c(N_k)}{\sqrt{\log N_k}},$$

and it follows from (1.10) that, if RH does not hold, then

$$\liminf_{n \rightarrow \infty} c(n) = -\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} c(n) = +\infty.$$

Therefore, from Theorem 1.1, we deduce :

Corollary 1.1 *Each of the four assertions (1.4), (1.5), (1.6), (1.7) is equivalent to the Riemann hypothesis.*

1.1 Notation and results used

If $\theta(x)$ and $\psi(x)$ are the Chebichev functions defined by (1.8), we set

$$(1.11) \quad R(x) = \psi(x) - x \quad \text{and} \quad S(x) = \theta(x) - x.$$

Under RH, we shall use the upper bound (cf. [10, (6.3)])

$$(1.12) \quad x \geq 599 \quad \implies \quad |S(x)| \leq T(x) \stackrel{def}{=} \frac{1}{8\pi} \sqrt{x} \log^2 x$$

P. Dusart (cf. [1, Table 6.6]) has shown that

$$(1.13) \quad \theta(x) < x \quad \text{for} \quad x \leq 8 \cdot 10^{11}$$

thus improving the result of R. P. Brent who has checked (1.13) for $x < 10^{11}$ (cf. [10, p. 360]). We shall also use (cf. [9, Theorem 10])

$$(1.14) \quad \theta(x) \geq 0.84 x \geq \frac{4}{5} x \quad \text{for} \quad x \geq 101.$$

As in [6], we define the following integrals

$$(1.15) \quad K(x) = \int_x^\infty \frac{S(t)}{t^2} \left(\frac{1}{\log t} + \frac{1}{\log^2 t} \right) dt,$$

$$(1.16) \quad J(x) = \int_x^\infty \frac{R(t)}{t^2} \left(\frac{1}{\log t} + \frac{1}{\log^2 t} \right) dt,$$

and, for $\Re(z) < 1$,

$$(1.17) \quad F_z(x) = \int_x^\infty t^{z-2} \left(\frac{1}{\log t} + \frac{1}{\log^2 t} \right) dt.$$

We also set for $x \geq 1$

$$(1.18) \quad W(x) = \sum_{\rho} \frac{x^{i\Im(\rho)}}{\rho(1-\rho)}$$

so that, under RH, from (1.3) we have

$$(1.19) \quad |W(x)| \leq \beta = \sum_{\rho} \frac{1}{\rho(1-\rho)}.$$

We often implicitly use the following result : for a and b positive, the function

$$(1.20) \quad t \mapsto \frac{\log^a t}{t^b} \quad \text{is decreasing for } t > e^{a/b}$$

and

$$(1.21) \quad \max_{t \geq 1} \frac{\log^a t}{t^b} = \left(\frac{a}{eb}\right)^a.$$

1.2 Organization of the article

In Section 2, the results of [6] about $f(x)$ are revised so as to get effective upper and lower bounds for both $\log f(x)$ and $1/f(x) - 1$ under RH (cf. Proposition 2.1).

In Section 3, we study $c(N_k)$ and $c(n)$ in terms of $f(p_k)$.

Section 4 is devoted to the proof of Theorem 1.1.

2 Estimate of $\log(f(x))$

The following lemma is Proposition 1 of [6].

Lemma 2.1 *For $x \geq 121$, we have*

$$(2.1) \quad K(x) - \frac{S^2(x)}{x^2 \log x} \leq \log f(x) \leq K(x) + \frac{1}{2(x-1)}.$$

The next lemma is a slight improvement of Lemma 1 of [6].

Lemma 2.2 *Let x be a real number, $x > 1$. For $\Re z < 1$, we have*

$$(2.2) \quad F_z(x) = \frac{x^{z-1}}{(1-z) \log x} + r_z(x) \quad \text{with } r_z(x) = \int_x^\infty -\frac{zt^{z-2}}{(1-z) \log^2 t} dt$$

and, if $\Re z = 1/2$,

$$(2.3) \quad |r_z(x)| \leq \frac{1}{|1-z|\sqrt{x} \log^2 x} \left(1 + \frac{4}{\log x}\right).$$

Moreover, for $z = 1/2$, we have

$$(2.4) \quad \frac{2}{\sqrt{x} \log x} - \frac{2}{\sqrt{x} \log^2 x} \leq F_{1/2}(x) \leq \frac{2}{\sqrt{x} \log x} - \frac{2}{\sqrt{x} \log^2 x} + \frac{8}{\sqrt{x} \log^3 x}$$

and, for $z = 1/3$,

$$(2.5) \quad 0 \leq F_{1/3}(x) \leq \frac{3}{2x^{2/3} \log x}.$$

Proof : The proof of (2.2) is easy by taking the derivative. By partial summation, we get

$$(2.6) \quad r_z(x) = -\frac{z}{1-z} \left(\frac{x^{z-1}}{(1-z) \log^2 x} + \int_x^\infty \frac{2 t^{z-2}}{(z-1) \log^3 t} dt \right).$$

If we assume $\Re z = 1/2$, we have $1-z = \bar{z}$ and

$$|r_z(x)| \leq \frac{1}{|1-z| \sqrt{x} \log^2 x} + \frac{2}{|1-z| \log^3 x} \int_x^\infty t^{-3/2} dt$$

which yields (2.3). The proof of (2.4) follows from (2.2) and (2.6) by choosing $z = 1/2$. The proof of (2.5) follows from (2.2) since $r_{1/3}$ is negative. \square

To estimate the difference $J(x) - K(x)$, we need Lemma 2.4 which, under RH, is an improvement of Propositions 3.1 and 3.2 of [1] (obtained without assuming RH). The following lemma will be useful for proving Lemma 2.4.

Lemma 2.3 Let $\kappa = \kappa(x) = \lfloor \frac{\log x}{\log 2} \rfloor$ the largest integer such that $x^{1/\kappa} \geq 2$. For $x \geq 16$, we set

$$H(x) = 1 + \sum_{k=4}^{\kappa} x^{1/k-1/3}$$

and for $x \geq 4$

$$L(x) = \sum_{k=2}^{\kappa} \ell_k(x) \quad \text{with} \quad \ell_k(x) = \frac{T(x^{1/k})}{x^{1/3}} = \frac{\log^2 x}{8\pi k^2 x^{1/3-1/(2k)}}.$$

(i) For $j \geq 9$ and $x \geq 2^j$, $H(x) \leq H(2^j)$ holds.

(ii) For $j \geq 35$ and $x \geq 2^j$, $L(x) \leq L(2^j)$ holds.

Proof : The function H is continuous and decreasing on $[2^j, 2^{j+1})$; so, to show (i), it suffices to prove for $j \geq 9$

$$(2.7) \quad H(2^j) \geq H(2^{j+1}).$$

If $9 \leq j \leq 19$, we check (2.7) by computation. If $j \geq 20$, we have

$$\begin{aligned} H(2^j) - H(2^{j+1}) &= \sum_{k=4}^j 2^{j(\frac{1}{k}-\frac{1}{3})} \left(1 - 2^{(\frac{1}{k}-\frac{1}{3})}\right) - 2^{(j+1)(\frac{1}{j+1}-\frac{1}{3})} \\ &\geq 2^{j(\frac{1}{4}-\frac{1}{3})} \left(1 - 2^{(\frac{1}{4}-\frac{1}{3})}\right) - 2^{(j+1)(\frac{1}{j+1}-\frac{1}{3})} \\ &= 2^{-\frac{j}{3}} \left[\left(1 - 2^{-\frac{1}{12}}\right) 2^{\frac{j}{4}} - 2^{\frac{2}{3}} \right] \end{aligned}$$

which proves (2.7) since the above bracket is $\geq (1 - 2^{-\frac{1}{12}}) 2^{\frac{20}{4}} - 2^{\frac{2}{3}} = 0.208 \dots$ and therefore positive.

Let us assume that $j \geq 35$ so that $2^j \geq e^{24}$ holds. From (1.20), for each $k \geq 2$, $x \mapsto \ell_k(x)$ is decreasing for $x \geq 2^j$ so that L is decreasing on $[2^j, 2^{j+1})$ and, to show (ii), it suffices to prove

$$(2.8) \quad L(2^j) \geq L(2^{j+1}).$$

We have

$$\begin{aligned} L(2^j) - L(2^{j+1}) &= \sum_{k=2}^j \{ \ell_k(2^j) - \ell_k(2^{j+1}) \} - \ell_{j+1}(2^{j+1}) \\ &\geq \ell_2(2^j) - \ell_2(2^{j+1}) - \ell_{j+1}(2^{j+1}) \\ &= \frac{\log^2 2}{32\pi} 2^{-\frac{j}{3}} \left\{ 2^{\frac{j}{4}} \left[j^2 - 2^{-\frac{1}{12}}(j+1)^2 \right] - 4 \cdot 2^{\frac{1}{6}} \right\}. \end{aligned}$$

For $j \geq \frac{1}{2^{1/12}-1} = 16.81 \dots$, the above square bracket is increasing on j and it is positive for $j = 35$. Therefore, the curly bracket is increasing for $j \geq 35$ and, since its value for $j = 35$ is equal to $744.17 \dots$, (2.8) is proved for $j \geq 35$. \square

Lemma 2.4 *Under RH, we have*

$$(2.9) \quad \psi(x) - \theta(x) \geq \sqrt{x}, \quad \text{for } x \geq 121$$

and, for $x \geq 1$,

$$(2.10) \quad \frac{\psi(x) - \theta(x) - \sqrt{x}}{x^{1/3}} \leq 1.332768 \dots \leq \frac{4}{3}.$$

Proof : For $x < 599^3$, we check (2.9) by computation. Note that 599 is prime. Let $q_0 = 1$, and let $q_1 = 4, q_2 = 8, q_3 = 9, \dots, q_{1922} = 599^3$ be the sequence of powers (with exponent ≥ 2) of primes not exceeding 599^3 . On the intervals $[q_i, q_{i+1})$, the function $\psi - \theta$ is constant and $x \mapsto (\psi(x) - \theta(x))/\sqrt{x}$ is decreasing. For $11 \leq i \leq 1921$ (i.e. $121 \leq q_i < q_{i+1} \leq 599^3$), we calculate $\delta_i = (\psi(q_i) - \theta(q_i))/\sqrt{q_{i+1}}$ and find that $\min_{11 \leq i \leq 1921} \delta_i = \delta_{1886} = 1.0379 \dots$ ($q_{1886} = 206468161 = 14369^2$) while $\delta_{10} = 0.9379 \dots < 1$ ($q_{10} = 81$).

Now, we assume $x \geq 599^3$, so that, by (1.12), we have

$$(2.11) \quad \psi(x) - \theta(x) \geq \theta(x^{1/2}) + \theta(x^{1/3}) \geq x^{1/2} + x^{1/3} - T(x^{1/2}) - T(x^{1/3}).$$

By using (1.21), we get

$$\frac{T(x^{1/2})}{x^{1/3}} + \frac{T(x^{1/3})}{x^{1/3}} = \frac{1}{8\pi} \left(\frac{\log^2 x}{4x^{1/12}} + \frac{\log^2 x}{9x^{1/6}} \right) \leq \frac{20}{\pi e^2} = 0.86157\dots$$

which, with (2.11), implies

$$(2.12) \quad \psi(x) - \theta(x) \geq \sqrt{x} + \left(1 - \frac{20}{\pi e^2}\right) x^{1/3} \geq \sqrt{x}.$$

The inequality (2.10) is Lemma 3 of [8]. We give below another proof by considering three cases according to the values of x .

Case 1, $1 \leq x < 2^{32}$. The largest q_i smaller than 2^{32} is $q_{6947} = 4293001441 = 65521^2$. On the intervals $[q_i, q_{i+1})$, the function

$$G(x) \stackrel{\text{def}}{=} \frac{\psi(x) - \theta(x) - \sqrt{x}}{x^{1/3}}$$

is decreasing. By computing $G(q_0), G(q_1), \dots, G(q_{6947})$ we get

$$G(x) \leq G(q_{103}) = 1.332768\dots \quad [q_{103} = 80089 = 283^2].$$

Case 2, $2^{32} \leq x < 64 \cdot 10^{22}$. By using (1.13), we get

$$\psi(x) - \theta(x) = \sum_{k=2}^{\kappa} \theta(x^{1/k}) \leq \sum_{k=2}^{\kappa} x^{1/k}$$

so that Lemma 2.3 implies $G(x) \leq H(x) \leq H(2^{32}) = 1.31731\dots$

Case 3, $x \geq 64 \cdot 10^{22} \geq 2^{79}$. By (1.12) and (1.13), we get

$$\psi(x) - \theta(x) = \sum_{k=2}^{\kappa} \theta(x^{1/k}) \leq \sum_{k=2}^{\kappa} \{x^{1/k} + T(x^{1/k})\},$$

whence, from Lemma 2.3, $G(x) \leq H(x) + L(x) \leq H(2^{79}) + L(2^{79}) = 1.32386\dots$ \square

Corollary 2.1 *For $x \geq 121$, we have*

$$(2.13) \quad F_{1/2}(x) \leq J(x) - K(x) \leq F_{1/2}(x) + \frac{4}{3}F_{1/3}(x).$$

The following lemma is an improvement of [6, Proposition 2].

Lemma 2.5 *Let us assume that RH holds. For $x > 1$, we may write*

$$(2.14) \quad J(x) = -\frac{W(x)}{\sqrt{x} \log x} - J_1(x) - J_2(x)$$

with

$$(2.15) \quad 0 < J_1(x) \leq \frac{\log(2\pi)}{x \log x} \quad \text{and} \quad |J_2(x)| \leq \frac{\beta}{\sqrt{x} \log^2 x} \left(1 + \frac{4}{\log x}\right).$$

Proof : In [6, (17)–(19)], for $x > 1$, it is proved that

$$J(x) = -\sum_{\rho} \frac{1}{\rho} F_{\rho}(x) - J_1(x)$$

with J_1 satisfying $0 < J_1(x) \leq \frac{\log(2\pi)}{x \log x}$.

Now, by Lemma 2.2, we have $F_{\rho}(x) = \frac{x^{\rho-1}}{(1-\rho) \log x} + r_{\rho}(x)$ which yields (2.14) by setting $J_2(x) = \sum_{\rho} \frac{1}{\rho} r_{\rho}(x)$. Further, from (2.3) and (1.3), we get the upper bound for $|J_2(x)|$ given in (2.15). \square

Proposition 2.1 *Under RH, for $x \geq x_0 = 10^9$, we have*

$$(2.16) \quad -\frac{2+W(x)}{\sqrt{x} \log x} + \frac{0.055}{\sqrt{x} \log^2 x} \leq \log f(x) \leq -\frac{2+W(x)}{\sqrt{x} \log x} + \frac{2.062}{\sqrt{x} \log^2 x}$$

and

$$(2.17) \quad \frac{2+W(x)}{\sqrt{x} \log x} - \frac{2.062}{\sqrt{x} \log^2 x} \leq \frac{1}{f(x)} - 1 \leq \frac{2+W(x)}{\sqrt{x} \log x} - \frac{0.054}{\sqrt{x} \log^2 x}.$$

Proof : By collecting the information from (2.1), (1.12), (2.13), (2.14), (2.15), (2.4) and (2.5), for $x \geq 599$, we get

$$(2.18) \quad \log f(x) \geq -\frac{W(x)+2}{\sqrt{x} \log x} + \frac{2-\beta}{\sqrt{x} \log^2 x} - \frac{8+4\beta}{\sqrt{x} \log^3 x} - \frac{\log(2\pi)}{x \log x} - \frac{2}{x^{2/3} \log x} - \frac{\log^3 x}{64\pi^2 x}$$

and

$$(2.19) \quad \log f(x) \leq -\frac{W(x)+2}{\sqrt{x} \log x} + \frac{2+\beta}{\sqrt{x} \log^2 x} + \frac{4\beta}{\sqrt{x} \log^3 x} + \frac{1}{2(x-1)}.$$

Since $x \geq x_0 = 10^9$ holds, (2.18) and (2.19) imply respectively

$$(2.20) \quad \log f(x) \geq -\frac{W(x)+2}{\sqrt{x} \log x} + \frac{1}{\sqrt{x} \log^2 x} \left(2 - \beta - \frac{8+4\beta}{\log x_0} - \frac{\log(2\pi) \log x_0}{\sqrt{x_0}} - \frac{2 \log x_0}{x_0^{1/6}} - \frac{\log^5 x_0}{64\pi^2 \sqrt{x_0}}\right)$$

and

$$(2.21) \quad \log f(x) \leq -\frac{W(x)+2}{\sqrt{x} \log x} + \frac{1}{\sqrt{x} \log^2 x} \left(2 + \beta + \frac{4\beta}{\log x_0} + \frac{\sqrt{x_0} \log^2 x_0}{2(x_0 - 1)} \right)$$

which prove (2.16).

Setting $v = -\log f(x)$, it follows from (2.16), (1.19) and (1.3) that

$$v \leq \frac{W(x)+2}{\sqrt{x} \log x} \leq \frac{2+\beta}{\sqrt{x} \log x} \leq v_0 \stackrel{\text{def}}{=} \frac{2+\beta}{\sqrt{x_0} \log x_0} = 0.00000312\dots$$

By Taylor's formula, we have $e^v - 1 \geq v$ (which, with (2.16), provides the lower bound of (2.17)) and

$$e^v - 1 - v \leq \frac{e^{v_0}}{2} v^2 \leq \frac{e^{v_0}(2+\beta)^2}{2x \log^2 x} \leq \frac{e^{v_0}(2+\beta)^2}{2\sqrt{x_0}\sqrt{x} \log^2 x} = \frac{0.0000662\dots}{\sqrt{x} \log^2 x}$$

(which implies the upper bound in (2.17)). \square

3 Bounding $c(n)$

Lemma 3.1 *Let n and k be two integers satisfying $n \geq 2$ and $k \geq 1$. Let us assume that either the number $j = \omega(n)$ of distinct prime factors of n is equal to k or that $N_k \leq n < N_{k+1}$ holds. We have*

$$(3.1) \quad c(n) \leq c(N_k).$$

Proof : It follows from our hypothesis that $n \geq N_k$ and $j \leq k$ hold. Let us write $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_j^{\alpha_j}$ (with $q_1 < q_2 < \dots < q_j$ as defined in the proof of Lemma 2.4). We have

$$\frac{n}{\varphi(n)} = \prod_{i=1}^j \frac{1}{1-1/q_i} \leq \prod_{i=1}^j \frac{1}{1-1/p_i} \leq \prod_{i=1}^k \frac{1}{1-1/p_i} = \frac{N_k}{\varphi(N_k)}$$

which yields

$$(3.2) \quad c(n) \leq \left(\frac{N_k}{\varphi(N_k)} - e^\gamma \log \log n \right) \sqrt{\log n} \stackrel{\text{def}}{=} h(n)$$

and $h(n)$ can be extended to a real number n . Further,

$$\begin{aligned} \frac{d}{dn} h(n) &= \frac{1}{2n\sqrt{\log n}} \left(\frac{N_k}{\varphi(N_k)} - e^\gamma \log \log n - 2e^\gamma \right) \\ &\leq \frac{1}{2n\sqrt{\log n}} \left(\frac{N_k}{\varphi(N_k)} - e^\gamma \log \log N_k - 2e^\gamma \right). \end{aligned}$$

If $k = 1$ or 2 , it is easy to see that the above parenthesis is negative, while, if $k \geq 3$, by (1.1), it is smaller than $\frac{2.51}{\log \log N_k} - 2e^\gamma$ which is also negative because $\log \log N_k \geq \log \log 30 = 1.22\dots$. Therefore, we get $h(n) \leq h(N_k) = c(N_k)$, which, with (3.2), completes the proof of Lemma 3.1. \square

Proposition 3.1 *Let us assume that $x_0 = 10^9 \leq p_k \leq x < p_{k+1}$ holds. Under RH, we have*

$$(3.3) \quad c(N_k) \leq e^\gamma(2 + W(x)) - \frac{0.07}{\log x} \leq e^\gamma(2 + \beta) - \frac{0.07}{\log x}$$

and

$$(3.4) \quad c(N_k) \geq e^\gamma(2 + W(x)) - \frac{3.7}{\log x} \geq e^\gamma(2 - \beta) - \frac{3.7}{\log x}.$$

Proof : From (1.2) and (1.9), we get

$$(3.5) \quad c(N_k) = e^\gamma \sqrt{\theta(x)} \log \theta(x) \left(\frac{1}{f(x)} - 1 \right).$$

By the fundamental theorem of calculus, (1.14) and (1.12), we have

$$\begin{aligned} |\sqrt{\theta(x)} \log \theta(x) - \sqrt{x} \log x| &= \left| \int_x^{\theta(x)} \frac{\log t + 2}{2\sqrt{t}} dt \right| \leq |\theta(x) - x| \frac{\log(4x/5) + 2}{2\sqrt{4x/5}} \\ &\leq \frac{\sqrt{5}}{4} T(x) \frac{\log x + 2}{\sqrt{x}} = \frac{\sqrt{5}}{32\pi} \log^2 x (\log x + 2) \end{aligned}$$

whence

$$\left| \frac{\sqrt{\theta(x)} \log \theta(x)}{\sqrt{x} \log x} - 1 \right| \leq \frac{\sqrt{5} \log^2 x (\log x + 2)}{32\pi \sqrt{x} \log x} \leq \frac{\sqrt{5} \log^2 x_0 (\log x_0 + 2)}{32\pi \sqrt{x_0} \log x} \leq \frac{0.0069}{\log x}.$$

Therefore, (3.5), (2.17) and (1.19) yield

$$\begin{aligned} c(N_k) &\leq e^\gamma \left(2 + W(x) - \frac{0.054}{\log x} \right) \left(1 + \frac{0.0069}{\log x} \right) \\ &\leq e^\gamma(2 + W(x)) - \frac{e^\gamma}{\log x} (0.054 - 0.0069(2 + \beta)) \end{aligned}$$

which proves (3.3). The proof of (3.4) is similar. \square

4 Proof of Theorem 1.1

It follows from (3.1), (3.3) and (3.4) that

$$\limsup_{n \rightarrow \infty} c(n) = e^\gamma(2 + \limsup_{x \rightarrow \infty} W(x)).$$

As observed in [6, p. 383], by the pigeonhole principle (cf. [3, §2.11] or [4, §11.12]), one can show that $\limsup_{x \rightarrow \infty} W(x) = \beta$, which proves (1.4).

To show the other points of Theorem 1.1, we first consider $k_0 = 50847534$, the number of primes up to $x_0 = 10^9$. For all $k \leq k_0$, we have calculated

$c(N_k)$ in Maple with 30 decimal digits, so that we may think that the first ten are correct.

We have found that for $k_1 = 120568 < k \leq k_0$, $c(N_k) < e^\gamma(2 + \beta)$ holds (while $c(N_{k_1}) = 3.6444180\dots > e^\gamma(2 + \beta)$) and for $1 \leq k \leq k_0$, we have $c(N_1) = c(2) \leq c(N_k) \leq c(N_{66})$.

Further, for $k > k_0$, (3.3) implies $c(N_k) < e^\gamma(2 + \beta) < c(N_{66})$ which, together with Lemma 3.1, proves (1.5) and (1.6).

As a challenge, for $k_1 = 120568$, I ask to find the largest number M such that $M < N_{k_1+1}$ and $c(M) \geq e^\gamma(2 + \beta)$. Note that $M > N_{k_1}$ holds since, for $n = N_{k_1-1}p_{k_1+1}$, we have $c(n) = 3.6444178\dots > e^\gamma(2 + \beta)$. Another challenge is to determine all the n 's satisfying $n < N_{k_1+1}$ and $c(n) > e^\gamma(2 + \beta)$.

Finally, for $k > k_0$, (3.4) implies

$$c(N_k) \geq e^\gamma(2 - \beta) - \frac{3.7}{\log(10^9)} = 3.30\dots > c(2)$$

which completes the proof of (1.7) and of Theorem 1.1. \square

It is not known if $\liminf_{x \rightarrow \infty} W(x) = -\beta$. Let $\rho_1 = 1/2 + i t_1$ with $t_1 = 14.13472\dots$ the first zero of ζ . By using a theorem of Landau (cf. [3, Th. 6.1 and §2.4]), it is possible to prove that $\liminf_{x \rightarrow \infty} W(x) \leq -1/(\rho_1(1 - \rho_1)) = -0.00499\dots$. A smaller upper bound is wanted.

An interesting question is the following : assume that RH fails. Is it possible to get an upper bound for k such that $k > k_0$ and either $c(N_k) > e^\gamma(2 + \beta)$ or $c(N_k) < c(2)$?

Acknowledgements. We would like to thank the anonymous referee for his or her careful reading of our article and his or her valuable suggestions.

References

- [1] P. DUSART. Estimates of some functions over primes without R. H., to be published, cf. <http://arxiv.org/abs/1002.0442v1>, 2010.
- [2] H. M. EDWARDS. Riemann's Zeta function, Academic Press, 1974.
- [3] W. J. ELLISON et M. MENDES-FRANCE. Les nombres premiers, Hermann, Paris, 1975. Actualités Scientifiques et Industrielles, No 1366.
- [4] G. H. HARDY and E. M. WRIGHT. An introduction to the theory of numbers, 4th edition, Oxford at the Clarendon Press, 1964.
- [5] E. LANDAU. Handbuch der Lehre von der Verteilung der Primzahlen, I, 2nd ed, Chelsea, New-York, 1953.
- [6] J.-L. NICOLAS. Petites valeurs de la fonction d'Euler, *J. Number Theory*, 17, 1983, 375–388.
- [7] J.-L. NICOLAS. Petites valeurs de la fonction d'Euler et hypothèse de Riemann, *Séminaire de Théorie des nombres D.P.P., Paris*, 1981–82, 207–218, Progress in Mathematics, vol. 38, Birkhäuser, 1983.

- [8] G. ROBIN. Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, *J. Math. Pures Appl.*, 63, 1984, 187–213.
- [9] J. B. ROSSER and L. SCHOENFELD. Approximate Formulas for Some Functions of Prime Numbers, *Illinois. J. Math.*, 6, (1962), 64–94.
- [10] L. SCHOENFELD. Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II, *Math. Comp.*, 30, 1976, 337–360.

Jean-Louis Nicolas, Université de Lyon, Université de Lyon 1, CNRS, Institut Camille Jordan, Mathématiques, Bât. Doyen Jean Braconnier, Université Claude Bernard (Lyon 1), 21 Avenue Claude Bernard, F-69622 Villeurbanne cédex, France.
jlnicola@in2p3.fr, <http://math.univ-lyon1.fr/~nicolas/>