The Spread of a Catalytic Branching Random Walk

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Abstract

We consider a random walk on $\mathbb{Z}$ that branches at the origin only. In the supercritical regime we establish a law of large number for the maximal position M_n. Then we prove convergence in distribution for the sequence $M_n - \alpha n$ where α is a deterministic constant.

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1 Introduction

A CBRW (Catalytic Branching Random Walk) on $\mathbb{Z}$ branching at the origin only is the following particle system.
When a particle location x is not the origin, the particle evolves as a random walk $(S_n)_{n\in\mathbb{N}}$ whose law is denoted by $\mathbb{P}_x$ when it starts from x (we let $\mathbb{P}=\mathbb{P}_0$).
When a particle reaches the origin, say at time t, then a time $t+1$ it dies and gives birth to new particles positioned according to a point process $\mathscr{D}_0$.
The system starts with an initial ancestor particle located at the origin. The system goes on indefinitely, as long as there are particle that are alive. We assume that each particle (at the origin at time t) produces new particles independently form every particle living in the system up to time t.
Let $(X_u, |u| = n)$ denote the positions of the particles at time n (here $|u| = n$ means that the generation of the particle u in the *Ulam-Harris* tree is n). We assume that

$$\mathscr{D}_0 = (X_u, |u| = 1) \stackrel{d}{=} (S_1^{(i)}, 1 \le i \le N)$$

where N is an integer random variable describing the offspring of a branching particle, with mean $m = \mathbb{E}[N]$, and $(S_n^{(i)}, n \ge 0)_{i\ge 1}$ are IID random walks, distributed as $(S_n, n \ge 0)$ and independent from N.
We assume that we are in the *supercritical regime*, that is

$$m(1-q_{esc}) > 1 \tag{1.1}$$

where q_{esc} is the escape probability :

$$q_{esc} := \mathbb{P}\big(\forall n \ge 1, S_n \ne 0\big) .$$

An explanation of assumption (1.1) is given in section 7, Lemma 7.3.
Eventually, we also assume, for sake of simplicity, some aperiodicity, that is $\gcd\{n \ge 1 : \mathbb{P}(\tau = n) > 0\} = 1$ where τ is the first return time to the origin

$$\tau := \inf\big\{n \ge 1 : S_n = 0\big\} .$$

Let $M_n := \sup_{|u|=n} X_u$ we the maximal position at time n of a living particle. Our first result is

Theorem 1.1 (Law of large numbers)**.** *There exists a constant α, depending only on characteristics of the random walk $(S_n)_n \ge 0$ and on the mean offspring m, such that on the set of non extinction $\mathscr{S}$,*

$$\lim_{n\to+\infty} \frac{M_n}{n} = \alpha \quad a.s.$$

The exact value of the constant $alpha$ is given at the beginning of section 5.

In order to refine this convergence by centering M_n, we shall assume for this proof that $(S_n)_{n\in\mathbb{N}}$ is a nearest neighbor random walk.

Theorem 1.2. *There exists a constant $c_* > 0$, a random variable Λ_∞ and $t_0 > 0$ such that*

$$\{\Lambda_\infty > 0\} = \mathscr{S} \quad a.s. \tag{1.2}$$

$$\lim_{n\to+\infty} \mathbb{P}(M_n - \alpha n > y) = \mathbb{E}\left[1 - e^{-c_* e^{-t_0 y}\Lambda_\infty}\right] \qquad (\forall y \in \mathbb{R}). \tag{1.3}$$

The random variable Λ_∞ is the limit of the positive fundamental martingale of section 4. Constant t_0 is defined at the beginning of section 5. The value of constant c_* is given at the beginning of section 6.

Theorems 1 and 2 are new, even though a lot of attention has been given to CBRW in continuous time. In papers [1–4, 13–16] very precise asymptotics are established for the moments of $\eta_t(x)$ the number of particles located at x at time t, in every regime (sub/super/critical).
Elaborate limit theorems were obtained for the critical case by Vatutin, Topchii and Yarovaya in [13–16]. We especially acknowledge paper [6] that introduced us to the magic technique of multiple spines. To confront our results to the litterature on (non catalytic) Branching Random Walk, we refer to [12].

We first give in section 2 the heuristics explaining the differences between CBRW and ordinary BRW (branching random walk). Then we proceed (in section 3) to establish many to few lemmas, we exhibit a fundamental martingale (in section 4) and prove Theorems 1 and 2 with the help of sharp asymptotics derived from renewal theory.
Finally, section 7 is devoted to an extension to the case of multiple catalysts. There the supercritical assumption (1.1) appears in a very natural way.

2 Heuristics

Assume for sake of simplicity that we are on $\mathbb{Z}$, with a single catalyst at the origin and a Simple Random Walk. The sheer existence of the fundamental martingale $\Lambda_n = e^{-rn}\sum_{|u|=n}\phi(X_u)$, see section 4, such that $\{\Lambda_\infty > 0\} = \mathscr{S}$ shows that on the set of non extinction $\mathscr{S}$, we have roughly e^{rn} particles at time n.

If we apply the usual heuristic for Branching Random Walk, then we say that we have approximately e^{rn} independent random walks positioned at time n, and therefore the expected population above level $an > 0$ is roughly:

$$\mathbb{E}\left[\sum_{i=1}^{e^{rn}} \mathbf{1}_{(S_n^{(i)} \geq an)}\right] = e^{rn}\mathbb{P}(S_n \geq an) = e^{-n(I(a)-r)(1+o(1))}$$

where $I(a) = \psi^*(a) = \sup_{t\geq 0}(ta - \psi(t))$ is the large deviation rate function (for Simple Random Walk, $e^{\psi(t)} = \mathbb{E}\left[e^{tS_1}\right] = \mathrm{ch}(t)$).
This expected population is of order 1 when $I(a) = r$ and therefore we would expect to have $\frac{M_n}{n} \to \gamma$ on $\mathscr{S}$, where $I(\gamma) = r$.
However, for CBRW, this is not the right speed, since the positions of the independent particles cannot be assumed to be distributed as random walks. Instead, the e^{rn} independent particles may be assumed to be distributed as a fixed probability distribution, namely $\nu(x) = c\phi(x)$.
Since here $\phi(x) = \mathbb{E}_x\left[e^{-rT_0}\right] = e^{-t_0|x|}$, ν is a symmetric geometric distribution. Therefore the expected population with distance to the origin at least an is roughly

$$e^{rn}\nu(|x| \geq an) = c' e^{rn} e^{-nat_0}\,.$$

This expectation is of order 1 when $a = \frac{r}{t_0} = \frac{\psi(t_0)}{t_0} = \alpha$, and this yields the right asymptotics

$$\frac{M_n}{n} \to \alpha \quad \text{a.s. on } \mathscr{S}\,.$$

Furthermore, we can even obtain the refined asymptotics if we use a slightly refined heuristic. We assume that conditionally on Λ_n, we have $e^{rn}\Lambda_n$ independent random variables $(\xi_i)_i$ distributed as ν describing the positions of the particles. Then with $Y_n = \max_{i\leq e^{rn}\Lambda_n}\left|\xi_i\right|$ we have

$$\mathbb{P}(Y_n - \alpha n \geq y \mid \Lambda_n) = 1 - (1 - \mathbb{P}(\xi_i > y + \alpha n))^{e^{rn}\Lambda_n}.$$

Therefore

$$\mathbb{P}(Y_n - \alpha n \geq y) = \mathbb{E}\left[1 - (1 - c'\frac{1}{e^{rn}}e^{-t_0 y})^{e^{rn}\Lambda_n}\right] \to \mathbb{E}\left[1 - e^{-c'\Lambda_\infty e^{-t_0 y}}\right].$$

This is what states Theorem 1.2, up to constants.

3 Many to few formulas

We have found it easier to write many to one and many to two formulas in a fairly general setting, and then specialize them to our needs, than trying to write them directly for CBRW.
For the spine construction we refer to[10, 11] and the references therein. However, we feel necessary to give some details here, since we are in a discrete time setup.

3.1 Trees

We use the *Ulam-Harris* labeling system. The set of labels is

$$\mathcal{U} := \{\emptyset\} \cup \cup_{n\in\mathbb{N}^*}(\mathbb{N}^*)^n \quad \text{with} \quad \mathbb{N}^* = \{1,2,3,\ldots\}\,.$$

The elements of $\mathcal{U}$ are called particles/labels/nodes/words. We think of $\emptyset$ as the initial ancestor. For $u \in \mathcal{U}$, if $u = (u_1, u_2, \ldots, u_n)$ then $|u| = n$ is the generation of u (by convention $|\emptyset| = 0$). We write uv for the concatenation of the words u, v and we set $u\emptyset = \emptyset u = u$.
We say that v is an ancestor of u and we write $v \le u$ if there exists $w \in \mathcal{U}$ such that $u = vw$.
We define a *tree* to be a subset $\tau \subset \mathcal{U}$ such that

- $\emptyset \in \tau$.
- $uv \in \tau \Rightarrow u \in \tau$ (if a particle is in the tree, the its ancestor are in the tree).
- For each $u \in \tau$ there exists $N_u \in \mathbb{N}$ such that $uj \in \tau \iff 1 \le j \le N_u$ (each particle u has a finite number N_u of children).

The set of particles of generation n is

$$\mathcal{N}_n = \{u \in \tau : |u| = n\}\,.$$

We let $\mathbb{T}$ be the set of such trees.

3.2 Marked trees

A marked tree is a set T of pairs (u, X_u) such that $u \in \mathcal{U}$, the set

$$\text{tree}(T) = \left\{u \in \mathcal{U} : \exists X_u, (u, X_u) \in \mathcal{U}\right\}$$

is a tree. We think of $X_u \in J$ as describing the position of the particle u. The state space J is usually $J = \mathbb{Z}^d$. The set of particles of generation n is

$$\mathcal{N}_n = \{u \in \mathrm{tree}(T) : |u| = n\} .$$

Let $\mathcal{T}$ be the set of marked trees.

3.3 Marked trees with spine

A spine of a tree $\tau = \mathrm{tree}(T)$ is a single maximal distinguished line of descent. It is a subset ξ of τ such that

- $\emptyset \in \xi$.
- $\xi \cap \mathcal{N}_n$ contains at most one point.
- $if\, v \in \xi$ and $u < v$ then $u \in \xi$.
- $if\, v \in \xi$ and $A_v > 0$, then there exists one and only one child on the spine,i.e. $\exists! j \in \{1, \ldots, A_v\}, vj \in \xi$.

When $\xi \cap \mathcal{N}_n$ is a singleton, we let ξ_n be the node on the spine of generation n : $\{\xi_n\} = \xi \cap \mathcal{N}_n$.
Let $\tilde{\mathcal{T}}$ be the set of marked trees with a spine (T, ξ).

3.4 Filtrations

We shall work on $\tilde{\mathcal{T}}$.
The filtration

$$\mathcal{F}_n = \sigma\{\{(T,\xi) : u \in \mathrm{tree}(T)\}, |u| \leq n\}$$

is the natural filtration of the branching process. It does not carry information about the spine.
The filtration

$$\tilde{\mathcal{F}}_n := \sigma(\mathcal{F}_n \cup \{\{(T,\xi) : u \in \xi\}, |u| \leq n\})$$

carries information about the branching process and the first n particles of the spine.
The filtration

$$\mathcal{G}_n = \sigma(X_u : u \in \xi \cap \mathcal{N}_p, p \leq n)$$

carries information about the location of the spine up to generation n.

3.5 The CBRW probability measure

We are given a family of point processes $(\mathscr{D}_y, y \in J)$ such that $\mathscr{D}_y$ describes the progeny (number and positions) of a particle located at y.
For every y there exists a probability measure $\mathbb{P}_y$ on $(\tilde{\mathscr{T}}, \tilde{\mathscr{F}}_\infty)$ such that under $\mathbb{P}_y$ the process start at time 0 with one particle located at y.
Any particle located at z in generation n gives rise, independently of the other particles of generation n, to a family of particles whose location is described by an independent copy of $\mathscr{D}_z$. In particular, $\mathscr{D}_y$ has the same distribution as $\{X_u, u \in \mathscr{N}_1\}$ under $\mathbb{P}_y$. The restriction of $\mathbb{P}_y$ on $\mathscr{F}_\infty$ is called the Catalytic Branching Random Walk (CBRW) driven by $(\mathscr{D}_z, z \in J)$ and starting from y.

Example 1. For example, the CBRW on $\mathbb{Z}$ branching at the origin only, driven by the reproduction random variable N, is obtained with

$$\mathscr{D}_0 = \sum_{i=1}^{N} \delta_{S_1^{(i)}}$$
$$\mathscr{D}_x = \delta_{S_1^{(1,x)}} \quad (x \in \mathbb{Z}, x \neq 0),$$

with $(S^{(i)})_{i\geq 1}$ an independent family of simple random walks starting from 0, and $(S^{(i,x)})_{i\geq 1, x\in\mathbb{Z}}$ an independent family of simple random walks starting from x.

Example 2. If the set of catalysts is a subset $\mathscr{C} \subset \mathbb{Z}$ and we have random variables $(N_x, x \in \mathscr{C})$ describing the offsprings at the different sites, then we set

$$\mathscr{D}_x = \sum_{i=1}^{N_x} \delta_{S_1^{(i,x)}} \quad (x \in \mathscr{C}),$$
$$\mathscr{D}_x = \delta_{S_1^{(1,x)}} \quad (x \notin \mathscr{C}).$$

The classical BRW is obtained with $\mathscr{C} = \mathbb{Z}$ and the N_c IID.

Example 3. We obtain a Branching Markov Process, see e.g.[9], by assuming in the previous example $(S^{(i,x)})_{i\geq 1, x\in\mathbb{Z}}$ to be an independent family of Markov chains (starting from x for $S^{(i,x)}$), with a fixed Markov kernel say $p(x,y)$, and thus setting

$$\mathscr{D}_x = \sum_{i=1}^{N_x} \delta_{S_1^{(i,x)}}.$$

An associated martingale.

For some $\beta \in \mathbb{R}$, and $y \in J$, we assume that

$$e^{\psi_y(\beta)} := \mathbb{E}_y\left[\sum_{u\in\mathcal{N}_1} e^{\beta(X_u-y)}\right] = \mathbb{E}\left[\sum_{z\in\mathcal{D}_y} e^{\beta(z-y)}\right] < +\infty$$

Then,

$$W_n(\beta) := \sum_{u\in\mathcal{N}_n} e^{\beta(X_u-y)-\sum_{\emptyset\le v<u}\psi_{X_v}(\beta)}$$

is a $(\mathbb{P}_y, \mathscr{F}_n)$ martingale.
Indeed, if we have a family $(\theta_{u,y}, u \in \mathscr{U}, y \in J)$ of independent point measures such that $\theta_{u,y}$ is distributed as $\mathscr{D}_y$, then we can construct recursively the point measures by

$$\sum_{v\in\mathcal{N}_{n+1}} \delta_{X_v} = \sum_{u\in\mathcal{N}_n}\sum_{z\in\theta_{u,X_u}} \delta_z\,.$$

Therefore,

$$\mathbb{E}_{\mathbb{P}_y}\left[W_{n+1}(\beta) \mid \mathscr{F}_n\right] = \sum_{u\in\mathcal{N}_n, x\in J} \mathbf{1}_{(X_u=x)} e^{-\sum_{v<u}\psi_{X_v}(\beta)} \mathbb{E}_{\mathbb{P}_y}\left[\sum_{z\in\theta_{u,x}} e^{\beta(z-y)-\psi_x(\beta)} \mid \mathscr{F}_n\right] = W_n(\beta)$$

3.6 The size biased CBRW

The size biased CBRW is constructed via a probability measure $\mathbb{Q}_y^\beta$ on $\tilde{\mathscr{F}}_\infty$.
This is done by setting consistently the values of $\mathbb{E}_{\mathbb{Q}_y^\beta}[Y]$ for Y a $\tilde{\mathscr{F}}_n$ measurable positive random variable. Indeed, we can write

$$Y = Z\sum_{v\in\mathcal{N}_n} \Gamma(v)\,\mathbf{1}_{(\xi_n=v)}$$

where Z and each $\Gamma(v)$ is $\mathscr{F}_n$ measurable. We set,

$$\mathbb{E}_{\mathbb{Q}_y^\beta}[Y] = \mathbb{E}_{\mathbb{P}_y}\left[Z\sum_{u\in\mathcal{N}_n}\Gamma(u)e^{\beta(X_u-y)-A_\beta(u)}\right] \tag{3.1}$$

with

$$A_\beta(u) := \sum_{\emptyset\le v<u} \psi_{X_v}(\beta)\,. \tag{3.2}$$

It is really easy to check that the preceding formula defines a consistent family of measures on the filtration $\tilde{\mathscr{F}}_n$, and Kolmogorov extension Lemma yields the

existence of $\mathbb{Q}_y^\beta$. For simplicity of notations, we intentionally omit the dependence on parameter β.
Furthermore, we can describe the evolution of the particle system in $\mathbb{Q}_y^\beta$ in the following way :

- There is initially one particle located at y.
- The offspring of this particle is generated under the size biased law $\hat{\mathscr{D}}_y$ defined by

$$\mathbb{E}\left[F(\hat{\mathscr{D}}_y)\right] = \mathbb{E}\left[F(\mathscr{D}_y)(\sum_{z\in\mathscr{D}_y} e^{\beta(z-y)-\psi_y(\beta)})\right]$$

- Pick one of the offspring ξ_1 at random : the probability that u is picked is proportional to $e^{\beta X_u}$.
- The particles other than ξ_1 give rise to an ordinary CBRW. The spine particle ξ_1 has an offspring according to $\hat{\mathscr{D}}_{X_{\xi_1}}$, and we go on by choosing ξ_2 among the children of ξ_1 with probability proportional to $e^{\beta X_u}$ of picking child u, and we go on ...

Remark 4. When we restrict the measure $\mathbb{Q}_y^\beta$ to the filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$ we obtain the change of measure by the martingale $W_n(\beta)$. In other words

$$\frac{d\mathbb{Q}_y^\beta}{d\mathbb{P}_y}|_{\mathscr{F}_n} = W_n(\beta)\,.$$

Indeed, if Z is $\mathscr{F}_n$ measurable, then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_y^\beta}[Z] &= \mathbb{E}_{\mathbb{Q}_y^\beta}\left[Z\sum_{v\in\mathscr{N}_n}\mathbf{1}_{(\xi_n=v)}\right]\\
&= \mathbb{E}_{\mathbb{P}_y}\left[Z\sum_{u\in\mathscr{N}_n} e^{\beta(X_u-y)-A_\beta(u)}\right]\\
&= \mathbb{E}_{\mathbb{P}_y}\left[ZW_n(\beta)\right].
\end{aligned}$$

3.7 The many to one formula

The many to one formula is obtained by specializing equation (3.1) to $\beta=0$, $Z=1$ and $\Gamma(u)=f(X_u)e^{A_0(u)}$. We set $S_n=X_{\xi_n}$ the position of the spine at time

n.

$$\mathbb{E}_{\mathbb{P}_y}\left[\sum_{u\in\mathcal{N}_n} f(X_u)\right] = \mathbb{E}_{\mathbb{Q}_y^\beta}\left[f(S_n)e^{A_0(\xi_n)}\right] \tag{3.3}$$

For CBRW on $\mathbb{Z}$ branching only at the origin, we have $\psi_y(\beta) = \psi(\beta)+\ln(m)\,\mathbf{1}_{(y=0)}$. We shall use the now *standard* notation

$$\sum_{u\in\mathcal{N}_n} f(u) = \sum_{|u|=n} f(u)\,.$$

Therefore, when the CBRW is started from zero,

Lemma 3.1 (Many to one formula)**.** *For CBRW branching only at the origin of* $\mathbb{Z}$

$$\mathbb{E}\left[\sum_{|u|=n} f(X_u)\right] = \mathbb{E}\left[f(S_n)m^{L_{n-1}}\right] \tag{3.4}$$

where $L_{n-1} = \sum_{k=0}^{n-1} \mathbf{1}_{(S_k=0)}$ *is the local time at level* 0.

3.8 The many to two formula

We consider now marked trees with two spines (T,ξ^1,ξ^2) : T is a marked tree, and ξ^i, $i = 1,2$ are two maximal distinguished lines of descent. When $\xi^i \cap \mathcal{N}_n \neq \emptyset$, then $\left\{\xi^i_n\right\} = \xi^i \cap \mathcal{N}_n$.
The filtration

$$\mathscr{F}_n = \sigma\left\{\left\{(T,\xi^1,\xi^2) : u \in \text{tree}(T)\right\}, |u| \le n\right\}$$

is the natural filtration of the branching process. It does not carry information about the spine.
The filtration

$$\tilde{\mathscr{F}}^2_n := \sigma(\mathscr{F}_n \cup \left\{\left\{(T,\xi^1,\xi^2) : u \in \xi^i\right\}, |u| \le n, i = 1,2\right\})$$

carries information about the branching process and the first n particles of the two spines.
For a given $\beta \in \mathbb{R}$ and $y \in J$ we assume that

$$e^{\psi^{(2)}_y(\beta)} := \ln\mathbb{E}\left[\sum_{z,z'\in\mathscr{D}_y} e^{\beta((z-y)+(z'-y))}\right] < +\infty$$

We construct a new probability measure on $\tilde{\mathscr{F}}^2_\infty$ by prescribing consistently its value on the sigma fields $\tilde{\mathscr{F}}^2_n$. Indeed a general $\tilde{\mathscr{F}}^2_\infty$ bounded (or positive) measurable random variable Y can be written

$$Y = Z\Gamma(\xi^1_n, \xi^2_n) = Z \sum_{u,v \in \mathscr{N}_n} \Gamma(u,v)\, \mathbf{1}_{(\xi^1_n = u, \xi^2_n = v)}$$

with $Z, (\Gamma(u,v))_{|u|=|v|=n}$ $\mathscr{F}_n$ measurable random variables.
We set,

$$\mathbb{E}_{\mathbb{Q}^2_y}[Y] := \mathbb{E}_{\mathbb{P}_y}\left[Z \sum_{u,v\in\mathscr{N}_n} \Gamma(u,v) e^{\beta((X_u-y)+(X_v-y))-A_\beta(u,v)} \right], \tag{3.5}$$

with $u \wedge v$ the greatest common ancestor of u and v and

$$A_\beta(u,v) := \sum_{w\le u\wedge v, w<u, w<v} \psi^{(2)}_{X_W}(\beta) + \sum_{u\wedge v<w<u} \psi_{X_w}(\beta) + \sum_{u\wedge v<w<v} \psi_{X_w}(\beta).$$

It is clear that by construction, the restriction on the filtration $\mathscr{F}_n$ of $\mathbb{Q}^2_y$ admits a Radon-Nikodym density which is the martingale:

$$W^2_n(\beta) = \frac{d\mathbb{Q}^2_y}{d\mathbb{P}_y} = \sum_{u,v\in\mathscr{N}_n} e^{\beta((X_u-y)+(X_v-y))-A_\beta(u,v)} \qquad (\text{on } \mathscr{F}_n).$$

We can describe the evolution of the particle system with two spines under $\mathbb{Q}^2_y$ as follows:

- Initially there is one particle located at y and it bears the two spines $\xi^1_0 = \xi^2_0 = y$.
- The offspring of this particle is generated according to the size biased law:
$$\mathbb{E}\left[F(\mathscr{D}^2_y)\right] = \mathbb{E}\left[F(\mathscr{D}_y) \sum_{z,z'\in\mathscr{D}_y} e^{\beta(z+z'-2y)-\psi^{(2)}_y(\beta)} \right]$$
- Pick two offspring (ξ^1_1, ξ^2_1) at random among the couples of children (they may be the same) : the probability that (u,v) is picked is proportional to $e^{\beta(X_u+X_v)}$.. The particles other than the spines evolve as a standard BRW. If the spine particles are identical $\xi^1_1 = \xi^2_1$, then they evolve again as if started form the same origin located at $y' = X_{\xi^1_1}$. If the spine particles are different, they each evolve independently as a CBRW with one spine.

Observe also that we obtain the law of (ξ_n^1, ξ_n^2) conditionally on $\mathscr{F}_n$:

$$\mathbb{Q}_y^2\left(\xi_n^1 = u, \xi_n^2 = v \mid \mathscr{F}_n\right) = \frac{e^{\beta((X_u - y)+(X_v - y)) - A_\beta(u,v)}}{W_n^2(\beta)}$$

The Harris-Robert's many to two formula (see[11]) can be obtained by specializing the equality (3.5) to $Z = 1$, $\beta = 0$ and $\Gamma(u, v) = f(X_u, X_v)e^{A_0(u,v)}$, by letting $S_n^i = X_{\xi_n^i}$ be the location of the ith spine at time n,:

$$\mathbb{E}_{\mathbb{P}_y}\left[\sum_{u,v\in\mathcal{N}_n} f(X_u, X_v)\right] = \mathbb{E}_{\mathbb{Q}_y^2}\left[f(S_n^1, S_n^2)e^{A_0(\xi_n^1,\xi_n^2)}\right]. \tag{3.6}$$

We have,

$$A_0(\xi_n^1, \xi_n^2) = \sum_{k=0}^{n-1} \psi^{(2)}_{S_k^1}(\beta)\, \mathbf{1}_{(\xi_k^1=\xi_k^2)} + (\psi_{S_k^1}(\beta) + \psi_{S_k^2}(\beta))\, \mathbf{1}_{(\xi_k^1\neq\xi_k^2)}$$

If we specialize a bit more to the CBRW on $\mathbb{Z}$ branching only at the origin we get

$$\psi_y(\beta) = (\ln m + \psi(\beta))\, \mathbf{1}_{(y=0)} + \psi(\beta)\, \mathbf{1}_{(\neq 0)} \qquad \psi_y(0) = \ln(m)\, \mathbf{1}_{(y=0)}$$

$$\psi_y^{(2)}(\beta) = \ln\left(me^{\psi(2\beta)} + (m_2 - m)e^{2\psi(\beta)}\right) \qquad \psi_y^{(2)}(0) = \ln(m_2)\, \mathbf{1}_{(y=0)}$$

with $m_2 = \mathbb{E}\left[N^2\right]$ the second moment of the reproduction law at 0. Therefore, letting $T = \inf\left\{n \geq 1 : \xi_n^1 \neq \xi_n^2\right\}$ we obtain

$$A_0(\xi_n^1, \xi_n^2) = \sum_{k=0}^{n-1} \ln(m_2)\, \mathbf{1}_{(k\leq T, S_k^1=S_k^2=0)} + \ln(m)\, \mathbf{1}_{(k>T)}\left(\mathbf{1}_{(S_k^1=0)} + \mathbf{1}_{(S_k^2=0)}\right) \tag{3.7}$$

Lemma 3.2 (Many to two formula). *For Catalytic Branching Random Walk branching at the origin only, we have*

$$\mathbb{E}\left[\sum_{|u|=|v|=n} f(X_u, X_v)\right] = \mathbb{Q}^2\left(f(S_n^1, S_n^2)e^{A_0(\xi_n^1,\xi_n^2)}\right).$$

where $A_0(\xi_n^1, \xi_n^2)$ is given by (3.7) *and the law of the coupled random walks $(S_n^1, S_n^2)_{n\geq 0}$ is described below.*

The process $(S^1_n, S^2_n, \mathbf{1}_{(\xi^1_n=\xi^2_n)})_{n\geq 0}$ is distributed as the process $(U_n, V_n, \sigma_n)_{n\geq 0}$ constructed as follows. We are given two independent simple random walks U and U' on $\mathbb{Z}$, starting from 0 and an independent family of Bernoulli random variables η_n with Bernoulli law $\mathbb{P}(\eta_n = 1) = 1 - \mathbb{P}(\eta_n = 0) = \frac{m}{m_2}$. Given $(U_n)_{n\geq 0}$ we set recursively V_n and σ_n in the following way.

- $V_0 = 0, \sigma_0 = 1$.
- if $\sigma_n = 1$ and $U_n \neq 0$ then $V_{n+1} = U_{n+1}$ and $\sigma_{n+1} = 1$ (there is only one child so the spine stay together).
- if $\sigma_n = 1$ and $U_n = 0$ and $\eta_n = 1$, then $V_{n+1} = U_{n+1}$ and $\sigma_{n+1} = 1$ (the spines stay together with probability $\frac{m}{m_2}$)
- if $\sigma_n = 1$ and $U_n = 0$ and $\eta_n = 0$, then for all $p \geq 1$, $V_{n+p} = U'_p$ and $\sigma_{n+p} = 0$ (the spines separate).
- if $\sigma_n = 0$ and $T = \inf\{n \geq 1 : \sigma_n = 0\}$, then $V_{n+1} = U'_{n-T+1}$ (the spines stay distinct once they split).

4 A fundamental Martingale

Let us consider the function, defined on $(0,\infty)$, $r \to \rho^{(r)} = m\mathbb{E}\left[e^{-r\tau}\right]$. It is of class C^∞, strictly decreasing, $\lim_{r\to 0} \rho^{(r)} = m\mathbb{P}(\tau < +\infty) = m(1 - q_{esc}) > 1$ and $\lim_{r\to+\infty} \rho^{(r)} = 0$. Therefore there exists a unique $r > 0$, a *Malthusian parameter* such that

$$m\mathbb{E}\left[e^{-r\tau}\right] = 1\,. \tag{4.1}$$

Let $T = \inf\{n \geq 0 : S_n = 0\}$ be the first hitting time of 0, recall that $\tau = \inf\{n \geq 1 : S_n = 0\}$ and let

$$\phi(x) := \mathbb{E}_x\left[e^{-rT}\right] \quad (x \in \mathbb{Z}^d)\,.$$

Finally let $p(x, y) = \mathbb{P}_x(S_1 = y)$ and $Pf(x) = \sum_y p(x, y)f(y)$ be the kernel and semi group of the random walk.

Proposition 4.1. *1. The function ϕ satisfies*

$$P\phi(x) = e^r \phi(x)\left(\frac{1}{m}\mathbf{1}_{(x=0)} + \mathbf{1}_{(x\neq 0)}\right).$$

2. The process

$$\Delta_n := e^{-rn}\phi(S_n)m^{L_{n-1}}$$

is a martingale, where $L_{n-1} = \sum_{0\leq k\leq n-1} \mathbf{1}_{(S_k=0)}$ is the local time at level 0.

3. *The process*

$$\Lambda_n := e^{-rn} \sum_{|u|=n} \phi(X_u)$$

is a martingale called the fundamental martingale.

4. *The process Λ_n is bounded in L^2, and therefore is a Uniformly Integrable martingale.*

Proof. (1) If $x \neq 0$, then $T \geq 1$, therefore, by conditioning on the first step:

$$\phi(x) = \sum_y p(x,y) e^{-r} \mathbb{E}_y \left[e^{-rT} \right] = e^{-r} P\phi(x) .$$

On the other hand, $\tau \geq 1$ so conditioning by the first step again,

$$\phi(0) = 1 = m\mathbb{E}\left[e^{-r\tau}\right] = m \sum_y p(0,y) e^{-r} \mathbb{E}_y \left[e^{-rT} \right] = me^{-r} P\phi(0) .$$

(2) We have,

$$\begin{aligned} \mathbb{E}\left[\Delta_{n+1} \mid \mathscr{F}_n\right] &= e^{-r(n+1)} m^{L_n} \mathbb{E}\left[\phi(S_{n+1}) \mid \mathscr{F}_n\right] = e^{-r(n+1)} m^{L_n} P\phi(S_n) \\ &= e^{-r(n+1)} m^{L_n} e^r \phi(S_n) (\frac{1}{m} \mathbf{1}_{(S_n=0)} + \mathbf{1}_{(S_n \neq 0)}) = \Delta_n . \end{aligned}$$

(3) By the many to one formula, if Z is $\mathscr{F}_{n-1}$ measurable positive, using the martingale property of Δ_n,

$$\begin{aligned} \mathbb{E}\left[\Lambda_n Z\right] &= e^{-rn} \mathbb{E}\left[\sum_{|u|=n} \phi(X_u) Z \right] \\ &= e^{-rn} \mathbb{E}\left[Z\phi(S_n) m^{L_{n-1}} \right] = \mathbb{E}\left[Z\Delta_n\right] \\ &= \mathbb{E}\left[Z\Delta_{n-1}\right] = \mathbb{E}\left[\Lambda_{n-1} Z\right] . \end{aligned}$$

(4) The proof is given in Section 7 in the case of multiple catalysts and uses heavily the many to two formula. It is tedious but straightforward to rewrite it for only one catalyst and obtain the desired result. □

Let us introduce $\eta_n(x)$ the number of particles located at x at time n:

$$\eta_n(x) := \sum_{|u|=n} \mathbf{1}_{(X_u=n)} .$$

Corollary 4.2. 1. *We have* $\sup_{x,n} e^{-rn} \phi(x) \eta_n(x) < +\infty$ *a.s.*

2. *There exists a constant* $0 < C < \infty$ *such that*

$$\mathbb{E}\left[\eta_n(x)\eta_m(y)\right] \le \frac{C}{\phi(x)\phi(y)}\, e^{r(n+m)} \quad (n, m \in \mathbb{N}, x, y \in \mathbb{Z}^d).$$

Proof. (1) Let us write $\Lambda_n = e^{-rn}\sum_x \phi(x)\eta_n(x)$. Since it is a positive martingale it converges almost surely to a finite integrable positive random variable Λ_∞. Therefore $\Lambda^*_\infty := \sup \Lambda_n < +\infty$ a.s.and

$$\sup_{x,n} e^{-rn}\phi(x)\eta_n(x) \le \Lambda^*_\infty\,.$$

(2) Assume for example that $n \le m$ and let $C = \sup_n \mathbb{E}\left[\Lambda_n^2\right] < +\infty$. We have, since Λ_n is a martingale,

$$\begin{aligned}\phi(x)\phi(y)\mathbb{E}\left[\eta_n(x)\eta_n(y)\right] &\le \mathbb{E}\left[\Lambda_n\Lambda_m\right]\\ &= \mathbb{E}\left[\Lambda_n\mathbb{E}\left[\Lambda_m \mid \mathscr{F}_n\right]\right] = \mathbb{E}\left[\Lambda_n^2\right] \le C.\end{aligned}$$

□

For the proof of the following result instead of using large deviations for L_n, we use renewal theory, in the spirit of [5,7].

Proposition 4.3. *For every* $x \in \mathbb{Z}^d$ *there exists a constant* $c_x \in (0, \infty)$ *such that*

$$\lim_{n\to+\infty} e^{-rn}\mathbb{E}\left[\eta_n(x)\right] = c_x\,.$$

Proof. By the many to one formula

$$\begin{aligned}v_n(x) := \mathbb{E}\left[\eta_n(x)\right] &= \mathbb{E}\left[\sum_{|u|=n} \mathbf{1}_{(X_u=x)}\right]\\ &= \mathbb{Q}\Big(\mathbf{1}_{(S_n=x)}e^{A_0(\xi_n)}\Big)\\ &= \mathbb{E}\left[\mathbf{1}_{(S_n=x)}m^{L_{n-1}}\right].\end{aligned}$$

We decompose this expectation with respect to the value of $\tau = \inf\{n \ge 1 : S_n = 0\}$:

$$v_n(x) = m\mathbb{E}\left[\mathbf{1}_{(S_n=x)}\,\mathbf{1}_{(\tau\ge n)}\right] + \sum_{1\le k\le n-1}\mathbb{E}\left[\mathbf{1}_{(S_n=x)}m^{L_{n-1}}\,\mathbf{1}_{(\tau=k)}\right].$$

By the Markov property, if $u_k := \mathbb{P}(\tau = k)$, then

$$v_n(x) = m\mathbb{P}(\tau \ge n, S_n = x) + \sum_{1\le k\le n-1} m u_k v_{n-k}(x) = m\mathbb{P}(\tau \ge n, S_n = x) + m v_.(x) * u(n)\,,$$

Recall that the Malthusian parameter r is defined by

$$1 = m\mathbb{E}\left[e^{-r\tau}\right] = m\sum_{k\geq 1} e^{-rk} u_k\,.$$

Hence if we let $\tilde{v}_n(x) = e^{-rn} v_n(x)$ and $\tilde{u}_k = m e^{-rk} u_k$ then,

$$\tilde{v}_n(x) = m e^{-rn}\mathbb{P}\big(\tau \geq n, S_n = x\big) + \tilde{v} * \tilde{u}(n)\,.$$

For $x = 0$, this yields

$$\tilde{v}_n = \tilde{u}_n + \tilde{v} * \tilde{u}(n)\,.$$

By assumption, we have aperiodicity : $\gcd\left\{k \geq 1 : u_k > 0\right\} = 1$ and therefore,

$$e^{-rn}\mathbb{E}\left[\eta_n(0)\right] = \tilde{v}_n \to \frac{\sum_k \tilde{u}_k}{\sum_k k\tilde{u}_k} = \frac{1}{\sum_k k\tilde{u}_k} =: c_0\,.$$

And similarly,

$$e^{-rn}\mathbb{E}\left[\eta_n(x)\right] = \tilde{v}_n(x) \to c_x := c_0 m \sum_k e^{-rk}\mathbb{P}\big(\tau \geq k, S_k = x\big)\,.$$

□

Remark 5. A simple use of the Markov property shows that if $x \neq 0$, then

$$c_x = e^{-r}\sum_y c_y p(y,x) = e^{-r} P^* c(x)$$

an equation dual to the one satisfied by the function ϕ.

Remark 6 (The periodic case). Let us look at a periodic case, for example simple random walk on $\mathbb{Z}$. We have $\gcd\left\{k \geq 1 : u_k > 0\right\} = 2$, and $\eta_{2n+1}(0) = 0$. Therefore, by looking only even numbers, we obtain

$$\lim_{n\to+\infty} e^{-r2n}\eta_{2n}(0) = c_0,\,,$$

and similar results for $\eta_n(x)$ by looking only at odd n if x is odd,and even n when x is even. The law of large numbers of Theorem 1.1 can then be proved *mutatis mutandis* as in section 5.

We end this section by proving the first part of Theorem 1.2.

Lemma 4.4. *We have* $\left\{\Lambda_\infty > 0\right\} = \mathscr{S}$ *a.s.*

On the extinction set $\mathscr{S}^C$, there exists $n_0 = n_0(\omega)$ such that for all $n \geq n_0$, and all x, $\eta_n(x) = 0$. Therefore, if $n \geq n_0$ on $\mathscr{S}^C$, then $\Lambda_n = e^{-rn} \sum_x \phi(x)\eta_n(x) = 0$ and $\Lambda_\infty = 0$ a.s. We have just proved that $\mathscr{S}^C \subset \{\Lambda_\infty = 0\}$ a.s.

Let $s = \mathbb{P}(\Lambda_\infty = 0)$. Let $\tau^{(i)}$ be the time of return to 0 for the ith particle born to the ancestor, and let $\Lambda_n^{(i)}$ the process built from this particle, after the time $\tau^{(i)}$, when it is finite. We shall condition on the number of children of the initial ancestor N, a random variable independent of the rv's $\tau^{(i)}$ and of $\Lambda^{(i)}$.

$$\begin{aligned} s = \mathbb{P}(\Lambda_\infty = 0) &= \sum_k \mathbb{P}(N = k)\mathbb{P}\Big(\forall i \leq k, \tau^{(i)} < +\infty, \Lambda_\infty^{(i)} = 0\Big) \\ &= \sum_k \mathbb{P}(N = k)\mathbb{P}(\tau < \infty)^k s^k = f(s(1 - q_{esc})). \end{aligned}$$

with $f(x) = \mathbb{E}\left[x^N\right]$ the generating function of the reproduction law.
In the recurrent case, $q_{esc} = 0$, we have $s = f(s)$ and since we are in the supercritical regime, $m = f'(1) > 1$ so $s = 1$ or $s = \mathbb{P}\Big(\mathscr{S}^C\Big)$. The case $s = 1$ is impossible since Λ_n is uniformly integrable and therefore $\mathbb{E}\left[\Lambda_\infty\right] = 1$.
In the case $q_{esc} > 0$, by the same arguments, we prove that $t = \mathbb{P}\Big(\mathscr{S}^C\Big)$ satisfies the same equation $t = f(t(1-q_{esc}))$. It remains to show that this equation has a unique solution on $[0,1)$. It is easy to see that the function $g(x) = f(x) - \frac{x}{1-q_{esc}}$ has a unique zero on $[0, 1 - q_{esc})$ since g is convex, $g(0) = f(0) > 0$ (otherwise the extinction probability is 0 as is s), $g(1) = 1 - \frac{1}{1-q_{esc}} < 0$, $g'(1) = m - \frac{1}{1-q_{esc}} > 0$.

5 The law of large numbers : proof of Theorem 1.1.

We assume that for all real t : $e^{\psi(t)} = \mathbb{E}\left[e^{tS_1}\right] < +\infty$. We assume that $\mathbb{P}(S_1 > 0) > 0$ since otherwise the random walk never goes to the right and we have trivially $M_n = 0$ for all n.
We have then $e^{\psi(t)} \geq e^t \mathbb{P}(S_1 > 0) \to +\infty$ and therefore consider the unique $t_0 > 0$ such that

$$\psi(t_0) = r\,,$$

with r the Malthusian parameter given by (4.1).
We are going to prove that on the survival set $\mathscr{S}$, almost surely,

$$\lim_{n \to +\infty} \frac{M_n}{n} = \alpha := \frac{r}{t_0}\,.$$

5.1 Proof of the upper bound

Let $\theta > 0$, $x > 0$. By the many to one formula,

$$\begin{aligned}
\mathbb{P}(M_n > xn) &= \mathbb{P}\left(\left(\sum_{|u|=n} \mathbf{1}_{(X_u > xn)}\right) > 0\right) \\
&\le \mathbb{E}\left[\sum_{|u|=n} \mathbf{1}_{(X_u > xn)}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{(S_n > nx)} m^{L_{n-1}}\right] \\
&\le \mathbb{E}\left[e^{\theta(S_n - xn)} m^{L_{n-1}}\right] = e^{-\theta nx} v_n \quad , \text{with } v_n = \mathbb{E}\left[e^{\theta S_n} m^{L_{n-1}}\right].
\end{aligned}$$

As in Proposition 4.3, we are going to use renewal theory to study the asymptotics of v_n. Let us condition on $\tau = \inf\{n \ge 1 : S_n = 0\}$:

$$\begin{aligned}
v_n &= \mathbb{E}\left[e^{\theta S_n} m^{L_{n-1}} \mathbf{1}_{(\tau \ge n)}\right] + \sum_{1 \le k \le n-1} \mathbb{E}\left[e^{\theta S_n} m^{L_{n-1}} \mathbf{1}_{(\tau = k)}\right] \\
&= \mathbb{E}\left[e^{\theta S_n} \mathbf{1}_{(\tau \ge n)}\right] + \sum_{1 \le k \le n-1} m\mathbb{P}(\tau = k) v_{n-k} \\
&= y_n + mv * u(n),
\end{aligned}$$

with $y_n := \mathbb{E}\left[e^{\theta S_n} \mathbf{1}_{(\tau \ge n)}\right]$ and $u_n := \mathbb{P}(\tau = n)$.
Assume now that $\theta > t_0$ so that $\psi(\theta) > \psi(t_0) = r$. We let

$$\tilde{v}_n := e^{-n\psi(\theta)} v_n, \quad \tilde{y}_n := e^{-n\psi(\theta)} y_n, \quad \tilde{u}_n := me^{-n\psi(\theta)} u_n.$$

On the one hand, by definition of the Malthusian parameter we have $1 = m\mathbb{E}\left[e^{-r\tau}\right] = \sum m_n e^{-rn} u_n$ so that $\sum_k \tilde{u}_k < 1$.
On the other hand,

$$\tilde{y}_n = \mathbb{E}\left[e^{\theta S_n - n\psi(\theta)} \mathbf{1}_{(\tau \ge n)}\right] = \mathbb{P}_\theta(\tau \ge n)$$

with $\mathbb{P}_\theta$ defined by the martingale change of probability

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} = e^{\theta S_n - n\psi(\theta)} \quad (\text{on } \mathscr{F}_n).$$

Since under $\mathbb{P}_\theta$, $(S_n)_{n \ge 0}$ is a random walk with mean $\mathbb{E}_\theta\left[S_1\right] = \psi'(\theta) > 0$, we have

$$\tilde{y}_n \to \tilde{y}_\infty := \mathbb{P}_\theta(\tau = +\infty).$$

Recall the aperiodicity assumption $\gcd\{u_k : k \geq 1\} = 1$. By the renewal theorem, we have

$$\tilde{v}_n \to \frac{\tilde{y}_\infty}{1 - \sum_k \tilde{u}_k}\,.$$

Therefore, if $x > \frac{\psi(\theta)}{\theta}$

$$\mathbb{P}\big(M_n > xn\big) \leq e^{-n(\theta x - \psi(\theta))}\tilde{v}_n$$

satisfies $\sum_n \mathbb{P}\big(M_n > xn\big) < +\infty$ and by Borel Cantelli

$$\limsup_{n\to+\infty} \frac{M_n}{n} \leq x \quad a.s.$$

Hence, letting first $x \downarrow \frac{\psi(\theta)}{\theta}$ and then $\theta \downarrow t_0$ we obtain that

$$\limsup_{n\to+\infty} \frac{M_n}{n} \leq \frac{\psi(t_0)}{t_0} = \alpha \quad a.s.$$

5.2 Proof of the lower bound

Recall from Proposition 4.3 and Corollary 4.3 that

$$\lim_{n\to+\infty} e^{-rn}\eta_n(0) = c_0\,, \quad \sup_n e^{-2rn}\mathbb{E}\left[\eta_n(0)^2\right] + \infty\,.$$

Therefore Paley-Zygmund's inequality entails that

$$\mathbb{P}\Big(\eta_n(0) \geq c'\,\lambda^n\Big) \geq c',$$

for some constant $c' > 0$. The following lemma aims at the a.s. behavior of $\eta_n(0)$:

Lemma 5.1. *Almost surely on $\mathscr{S}$,*

$$\eta_n(0) \geq \frac{c'}{2}e^{rn},$$

for all large n.

Proof. Let $m_L := m\,\mathbb{P}(\tau \leq L)$. We have

$$\lim_{L\to+\infty} m_L = m\mathbb{P}(\tau < +\infty) = m(1 - q_{esc}) > 1$$

so we can pick up an integer L large enough so that $m_L > 1$.
Now, we shall construct a supercritical Galton-Watson process GW_L: By the construction of the branching walks system, at time 0 there is a single particle which branches according to the law (p_i). Amongst the set of children, we only choose those particles which return to 0 before time L. This forms the first generation of GW_L. We repeat the above selection independently and we get a Galton-Watson process GW_L. Denote by $(p_i^{(L)})$ the reproduction law GW_L. Plainly

$$p_i^{(L)} = \sum_{j=i}^{\infty} p_j \, C_j^i \, (\mathbb{P}(\tau \le L))^i (\mathbb{P}(\tau > L))^{j-i}, \qquad i \ge 0.$$

Note that $\sum i p_i^{(L)} = m_L > 1$. Denote by $\mathscr{S}_L$ the survival set of GW_L. Then $\mathscr{S}_L$ is increasing on L and

$$\mathscr{S} = \cup_{L \ge 2} \mathscr{S}_L.$$

Let $0 < \varepsilon < m_L - 1$. Let $GW_L(n)$ be the number of individuals of nth generation of GW_L. On $\mathscr{S}_L$, a.s., $GW_L(n) \ge (m_L - \varepsilon/2)^n$ for all large n. Observe that for any $n \ge 1$,

$$GW_L(n) \le \sum_{k \le Ln} \eta_k(0).$$

It follows that on $\mathscr{S}_L$, a.s, for all large n,

$$\max_{k \le Ln} \eta_k(0) \ge (m_L - \varepsilon)^n.$$

Pick up a constant $1 < \gamma := \gamma(\varepsilon, L) < (m_L - \varepsilon)^{1/L}$. Considering the stopping time (for the branching system endowed with natural filtration)

$$T_\gamma := \inf\{n : \eta_n(0) > \gamma^n\}.$$

We have shown that on $\mathscr{S}_L$, $T_\gamma < \infty$ a.s. It follows from the branching property that

$$\begin{aligned} \mathbb{P}\Big(\eta_{n+T_\gamma}(0) \le c' \, e^{rn}, \, \mathscr{S}_L\Big) &\le \mathbb{P}\Big(\eta_n(0) \le c' \, e^{rn}\Big)^{\gamma^n} \\ &\le (1 - c')^{\gamma^n}, \end{aligned}$$

whose sum on n converges. By Borel-Cantelli's lemma, on $\mathscr{S}_L$, a.s. for all large n,

$$\eta_n(0) \ge c' e^{r(n - T_\gamma)} \ge \frac{c'}{2} e^{rn}.$$

Since $\mathscr{S} = \cup_L \mathscr{S}_L$, we get the lemma. $\square$

Proof of the lower bound of M_n. Let $0 < s < 1$ and $0 < a < 1$. On the survival set $\mathscr{S}$, at time sn, there are $\frac{c'}{2}e^{rsn}$ particles at 0, which moves independently. Letting these particles move as a simple random walk staying positive up to time $(1-r)n$, then M_n is bigger than $\frac{c'}{2}e^{rsn}$ i.i.d. copies of $S_{(1-r)n}$ with $S_1 > 0, ..., S_{(1-r)n} > 0$, where S is a simple symmetric walk on $\mathbb{Z}$. By Large Deviations Property,

$$\mathbb{P}\Big(S_{(1-r)n} > a(1-r)n, S_1 > 0, ..., S_{(1-r)n} > 0\Big) = e^{-(1-r)nI(a)+o(n)},$$

with

$$I(a) := \sup_{\theta>0}(a\theta - \psi(\theta)).$$

It follows that

$$\begin{aligned} & \mathbb{P}\Big(M_n \le (1-r)an, \eta_{rn}(0) \ge \frac{c'}{2}e^{rsn}\Big) \\ \le & \quad \Big(1-\mathbb{P}(S_{(1-r)n} > a(1-r)n, S_1 > 0, ..., S_{(1-r)n} > 0\Big)\Big)^{\frac{c'}{2}e^{rsn}} \\ = & \quad \exp(-e^{rsn}e^{-I(a)(1-r)n+o(n)}). \end{aligned}$$

Choose $(a,s) \in (0,1)^2$ such that

$$rs > I(a)(1-r),$$

we apply Borel-Cantelli's lemma and get that a.s. for all large n, either $M_n > (1-s)an$ or $\eta_{sn}(0) < \frac{c'}{2}e^{rsn}$. Hence on the survival set $\mathscr{S}$,

$$\liminf_{n\to\infty}\frac{M_n}{n} \ge \sup\{(1-s)a : (a,s) \in (0,1)^2, rs > I(a)(1-r)\}, \qquad a.s.$$

But $r = \psi(t_0)$, therefore

$$\sup\{(1-s)a\} = \sup_{0<a<1}\frac{a\psi(t_0)}{I(a)+\psi(t_0)}.$$

Considering the derivative of $a \to \frac{a\psi(t_0)}{I(a)+\psi(t_0)}$: $I(a) = a\theta(a) - \psi(\theta(a))$ with $a = \psi'(\theta(a))$, then $I'(a) = \theta(a)$ and that the derivative of $a \to \frac{a\psi(t_0)}{I(a)+\psi(t_0)}$ has the same sign as $I(a)+\psi(t_0)-aI'(a) = \psi(t_0)-\psi(\theta(a))$, which is negative if $a > \psi'(t_0)$ (i.e. $\theta(a) > t_0$), positive if $a < \psi'(t_0)$ and vanishes at $\psi'(t_0)$, then

$$\sup\{(1-s)a\} = \frac{\psi(t_0)}{t_0},$$

as desired. □

6 Refining the Convergence : proof of Theorem 1.2.

Proposition 6.1. *There exists some positive constant $c_* > 0$ such that*

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|e^{t_0 z}\mathbb{P}\Big(M_n > \alpha n + z\Big) - c_*\right| = 0,$$

where $\alpha := \frac{\psi(t_0)}{t_0}$.

The value of c_* is given by $c_* := \frac{c_+ c_1}{\widetilde{\mathbb{E}}(H_1)}$, where $c_+ := \frac{1}{\mathbb{P}(S_1>0)}$, c_1 is such that $\mathbb{E}(\eta_n(1)) \sim c_1 e^{n\psi(t_0)}$ and $\widetilde{\mathbb{E}}(H_1)$ is given in equation (6.6).

6.1 Upper bound in Proposition 6.1

We are going to prove that for any $z \in \mathbb{R}$,

$$\limsup_{n\to\infty}\mathbb{P}(M_n > \alpha n + z) \leq \frac{c_+ c_1}{\widetilde{\mathbb{E}}(H_1)} e^{-t_0 z}.$$

Denote as before by $\eta_n(x)$ the number of particles at x at time n. Let $\alpha := \frac{\psi(t_0)}{t_0}$ be the velocity of M_n. We prove the following upper bound : for all $z \in \mathbb{R}$,

$$\limsup_{n\to\infty}\mathbb{P}\Big(M_n > \alpha n + z\Big) \leq c_*\, e^{-t_0 z}.$$

Start from $\mathbb{P}\Big(M_n > \alpha n + z\Big) = \mathbb{P}\Big(\exists |u| = n : X_u > \alpha n + z\Big)$. For $|u| = n$, denote by $u_0 = o < u_1 < \ldots < u_n = u$ the shortest path relating o to u such that $|u_k| = k$ for any $k \leq n$. For $|u| = n$ with $X_u > \alpha n + z > 0$ (as n is large), there exits some $k < n$ such that $X_{u_{k-1}} = 0$ and $X_{u_j} > 0$ for all $k \leq j \leq n$; Moreover, $X_{u_k} = 1$ as we consider nearest neighbor walks. Denote by

$$\begin{aligned} B_k &:= \bigcup_{|v|=k}\Big\{\exists |u| = n : v = u_k, X_v = 1, X_{u_j} > 0, \forall k < j \leq n, X_u > \alpha n + z\Big\} \\ &:= \bigcup_{|v|=k} A_v(k,n). \end{aligned}$$

Then conditioning on $\mathscr{F}_k$, B_k is an union of $\eta_k(1)$ i.i.d. events, and each event holds with probability

$$\begin{aligned} p(k,n) &:= \mathbb{P}_1\Big(S_1 > 0, \ldots, S_{n-k-1} > 0, S_{n-k-1} > \alpha n + z\Big) \\ &= c_+\,\mathbb{P}\Big(S_1 > 0, \ldots, S_{n-k} > 0, S_{n-k} > \alpha n + z\Big), \end{aligned}$$

with $c_+ := \frac{1}{\mathbb{P}(S_1>0)}$, by the Markov property of S at time 1. It follows that

$$\mathbb{P}\Big(B_k\Big) \le \mathbb{E}\Big(\eta_k(1)p(k,n)\Big) = (c_1 + o_k(1))e^{\psi(t_0)k}\,p(k,n),$$

where $o_k(1) \to 0$ as $k \to \infty$. Hence for any $j > 1$ and $n > j$,

$$\mathbb{P}\Big(M_n > \alpha n + z\Big) \le (c_1 + o_j(1))\sum_{k=j}^{n} e^{\psi(t_0)k}\,p(k,n) + C\sum_{k=1}^{j-1} e^{\psi(t_0)k}\,p(k,n), \quad (6.1)$$

where $o_j(1) \to 0$ as $j \to \infty$. It will be clear that the above sum $\sum_{k=1}^{j-1}$ is negligible as $n \to \infty$ [in fact, this sum goes to 0 exponentially fast for any j fixed]. To estimate the sum $\sum_{k=j}^{n}$, we introduce

$$a := \psi'(t_0), \qquad r := \frac{\psi^*(a)}{a t_0} < 1, \quad (6.2)$$

since $\psi^*(a) = a t_0 - \psi(t_0)$. Note that $\alpha = a(1-r)$. Define a new probability

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\sigma\{S_0,\ldots,S_n\}} = e^{t_0 S_n - n\psi(t_0)}.$$

Under $\widetilde{\mathbb{P}}$, S_1 has the mean a. Let $\widetilde{S}_j := S_j - aj$ for $j \ge 0$. Therefore for $1 \le k \le n$,

$$\begin{aligned} p(k,n) &= c_+\,\widetilde{\mathbb{E}}\Big(e^{-t_0 S_{n-k} + (n-k)\psi(t_0)} 1_{(S_j>0,\forall j\le n-k, S_{n-k}>\alpha n+z)}\Big) \\ &= c_+\,e^{-\psi^*(a)(n-k)}\,\widetilde{\mathbb{E}}\Big(e^{-t_0\widetilde{S}_{n-k}} 1_{(\widetilde{S}_j>-aj,\forall j\le n-k, \widetilde{S}_{n-k}>-arn+ak+z)}\Big). \end{aligned}$$

Write $k = rn + \ell$ [ℓ could be a negative real number]. Then

$$\begin{aligned} e^{\psi(t_0)k}p(k,n) &= c_+\,\widetilde{\mathbb{E}}\Big(e^{-t_0(\widetilde{S}_{n-k}-a\ell)} 1_{(\widetilde{S}_j>-aj,\forall j\le n-k, \widetilde{S}_{n-k}-a\ell>z)}\Big) \\ &= c_+\,e^{-t_0 z}\,t_0 \int_0^\infty ds e^{-t_0 s}\,\widetilde{\mathbb{P}}\Big(\widetilde{S}_j > -aj, \forall j \le n-k, z \le \widetilde{S}_{n-k} - a\ell < z+s\Big). \end{aligned} \quad (6.3)$$

Re-writing $a\ell = a(1-r)n - a(n-k)$, we have $\widetilde{S}_{n-k} - a\ell = \widetilde{S}_{n-k} + a(n-k) -$

$a(1-r)n$. Hence

$$\begin{aligned}
&\sum_{k=1}^{n} e^{\psi(t_0)k}\, p(k,n) \\
&= \quad c_+\, e^{-t_0 z}\, t_0 \sum_{k=1}^{n} \int_0^\infty ds e^{-t_0 s}\, \widetilde{\mathbb{P}}\Big(\widetilde{S}_j > -aj, \forall j \le n-k, \\
&\qquad\qquad z \le \widetilde{S}_{n-k} + a(n-k) - a(1-r)n < z+s\Big) \qquad (6.4)\\
&\le \quad c_+\, e^{-t_0 z}\, t_0 \int_0^\infty U(a(1-r)n+z, a(1-r)n+z+s] e^{-t_0 s} ds, \qquad (6.5)
\end{aligned}$$

where for any $x < y$,

$$U(y) := \sum_{j\ge 0}^{\infty} \widetilde{\mathbb{P}}\Big(ai+\widetilde{S}_i > 0, \forall 1 \le i \le j,\ aj+\widetilde{S}_j \le y\Big), \quad U(x,y] := U(y)-U(x).$$

Under $\widetilde{\mathbb{P}}$, $S_j \equiv aj + \widetilde{S}_j$ is a random walk with positive mean a. If we denote by $T_0 = 0 < T_1 < \ldots < T_n < \ldots$ and $H_0 = 0 < H_1 < \cdots < H_n < \cdots$ the strict ladder epochs and ladder heights of the random walk S (under $\widetilde{\mathbb{P}}$), then the duality lemma says that for any $y > 0$,

$$U(y) = \sum_{l=0}^{\infty} \widetilde{\mathbb{P}}\Big(H_l \le y\Big).$$

Since $\widetilde{\mathbb{E}}\left[S_1^2\right] < +\infty$, $\widetilde{\mathbb{E}}(H_1) < \infty$ and we have the Wald identity (see[8] Feller Volume II, Chapter XVIII, Theorem 1)

$$\widetilde{\mathbb{E}}(H_1) = \widetilde{\mathbb{E}}(S_1)\widetilde{\mathbb{E}}(T_1). \qquad (6.6)$$

Hence , the renewal theorem (see[8] Feller, pp381, non-lattice case) implies that for any finite interval I , $U(I+t) \to \frac{|I|}{\widetilde{\mathbb{E}}(H_1)}$ as $t \to \infty$ (in particular $\sup_{t\ge 0} U(t,t+1] < \infty$). Moreover $U(y) \le C(1+y)$ for all $y > 0$.
The dominated convergence theorem implies that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty U(a(1-r)n+z, a(1-r)n+z+s] e^{-t_0 s} ds &= \frac{1}{\widetilde{\mathbb{E}}(H_1)} \int_0^\infty s e^{-t_0 s} ds \\
&= \frac{1}{t_0 \widetilde{\mathbb{E}}(H_1)}. \qquad (6.7)
\end{aligned}$$

Going back to (6.4) and (6.5), we mention that the upper bound (6.5) is optimal as $n \to \infty$, which is equivalent to say that

$$x_n := \sum_{l=n+1}^{\infty} \int_0^{\infty} ds e^{-t_0 s} \widetilde{\mathbb{P}}\Big(\widetilde{S}_j > -aj, \forall j \le l, z \le \widetilde{S}_l + al - a(1-r)n < z+s\Big) \to 0.$$

Indeed, for $l > n$, the probability term in x_n is less than $\widetilde{\mathbb{P}}(\widetilde{S}_l < -arl + z + s) \le e^{-abrl + b(z+s)} \widetilde{\mathbb{E}} e^{b\widetilde{S}_l}$ for any $b > 0$. Since $\widetilde{\mathbb{E}}(\widetilde{S}_1) = 0$, we may choose a sufficiently small but fixed $0 < b < t_0/2$ such that $\widetilde{\mathbb{P}}(\widetilde{S}_l < -arl + z + s) \le e^{-\frac{arbl}{2} + b(z+s)}$, from which we get immediately that x_n tends exponentially fast to 0 as $n \to \infty$. In the same way, we get that

$$\max_{1 \le k \le \sqrt{n}} e^{\psi(t_0)k} p(k,n) \to 0, \qquad \text{exponentially fast when } n \to \infty. \tag{6.8}$$

Assembling the above estimates to (6.1), we obtain that for any $z \in \mathbb{R}$,

$$\limsup_{n\to\infty} \mathbb{P}(M_n > \alpha n + z) \le \frac{c_+ c_1}{\widetilde{\mathbb{E}}(H_1)} e^{-t_0 z}.$$

Furthermore, we see that for some constant $C > 0$,

$$\mathbb{P}(M_n > \alpha n + z) \le C e^{-t_0 z}, \qquad \forall z \in \mathbb{R}, n \ge 1, \tag{6.9}$$

and for any fixed $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} e^{\psi(t_0)k} p(k,n) = \frac{c_+}{\widetilde{\mathbb{E}}(H_1)} e^{-t_0 z}. \tag{6.10}$$

6.2 Lower bound in Proposition 6.1

Recall (6.2). Let $\varepsilon > 0$ be small. Consider

$$E_n := \bigcup_{k=1}^{n} B'_k,$$

with $B'_k := B_k \cap \{\eta_k(1) \le \frac{1}{\varepsilon} e^{k\psi(t_0)}\} := B_k \cap F_k$. By using Corollary 4.2

$$\sup_k e^{-k\psi(t_0)} \eta_k(1) < \infty, \qquad a.s.$$

Hence

$$\mathbb{P}\Big(M_n > \alpha n + z\Big) \ge \mathbb{P}\Big(E_n\Big) + o_\varepsilon(1),$$

with $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. By Cauchy-Schwarz' inequality,

$$\mathbb{P}\Big(E_n\Big) \geq \frac{\Big(\sum_{1\leq k\leq n} \mathbb{P}(B'_k)\Big)^2}{\sum_{1\leq k_1,k_2\leq n} \mathbb{P}(B'_{k_1} \cap B'_{k_2})}. \tag{6.11}$$

Conditioning on $\mathscr{F}_k$, B_k is an union of $\eta_k(1)$ i.i.d. events,

$$\mathbb{P}\Big(B_k\Big|\mathscr{F}_k\Big) = 1-(1-p(k,n))^{\eta_k(1)}.$$

On B'_k, $\eta_k(1) \leq e^{k\psi(t_0)}/\varepsilon$. By (6.3), $p(k,n)\eta_k(1) \leq e^{-t_0 z}/\varepsilon \to 0$ as $z \to \infty$. Then for all $z \geq z_0(\varepsilon)$, uniformly for all $k \leq n$,

$$1-(1-p(k,n))^{\eta_k(1)} \geq (1-\varepsilon)p(k,n)\eta_k(1)$$

hence

$$\mathbb{P}\Big(B'_k\Big|\mathscr{F}_k\Big) \geq (1-\varepsilon)p(k,n)\eta_k(1)1_{F_k}.$$

In particular,

$$\sum_{k=1}^n \mathbb{P}(B'_k) \geq (1-\varepsilon)\sum_{k=1}^n p(k,n)\mathbb{E}\Big(\eta_k(1)1_{F_k}\Big).$$

Recall (6.8). Since $\mathbb{E}\Big(\eta_k(1)1_{F_k}\Big) = (1+o_\varepsilon(1))\mathbb{E}\Big(\eta_k(1)\Big) = (c_1+o_\varepsilon(1))e^{k\psi(t_0)}$ as k large, we get that

$$\sum_{k=1}^n \mathbb{P}(B'_k) = (c_1+o_\varepsilon(1))\sum_{k=1}^n p(k,n)e^{k\psi(t_0)} = (c_*+o_\varepsilon(1))e^{-t_0 z},$$

where $c_* = \frac{c_1 c_+}{\widetilde{\mathbb{E}}(H_1)}$ and $o_\varepsilon(1)$ does not depend on z.
Let $k_1 < k_2$. On $B_{k_1} \cap B_{k_2}$, there are at least two different $v \neq v'$ at generation k_1 such that $A_v(k_1,n)$ holds and for v', there exists some descend u at generation k_2 such that $A_u(k_2,n)$ holds. Then,

$$B_{k_1} \cap B_{k_2} \subset \bigcup_{v\neq v', |v|=|v'|=k_1} \Big\{A_v(k_1,n) \cap \{\exists u : |u| = k_2, u > v' : A_u(k_2,n)\}\Big\}.$$

Since different particles branch independently, we get that

$$\mathbb{P}\Big(B_{k_1} \cap B_{k_2}\Big|\mathscr{F}_{k_1}\Big) \leq \sum_{v\neq v', |v|=|v'|=k_1} p(k_1,n)\mathbb{E}\Big(\sum_{|u|=k_2, u>v'} p(k_2,n)\Big|\mathscr{F}_{k_1}\Big).$$

Taking the expectations, we get that for $k_1 < k_2$,

$$\mathbb{P}\Big(B_{k_1} \cap B_{k_2}\Big) \le p(k_1,n)p(k_2,n)\mathbb{E}(\eta_{k_1}(1)\eta_{k_2}(1)) \le C\, p(k_1,n)p(k_2,n)e^{k_1\psi(t_0)+k_2\psi(t_0)},$$

by Corollary 4.2.
Therefore, we get that

$$\begin{aligned}\sum_{1\le k_1,k_2\le n} \mathbb{P}\Big(B_{k_1} \cap B_{k_2}\Big) &\le \sum_{k=1}^{n} \mathbb{P}(B'_k) + Ce^{-2t_0 z} \\ &\le (c_* + o_\varepsilon(1))e^{-t_0 z} + Ce^{-2t_0 z},\end{aligned}$$

for some constant $C > 0$. It follows that for all large $z \ge z_0(\varepsilon)$,

$$\liminf_{n\to\infty} e^{t_0 z}\mathbb{P}\Big(M_n > \alpha n + z\Big) \ge \frac{(c_*^2 + o_\varepsilon(1))e^{-t_0 z}}{(c_* + o_\varepsilon(1))e^{-t_0 z} + Ce^{-2t_0 z}}.$$

Letting $z \to \infty$ and $\varepsilon \to 0$ in order, we get that

$$\liminf_{z\to\infty}\liminf_{n\to\infty} e^{t_0 z}\mathbb{P}\Big(M_n > \alpha n + z\Big) \ge c_* o_\varepsilon(1).$$

giving the lower bound by letting $\varepsilon \to 0$. □

Proposition 6.2. *For any x,*

$$\limsup_{z\to\infty}\limsup_{n\to\infty} \left| e^{t_0 z}\mathbb{P}_x\Big(M_n > \alpha n + z\Big) - c_*\phi(x) \right| = 0\,.$$

Proof. Let $S^* = \max_{0\le i\le \tau_0} S_i$. Then

$$\begin{aligned}&\mathbb{P}_x\Big(M_n > \alpha n + z\Big) \\ =\ & O\Big(\mathbb{P}_x(S^* > \alpha n + z)\Big) + \sum_{k=1}^{n} \mathbb{P}_x(\tau = k)\mathbb{P}\Big(M_{n-k} > \alpha n + z\Big)\end{aligned}$$

Applying Proposition 6.1, we have

$$e^{t_0(z+\alpha k)}\mathbb{P}_x\Big(M_n > \alpha(n-k) + z + \alpha k\Big) \sim c_*.$$

Therefore,

$$e^{t_0 z}\mathbb{P}_x\big(M_n > \alpha n + z\big) \sim c_* \lim_{n\to+\infty} \sum_{1\le k\le n} e^{-t_0\alpha k}\mathbb{P}_x(\tau = k) \to c_* \sum_k e^{-rk}\mathbb{P}_x(\tau = k) = c_*\phi(x)\,.$$

□

6.3 Proof of Theorem 1.2

Let k be a large constant ($k \gg |y|$) and $n > k$. Conditioning on $\mathscr{F}_k$, the η_k particles move independently, hence

$$\mathbb{P}\Big(M_n > \alpha n + y\Big) = \mathbb{E}\Big(1 - \prod_{x\in\mathbb{Z}} \mathbb{P}_x\Big(M_{n-k} \le \alpha n + y\Big)^{\eta_k(x)}\Big). \tag{6.12}$$

Applying Proposition 6.2, we see that

$$\mathbb{P}_x\Big(M_{n-k} \le \alpha n + y\Big) = 1 - (c_* + o_k(1))\phi(x)e^{-\psi(t_0)k - t_0 y},$$

with $o_k(1) \to 0$ as $k \to \infty$. The product in (6.12) is in fact taken over a finite set of x (the walk has bounded jumps), hence both $\limsup_{n\to\infty} \mathbb{P}(M_n > \alpha n + y)$ and $\liminf_{n\to\infty} \mathbb{P}(M_n > \alpha n + y)$ are equal to

$$\mathbb{E}\left[1 - \exp\Big(-(c_* + o_k(1))\sum_{x\in\mathbb{Z}} \phi(x)e^{-t_0 y - \psi(t_0)}\eta_k(x)\Big)\right] = \mathbb{E}\left[1 - e^{-(c_* + o(1))e^{-t_0 y}\Lambda_k}\right],$$

where $o_k(1)$ may be different according to $\limsup_{n\to\infty}$ or $\liminf_{n\to\infty}$. But since Λ_k is a positive martingale, we have $\Lambda_k \to \Lambda_\infty$ a.s. and this implies that the above expectation converges to

$$\mathbb{E}\left[1 - e^{-c_* e^{-t_0 y}\Lambda_\infty}\right].$$

7 Extension to multiple catalysts

The set of catalysts is a finite subset $\mathscr{C}$ of $\mathbb{Z}^d$. Outside of $\mathscr{C}$ a particle performs a standard (fixed) random walk. When a particle reaches a catalyst $x \in \mathscr{C}$ it dies and gives birth to new particles according to the point process

$$\mathscr{D}_x \stackrel{d}{=} (S_1^{(i)}, 1 \le i \le N_x)$$

where $(S_n^{(i)}, n \in \mathbb{N})_{i\ge1}$ are IID random walk starting form x, independent from the random variable N_x, assumed to be square integrable. Each particle produces new particles independently from the other particles living in the system.

The underlying Galton-Watson process is obtained by forgetting/erasing the time spent between the catalysts. Let us assume first that the random walk is recurrent. Then *the Galton-Watson process is multitype* with the moment matrix

$$\begin{aligned} M_{xy} &:= \text{mean number of particles born at } x \text{ that reach site } y \\ &= m_x \mathbb{P}_x\Big(\tau = \tau_y\Big) \qquad\qquad (x, y \in \mathscr{C}), \end{aligned}$$

where $m_x = \mathbb{E}\left[N_x\right]$ is the mean offspring at site x, $\tau_y := \inf\{n \geq 1 : S_n = y\}$ is the first return time at y, and $\tau = \tau_{\mathscr{C}} = \inf_{y \in \mathscr{C}} \tau_y$ is the first return time to $\mathscr{C}$.
We assume to be in the supercritical regime, that is $\rho > 1$, where ρ is the maximal eigenvalue of matrix M, which by assumption is irreducible. We let $\rho^{(r)}$ be the maximum eigenvalue of the matrix

$$M^{(r)}_{xy} := m_x \mathbb{E}_x \left[e^{-r\tau}\, \mathbf{1}_{(\tau=\tau_y)}\right] \quad (x, y \in \mathscr{C}).$$

The function $r \to \rho^{(r)}$ is continuous, strictly decreasing, C^∞ on $(0, +\infty)$, $\rho^{(0)} = \rho > 1$ and $\lim_{r\to+\infty} \rho^{(r)} = 0$ since $M^{(r)}_{xy} \leq m_x e^{-r}$. Therefore there exists a unique $r > 0$, a *Malthusian parameter*, such that $\rho^{(r)} = 1$. We shall fix this value of r in the sequel.
Let $v = v^{(r)}$ be a right eigenvector of $M^{(r)}$ associated to $\rho^{(r)} = 1$:

$$v = M^{(r)} v \quad \text{i.e.} \quad v(x) = \sum_{a\in\mathscr{C}} m_x \mathbb{E}_x \left[e^{-r\tau}\, \mathbf{1}_{(\tau=\tau_a)}\right] v(a) \quad (x \in \mathscr{C}).$$

Let us denote by $p(x, y) = \mathbb{E}_x \left[S_1 = y\right]$ and $Pf(x) = \sum_y p(x, y) f(y)$ the random walk kernel and semigroup. Let us consider the hitting times

$$T_x := \inf\{n \geq 0 : S_n = x\}, \quad T = T_{\mathscr{C}} = \inf_{x\in\mathscr{C}} T_x = \inf\{n \geq 0 : S_n \in \mathscr{C}\}.$$

Lemma 7.1. *The function*

$$\phi(x) := \sum_{a\in\mathscr{C}} v(a) \mathbb{E}_x \left[e^{-rT}\, \mathbf{1}_{(T=T_a)}\right]$$

is a solution of

$$P\phi(x) = e^r \phi(x) \left(\frac{1}{m_x} \mathbf{1}_{(x\in\mathscr{C})} + \mathbf{1}_{(x\notin\mathscr{C})} \right).$$

Proof. Indeed, by conditioning on the first step of the random walk, if $x \notin \mathscr{C}$ then $T \geq 1$ and

$$\phi(x) = \sum_{a\in\mathscr{C}} v(a) \sum_y p(x, y) e^{-r} \mathbb{E}_y \left[e^{-rT}\, \mathbf{1}_{(T=T_a)}\right] = e^{-r} P\phi(x).$$

On the other hand, if $x \in \mathscr{C}$ then $T = T_x = 0$ and by definition of v

$$\phi(x) = v(x) = \sum_{a\in\mathscr{C}} v(a) m_x \mathbb{E}_x \left[e^{-r\tau}\, \mathbf{1}_{(\tau=\tau_a)}\right].$$

We condition again on the first step, since $\tau \geq 1$, and get

$$\phi(x) = m_x \sum_{a\in\mathscr{C}} \nu(a) \sum_y p(x,y) e^{-r} \mathbb{E}_y \left[e^{-rT} \mathbf{1}_{(T=T_a)} \right] = m_x e^{-r} P\phi(x).$$

□

We are now ready to introduce the *fundamental martingale*.

Lemma 7.2. *(1) For the CBRW process with multiple catalysts, the process*

$$\Lambda_n := e^{-rn} \sum_{|u|=n} \phi(X_u)$$

is a martingale.
(2) For the random walk, the process

$$\Delta_n := e^{-rn} \phi(S_n) \prod_{x\in\mathscr{C}} m_x^{L_{n-1}^x}$$

is a martingale where $L_{n-1}^x = \sum_{0\leq k\leq n-1} \mathbf{1}_{(S_k=x)}$ is the local time at level x at time $n-1$.
(3) The process Λ_n is bounded in L^2 and therefore a Uniformly Integrable martingale.

Proof. Let us prove (2) first.

$$\begin{aligned}
\mathbb{E}\left[\Delta_{n+1} \mid \mathscr{F}_n\right] &= e^{-r(n+1)} \prod_{x\in\mathscr{C}} m_x^{L_n^x} \mathbb{E}\left[\phi(S_{n+1}) \mid \mathscr{F}_n\right] \\
&= e^{-r(n+1)} \prod_{x\in\mathscr{C}} m_x^{L_n^x} P\phi(S_n) \\
&= e^{-r(n+1)} \prod_{x\in\mathscr{C}} m_x^{L_n^x} e^r \phi(S_n) \left(\frac{1}{m_{S_n}} \mathbf{1}_{(S_n\in\mathscr{C})} + \mathbf{1}_{(S_n\notin\mathscr{C})} \right) \\
&= \Delta_n\,.
\end{aligned}$$

To establish (1), we shall need the many to one lemma. Since

$$e^{\psi_x(0)} = \mathbb{E}_x \left[\sum_{|u|=1} 1 \right] = m_x \mathbf{1}_{(x\in\mathscr{C})} + \mathbf{1}_{(x\notin\mathscr{C})} = \prod_{a\in\mathscr{C}} m_a^{\mathbf{1}_{(x=a)}}$$

we have for $|u| = n$

$$e^{A_0(u)} = \prod_{v<u} e^{\psi_{X_v}(0)} = \prod_{v<u} \prod_{a\in\mathscr{C}} m_a^{\mathbf{1}_{(X_v=a)}} = \prod_{a\in\mathscr{C}} m_a^{\sum_{v<u} \mathbf{1}_{(X_v=a)}}\,.$$

Therefore, if Z is $\mathscr{F}_{n-1}$ measurable positive or bounded,

$$\begin{aligned}
\mathbb{E}\left[\Lambda_n Z\right] &= e^{-rn}\mathbb{E}\left[\sum_{|u|=n}\phi(X_u)Z\right] \\
&= e^{-rn}\mathbb{Q}\Big(Z\phi(S_n)e^{A_0(\xi_n)}\Big) && \text{many to one lemma} \\
&= e^{-rn}\mathbb{E}\left[Z\phi(S_n)\prod_{x\in\mathscr{C}} m_x^{L_{n-1}^x}\right] \\
&= \mathbb{E}\left[Z\Delta_n\right] \\
&= \mathbb{E}\left[Z\Delta_{n-1}\right] && \text{since } \Delta_n \text{ is a martingale} \\
&= \mathbb{E}\left[Z\Lambda_{n-1}\right].
\end{aligned}$$

(3) To compute second moments, we use the many to two formula of section 3

$$\begin{aligned}
\mathbb{E}\left[\Lambda_n^2\right] &= e^{-2rn}\mathbb{E}\left[\sum_{|u|=|v|=n}\phi(X_u)\phi(X_v)\right] \\
&= e^{-2rn}\mathbb{Q}^2\left(\phi(S_n^1)\phi(S_n^2)e^{A_0(\xi_n^1,\xi_n^2)}\right).
\end{aligned}$$

We let $T=\inf\left\{n\geq 1 : S_n^1\neq S_n^2\right\}$ be the splitting time of the coupled random walks, and $m_{x,p} := \mathbb{E}\left[N_x^p\right]\mathbf{1}_{(x\in\mathscr{C})}+\mathbf{1}_{(x\notin\mathscr{C})}$. Since

$$e^{A_0(\xi_n^1,\xi_n^2)} = \prod_{0\leq l\leq (T-1)\wedge n-1} m_{S_l^1,2}\prod_{T-1<l\leq n-1} m_{S_l^1,1}m_{S_l^2,1}$$

and

$$\mathbb{Q}^2\left(T\geq n+1\mid \mathscr{F}_n\right) = \prod_{0\leq l\leq n-1}\frac{m_{S_l^1,1}}{m_{S_l^2,2}},$$

we have

$$\begin{aligned}
\mathbb{E}\left[\Lambda_n^2\right] &= e^{-2rn}\mathbb{Q}^2\left(\phi(S_n^1)\phi(S_n^2)e^{A_0(\xi_n^1,\xi_n^2)}\mathbf{1}_{(T\geq n)}\right)+ \\
&\quad + e^{-2rn}\sum_{1\leq k\leq n-1}\mathbb{Q}^2\left(\phi(S_n^1)\phi(S_n^2)e^{A_0(\xi_n^1,\xi_n^2)}\mathbf{1}_{(T=k)}\right) \\
&= e^{-2rn}\mathbb{Q}^2\left(\phi(S_n^1)^2\prod_{x\in\mathscr{C}} m_x^{L_{n-1}^x}\right) \\
&\quad + e^{-2rn}\sum_{1\leq k\leq n-1}\mathbb{Q}^2\left(\prod_{0\leq l\leq k-2}\frac{m_{S_l^1,1}}{m_{S_l^2,2}}\left(1-\frac{m_{S_{k-1}^1,1}}{m_{S_{k-1}^2,2}}\right)\mathbb{E}_{S_{k-1}^1}\left[\Delta_{n-(k-1)}\right]^2 e^{2r(n-(k-1))}\right).
\end{aligned}$$

Observe that since $0 \leq \phi \leq 1$ we have $\phi(x)^2 \leq \phi(x)$, and combine it with $\mathbb{E}_x\left[\Delta_p\right] = \phi(x)m_{x,1} \leq c_1$ to obtain the upper bound

$$\begin{aligned}\mathbb{E}\left[\Lambda_n^2\right] &\leq 1 + c_2 \sum_{1\leq k\leq n-1} e^{-r(k-1)}\mathbb{Q}^2\left(\phi^2(S_{k-1}^1)\prod_{x\in\mathscr{C}} m_x^{L_{k-2}^x}\right) \\ &\leq c_3(1+\sum_{k\geq 1} e^{-r(k-1)}) = c_4 < +\infty.\end{aligned}$$

□

When the random walk is transient, we have a continuity defect at zero. Hence we modify the definition of M

$$M_{x,y} := \lim_{r\downarrow 0} M_{xy}^{(r)} = m_x \mathbb{P}_x\left(\tau = \tau_y, \tau < +\infty\right).$$

The rest of the proof then goes unchanged.

We are now able to give an explanation of the supercritical regime assumption of the introduction.

Lemma 7.3. *When there is only one catalyst at the origin, the supercritical regime is $m(1-q_{esc}) > 1$.*

Proof. Indeed, M is then a one dimensional matrix and

$$\rho = M_{00} = m\mathbb{P}(\tau <= \infty) = m(1-q_{esc}).$$

□

We end this section by stating the law of large numbers. Intuitively, if c is the rightmost catalyst, the maximal position at time n comes from particles born to c.

Proposition 7.4 (Law of large numbers)**.** *On the set of non extinction $\mathscr{S}$ we have*

$$\lim_{n\to+\infty} \frac{M_n}{n} = \alpha \quad a.s.$$

with $\alpha = \frac{r}{t_0}$, r the Malthusian parameter and $t_0 > 0$ defined by $\psi(t_0) = r$.

Proof. First observe that the heuristics does not changes at all since by applying the optional stopping theorem to the martingale $e^{t_0 S_n - nr}$ to the time T_c, we obtain that

$$e^{t_0 x} = e^{t_0 c}\mathbb{E}_x\left[e^{-rT}\right].$$

Therefore, for $x > c$,

$$\phi(x) = \nu(c)\mathbb{E}_x\left[e^{-rT_c}\right] = \nu(c)e^{t_0(x-c)},$$

and we compute the expected number of particles above level an in the same way, and hence obtain the same guess for the asymptotics.
Furthermore, the proofs are *mutatis mutandis* the same as the one given in section 5. □

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