Lamb shift of non-degenerate energy level systems placed between two infinite parallel conducting plates

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Abstract

The issue of the observability of the Lamb shift in systems with non-degenerate energy levels is put to question. To this end, we compute the Lamb shift of such systems in the electromagnetic environment provided by two infinite parallel conducting plates, which is instrumental in demonstrating the existence of the so-called Casimir effect. A formula giving the relative change in the Lamb shift (as compared to the standard one in vacuum) is explicitly obtained for spherical semiconductor Quantum Dots (QD). It suggests a possibility of QD non-degenerate energy spectrum fine-tuning for experimental purposes as well as a *Gedankenexperiment* to observe the Lamb shift in spherical semiconductor quantum dots.

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1 Introduction

Both Lamb and Casimir effects were discovered or predicted in the late 1940s [1, 2], and have been actively studied ever since [3, 4]. These two phenomena are the most striking effects in standard Quantum Electrodynamics (QED) for they provide the strongest experimental support to the quantization of the electromagnetic field.

In free space, the ground state of the quantized electromagnetic field is not just a empty state. It is the siege of quantum fluctuations which contribute to the so-called zero-point energy and lead to observable effects. In hydrogen-like atoms, the s-levels are the only ones getting dressed by the electromagnetic field quantum fluctuations and thereby are split from other states, which should have been degenerate with them according to Dirac theory, giving rise to the Lamb shift [5, 6].

Quantum fluctuations also induce a Casimir effect. In general, the summation of the zero-point energy fluctuations of the electromagnetic field yields a divergent ground state energy. In the absence of coupling with gravity, this divergence is usually subtracted off in an additive renormalization scheme. However, a careful analysis on its volume dependence reveals the occurrence of a finite and observable force, known as Casimir force. The modification of the electromagnetic field boundary conditions in a fixed spatial volume, for example by the presence of two parallel perfectly conducting plates, should manifest itself as an attractive force between these plates [7]. Despite a previous (but not totally satisfactory) attempt at detecting the Casimir force [8], an experimental convincing proof of its existence has been only brought to light in the late 1990s for separation distances in the range of respectively 0.6-6μm [9] and 0.1-0.9μm [10]. In these experiments, one plate has been replaced by a perfectly conducting sphere. In addition of enhancing the amplitude of the Casimir force by a geometrical factor π, the replacement of one of the plates by a sphere allows to weaken the boundary effects due to the finite size of the plates, and thereby have facilitated its observation.

In the context of Cavity Quantum Electrodynamics (CQED) [11, 12], our recent investigation on cavity-induced effects on atomic radiative properties of quantum dots (QDs) has led us to wonder how much the Lamb shift of a system would change if put inside a *Casimir device*, an environment provided by two parallel conducting plates. The change of the atomic Lamb shift induced by a modification of the zero-point energy due to new electromagnetic field boundary conditions has been studied in [13]. It has been also shown how hermitian conditions are sufficient to separate the contributions of vacuum fluctuations from those of self-radiation reaction to the energy level shifts [14]. The coupling of an electric dipole to a conducting surface through absorption and emission of its own radiative field is a well-known problem from a classical as well as from a quantum-mechanical point of view [15, 16]. However, the understanding of the respective role of vacuum fluctuations and radiative field on the energy level shifts between two parallel plates seems to remain ambiguous [17].

In this paper, to pursue investigations on the analogy between semiconductor QDs and real atoms, we propose to sketch a protocol based on the Lamb shift due to the modification of the zero-point energy to observe the atomic-like Lamb shift in spherical semiconductor QDs, established recently in [18]. This issue arises from the fact that, exception made of the azimuthal quantum number degeneracy for obvious symmetry reasons, the energy levels of an electron-hole pair confined in a spherical semiconductor QD are non-degenerate [19]. Then, the observability of Lamb shift in QDs is a conceptual problem, because the s- and p-levels are both shifted by the Lamb effect, as opposed to real atoms. This may explain the lack for theoretical works addressing the Lamb effect in such confinement structures. To this end, in section **2**, a formula expressing the change in Lamb shift for a system placed in a Casimir device is established by a careful mathematical treatment. Then in the following section **3**, we apply the obtained results to the case of a spherical semiconductor QD, described in the formalism of the effective mass approximation (EMA) [20, 21]. The explicit dependence of additional Lamb shift on the separation distance of the plates in the Casimir device suggests a convenient way to modulate the QD energy spectrum when needed in experimental contexts. We then propose a *Gedankenexperiment* to discuss the possibility of observing the Lamb shift in semiconductor QDs, at least in a so-called strong confinement regime and for a judiciously chosen semiconductor. A concluding section summarizes our main results.

2 Modification of the Lamb shift between Casimir plates

Let us consider two parallel perfectly conducting squared plates of linear size L placed at a separation distance $d \ll L$ in vacuum. Intuitively, because the zero-point energy fluctuations are more important outside than inside the volume defined by the plates, a Casimir effect arises as an attractive force between them [2]. According to the phenomenological Welton argument for atoms [6], the energy level Lamb shift may be viewed as due to the particle position fluctuations induced by the zero-point energy of the unrestricted surrounding electromagnetic field. Thus, when this surrounding electromagnetic environment is changed to that of a Casimir device, one should expect that the Lamb shift takes a different form and value for a given quantum state specified by a same set of quantum numbers. We argue that this difference in Lamb shifts may be exploited to make the Lamb shift observable in systems with non-degenerate energy-levels.

The Lamb shift in real atoms placed in a Casimir device has been calculated with the Bethe approach for a relativistic electron in [13], as well as for a non-relativistic electronic motion in [17]. They both predict an additional shift to the standard Lamb shift, which depends on the separation distance d between the plates and goes to zero in the limit of $d \to \infty$. One can reasonably expect a similar result in the Welton approach.

However, there is a discrepancy in the leading contribution to the additional Lamb shift in these results [13, 17]. For hydrogen-like atoms, it is shown in [13] that this additional shift is inversely proportional to the separation distance d, and that it is the sum of a non-relativitic contribution, scaling as the atomic Rydberg energy E^*, and of a relativistic correction inversely proportional to the electronic bare mass m_e — in units with $\hbar = c = 1$. In the non-relativistic limit, where the typical binding energy of theelectron to the nucleus is negligible against its typical rest energy, *i.e.* if $E^* \ll m_\mathrm{e}$, only the non-relativistic contribution survives, so that the additional shift finally scales as $\propto \frac{E^*}{d}$. This is in flagrant contradiction with the result of [17], which predicts, in the non-relativistic limit, that the leading contribution scales as $\propto \frac{1}{d^2}$. In this work, we succeed to remove this discrepancy and show that the correct behavior goes as $\propto \frac{1}{d^2}$, bringing to light the reasoning mistake made in [13].

2.1 Statement of the problem and assumptions

Let us consider a theoretical non-relativistic spinless particle of mass m^* and of charge $\pm e$, described by an Hamiltonian H_0, which is diagonal in some orthonormal basis of its eigenvectors $\{|\mathbf{n}\rangle\}_\mathbf{n}$ with energy eigenvalues $\{E_\mathbf{n}\}_\mathbf{n}$. The interaction of this particle with a quantized electromagnetic field is given by the Pauli-Fierz Hamiltonian in the Coulomb gauge [22]. In this section, we will make use of a convenient way to retrieve the standard Casimir force between the plates, which is inspired by [23] but formulated and handled in the framework of distribution theory, in order to deduce the density of states modified by the Casimir device. The detailed and involved mathematical steps are presented in the two appendices **A** and **B**. This modification of the density of states induces the expected modification of the Lamb shift undergone by the energy levels of the considered particle.

To perform such calculations, it is only needed to evaluate the energy shift induced by the vacuum fluctuations, the effect of the particle self-radiation reflected by the plates being neglected. To this end, we choose a regime in which the separation distance d is sufficiently small to allow the emergence of the Casimir effect between the plates, *i.e.* d should be at most of the order of magnitude of μm [10]. Furthermore, we will assume that the typical wavelength, scaling as $\propto d$, due to the confinement between the Casimir plates should be greater than the order of magnitude of the wavelength associated with an authorized radiative transition between two energy levels of the particle. In such a weak coupling regime, the coupling of a two-level quantum atom with itself through absorption and emission of dipolar radiation reflected by the Casimir plates is dominated by the coupling of the two-level atom with the electromagnetic field vacuum fluctuations [15, 16]. In a hydrogen-like atom, the associated Rydberg energy E^* typically characterizes the radiative transitions, their wavelength scaling as $\propto \frac{1}{E^*}$. Then, the weak coupling regime is fulfilled if we impose that

$$\frac{\kappa_d}{2\pi} < \kappa^*, \tag{1}$$

where $\kappa_d = \frac{\pi}{d}$ is the ground state energy in the presence of the Casimir plates. The quantity $\kappa^* \propto E^*$, being the usual Bethe average excitation energy used as the IR cut-off for the Lamb effect [5, 6], is surprisingly higher than the associated Rydberg energy E^*, and though represents the maximal excitation energy the atom could access [24].

The coupling of the quantum particle placed between the Casimir plates with its own radiation field shall be also neglected when applied to semiconductor QDs. The validity of the condition Eq. (1) in spherical semiconductor QDs will be dealt in more details in section **3**. As we shall see in this particular case, the Bethe cut-off κ^* is also proportional but is of the same order of magnitude than the typical QD radiative transition energy. It is then also the natural candidate to depict the characteristic properties of the quantum system under study, as far as Lamb effect is concerned. Contrary to real atoms, wavefunctions of an electron-hole pair confined in a QD, described in the standard effective mass approximation (EMA), are restricted to the region of space defined by the QD boundary surface [19, 20]. Then the probability for the electron, and to a lesser extent for the hole, to tunnel from the inside part of the QD to its outside surrounding is vanishingly small. Actually, even if the confinement potential exerted on the electron-hole trapped in the semiconductor QD is more appropriately represented by a finite potential step [25] instead of an infinite potential well, this assumption remains reasonable, since the tunnel effect probability remains exponentially small. Thus, by construction, the probability of interaction between a photon emitted by the QD inside the Casimir plates and the QD itself is negligible in comparison to the probability of its interaction with an excitation of the surrounding quantized electromagnetic field.

2.2 Standard Casimir effect

As described in [23], the limit of perfectly conducting plates allows to consider the Casimir effect as a manifestation only of the electromagnetic field vacuum fluctuations by uncoupling non-ambiguously the zero-point energy of the electromagnetic field from the surrounding matter. The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are now supposed to satisfy the continuity boundary conditions on the plates

$$\mathbf{E}_{\parallel} = \mathbf{B}_{\perp} = \mathbf{0},$$

where the indices "$\parallel$" and "$\perp$" stand for the tangential and orthogonal components of the fields with respect to the plates. A convenient way to describe the Casimir effect consists in considering that the rectangular box $\mathcal{B}$ of volume $V = L^2 d$, built from the Casimir plates, is a waveguide in the direction orthogonal to the plates, which shall be noted z-direction from now on. Then, the set of TM and TE modes of the electromagnetic field is a natural functional basis[1]. This means that, except for modes for which $n_\perp = 0$, each mode of wave-number $k_{\mathbf{n}_\parallel n_\perp} = \sqrt{\mathbf{k}_\parallel^2 + (\kappa_d n_\perp)^2}$ of quantum numbers $(\mathbf{n}_\parallel, n_\perp) \in \mathbb{Z}^2 \times \mathbb{N}^*$ — the tangential wave vector is $\mathbf{k}_\parallel = \frac{2\pi}{L}\mathbf{n}_\parallel$ and the orthogonal wave vector $k_\perp = \kappa_d n_\perp$ is a natural cut-off wave number due to the finite size of the waveguide in the z-direction — has two possible polarizations. If periodic boundary conditions along the plates are also imposed, the zero-point energy of such electromagnetic field is then given by the divergent series

$$E(L,d) = \sum_{(\mathbf{n}_\parallel, n_\perp) \in \mathbb{Z}^2 \times \mathbb{N}^*} k_{\mathbf{n}_\parallel n_\perp} + \sum_{\mathbf{n}_\parallel \in \mathbb{Z}^2} \frac{k_{\mathbf{n}_\parallel 0_\perp}}{2} = \sum_{(\mathbf{n}_\parallel, n_\perp) \in \mathbb{Z}^2 \times \mathbb{Z}} \frac{k_{\mathbf{n}_\parallel n_\perp}}{2}.$$

Reference [23] prescribes how it should be regularized. Recalling that even if the plates are supposed to be perfectly conducting, any conducting material is transparent to radiation at sufficiently high frequencies. Then, modes of arbitrary high frequencies do not actually contribute to the Casimir force between the plates. This can be described by introducing a dimensionless cut-off function $k \longmapsto \phi\left(\frac{k}{\kappa_\phi}\right)$, where κ_ϕ is a UV cut-off, in the expression of the zero-point energy $E(L,d)$, as follows

$$E_\phi(L,d) = \sum_{(\mathbf{n}_\parallel, n_\perp) \in \mathbb{Z}^2 \times \mathbb{Z}} \frac{k_{\mathbf{n}_\parallel n_\perp}}{2} \phi\left(\frac{k_{\mathbf{n}_\parallel n_\perp}}{\kappa_\phi}\right). \tag{2}$$

The function ϕ is assumed to be positive, without loss of generality, and should verify $f(0) = 1$, so that, in the limit of a perfect conductor, *i.e.* $k \ll \kappa_\phi$, each term appearing in the sum defining the regularized $E_\phi(L,d)$ goes to the term with the same quantum numbers of the sum defining non-regularized $E(L,d)$. It is very useful to suppose moreover that $\phi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}$. This assumption allows us to rigourously justify our calculation in the distribution sense, based on a generalized and corrected version of the one given in [13]. The details of the calculations are given in appendix **B**, and lead to the following expression of the zero-point energy of the electromagnetic field in presence of the Casimir device

$$E_\phi(L,d) = V \int_{\mathbb{R}_+} \mathrm{d}k\, \frac{k^3}{2\pi^2} \phi\left(\frac{k}{\kappa_\phi}\right) + V \int_{\mathbb{R}_+} \mathrm{d}k\, \frac{\kappa_d k^2}{2\pi^3} g\left(\frac{k}{\kappa_d}\right) \phi\left(\frac{k}{\kappa_\phi}\right), \tag{3}$$

the standard Casimir energy between the two plates being

$$E^\phi_{\text{Casimir}}(L,d) = -\frac{\pi^2}{720} \frac{L^2}{d^3},$$

where the expression of the function $g\colon\ s \longmapsto \arctan\tan\pi(\frac{1}{2} - s)\chi_{\mathbb{R}\setminus\mathbb{Z}}(s)$ is proved in appendix **A**, and where the function $\chi_{\mathbb{A}}\colon\ x \longmapsto \begin{cases} 1 & \text{if } x \in \mathbb{A} \\ 0 & \text{otherwise} \end{cases}$ is the so-called characteristic function of the subset $A \subseteq \mathbb{R}$. This calculation of the zero-point energy of the electromagnetic field shows how its density of states is modified by the presence of the plates. This modification can be viewed as a perturbation on the case of free-space, *i.e.* without Casimir plates. More precisely, the first term of Eq. (3) consists of the well-known contribution of the zero-point energy of the electromagnetic field in a region of free-space of volume V, because of its characteristic behavior as the third power of the mode eigenenergy k. The second term, which is the only one depending on the separation distance d, is then the contribution due to the presence of the

[1]For more details, one can refer to sections **8.1** and **8.2** of [26]. In particular, there also exists a TEM mode, which is non-trivial only if the waveguide section is simply connected.

plates, denoted by $E^{\phi}_{\text{Casimir}}(L,d)$. In particular, it is actually, as expected, independent from the regularization function ϕ in the limit $\frac{\kappa_d}{\kappa_\phi} \to 0$. This is coherent with the fact that the Casimir energy is a physical quantity, as opposed to the first term, which is explicitly regularized by it. Therefore, the second term of Eq. (3) suggests that the modification of the zero-point energy due to the Casimir device can be indeed entirely determined by a correction to the standard density of states $\rho(k) = \frac{k^2}{2\pi^2}$ in free-space [33], defined by

$$\rho_d(k) = \frac{\kappa_d k}{2\pi^3} g\left(\frac{k}{\kappa_d}\right). \tag{4}$$

We insist on the fact that this function is not strictly a density, since it is not positive. However, this should be considered as a perturbation to the case of the absence of the plates in the sense that, for a fixed mode of energy $k > 0$ and in the limit of infinite separation distance between the plates, the correction ρ_d vanishes, and the free-space density of states ρ is recovered in this limit

$$\left|\frac{\rho_d(k)}{\rho(k)}\right| = \frac{1}{\pi}\frac{\kappa_d}{k}\left|g\left(\frac{k}{\kappa_d}\right)\right| \leq \frac{\kappa_d}{2k} \xrightarrow[d\to\infty]{} 0.$$

This insures that the usual free-space properties should be retrieved in the limit $d \to \infty$. This is the reason why we will refer to the function ρ_d as a density by language abuse. Furthermore, it is shown in appendix **B** that the computation is more general, and applies to any function F of the eigenmode energy k, when its mean value between the Casimir plates is to be evaluated. The total density of states between the two parallel plates is then the sum of the density of states ρ and a perturbation coming from the density of states ρ_d, due to the plates. Therefore, to study the modification of a physical quantity due to the presence of the Casimir plates, we shall focus on the contribution due to the correction ρ_d, given formally by the relation

$$\int \mathrm{d}k\, F(k)\rho_d(k)\phi\left(\frac{k}{\kappa_\phi}\right) = \frac{\kappa_d^{\gamma+2}}{2\pi^3}\int \mathrm{d}s\, g(s) f^\beta(s) f_\gamma(s)\phi\left(\frac{\kappa_d}{\kappa_\phi}s\right),$$

the free-space contribution of the standard density of states ρ being set aside. The functions f^β and f_γ, as well as the indices β and γ are defined in appendix **B**, intuitively they respectively contain the IR divergent part, to be appropriately regularized by appropriate physical arguments, and the regular part of the function F under study. This argumentation will allow to directly identify the modification of the Lamb shift coming from the presence of the plates in both Bethe [5] and Welton [6] approaches, as discussed in the two following subsections.

2.3 Bethe et Welton approaches

The Bethe approach to the Lamb effect is purely pertubative. The quantum second order time independent degenerate perturbation theory is applied to the Pauli-Fierz Hamiltonian H_{PF}, where the electromagnetic field is treated in the weak field limit [5]. Using renormalization arguments, in free-space, the Lamb shift undergone by any energy eigenstate $|\mathbf{n}\rangle$ of the quantum system is found to be

$$\Delta E_{\mathbf{n}} = \frac{\alpha}{3\pi}\frac{q^2}{m^{*2}}\log\frac{m^*}{\kappa^*}\langle\mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle, \tag{5}$$

where m^* is used as a natural UV cut-off, since the assumption of non-relativistic particle is assumed here. Historically, this predicts a Lamb shift for the hydrogen atom $2s$-level, which is in excellent agreement with experimental values [5, 24].

The Welton approach is also a perturbative approach, but is more phenomenological. It has the merit of giving a physical picture of the origin of the Lamb effect. More precisely, the Lamb shift is interpreted as a fluctuation effect on the particle position due to its interaction with the surrounding electromagnetic field. These fluctuations $\Delta\mathbf{r}$ can be described as a continuous random variable, whose probability density is a three-dimensional centered isotropic Gaussian distribution of variance $\langle(\Delta\mathbf{r})^2\rangle = \frac{2\alpha}{\pi}\frac{q^2}{m^{*2}}\log\frac{m^*}{\kappa^*}$, where κ^* is the Bethe IR cut-off, and m^* is used as a natural UV cut-off consistent with non-relativistic assumption, discarding fluctuation modes of order of the particle Compton wavelength [6]. The particle then moves in a new effective potential $\langle V(\mathbf{r}+\Delta\mathbf{r})\rangle$, averaged on the fluctuation distribution, whose first corrective term $\Delta V(\mathbf{r})$ in the fine structure constant α is precisely the term giving rise to the Lamb shift

$$\Delta E_{\mathbf{n}} = \int \mathrm{d}^3\mathbf{r}\,|\langle\mathbf{r}|\mathbf{n}\rangle|^2\,\Delta V(\mathbf{r}) = \frac{\alpha}{3\pi}\frac{q^2}{m^{*2}}\log\frac{m^*}{\kappa^*}\langle\mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle.$$

2.4 Modification of the Lamb shift

Eq (5) is a regularized version of Eq. (5) of Bethe original paper [5], written in the formalism we have introduced previously. In the non-relativistic limit $\kappa^* \ll m^*$

$$\Delta E_{\mathbf{n}} = \frac{2\pi\alpha}{3}\frac{q^2}{m^{*2}}\langle\mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle\int_{\kappa^*}^{m^*}\frac{\mathrm{d}k}{k^3}\rho(k).$$

Therefore, invoking the remarks we have made in subsection **2.2**, the correction to the Lamb shift due to the presence of the Casimir device should be evaluated by replacing the density of states ρ in absence of the device by the corrective term ρ_d, and the divergent integral is regularized following the prescriptions of appendix **B** with the test function $\phi \in \mathcal{S}(\mathbb{R})$. By direct identification, we recognize that $f^\beta(s) = \frac{1}{(\kappa_d s)^2+\kappa^{*2}}$, $f_\gamma = 1$, $\beta = 2$ and $\gamma = 0$. Theorem **1**, but not theorem **2**, applies, and particularly a IR cut-off is needed here $\eta = \frac{\kappa^*}{\kappa_d} > \frac{1}{2\pi}$

$$\begin{aligned}\Delta E^{\mathbf{n}}_{\text{Casimir}}(d) &= \frac{2\pi\alpha}{3}\frac{q^2}{m^{*2}}\langle\mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle\int_{\mathbb{R}_+}\frac{\mathrm{d}k}{k}\frac{\rho_d(k)}{k^2+\kappa^{*2}}\phi\Big(\frac{k}{m^*}\Big)\\ &= \frac{\alpha}{3\pi^2}\frac{q^2}{m^{*2}}\langle\mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle\int_{\mathbb{R}_+}\mathrm{d}s\,\frac{(g\chi_{\mathbb{R}_+})(s)}{s^2+\eta^2}\phi\Big(\frac{\kappa_d}{m^*}s\Big).\end{aligned}$$

This reasoning is a shortcut to adapt Bethe or Welton original arguments to this new framework. Let us insist on the fact that both methods are equivalent and give the same results. In particular, they prescribe the same way to regularized the IR divergence of the previous integral.

In the non-relativistic limit, the scale cut-off κ_ϕ controlling the UV divergence should be appropriately chosen to be the particle mass $m^* \gg \kappa^* > \frac{\kappa_d}{2\pi}$. This integral should be seen as the action of the distribution $g\chi_{\mathbb{R}_+} \in \mathcal{S}'(\mathbb{R})$ on the test function $s \longmapsto (s^2+\eta^2)^{-1}\phi\big(\frac{\kappa_d}{m^*}s\big) \in \mathcal{S}(\mathbb{R})$. We remark that the action of its derivative $(g\chi_{\mathbb{R}_+})' \in \mathcal{S}'(\mathbb{R})$ is simpler. Denoting Φ_η: $s \longmapsto \Phi_\eta(s) = \int_s^\infty \mathrm{d}t\,(t^2+\eta^2)^{-1}\phi\big(\frac{\kappa_d}{m^*}t\big) \in \mathcal{C}^\infty(\mathbb{R})$, the problem is that $\Phi_\eta \notin \mathcal{S}(\mathbb{R})$. However, because the Schwartz class $\mathcal{S}(\mathbb{R})$ is dense in the set $\mathcal{B}(\mathbb{R})$ of continuously bounded functions on $\mathbb{R}$, and by continuity of the action of tempered distributions, it is common to extend the action of any element of $\mathcal{S}'(\mathbb{R})$ on $\mathcal{S}(\mathbb{R})$ to any element of $\mathcal{B}(\mathbb{R})$, such that the action of $(g\chi_{\mathbb{R}_+})' \in \mathcal{S}'(\mathbb{R})$ on $\Phi_\eta \in \mathcal{B}(\mathbb{R})$ has a meaning, and we get

$$\Delta E^{\mathbf{n}}_{\text{Casimir}}(d) = \frac{\alpha}{3\pi}\frac{q^2}{m^{*2}}\langle\mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle\left[\sum_{p\in\mathbb{N}^*}\Phi_\eta(p) + \frac{\Phi_\eta(0)}{2} - \int_{\mathbb{R}_+}\mathrm{d}s\,\Phi_\eta(s)\right].$$

Since $\Phi_\eta \in \mathcal{C}^\infty(\mathbb{R})$, the Euler-Maclaurin formula can be applied to the previous expression for any $r \in \mathbb{N}^*$. We deduce that $\Phi'_\eta(0) = -\frac{1}{\eta^2}$, and $\Phi^{(2k-1)}_\eta(0) = \frac{(-1)^k}{\eta^{2k}} + O\big(\frac{\kappa_d}{m^*}\big)$, for any $k \geq 2$, by performing the Taylor expansion of the function Φ'_η, and for any $r \in \mathbb{N}^*$, the remainder satisfies

$$\left|\int_{\mathbb{R}_+}\mathrm{d}t\,\frac{\widetilde{B}_{2r}(t)}{(2r)!}\Phi^{(2r)}_\eta(t)\right| \leq \frac{|b_{2r}|}{\eta^{2r}(2r)!}\int_{\mathbb{R}}\mathrm{d}s\left|\frac{\mathrm{d}^{2r-1}}{\mathrm{d}s^{2r-1}}\frac{\phi\left(\frac{\kappa^*}{m^*}s\right)}{s^2+1}\right| = \frac{2\zeta(2r)}{(2\pi\eta)^{2r}}C_{2r-1} = O\Big(\frac{1}{\eta^{2r}}\Big),$$

where the dimensionless constant $C_r = \int_{\mathbb{R}_+} \mathrm{d}s \left| \frac{\mathrm{d}^r}{\mathrm{d}s^r}(s^2+1)^{-1}\phi\left(\frac{\kappa^*}{m^*}s\right)\right| = \int_{\mathbb{R}_+} \mathrm{d}s \left| \frac{\mathrm{d}^r}{\mathrm{d}s^r}(s^2+1)^{-1}\right| \phi\left(\frac{\kappa^*}{m^*}s\right) + O\left(\frac{\kappa^*}{m^*}\right)$ is independent from η, and where the asymptotic behavior of the Bernoulli numbers $|b_{2r}| = 2\frac{(2r)!}{(2\pi)^{2r}}\zeta(2r)$ is used, ζ being the Riemann-Zeta function[2]. This implies that, in the meaning of asymptotic series[3], and after having taken the limits $\frac{\kappa^*}{m^*} \to 0$, which is well defined since $\phi \in \mathcal{S}(\mathbb{R})$ and $s \longmapsto \left|\frac{\mathrm{d}^r}{\mathrm{d}s^r}(s^2+1)^{-1}\right| \in \mathcal{L}^1(\mathbb{R}_+, \mathrm{d}s) = \{h : \mathbb{R}_+ \longrightarrow \mathbb{R} \,/\, \int_{\mathbb{R}_+} \mathrm{d}s\, |h(s)| < \infty\}$,

$$\Delta E^{\mathbf{n}}_{\text{Casimir}}(d) = \frac{\alpha}{3\pi}\frac{q^2}{m^{*2}} \langle \mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle \sum_{k\in\mathbb{N}^*} \frac{b_{2k}}{(2k)!}\frac{(-1)^k}{\eta^{2k}} = \frac{\alpha}{3\pi}\frac{q^2}{m^{*2}} \langle \mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle \left[\frac{1}{12\eta^2} + O\left(\frac{1}{\eta^4}\right)\right]$$
$$= \frac{\alpha}{3\pi}\frac{q^2}{m^{*2}} \langle \mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle \left[1 - \frac{\cot\frac{1}{2\eta}}{2\eta}\right],$$

where Eq. **1.411**.7 p. 41 [32] is used, because $\frac{1}{2\eta} < \pi$ by definition. Finally, when $\langle \mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle \neq 0$, *i.e.* when the energy level under study actually undergoes the Lamb effect, we deduce the relative modification to the Lamb shift due to the Casimir plates

$$\frac{\Delta E^{\mathbf{n}}_{\text{Casimir}}(d)}{\Delta E_{\mathbf{n}}} = \left[1 - \frac{\frac{\kappa_d}{2\kappa^*}}{\tan\frac{\kappa_d}{2\kappa^*}}\right] \log^{-1}\frac{m^*}{\kappa^*} \geq 0. \tag{6}$$

Because of the positive sign, this always actually gives rise to a enhancement of the Lamb shift undergone by the particle. This result calls for a physical explanation. When the separation distance d decreases, the amplitude of the electromagnetic modes inside the Casimir plates increases, while their number is fixed. This leads to the reinforcement of the interaction of the quantum system with the quantized electromagnetic field, implying a strengthening of the Lamb effect. Moreover, this relative enhancement does not depend on the quantum state under consideration. However, it shows an explicit concurrence between the different scales of energies $\frac{\kappa_d}{2\pi} < \kappa^* \ll m^*$, and then of characteristic lengths of the problem, as we shall see in the next section. This re-summed expression is in agreement with [17], because up to the smallest order in the dimensionless IR cut-off η, the correction $\Delta E^{\mathbf{n}}_{\text{Casimir}}(d)$ scales as $\propto \frac{1}{\eta^2} \propto \frac{1}{d^2}$. Moreover, contrary to this reference, in the regime where the quantum system does not interact with its own radiative field, our approach allows to explicitly compute the proportionality factor. In particular, it has been possible to factorize the mean value of the Laplace operator of the potential $\langle \mathbf{n}|\nabla^2 V(\mathbf{r})|\mathbf{n}\rangle$, which is considered as a characteristic of the Lamb effect according to Welton approach.

3 Observability of the Lamb shift in spherical semiconductor QDs

An experimental protocol, according to which the energy levels dressed by the zero-point fluctuations energy with or without the Casimir plates are compared, should allow to overcome the need of degenerate energy levels, or of exactly computed energy levels. As known, the experimental observability of the Lamb shift in hydrogen-like atom is due to the s- and p-level degeneracy, when the principal quantum number is $n \geq 2$, in the absence of interaction with

[2]For more details on Euler-Maclaurin formula, see appendix **A.2**.

[3]Asymptotic series consist in a Taylor expansion at all orders, which do not converge in the meaning of power series. This is the typical kind of objets that Quantum Electrodynamics allows us to access. In our case, the successive derivative will produce factors of the order at least of $(2r)!$, which dominate the geometric dependence in $\frac{1}{\eta^{2r}}$ in the remainder of Euler-Maclaurin formula for sufficiently large r. Particularly, this means that it is not possible to show that, for fixed $\eta > 1$, $\frac{2\zeta(2r)}{(2\pi\eta)^{2r}} C_{2r} \xrightarrow[r\to\infty]{} 0$.

the quantized electromagnetic field. The Lamb shift arises as a separation of the ns-spectral band from np-spectral band, while they should stay merged in absence of Lamb effect. So, how would an energy level Lamb shift be detected for quantum systems displaying no spectral band degeneracy such as a QD? In such systems each non-degenerate energy level is dressed by the quantum zero-point fluctuations of the electromagnetic field, forbidding the detection of the corresponding bare energy level. Eq. (6) gives the protocol of a *Gedankenexperiment*, which may allow to access the Lamb shift in semiconducting QDs, first calculated in [18].

The model we use to obtain this Lamb shift in QDs is an improved version of the standard EMA, where a pseudo-potential is introduced to partly remove the usual divergence of the QD ground state energy in such models with small radius. One electron and one hole, moving with their standard effective masses $m^*_{\mathrm{e,h}}$ in the considered semiconductor, are confined by an infinite spherical potential well of radius R placed at the QD surface, and interact with each other through the Coulomb potential. The common approach to treat the interplay of the Coulomb interaction of the electron-hole pair, which scales as $\propto \frac{1}{R}$, and the quantum confinement, which scales as $\propto \frac{1}{R^2}$ is to use a variational procedure, for which two regimes of electron-hole pair should be singled out according to the ratio of the Bohr radius $a^* = \frac{\kappa}{e^2\mu}$ of the exciton, μ being its reduced mass, to the QD radius R. First, in the strong confinement regime, valid for a QD radius $R \leq 2a^*$, the confinement potential sufficiently affects the relative electron-hole motion, so that the interactive electron-hole pair states should then consists of uncorrelated electron and hole states. In the weak confinement regime, valid for a QD radius $R \geq 4a^*$, the electron-hole relative motion is *quasi* left unchanged by the confinement potential, so that excitonic binding states appear, as if the electron-hole pair has not been confined. However, the exciton has to be treated as a confined quasi-particle of total mass M, and its center-of-mass motion should then be quantized.

For this simple model, in the strong confinement regime, it is proven that the predicted Lamb shift, in QDs of size experimentally synthetized and used, is of the same order of magnitude as the one in hydrogen-like atoms, at least for judiciously chosen semiconductors, such as for example InAs or GaAs, and then seems to be observable. Since this is the relative modification of the Lamb shift due to the presence of the Casimir device, the method of calculation of the Lamb shift in QDs [18] is not of interest. However, this provides the Bethe IR cut-offs $\kappa^*_{\mathrm{e,h}} = \frac{7\pi^2}{12 m^*_{\mathrm{e,h}} R^2}$ and $\kappa^* = \kappa^*_{\mathrm{e}} + \kappa^*_{\mathrm{h}} = \frac{7\pi^2}{12\mu R^2}$ respectively for the electron, the hole and the exciton. Let introduce the electron and the hole reduced Compton wavelengths $\lambda^*_{\mathrm{e,h}} = \frac{1}{m^*_{\mathrm{e,h}}}$ and the radius $R^*_{\mathrm{e,h}} = \frac{\pi}{2}\sqrt{\frac{7}{3}}\lambda^*_{\mathrm{e,h}}$. These are interpreted as the lower bound for the QD radius, allowing fluctuations of the charge carrier, according to the Welton approach, to be confined inside the QD. Then, it is supposed that $R^*_{\mathrm{e,h}} \leq R \leq d$, with the additional constraint $\frac{\kappa_d}{2\pi} < \kappa^*_{\mathrm{e,h}}(R)$ given by Eq. (1). In this context, we deduce from Eq. (6) the relative modification of the Lamb effect undergone by a semiconducting QD placed between the Casimir plates

$$\frac{\Delta E^{\mathrm{e,h}}_{\mathrm{Casimir}}(d)}{\Delta E_{\mathrm{e,h}}} = \frac{1}{2}\left[1 - \frac{\sqrt{\frac{3}{7}\frac{R^2}{R^*_{\mathrm{e,h}}d}}}{\tan\sqrt{\frac{3}{7}\frac{R^2}{R^*_{\mathrm{e,h}}d}}}\right]\log^{-1}\frac{R}{R^*_{\mathrm{e,h}}}, \quad \text{valid for } \frac{R^2}{R^*_{\mathrm{e,h}}d} < \sqrt{\frac{7}{3}}\pi. \tag{7}$$

Then, there is a competition between the dimensionless ratios $\frac{R}{R^*_{\mathrm{e,h}}} \geq 1$ and $\frac{R}{d} \leq 1$, characterising the problem under study in the strong confinement regime, which is characterized in turns by the ratio $\frac{R}{a^*} \leq 2$. Figure 1 shows the behavior of this modification inside InAs QDs for several values of the separation distance d. Only the electronic contribution to the Lamb shift is represented on this figure, because, in InAs, either the hole Lamb shift is negligible against the electronic contribution (heavy hole, $\frac{m^*_{\mathrm{h}}}{m_{\mathrm{e}}} \approx \frac{m^*_{\mathrm{e}}}{m_{\mathrm{e}}} \approx 0.026$) or both are almost equal (light hole, $\frac{m^*_{\mathrm{h}}}{m_{\mathrm{e}}} \approx 0.41$), the effective masses being themselves almost identical.

Figure 1: Modification of the Lamb shift in spherical InAs microcrystals for $d = 1\mu$m (—), 0.5μm (−−) or 0.25μm (−·−).

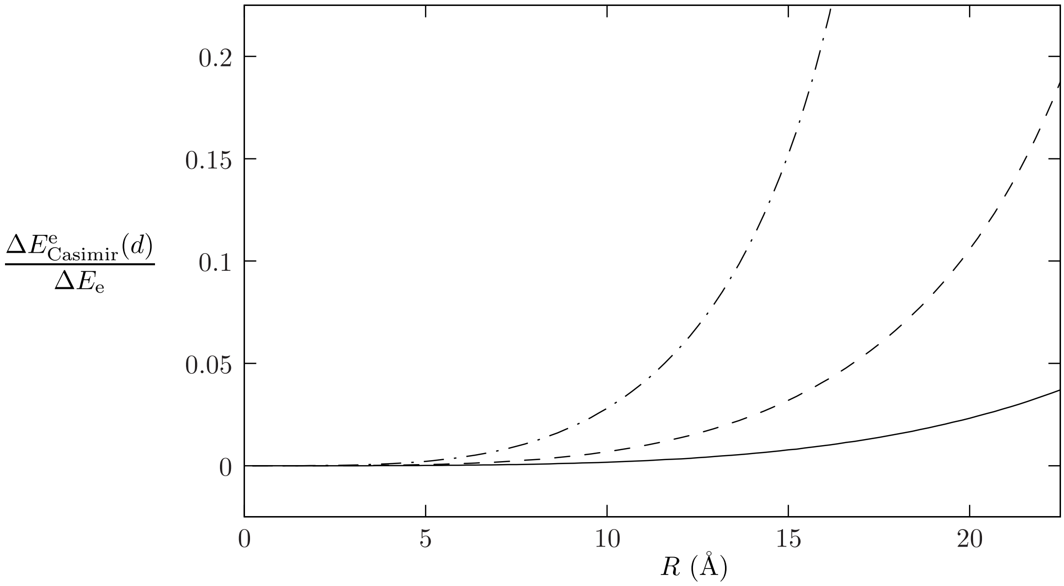

Figure 1 indicates that if the separation distance d is chosen to be 0.5μm, the experimental observation of the Casimir effect is possible. The modification of the electron-hole Lamb shift between the Casimir plates is of about 5-10% in spherical InAs QDs of radius in the range of 10-20nm, which is of reasonable experimental size [27]. It is possible to enhance the amplitude of this modification. The first idea is reducing the separation distance d until the order of a few tenth parts μm, the problem being then that the radius R should be reduced accordingly to still satisfy the weak coupling regime condition, which may not be experimentally easily manageable. It seems then simpler to act directly on the Casimir configuration geometry. For example, the use of a sphere of large radius $L' \gg d$ instead of one of the Casimir plates changes the Casimir energy from $E^{\mathrm{plates}}_{\mathrm{Casimir}}(L, d) = -\frac{\pi^2}{720}\frac{L^2}{d^3}$ to $E^{\mathrm{sphere}}_{\mathrm{Casimir}}(L', d) = -\frac{\pi^3}{720}\frac{L'}{d^2}$. Because $L' \gg d$, the volume limited by the plate and the sphere can be firstly approximated by $L'd^2$, which implies that the modification of the Lamb shift is enhanced by a factor π [10]. This effect alone almost leads to the total modification of the Lamb shift in free-space of at least 50%, which seems significant enough to be observable.

There finally exists other corrective effects, such as the reflectivity of the metal used, the roughness of the surfaces of plates and sphere, and the finite temperature, which has an impact on the Casimir effect, as described in [10]. By the same kind of reasoning as above, it seems also possible to account for them in our description. Since these are corrections scaling as $\propto \frac{1}{d^2}$ to the standard Casimir force, it is sufficient to consider the two first terms in Eqs. (6) or (7), which are the dominant terms of $\frac{\Delta E^{\mathrm{e,h}}_{\mathrm{Casimir}}(d)}{\Delta E_{\mathrm{e,h}}}$ and its first correction, also scaling as $\propto \frac{1}{d^4}$, is to be considered.

4 Conclusion

In this work, we have developed a comprehensive computation method to obtain mathematically rigorous results on the Lamb shift for non degenerate energy levels in semiconductor spherical quantum dots placed in a Casimir device. A significant deviation from the standard Lamb shift (*i.e.* in vacuum) found in [18] is revealed. The explicit formula giving this deviation suggests a way to finely adjust the spectrum of a quantum dot for experimental purposes, by changing the distance between the Casimir plates, provided that such micro-mechanical operations are realizable. Moreover, the energy shift order magnitude, brought to light in this *Gedankenexperiment*, at least for judiciously chosen semiconductor and QD sizes, seems to be sufficient to indicate that such QDs may be appropriate systems to observe this modification, and that the Lamb shift in non-degenerate systems may not be out of reach.

A Calculation of the function g

In this appendix, we propose to compute the value of the series

$$\sum_{p\in\mathbb{N}^*}\frac{\sin(2\pi ps)}{p},\quad \forall s\in\mathbb{R}.$$

These series are the Fourier series on $\mathbb{R}$ representing the function g: $s\longmapsto \arctan\tan\pi(\frac{1}{2}-s)\chi_{\mathbb{R}\smallsetminus\mathbb{Z}}(s)$. The function g is odd and 1-periodic on $\mathbb{R}$, and is equal to $g(s)=\arctan\tan\pi(\frac{1}{2}-s)=\pi(\frac{1}{2}-s)$ for $s\in]0,1[$. And, since the function g is of class $\mathcal{C}^1$ by parts on $\mathbb{R}$, we deduce from Dirichlet theorem [28] that the Fourier series of g is point-wise convergent on $\mathbb{R}$, and that

$$g(s)=\sum_{p\in\mathbb{N}^*}\frac{\sin(2\pi ps)}{p},\quad \forall s\in\mathbb{R}. \tag{8}$$

The error made in [13] is the statement that $g(s)=\pi(\frac{1}{2}-s)$ for any $s\in\mathbb{R}$, from Eq. (A14) in appendix **A** of reference [13]. But g is neither odd (because $g(0)\neq 0$) nor periodic. Only its restriction to $]0,1[$ is shown to be valid.

To be as most exhaustive as possible, let us prove rigourously the validity of Eq. (8), using the Euler-Maclaurin formula, as proposed in [13], which turns out to be really helpful to get a fundamental convergence result. This result will be established despite a cumbersome proof which involves tricky points of regularization theory of series and distribution theory.

A.1 Notations and convergences

For any fixed $\epsilon>0$ and any $s\in\mathbb{R}$, let the even functions $g_s^\epsilon\in\mathcal{S}(\mathbb{R})$ and g_s^0 be defined, for any $t\in\mathbb{R}$, by

$$g_s^\epsilon(t)=\frac{\sin(2\pi st)}{t}\mathrm{e}^{-\epsilon t^2}\quad\text{and}\quad g_s^0(t)=\frac{\sin(2\pi st)}{t},$$

such that $g_s^\epsilon \xrightarrow[\epsilon\to 0^+]{} g_s^0$, point-wise on $\mathbb{R}$. It is possible to justify directly the point-wise limit, for any $s\in\mathbb{R}^*$

$$\int_{\mathbb{R}_+}\mathrm{d}t\, g_s^\epsilon(t)=\frac{\pi}{2}\mathrm{sign}\, s\,\mathrm{erf}\frac{\pi}{\sqrt{\epsilon}}|s| \xrightarrow[\epsilon\to 0^+]{} \frac{\pi}{2}\mathrm{sign}\, s=\int_{\mathbb{R}_+}\mathrm{d}t\, g_s^0(t),$$ [4]

where the function sign $=\chi_{\mathbb{R}_+}-\chi_{\mathbb{R}_-}$ is the sign-function on $\mathbb{R}$, and the function erf: $x\longmapsto \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}t\,\mathrm{e}^{-t^2}$ is the error-function on $\mathbb{R}$, which verifies $\mathrm{erf}\, x \xrightarrow[x\to\infty]{} 1$ and $|\mathrm{erf}|\leq 1$ on $\mathbb{R}$. When $s=0$, it is obvious that $g_0^\epsilon=g_0^0=0$ on $\mathbb{R}$, and the previous expression is trivially satisfied.

Consider now the two point-wise convergent series on $\mathbb{R}$ defined by

$$g_\epsilon(s)=\sum_{p\in\mathbb{N}^*}g_s^\epsilon(p)=\sum_{p\in\mathbb{N}^*}\frac{\sin(2\pi ps)}{p}\mathrm{e}^{-\epsilon p^2}\quad\text{and}\quad g_0(s)=\sum_{p\in\mathbb{N}^*}g_s^0(p)=\sum_{p\in\mathbb{N}^*}\frac{\sin(2\pi ps)}{p}.$$

[4] *cf.* Eqs. **2.5.3.12** p. 387 and **2.5.36.6** p. 452 [31].

Proposition 1. $g_\epsilon \underset{\epsilon\to 0^+}{\longrightarrow} g_0$, *point-wise on* $\mathbb{R}$.

Proof. For any $\epsilon \geq 0$, the functions g_ϵ are odd and 1-periodic on $\mathbb{R}$, so that it is sufficient to restrict the study of the point-wise convergence of the function g_ϵ to g_0 in the limit $\epsilon \to 0^+$ on $[0, \frac{1}{2}]$.

For $s \in \{0, \frac{1}{2}\}$, we have $g_\epsilon(s) = 0$, $\forall \epsilon \geq 0$.

Let $\epsilon > 0$. For any $s \in]0, \frac{1}{2}[$, the series $\sum g_s^\epsilon(p)$ are absolutely convergent — because $|g_s^\epsilon(p)| \leq \mathrm{e}^{-\epsilon p}$, $\forall p \in \mathbb{N}^*$, where $\mathrm{e}^{-\epsilon} \in]0,1[$ —, and the series $\sum g_s^0(p)$ are semi-convergent thanks to Abel theorem [28]. Denoting $\|\cdot\|_\infty^{\mathbb{A}} = \sup_{\mathbb{A}} |\cdot|$ for any subset $\mathbb{A} \subseteq \mathbb{R}$, Abel theorem also yields, for any $N \in \mathbb{N}^*$, $\left\|\sum_{p\geq N} g_s^{\cdot}(p)\right\|_\infty^{\mathbb{R}_+} \leq \frac{1}{N \sin \pi s} \underset{N\to\infty}{\longrightarrow} 0$, implying that the series $\epsilon \longmapsto \sum g_s^\epsilon(p)$ are uniformly convergent on $\mathbb{R}_+$. Then, $\epsilon \longmapsto g_\epsilon(s) \in \mathcal{C}^0(\mathbb{R}_+)$, and in particular $g_\epsilon(s) \underset{\epsilon\to 0^+}{\longrightarrow} g_0(s)$.

By parity and periodicity extensions, one obtains finally $g_\epsilon \underset{\epsilon\to 0^+}{\longrightarrow} g_0$, point-wise on $\mathbb{R}$. ∎

As we shall see, this point-wise convergence is not sufficient, and we would rather have a convergence of g_ϵ to g_0 in the limit $\epsilon \to 0^+$ in the sense of tempered distributions $\mathcal{S}'(\mathbb{R})$, denoted as $g_\epsilon \underset{\epsilon\to 0^+}{\longrightarrow} g_0$ in $\mathcal{S}'(\mathbb{R})$.[5] The difficulty lies in the fact that the previous inequality $\|g_\cdot(s)\|_\infty^{\mathbb{R}_+} \leq \frac{1}{|\sin \pi s|}$, valid for any $s \in \mathbb{R} \setminus \mathbb{Z}$ and then almost surely on $\mathbb{R}$, forbids the use of dominated convergence theorem, because $s \longmapsto \frac{1}{|\sin \pi s|} \notin \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathrm{d}s) = \{h : \mathbb{R} \longrightarrow \mathbb{R} / \int_{\mathbb{K}} \mathrm{d}s\, |h(s)| < \infty,\ \forall \mathbb{K} \subset \mathbb{R} \text{ compact}\}$. Therefore, it is not possible to directly deduce that $g_\epsilon \underset{\epsilon\to 0^+}{\longrightarrow} g_0$ in $\mathcal{D}'(\mathbb{R})$. However, we prove in proposition **4** that there exists a constant $C > 0$ such that $\|g_\cdot(s)\|_\infty^{[0,1]} \leq C$, almost surely on $[0, \frac{1}{2}]$, to solve the problem.

Proposition 2. $g_\epsilon \underset{\epsilon\to 0^+}{\longrightarrow} g_0$ *in* $\mathcal{S}'(\mathbb{R})$.

Proof. By parity and periodicity and proposition **4**, $\|g_\cdot(s)\|_\infty^{[0,1]} \leq C$, almost surely on $\mathbb{R}$, then $C\phi \in \mathcal{L}^1(\mathbb{R}, \mathrm{d}s)$, for any test function $\phi \in \mathcal{S}(\mathbb{R})$. Hence, the functions g_ϵ are tempered distributions on $\mathbb{R}$, for any $\epsilon \geq 0$, and the dominated convergence theorem applies, leading to the expected convergence

$$\int_{\mathbb{R}} \mathrm{d}s\, g_\epsilon(s)\phi(s) \underset{\epsilon 0+}{\longrightarrow} \int_{\mathbb{R}} \mathrm{d}s\, g_0(s)\phi(s), \quad \forall \phi \in \mathcal{S}(\mathbb{R}) \quad \Leftrightarrow \quad g_\epsilon \underset{\epsilon 0+}{\longrightarrow} g_0 \text{ in } \mathcal{S}'(\mathbb{R}).$$

∎

A.2 Euler-Maclaurin formula and consequences

The Euler-Maclaurin summation formula is an important tool in analysis. It provides an estimation of the sum $\sum_{p=0}^N h(p)$ by the integral $\int_{[0,N]} \mathrm{d}t\, h(t)$, h being a sufficiently regular function on $[0, N]$ with $N \in \mathbb{N}$. Let us assume, for convenience, that $h \in \mathcal{C}^\infty(\mathbb{R})$, then for any $N \in \mathbb{N}$ and $r \in \mathbb{N}^*$, we have

$$\sum_{p=0}^N h(p) = \int_{[0,N]} \mathrm{d}t\, h(t) + \frac{h(N) + h(0)}{2} + \sum_{k=1}^r \frac{b_{2k}}{(2k)!} \left\{h^{(2k-1)}(N) - h^{(2k-1)}(0)\right\} + R_r^N(h),$$

[5] For more details on theories of the Lebesgue integral and of distributions, one can respectively refer to [29] and [30].

where the remainder is expressed as

$$R_r^N(h) = -\int_{[0,N]} \mathrm{d}t\, \frac{\widetilde{B}_{2r}(t)}{(2r)!} h^{(2r)}(t).$$

Here, the function $\widetilde{B}_{2k}$ is the unique 1-periodic function which coincide on $[0,1]$ with the Bernoulli polynomial B_{2k}, and the real numbers b_{2k} are the non-vanishing Bernoulli numbers, defined by $b_{2k} = B_{2k}(0) = B_{2k}(1)$ [28]. There exists an useful Fourier series representation of the function $\widetilde{B}_{2r}$ (*cf.* Eq. **6.22** p. 1032 [32])

$$\widetilde{B}_{2r}(t) = 2(-1)^{r-1}(2r)! \sum_{k\in\mathbb{N}^*} \frac{\cos 2k\pi t}{(2k\pi)^{2r}}, \quad \forall t \in \mathbb{R} \text{ and } \forall r \in \mathbb{N}^*.$$

Let $s \in]0, \frac{1}{2}]$ be fixed for the rest of the section until further notice, the case $s = 0$ being trivial. Let $\epsilon > 0$ be also fixed. In the case of $g_s^\epsilon \in \mathcal{S}(\mathbb{R})$, since g_s^ϵ is even on $\mathbb{R}$, $(g_s^\epsilon)^{(2k-1)}(0) = 0$ for any $k \in \mathbb{N}^*$, and $g_s^\epsilon(0) = 2\pi s$. Then, the Euler-Maclaurin formula is, for any $N \in \mathbb{N}$ and any $r \in \mathbb{N}^*$

$$\sum_{p=1}^{N} g_s^\epsilon(p) = \int_{[0,N]} \mathrm{d}t\, g_s^\epsilon(t) + \frac{g_s^\epsilon(N)}{2} - \pi s + \sum_{k=1}^{r} \frac{b_{2k}}{(2k)!}(g_s^\epsilon)^{(2k-1)}(N) + R_r^{N,\epsilon}(s), \qquad (\bigstar_{\epsilon,r,N})$$

where the remainder is given by

$$R_r^{N,\epsilon}(s) = -\int_{[0,N]} \mathrm{d}t\, \frac{\widetilde{B}_{2r}(t)}{(2r)!}(g_s^\epsilon)^{(2r)}(t).$$

The mistake in [13] lies on the fact that both limits $r, N \to \infty$ are taken without justifications for any $s \in \mathbb{R}^*$. For $s = 0$, this limit yields a vanishing remainder. However, for $s \in \mathbb{R}^*$, care must be exercised, since the dominated convergent theorem does not actually apply in this case. As we shall see, it is easy to prove that it is possible to take the limit $N \to \infty$. But, the limit $R_r^\epsilon(s)$ of the remainder $R_r^{N,\epsilon}(s)$ does not go to zero in the limit $r \to \infty$ for any $s \in \mathbb{R}$, even after having taken the limit $\epsilon \to 0^+$.

Proposition 3. *The limit $N \to \infty$ of Eq. $(\bigstar_{\epsilon,r,N})$ exists, does not depend on $r \in \mathbb{N}^*$, and yields*

$$g_\epsilon(s) = \int_{\mathbb{R}_+} \mathrm{d}t\, g_s^\epsilon(t) - \pi s + R_1^\epsilon(s). \qquad (\bigstar_\epsilon)$$

Proof. Let $r \in \mathbb{N}^*$. First, $g_s^\epsilon \in \mathcal{S}(\mathbb{R})$ implies that $g_s^\epsilon(N), (g_s^\epsilon)^{(2k-1)}(N) \xrightarrow[N\to\infty]{} 0$, $\forall k \in \mathbb{N}^*$.

Second, $g_s^\epsilon \in \mathcal{L}^1(\mathbb{R}_+, \mathrm{d}\mu)$, where the set Ω is either $\mathbb{R}_+$ or $\mathbb{N}^*$, fitted with its natural measure μ, being respectively the Lebesgue measure or the so-called Dirac comb $\Delta = \sum_{p\in\mathbb{Z}} \delta_p$, δ_p being the Dirac measure at $p \in \mathbb{Z}$. Then, by the dominated convergence theorem, $\int_{\Omega\cap[0,N]} \mathrm{d}\mu\, g_s^\epsilon \xrightarrow[N\to\infty]{} \int_\Omega \mathrm{d}\mu\, g_s^\epsilon$, so that

$$\sum_{p=1}^{N} g_s^\epsilon(p) \xrightarrow[N\to\infty]{} \sum_{p\in\mathbb{N}^*} g_s^\epsilon(p) \quad \text{and} \quad \int_{[0,N]} \mathrm{d}t\, g_s^\epsilon(t) \xrightarrow[N\to\infty]{} \int_{\mathbb{R}_+} \mathrm{d}t\, g_s^\epsilon(t)$$

Third, $\|\widetilde{B}_{2r}\|_\infty^{\mathbb{R}} = |b_{2r}| < \infty$ and $g_s^\epsilon \in \mathcal{S}^1(\mathbb{R})$, hence $\widetilde{B}_{2r}(g_s^\epsilon)^{(2r)} \in \mathcal{L}^1(\mathbb{R}, \mathrm{d}t)$, and

$$R_r^{N,\epsilon}(s) \xrightarrow[N\to\infty]{} = -\int_{\mathbb{R}_+} \mathrm{d}t\, \frac{\widetilde{B}_{2r}(t)}{(2r)!}(g_s^\epsilon)^{(2r)}(t) = R_r^\epsilon(s).$$

Then, the limit $N \to \infty$ is well-defined, and yields, $\forall r \in \mathbb{N}^*$

$$g_\epsilon(s) = \int_{\mathbb{R}_+} \mathrm{d}t\, g_s^\epsilon(t) - \pi s + R_r^\epsilon(s) = \int_{\mathbb{R}_+} \mathrm{d}t\, g_s^\epsilon(t) - \pi s + R_1^\epsilon(s).$$

∎

Proposition 4. *The limit $\epsilon \to 0^+$ of Eq. $(\bigstar_\epsilon)$ exists, and*

$$g(s) = g_0(s), \quad \forall s \in [0, \tfrac{1}{2}]. \qquad (\bigstar)$$

Moreover, there exists $C > 0$, such that $\|g_\cdot(s)\|_\infty^{[0,1]} \leq C$, $\forall s \in [0, \frac{1}{2}]$, and $g_\epsilon \xrightarrow[\epsilon \to 0^+]{} g$ in $\mathcal{S}'(\mathbb{R})$.

Proof. First, we recall that it is proved in proposition **1** that the limit $\epsilon \to 0^+$ of the quantities $g_\epsilon(s)$ and $\int_{\mathbb{R}_+} \mathrm{d}t\, g_s^\epsilon(t)$ are defined, their respective limits being $g_0(s)$ and $\frac{\pi}{2}\mathrm{sign}\, s$. Only the limit $\epsilon \to 0^+$ of the remainder $R_1^\epsilon(s)$ is left to be justified. Since the Fourier series representation of the function $\widetilde{B}_2$ are normally convergent on $\mathbb{R}$, we can write

$$R_1^\epsilon(s) = -\int_{\mathbb{R}_+} \mathrm{d}t\, \frac{\widetilde{B}_2(t)}{2} g_s^{\epsilon\prime\prime}(t) = -2 \sum_{k \in \mathbb{N}^*} \int_{\mathbb{R}_+} \mathrm{d}t\, \frac{\cos 2k\pi t}{(2k\pi)^2} g_s^{\epsilon\prime\prime}(t) = \sum_{k \in \mathbb{N}^*} h_s^\epsilon(k).$$

After two integration by parts, for any $\epsilon > 0$ and any $k \in \mathbb{N}^*$, since $s \in [0, \frac{1}{2}]$, we get

$$h_s^\epsilon(k) = \frac{\pi}{2} \left\{ \mathrm{erf} \frac{\pi}{\sqrt{\epsilon}}(k+s) - \mathrm{erf} \frac{\pi}{\sqrt{\epsilon}}(k-s) \right\}.$$

Then, for any $k \in \mathbb{N}^*$ and for any $\epsilon > 0$, we have the following (in)equalities

$$0 \leq h_s^\epsilon(k) = \sqrt{\pi} \int_{\frac{\pi}{\sqrt{\epsilon}}(k-s)}^{\frac{\pi}{\sqrt{\epsilon}}(k+s)} \mathrm{d}t\, \mathrm{e}^{-t^2} \leq \sqrt{\frac{\pi^3}{\epsilon}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}t\, \mathrm{e}^{-\frac{\pi^2}{\epsilon}(t+k)^2} \leq \sqrt{\frac{\pi^3}{\epsilon}} \mathrm{e}^{-\frac{\pi^2}{\epsilon}k^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}t\, \mathrm{e}^{-\frac{\pi^2}{\epsilon}t^2} \mathrm{e}^{-2\frac{\pi^2}{\epsilon}kt}$$
$$\leq \sqrt{\frac{\pi^3}{\epsilon}} \mathrm{e}^{-\frac{\pi^2}{\epsilon}k(k-1)} \int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{-\frac{\pi^2}{\epsilon}t^2} = \pi \mathrm{e}^{-\frac{\pi^2}{\epsilon}(k-1)}.$$

Then we first deduce that the function $\epsilon \longmapsto h_s^\epsilon(k)$ is continuous on $\mathbb{R}_+$ by continuous extension with $h_s^0(k) = 0$, because it is continuous on $\mathbb{R}_+^*$, and $0 \leq h_s^\epsilon(k) \leq \pi \mathrm{e}^{-\frac{\pi^2}{\epsilon}(k-1)} \xrightarrow[\epsilon \to 0^+]{} 0$.

Second, we have $\|h_s^\cdot(k)\|_\infty^{[0,1]} \leq \pi \mathrm{e}^{-\pi^2(k-1)}$, $\forall k \in \mathbb{N}^*$, implying that the series $\epsilon \longmapsto \sum h_s^\epsilon(k)$ are normally convergent on $[0,1]$, and $\|R_1^\cdot(s)\|_\infty^{[0,1]} \leq \pi \sum_{k \in \mathbb{N}} \mathrm{e}^{-\pi^2 k} = \frac{\pi}{1-\mathrm{e}^{-\pi^2}}$. Finally, $\epsilon \longmapsto R_1^\epsilon(s) \in \mathcal{C}^0([0,1])$, and $R_1^\epsilon(s) \xrightarrow[\epsilon \to 0^+]{} \sum_{k \in \mathbb{N}^*} h_s^0(k) = 0$. The limit $\epsilon \to 0^+$ is therefore well-defined, and yields

$$g_0(s) = \pi(\tfrac{1}{2} - s) = g(s), \quad \forall s \in]0, \tfrac{1}{2}].$$

Since g_0 and $g : x \longmapsto \arctan \tan \pi(\frac{1}{2} - s)\chi_{\mathbb{R} \smallsetminus \mathbb{Z}}(x)$ are both 1-periodic odd functions on $\mathbb{R}$ and coincide on $[0, \frac{1}{2}]$, they are equal on $\mathbb{R}$. [6]

[6] In the case $s \in]0, \frac{1}{2}]$, we have proved that the remainder $R_1^\epsilon(s)$ goes to zero in the limit $\epsilon \to 0^+$. This becomes not true as soon as we consider $s \in \mathbb{R}^*$, but the same reasoning can be done to determine the new limit. Using the same notations, we have that for any $k \in \mathbb{N}^*$, $\epsilon > 0$ and $s \in \mathbb{R}^*$

$$h_s^\epsilon(k) = \frac{\pi}{2} \mathrm{sign}\, s \left\{ \mathrm{erf} \frac{\pi}{\sqrt{\epsilon}}(k + |s|) - \mathrm{sign}\,(k - |s|) \mathrm{erf} \frac{\pi}{\sqrt{\epsilon}} \big| k - |s| \big| \right\} \xrightarrow[\epsilon \to 0^+]{} \frac{\pi}{2} \mathrm{sign}\, s \{1 - \mathrm{sign}\,(k - |s|)\} = h_s^0(k).$$

Now, for any $k \geq |s|$, we have that $\|h_s^\cdot(k)\|_\infty^{[0,1]} \leq \pi \mathrm{e}^{-\pi^2(k-2|s|)}$. Reasoning as above, we conclude that $\epsilon \longmapsto$

We have also proved here the existence of the constant $C > 0$ introduced in proposition **2**. In fact, since for any $\epsilon > 0$, $\int_{\mathbb{R}_+} \mathrm{d}t\, g_s^\epsilon(t) = \frac{\pi}{2} \mathrm{sign}\, s\, \mathrm{erf} \frac{\pi}{\sqrt{\epsilon}} |s|$ and $\int_{\mathbb{R}_+} \mathrm{d}t\, g_s^0(t) = \frac{\pi}{2} \mathrm{sign}\, s$, we have that $\|\int_{\mathbb{R}_+} g_s^\cdot\|_\infty^{[0,1]} \leq \frac{\pi}{2}$, so that for any $s \in [0, \frac{1}{2}]$, $\|g_\cdot(s)\|_\infty^{[0,1]} \leq \left\|\int_{\mathbb{R}_+} g_s^\cdot\right\|_\infty^{[0,1]} + \pi s + \|R_1^\cdot(s)\|_\infty^{[0,1]} \leq \frac{2\pi}{1-\mathrm{e}^{-\pi^2}} = C$.

Finally, to sum up, we have just proved that $g_\epsilon \xrightarrow[\epsilon\to 0^+]{} g$, point-wise in $\mathbb{R}$, and in $\mathcal{S}'(\mathbb{R})$. ∎

B Mathematics of the Casimir effect

B.1 Density of states between Casimir plates

In this appendix, we present in details the general result on the density of states in vacuum or between the Casimir plates. To this end, the quantity of interest

$$H_\phi^F(L, d) = \sum_{(\mathbf{n}_\parallel, n_\perp)\in\mathbb{Z}^3} F(k_{\mathbf{n}_\parallel n_\perp}) \phi\left(\frac{k_{\mathbf{n}_\parallel n_\perp}}{\kappa_\phi}\right) = \sum_{(\mathbf{n}_\parallel, n_\perp)\in\mathbb{Z}^3} \frac{f(k_{\mathbf{n}_\parallel n_\perp})}{k_{\mathbf{n}_\parallel n_\perp}} f_\gamma(k_{\mathbf{n}_\parallel n_\perp}) \phi\left(\frac{k_{\mathbf{n}_\parallel n_\perp}}{\kappa_\phi}\right),$$

is totally formal, where the two functions f^β and f_γ satisfy the following assumptions.

i. f^β: $s \longmapsto f(\kappa_d s)$ is a dimensioned function of class $\mathcal{C}^0$ on $\mathbb{R}$ and $\mathcal{C}^\infty$ on $\mathbb{R}^*$, of dimension $k^{-\beta}$, where $\beta \geq 0$ is defined by the asymptotic behavior $f(k) = O(k^{-\beta})$ for $k \to \infty$.

ii. f_γ is a dimensionless function defined and of class $\mathcal{C}^\infty$ at least on $\mathbb{R}^*$, and is homogeneous of degree $\gamma \in \mathbb{R}$ such that $f_\gamma(\lambda s) = \lambda^\gamma f_\gamma(s)$, $\forall \lambda \in \mathbb{R}^*$ and $\forall s \in \mathbb{R}^*$.

In particular, f_γ has the asymptotic behavior $f_\gamma(s) = O(|s|^\gamma)$ for $s \to \pm\infty$. In practice, the successive derivatives of $f^\beta \in \mathcal{B}(\mathbb{R})$ are also bounded but only on subsets of $\mathbb{R}^*$ of the form $]-\infty, -\delta] \cup [\delta, \infty[$, $\forall \delta > 0$.

B.2 Integral representation of the quantity $H_\phi^F(L, d)$

Euler-Maclaurin or Poisson formulas [28] cannot be used in the previous expression, because the function $s_\perp \longmapsto F\big(\kappa_d s_{\parallel\perp}\big)\, \phi\Big(\frac{\kappa_d}{\kappa_\phi} s_{\parallel\perp}\Big)$, is not differentiable at $s_\perp = 0$, when $s_\parallel = 0$, with $s_{\parallel\perp} = \sqrt{\mathbf{s}_\parallel^2 + s_\perp^2}$. These are actually essentially the same equation once written in the tempered distributions formalism.

Let be $0 < \delta \leq \eta$, and consider the regularized expression

$$H_{\phi,\delta,\eta}^F(L, d) = \sum_{(\mathbf{n}_\parallel, n_\perp)\in\mathbb{Z}^3} \frac{f\left(\sqrt{\mathbf{k}_{\mathbf{n}_\parallel n_\perp}^2 + (\kappa_d\delta)^2}\right)}{\sqrt{\mathbf{k}_{\mathbf{n}_\parallel n_\perp}^2 + (\kappa_d\delta)^2}} f_\gamma\left(\sqrt{\mathbf{k}_{\mathbf{n}_\parallel n_\perp}^2 + (\kappa_d\eta)^2}\right) \phi\left(\frac{\sqrt{\mathbf{k}_{\mathbf{n}_\parallel n_\perp}^2 + (\kappa_d\eta)^2}}{\kappa_\phi}\right)$$

$R_1^\epsilon(s) \in \mathcal{C}^0([0,1])$, for any $s \in \mathbb{R}^*$, and

$$R_1^\epsilon(s) \xrightarrow[\epsilon\to 0^+]{} \sum_{k\in\mathbb{N}^*} h_s^0(k) = \sum_{k=1}^{[|s|]} h_s^0(k) = \pi \left\{ \begin{array}{ll} [s] & \text{if} \quad s \notin \mathbb{Z} \\ s - \frac{\mathrm{sign}\, s}{2} & \text{if} \quad s \in \mathbb{Z} \end{array} \right. \neq 0,$$

where the function $[\cdot]$: $x \longmapsto [x]$ is the integer part function on $\mathbb{R}$. The limit $\epsilon \to 0^+$ is stay well-defined, and yields

$$g_0(s) = \pi \left\{ \begin{array}{ll} \frac{\mathrm{sign}\, s}{2} + [s] - s & \text{if} \quad s \notin \mathbb{Z} \\ 0 & \text{if} \quad s \in \mathbb{Z} \end{array} \right. .$$

Once again, the functions g and g_0 are equal on $\mathbb{R}$, because they are both odd and 1-periodic on $\mathbb{R}$, and coincide with the function $x \longmapsto \pi(\frac{1}{2} - x)$ on $[0, \frac{1}{2}]$.

For $\mathbf{s}_\parallel \in \mathbb{R}^2$ and $s_\perp \in \mathbb{R}$, denoting for convenience $s_\delta^{\parallel\perp} = \sqrt{\mathbf{s}_\parallel^2 + s_\perp^2 + \delta^2}$, the regularized functions $(\mathbf{s}_\parallel, s_\perp) \longmapsto \frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}} \Phi_\gamma(s_\eta^{\parallel\perp})$ belong to $\mathcal{S}(\mathbb{R}^2 \times \mathbb{R})$, because the function Φ_γ: $s \longmapsto f_\gamma(s)\Phi(\frac{\kappa_d}{\kappa_\phi}s)$ is of class $\mathcal{C}^\infty$ and rapidly decreasing on $\mathbb{R} \smallsetminus [-\eta, \eta]$, for any $\eta > 0$. So that, Euler-Maclaurin and Poisson formulas hold. In the limit $L \gg d$, this allows first to replace the sum over the quantum numbers $\mathbf{n}_\parallel \in \mathbb{Z}^2$ in the previous expression by an integral over $\mathbb{R}^2$ with the dimensionless measure $\frac{L^2}{(2\pi)^2}\mathrm{d}^2\mathbf{k}_\parallel$. For simplicity, consider the function G: $k \longmapsto \frac{f\left(\sqrt{k^2+(\kappa_d\delta)^2}\right)}{\sqrt{k^2+(\kappa_d\delta)^2}} f_\gamma\left(\sqrt{k^2+(\kappa_d\eta)^2}\right)\phi\left(\frac{\sqrt{k^2+(\kappa_d\eta)^2}}{\kappa_\phi}\right)$, and denote G_1: $s \longmapsto \frac{f^\beta(s_\delta)}{s_\delta}\Phi_\gamma(s_\eta)$, with the notation $s_{\pm\delta} = \sqrt{s^2 \pm \delta^2}$. The Euler-Maclaurin formula yields, for any $r \in \mathbb{N}^*$

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} G\left(\frac{2\pi}{L}n\right) &= \int_{\mathbb{R}} \mathrm{d}t\, G\left(\frac{2\pi}{L}t\right) - \int_{\mathbb{R}} \mathrm{d}t\, \frac{\widetilde{B}_{2r}(t)}{(2r)!}\frac{\mathrm{d}^{2r}}{\mathrm{d}t^{2r}} G\left(\frac{2\pi}{L}t\right) \\
&= \frac{L}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\, G(k) - \left(\frac{2\pi}{L}\right)^{2r} \int_{\mathbb{R}} \mathrm{d}t\, \frac{\widetilde{B}_{2r}(t)}{(2r)!} G^{(2r)}\left(\frac{2\pi}{L}t\right).
\end{aligned}$$

Since we have $\frac{L}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\, G(k) = \kappa_d^\gamma \frac{L}{2d}\int_{\mathbb{R}} \mathrm{d}s\, G_1(s)$, the quantity of interest is

$$\left(\kappa_d^\gamma \frac{L}{2d}\right)^{-1} \left| \left(\frac{2\pi}{L}\right)^{2r} \int_{\mathbb{R}} \mathrm{d}t\, \frac{\widetilde{B}_{2r}(t)}{(2r)!} G^{(2r)}\left(\frac{2\pi}{L}t\right)\right| \le 2^{2r}\frac{|b_{2r}|}{(2r)!}\left(\frac{d}{L}\right)^{2r} \int_{\mathbb{R}} \mathrm{d}s \left|G_1^{(2r)}(s)\right| = \frac{2\zeta(2r)}{\pi^{2r}}\left(\frac{d}{L}\right)^{2r} C'_{2r},$$

where the dimensioned constant $C'_r = \int_{\mathbb{R}} \mathrm{d}s\, |G_1^{(r)}(s)|$ does not depend on L. This implies that for any $r \in \mathbb{N}^*$, the limit $\frac{d}{L} \to 0$ can be taken, leading to

$$\sum_{n\in\mathbb{Z}} G\left(\frac{2\pi}{L}n\right) = \kappa_d^\gamma \frac{L}{2d}\left\{\int_{\mathbb{R}} \mathrm{d}s\, G_1(s) + O\left[\left(\frac{d}{L}\right)^{2r}\right]\right\} \quad \Rightarrow \quad \sum_{n\in\mathbb{Z}} G\left(\frac{2\pi}{L}n\right) = \int_{\mathbb{R}} \mathrm{d}k\, G(k),$$

where the equality has to be understood in the meaning of asymptotic series, because it is not possible to show, for fixed $L \gg d$, that $\frac{C'_{2r}}{\pi^{2r}}\left(\frac{d}{L}\right)^{2r} \xrightarrow[r\to\infty]{} 0$.

The Poisson formula is now applied to the function $s_\perp \longmapsto \frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}}\Phi_\gamma(s_\eta^{\parallel\perp})$, for any $\mathbf{s}_\parallel \in \mathbb{R}^2$, and yields, after making the change of variables $\mathbf{k}_\parallel \to \kappa_d \mathbf{s}_\parallel$

$$\begin{aligned}
H^F_{\phi,\delta,\eta}(L,d) &= \frac{L^2}{4\pi^2}\kappa_d^{\gamma+1} \sum_{n_\perp\in\mathbb{Z}} \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_\parallel \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}} \Phi_\gamma\left(\frac{\kappa_d}{\kappa_\phi}\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right) \\
&= \frac{L^2}{4\pi^2}\kappa_d^{\gamma+1} \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_\parallel \sum_{p\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}s_\perp \frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}} \Phi_\gamma(s_\eta^{\parallel\perp}) \mathrm{e}^{2\mathrm{i}\pi p s_\perp}. \qquad (\bigstar^F_{\phi,\delta,\eta})
\end{aligned}$$

Lemma 1. *The function* $(\mathbf{s}_\parallel, n_\perp) \longmapsto \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2+n_\perp^2+\delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2+n_\perp^2+\delta^2}}\Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right)$ *is integrable on* $(\mathbb{R}^2 \times \mathbb{Z}, \mathrm{d}^2\mathbf{s}_\parallel \otimes \mathrm{d}\Delta(n_\perp))$. *The sum* $\sum_{n_\perp\in\mathbb{Z}}$ *and the integral* $\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_\parallel$ *in Eq.* $(\bigstar^F_{\phi,\delta,\eta})$ *can be inverted.*

Proof. For any $n_\perp \in \mathbb{Z}$, we can write[7], after performing the change of variables $s = \sqrt{s_\parallel^2 + n_\perp^2 + \eta^2}$

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_\parallel \left| \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right) \right| &\leq 2\pi \|f^\beta\|_\infty^{\mathbb{R}} \int_{\mathbb{R}_+} \frac{\mathrm{d}s_\parallel\, s_\parallel}{\sqrt{s_\parallel^2 + n_\perp^2}} \left|\Phi_\gamma\left(\sqrt{s_\parallel^2 + n_\perp^2 + \eta^2}\right)\right| \\
&\leq 2\pi \|f^\beta\|_\infty^{\mathbb{R}} \int_{\sqrt{n_\perp^2+\eta^2}}^\infty \frac{\mathrm{d}s\, s}{s_{-\eta}} \left|\Phi_\gamma(s)\right| \\
&\leq 2\pi \|f^\beta\|_\infty^{\mathbb{R}} \int_{\sqrt{n_\perp^2+\eta^2}}^\infty \mathrm{d}s \sqrt{\frac{s}{s-\eta}} \left|\Phi_\gamma(s)\right| \\
&\leq 2\pi \|f^\beta\|_\infty^{\mathbb{R}} \int_{|n_\perp|}^\infty \mathrm{d}s \left|\Phi_\gamma(s)\right|.
\end{aligned}$$

Furthermore, $s^3 \left|\Phi_\gamma(s)\right| \xrightarrow[s\to\infty]{} 0$, then $\exists N_\perp \in \mathbb{N}^*$, such that $\left|\Phi_\gamma(s)\right| \leq \frac{1}{s^3}$, $\forall s \geq N_\perp$. Then, for any $n_\perp \in \mathbb{Z}$ such that $|n_\perp| \geq N_\perp$, we have

$$\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_\parallel \left| \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right) \right| \leq \frac{2\pi}{3} \frac{\|f^\beta\|_\infty^{\mathbb{R}}}{n_\perp^2},$$

which shows that the series $\sum \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_\parallel \left| \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2+n_\perp^2+\delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2+n_\perp^2+\delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right) \right|$ are convergent, and by Fubini-Tonelli theorem

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{Z}^*} \mathrm{d}^2\mathbf{s}_\parallel \otimes \mathrm{d}\Delta(n_\perp) \left| \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right) \right| \\
&= \sum_{n_\perp\in\mathbb{Z}^*} \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_\parallel \left| \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right) \right| < \infty,
\end{aligned}$$

i.e. $(\mathbf{s}_\parallel, n_\perp) \longmapsto \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2+n_\perp^2+\delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2+n_\perp^2+\delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right) \in \mathcal{L}^1(\mathbb{R}^2 \times \mathbb{Z}, \mathrm{d}^2\mathbf{s}_\parallel \otimes \mathrm{d}\Delta(n_\perp))$, where $\mathrm{d}^2\mathbf{s}_\parallel \otimes \mathrm{d}\Delta(n_\perp)$ is the usual product mesure on $\mathbb{R}^2 \times \mathbb{Z}$. This leads to the inversion result by Fubini theorem. ∎

Corollary. The function $\delta \to H^F_{\phi,\delta,\eta}(L, d)$ is continuous on $[0, \eta]$.

Proof. The function $\delta \to \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2+n_\perp^2+\delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2+n_\perp^2+\delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right)$ is continuous on $[0, \eta]$. The result

[7] The last inequality is obtained by integration by parts

$$\begin{aligned}
\int_{\sqrt{n_\perp^2+\eta^2}}^\infty \mathrm{d}s \sqrt{\frac{s}{s-\eta}} \left|\Phi_\gamma(s)\right| &= 2\sqrt{\sqrt{n_\perp^2+\eta^2} - \eta} \int_{\sqrt{n_\perp^2+\eta^2}}^\infty \mathrm{d}t \sqrt{t} \left|\Phi_\gamma(t)\right| + 2 \int_{\sqrt{n_\perp^2+\eta^2}}^\infty \mathrm{d}s \sqrt{s-\eta} \int_s^\infty \mathrm{d}t \sqrt{t} \left|\Phi_\gamma(t)\right| \\
&\leq 2\sqrt{|n_\perp|} \int_{|n_\perp|}^\infty \mathrm{d}t \sqrt{t} \left|\Phi_\gamma(t)\right| + 2 \int_{|n_\perp|}^\infty \mathrm{d}s \sqrt{s} \int_s^\infty \mathrm{d}t \sqrt{t} \left|\Phi_\gamma(t)\right| = \int_{|n_\perp|}^\infty \mathrm{d}s \left|\Phi_\gamma(s)\right|.
\end{aligned}$$

is deduced directly from the proof of lemma **1**, which suggests that for any $\mathbf{s}_{\|} \in \mathbb{R}^2$,

$$\left| \frac{f^\beta\left(\sqrt{\mathbf{s}_{\|}^2 + n_\perp^2 + \delta^2}\right)}{\sqrt{\mathbf{s}_{\|}^2 + n_\perp^2 + \delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_{\|}^2 + n_\perp^2 + \eta^2}\right) \right| \leq \frac{\|f^\beta\|_\infty^{\mathbb{R}}}{\sqrt{\mathbf{s}_{\|}^2 + N_\perp^2 (\mathbf{s}_{\|}^2 + n_\perp^2 + \eta^2)^2}} \leq \frac{\|f^\beta\|_\infty^{\mathbb{R}}}{n_\perp^2 (\mathbf{s}_{\|}^2 + N_\perp^2)^{\frac{3}{2}}}, \ \forall |n_\perp| \geq N_\perp;$$

where $(\mathbf{s}_{\|}, n_\perp) \longmapsto n_\perp^{-2} (\mathbf{s}_{\|}^2 + N_\perp^2)^{-\frac{3}{2}} \chi_{\mathbb{R} \setminus]-N_\perp, N_\perp[}(n_\perp) \in \mathcal{L}^1(\mathbb{R}^2 \times \mathbb{Z}^*, \mathrm{d}\mathbf{s}_{\|} \otimes \mathrm{d}\Delta(n_\perp))$, and $(\mathbf{s}_{\|}, n_\perp) \longmapsto (\mathbf{s}_{\|}^2 + \eta^2)^{-\frac{1}{2}} \left| \Phi_\gamma\left(\sqrt{\mathbf{s}_{\|}^2 + n_\perp^2 + \eta^2}\right) \right| \chi_{[-N_\perp, N_\perp]}(n_\perp) \in \mathcal{L}^1(\mathbb{R}^2 \times \mathbb{Z}^*, \mathrm{d}\mathbf{s}_{\|} \otimes \mathrm{d}\Delta(n_\perp))$. ∎

Let now $\epsilon > 0$, and consider the fully regularized expression

$$H_{\phi,\delta,\eta,\epsilon}^F(L,d) = \frac{L^2}{4\pi^2} \kappa_d^{\gamma+1} \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_{\|} \sum_{p \in \mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}s_\perp \frac{f^\beta(s_\delta^{\|\perp})}{s_\delta^{\|\perp}} \Phi_\gamma(s_\eta^{\|\perp}) \mathrm{e}^{2\mathrm{i}\pi p s_\perp} \mathrm{e}^{-\epsilon p^2} \qquad (\bigstar_{\phi,\delta,\eta,\epsilon}^F)$$

As we will see later, the limits $(\delta, \epsilon) \to 0^+$ in Eq. $(\bigstar_{\phi,\delta,\eta,\epsilon}^F)$ are well-defined, whereas it will be not possible to take the limit $\eta \to 0^+$ in Eq. $(\bigstar_{\phi,\delta,\eta,\epsilon}^F)$ and then Eq. $(\bigstar_{\phi,\delta,\eta}^F)$, when the function Φ_γ will not satisfy the assumptions of theorem **2**. In this case, the regularization parameter $\eta > \frac{1}{2\pi}$ will be interpreted as a physical IR cut-off as mentioned in the body of this article.

Lemma 2. *The function* $(\mathbf{s}_{\|}, s_\perp, p) \longmapsto \frac{f^\beta(s_\delta^{\|\perp})}{s_\delta^{\|\perp}} \Phi_\gamma(s_\eta^{\|\perp}) \mathrm{e}^{2\mathrm{i}\pi p s_\perp} \mathrm{e}^{-\epsilon p^2}$ *is integrable on* $(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{Z}, \mathrm{d}^2\mathbf{s}_{\|} \otimes \mathrm{d}s_\perp \otimes \mathrm{d}\Delta(p))$. *And, the sum* $\sum_{n_\perp \in \mathbb{Z}}$ *and the integrals* $\int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_{\|}$ *and* $\int_{\mathbb{R}} \mathrm{d}s_\perp$ *in Eq.* $(\bigstar_{\phi,\delta,\eta,\epsilon}^F)$ *can be inverted.*

Proof. For any $\mathbf{s}_{\|} \in \mathbb{R}^2$, $s_\perp \in \mathbb{R}$ and $p \in \mathbb{Z}$, we have that

$$\left| \frac{f^\beta(s_\delta^{\|\perp})}{s_\delta^{\|\perp}} \Phi_\gamma(s_\eta^{\|\perp}) \mathrm{e}^{2\mathrm{i}\pi p s_\perp} \mathrm{e}^{-\epsilon p^2} \right| \leq \|f^\beta\|_\infty^{\mathbb{R}} \frac{\left|\Phi_\gamma(s_\eta^{\|\perp})\right|}{s^{\|\perp}} \mathrm{e}^{-\epsilon|p|}.$$

where, by Fubini-Tonelli theorem, and after performing the change of variable $s \to s_\eta$,

$$\int_{\mathbb{R}^2 \times \mathbb{R} \times \mathbb{Z}} \mathrm{d}^2\mathbf{s}_{\|} \otimes \mathrm{d}s_\perp \otimes \mathrm{d}\Delta(p) \left|\Phi_\gamma(s_\eta^{\|\perp})\right| \mathrm{e}^{-\epsilon|p|} \leq \frac{2}{1 - \mathrm{e}^{-\epsilon}} \int_{\mathbb{R}^2 \times \mathbb{R}} \mathrm{d}^2\mathbf{s}_{\|} \otimes \mathrm{d}s_\perp \frac{\left|\Phi_\gamma(s_\eta^{\|\perp})\right|}{s^{\|\perp}} \leq \frac{8\pi C_\eta}{1 - \mathrm{e}^{-\epsilon}},$$

with $C_\eta = \int_{\mathbb{R}_+} \mathrm{d}s\, s |\Phi_\gamma(s_\eta)| < \infty$. Then, $(\mathbf{s}_{\|}, s_\perp, p) \longmapsto \frac{f^\beta(s_\delta^{\|\perp})}{s_\delta^{\|\perp}} \Phi_\gamma(s_\eta^{\|\perp}) \mathrm{e}^{2\mathrm{i}\pi p s_\perp} \mathrm{e}^{-\epsilon p^2} \in \mathcal{L}^1(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{Z}, \mathrm{d}^2\mathbf{s}_{\|} \otimes \mathrm{d}s_\perp \otimes \mathrm{d}\Delta(p))$, and the inversion property is obtained by Fubini theorem. ∎

Lemma 3. *The family* $(H_\epsilon)_\epsilon$*, where* H_ϵ*:* $s \longmapsto \sum_{p \in \mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi p s} \mathrm{e}^{-\epsilon p^2}$*, for any* $\epsilon > 0$*, is convergent in* $\mathcal{S}'(\mathbb{R})$*, and goes to the Dirac comb* $\Delta = \widehat{\Delta}$*.*[8]

[8] This is a well known result in $\mathcal{S}'(\mathbb{R})$ that the Dirac comb Δ is equal to its Fourier transform $\widehat{\Delta}(s) = \sum_{p \in \mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi p s}$. Since $h \in \mathcal{S}(\mathbb{R})$ and $h_\epsilon(s) = \int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{2\mathrm{i}\pi s t} \mathrm{e}^{-\epsilon t^2}$, inversion theorem holds and yields $\mathrm{e}^{-\epsilon s^2} =$

Proof. Let the functions $h_\epsilon\colon\ s \longmapsto \frac{1}{\sqrt{\epsilon}} h(\frac{s}{\sqrt{\epsilon}})$ and $h\colon\ s \longmapsto \sqrt{\pi}\mathrm{e}^{-(\pi s)^2}$, which satisfies $\int_{\mathbb{R}} \mathrm{d}s\, h(s) = 1$, such that $h_\epsilon \xrightarrow[\epsilon\to 0^+]{} \delta_0$ in $\mathcal{S}'(\mathbb{R})$ [8]. We apply the same reasoning as in the proofs of propositions **1-4** to the series $H_\epsilon\colon\ s \longmapsto \sum_{p\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi p s}\mathrm{e}^{-\epsilon p^2}$. Using the Euler-Maclaurin formula, we have, for any $s \in \mathbb{R}$,

$$H_\epsilon(s) = \int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{2\mathrm{i}\pi s t}\mathrm{e}^{-\epsilon t^2} + 2\sum_{k\in\mathbb{N}^*}\int_{\mathbb{R}} \mathrm{d}t\, \cos 2k\pi t\, \mathrm{e}^{2\mathrm{i}\pi s t}\mathrm{e}^{-\epsilon t^2} = \sum_{k\in\mathbb{Z}} h_\epsilon(k-s) \geq 0.$$

Moreover, the function h_ϵ satisfies $H_\epsilon(s) \leq \frac{\pi^2}{3}\left[\sqrt{\frac{\pi}{\epsilon}}\left(s^2 + \frac{1}{\epsilon} + 1\right) + |s|\right]$, $\forall s \in \mathbb{R}$. Then, for any $\phi \in \mathcal{S}(\mathbb{R})$, $s \longmapsto H_\epsilon \phi \in \mathcal{L}^1(\mathbb{R}, \mathrm{d}s)$, *i.e.* $H_\epsilon \in \mathcal{S}'(\mathbb{R})$. And, by Fubini-Tonelli theorem,

$$\int_{\mathbb{R}\times\mathbb{Z}} \mathrm{d}s \otimes \mathrm{d}\Delta(k)\, h_\epsilon(k-s)|\phi(s)| = \int_{\mathbb{R}} \mathrm{d}s\, H_\epsilon(s)|\phi(s)| < \infty,$$

we obtain that $(s,k) \longmapsto h_\epsilon(k-s)\phi(s) \in \mathcal{L}^1(\mathbb{R}\times\mathbb{Z}, \mathrm{d}s\otimes\mathrm{d}\Delta(k))$. Then, by Fubini theorem,

$$\int_{\mathbb{R}} \mathrm{d}s H_\epsilon(s)\phi(s) = \int_{\mathbb{R}\times\mathbb{Z}} \mathrm{d}s\otimes\mathrm{d}\Delta(k) h_\epsilon(k-s)\phi(s) = \int_{\mathbb{R}\times\mathbb{Z}} \mathrm{d}s\otimes\mathrm{d}\Delta(k) h_\epsilon(s)\phi(s-k) = \int_{\mathbb{R}} h_\epsilon(s)\sum_{k\in\mathbb{Z}}\phi(s-k).$$

Since $s \longmapsto \sum_{k\in\mathbb{Z}}\phi(s-k) \in \mathcal{S}(\mathbb{R})$, the limit $\epsilon \to 0^+$ of the right-hand-side of this expression exists and we have

$$\int_{\mathbb{R}} \mathrm{d}s\, H_\epsilon(s)\phi(s) \xrightarrow[\epsilon\to 0^+]{} \sum_{k\in\mathbb{Z}}\phi(k) = \int_{\mathbb{R}} \mathrm{d}s\, \Delta(s)\phi(s) \quad\Leftrightarrow\quad H_\epsilon \xrightarrow[\epsilon\to 0^+]{} \Delta \text{ in } \mathcal{S}'(\mathbb{R}).$$

∎

Proposition 5. *The limit $\eta, \epsilon \to 0^+$ of Eq. $(\bigstar^F_{\phi,\delta,\eta,\epsilon})$ exists*

$$H^F_{\phi,\delta,\eta,\epsilon}(L,d) \xrightarrow[\eta,\epsilon\to 0^+]{} H^F_{\phi,0,\eta,0}(L,d) = H^F_{\phi,\eta}(L,d).$$

Proof. By lemma **2**, we have $(s_\perp, p) \longmapsto \frac{f^\beta(s^{\parallel\perp}_\delta)}{s^{\parallel\perp}_\delta}\Phi_\gamma(s^{\parallel\perp}_\eta)\mathrm{e}^{2\mathrm{i}\pi p s_\perp}\mathrm{e}^{-\epsilon p^2} \in \mathcal{L}^1(\mathbb{R}\times\mathbb{Z}, \mathrm{d}s_\perp\otimes\mathrm{d}\Delta(p))$, for any $\epsilon > 0$ and $\mathbf{s}_\parallel \in \mathbb{R}^2$, then it is possible to inverse the sum $\sum_{p\in\mathbb{Z}}$ and the integral $\int_{\mathbb{R}} \mathrm{d}s_\perp$

$$H^F_{\phi,\delta,\eta,\epsilon}(L,d) = \frac{L^2}{4\pi^2}\kappa_d^{\gamma+1}\int_{\mathbb{R}^2}\mathrm{d}^2\mathbf{s}_\parallel\int_{\mathbb{R}}\mathrm{d}s_\perp \frac{f^\beta(s^{\parallel\perp}_\delta)}{s^{\parallel\perp}_\delta} H_\epsilon(s_\perp)\Phi_\gamma(s^{\parallel\perp}_\eta).$$

$\int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{2\mathrm{i}\pi s t}h_\epsilon(t)$. Moreover, for any $\phi \in \mathcal{S}(\mathbb{R})$, by the dominated convergence theorem,

$$\int_{\mathbb{R}} \mathrm{d}s\, h_\epsilon(s)\phi(s) = \int_{\mathbb{R}} \mathrm{d}s\, h(s)\phi(\sqrt{\epsilon}s) \xrightarrow[\epsilon\to 0^+]{} \phi(0) = \int_{\mathbb{R}} \mathrm{d}s\, \delta_0(s)\phi(s) \quad\Leftrightarrow\quad h_\epsilon \xrightarrow[\epsilon\to 0^+]{} \delta_0 \text{ in } \mathcal{S}'(\mathbb{R}).$$

From this result and since $(s,t) \longmapsto h_\epsilon(t)\phi(s) \in \mathcal{L}^1(\mathbb{R}^2, \mathrm{d}s\mathrm{d}t)$ we get

$$\int_{\mathbb{R}} \mathrm{d}s\, \phi(s) \xleftarrow[\epsilon\to 0^+]{} \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{-\epsilon s^2}\phi(s) = \int_{\mathbb{R}} \mathrm{d}t\, h_\epsilon(t)\widehat{\phi}(t) \xrightarrow[\epsilon\to 0^+]{} \int_{\mathbb{R}} \mathrm{d}t\, \delta_0(t)\widehat{\phi}(t) \quad\Leftrightarrow\quad \widehat{\delta_0} = 1 \text{ in } \mathcal{S}'(\mathbb{R}).$$

Using the translation property of the Fourier transform, we have $\widehat{\delta_p}(s) = \mathrm{e}^{2\mathrm{i}\pi p s}$, $\forall k \in \mathbb{Z}$, and by linearity of the Fourier transform in $\mathcal{S}'(\mathbb{R})$, we obtain that $\widehat{\Delta}(s) = \sum_{p\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi p s}$. Finally, using Poisson formula for $\phi \in \mathcal{S}(\mathbb{R})$, we deduce

$$\int_{\mathbb{R}} \mathrm{d}s\, \Delta(s)\phi(s) = \sum_{n\in\mathbb{Z}}\phi(n) = \sum_{p\in\mathbb{Z}}\widehat{\phi}(p) = \int_{\mathbb{R}} \mathrm{d}s\, \widehat{\Delta}(s)\phi(s) \quad\Leftrightarrow\quad \Delta = \widehat{\Delta} \text{ in } \mathcal{S}'(\mathbb{R}).$$

Moreover, by lemma **3**, we have, for almost any $(\mathbf{s}_\parallel, s_\perp) \in \mathbb{R}^2 \times \mathbb{R}$,

$$\left| \frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}} H_\epsilon(s_\perp) \Phi_\gamma(s_\eta^{\parallel\perp}) \right| \le \frac{\pi^2}{3} \|f^\beta\|_\infty^{\mathbb{R}} \frac{\Phi(s_\eta^{\parallel\perp})}{s^{\parallel\perp}},$$

where $\Phi\colon\ s \longmapsto s\big[\sqrt{\frac{\pi}{\epsilon}}\left(s^2 + \frac{1}{\epsilon} + 1\right) + |s|\big]|\Phi_\gamma(s)| \in \mathcal{L}^1(\mathbb{R}, \mathrm{d}s)$. This upper-bound almost surely on $\mathbb{R}^2 \times \mathbb{R}$ is independent from $\delta \in]0,\eta]$ and is also valid for $\delta = 0$. Then, $(\mathbf{s}_\parallel, s_\perp) \longmapsto \frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}} H_\epsilon(s_\perp)\Phi_\gamma(s_\eta^{\parallel\perp}) \in \mathcal{L}^1(\mathbb{R}^2 \times \mathbb{R}, \mathrm{d}^2\mathbf{s}_\parallel \otimes \mathrm{d}s_\perp)$, so that Fubini theorem yields, for any $\epsilon > 0$ and any $\delta \in [0, \eta]$

$$H^F_{\phi,\delta,\eta,\epsilon}(L,d) = \frac{L^2}{4\pi^2}\kappa_d^{\gamma+1} \int_{\mathbb{R}^2\times\mathbb{R}} \mathrm{d}^2\mathbf{s}_\parallel \otimes \mathrm{d}s_\perp \frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}} H_\epsilon(s_\perp)\Phi_\gamma(s_\eta^{\parallel\perp}),$$

and the function $\delta \longmapsto H^F_{\phi,\delta,\eta,\epsilon}(L,d)$ is continuous $[0,\eta]$. Then, the limit $\delta \to 0^+$ exists

$$H^F_{\phi,\delta,\eta,\epsilon}(L,d) \xrightarrow[\delta\to 0^+]{} H^F_{\phi,0,\eta,\epsilon}(L,d).$$

But, for any $\delta \in [0,\eta]$, since $f^\beta \in \mathrm{C}^0(\mathbb{R})$ is bounded on $\mathbb{R}$, $(\mathbf{s}_\parallel, s_\perp) \longmapsto \frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}} H_\epsilon(s_\perp) \in \mathcal{S}'(\mathbb{R}^2 \times \mathbb{R})$. Then, by lemma **3**, $\frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}} H_\epsilon(s_\perp) \xrightarrow[\epsilon\to 0^+]{} f^\beta(s_\delta^{\parallel\perp})\Delta(s_\perp)$ in $S'(\mathbb{R}^2 \times \mathbb{R})$. Therefore, since $(\mathbf{s}_\parallel, s_\perp) \longmapsto \Phi_\gamma(s_\eta^{\parallel\perp}) \in \mathcal{S}(\mathbb{R}^2,\mathbb{R})$, $H^F_{\phi,\delta,\eta,\epsilon}(L,d) \xrightarrow[\epsilon\to 0^+]{} H^F_{\phi,\delta,\eta,0}(L,d)$, where by properties of the Dirac comb Δ, we have

$$\begin{aligned} H^F_{\phi,\delta,\eta,0}(L,d) &= \frac{L^2}{4\pi^2}\kappa_d^{\gamma+1} \int_{\mathbb{R}^2\times\mathbb{R}} \mathrm{d}^2\mathbf{s}_\parallel \otimes \mathrm{d}s_\perp \frac{f^\beta(s_\delta^{\parallel\perp})}{s_\delta^{\parallel\perp}} \Delta(s_\perp)\Phi_\gamma(s_\eta^{\parallel\perp}). \\ &= \frac{L^2}{4\pi^2}\kappa_d^{\gamma+1} \sum_{n_\perp\in\mathbb{Z}} \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{s}_\parallel \frac{f^\beta\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}\right)}{\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \delta^2}} \Phi_\gamma\left(\sqrt{\mathbf{s}_\parallel^2 + n_\perp^2 + \eta^2}\right) = H^F_{\phi,\delta,\eta}(L,d). \end{aligned}$$

This implies in particular that $H^F_{\phi,0,\eta,\epsilon}(L,d) \xrightarrow[\epsilon\to 0^+]{} H^F_{\phi,0,\eta,0}(L,d) = H^F_{\phi,0,\eta}(L,d)$, the reciprocal being directly given by the corollary of lemma **1**. ∎

Theorem 1. *For any* $\delta \in [0,\eta]$,

$$\begin{aligned} H^F_{\phi,\delta,\eta}(L,d) &= \frac{V}{\pi^2}\kappa_d^{\gamma+2} \int_{\mathbb{R}_+} \mathrm{d}s\, s^2 \left[1 + \frac{g(s)}{\pi s}\right] \frac{f^\beta(s_\delta)}{s_\delta}\Phi_\gamma(s_\eta) \\ &= V \int_{\mathbb{R}_+} \mathrm{d}k\, [\rho(k) + \rho_d(k)] \frac{f(\sqrt{k^2 + (\kappa_d\delta)^2})}{\sqrt{k^2+(\kappa_d\delta)^2}} f_\gamma\left(\sqrt{k^2 + (\kappa_d\eta)^2}\right)\phi\left(\frac{\sqrt{k^2+(\kappa_d\eta)^2}}{\kappa_\phi}\right). \end{aligned} \qquad (\bigstar^F_{\phi,\delta,\eta})$$

Proof. By lemma **2**, the sum $\sum_{p\in\mathbb{Z}}$ and the integral $\int_{\mathbb{R}} \mathrm{d}s_\perp$ can be inverted in Eq. $(\bigstar^F_{\phi,\delta,\eta,\epsilon})$, yielding

$$\begin{aligned} H^F_{\phi,\delta,\eta,\epsilon}(L,d) &= \frac{L^2}{\pi}\kappa_d^{\gamma+1} \int_{\mathbb{R}_+} \mathrm{d}s\, s^2 \frac{f^\beta(s_\delta)}{s_\delta}\Phi_\gamma(s_\eta) \sum_{p\in\mathbb{Z}} \frac{\sin 2\pi p s}{2\pi p s} \mathrm{e}^{-\epsilon p^2} \\ &= \frac{L^2}{\pi}\kappa_d^{\gamma+1} \int_{\mathbb{R}_+} \mathrm{d}s\, s^2 \left[1 + \frac{g_\epsilon(s)}{\pi s}\right] \frac{f^\beta(s_\delta)}{s_\delta}\Phi_\gamma(s_\eta). \end{aligned}$$

By the same reasoning as in the proof of proposition **5**, since by proposition **4**, $g_\epsilon \xrightarrow[\epsilon\to 0^+]{} g$ in $\mathcal{S}'(\mathbb{R})$, *i.e.* the limit $\epsilon \to 0^+$ of the right-hand-side of the previous expression exists

$$H^F_{\phi,\delta,\eta}(L,d) \xleftarrow[\epsilon\to 0^+]{} H^F_{\phi,\delta,\eta,\epsilon}(L,d) \xrightarrow[\epsilon\to 0^+]{} \frac{L^2}{\pi}\kappa_d^{\gamma+1}\int_{\mathbb{R}_+} \mathrm{d}s\, s^2\left[1+\frac{g(s)}{\pi s}\right]\frac{f^\beta(s_\delta)}{s_\delta}\Phi_\gamma(s_\eta).$$

∎

Theorem 2. *Assuming that* $\gamma > -1$, *then*

$$\begin{aligned} H^F_\phi(L,d) &= \frac{V}{\pi^2}\kappa_d^{\gamma+2}\int_{\mathbb{R}_+}\mathrm{d}s\, s\left[1+\frac{g(s)}{\pi s}\right]f^\beta(s)\Phi_\gamma(s) \\ &= V\int_{\mathbb{R}_+}\mathrm{d}k\,[\rho(k)+\rho_d(k)]\,F(k)\phi\left(\frac{k}{\kappa_\phi}\right). \end{aligned} \qquad (\bigstar^F_\phi)$$

Proof. If $\gamma > -1$, for $i = 0, 1$, the function h_i: $s \longmapsto s^i f_\gamma(s)$ is homogeneous of ratio $(\gamma+i) > 0$, then it is of class $\mathcal{C}^0$ on $\mathbb{R}$, and $h_i(s) = O(s^{\gamma+i})$ in the limit $s \to \infty$. Then, $\exists S \geq 0$ such that $|h_i(s)|\phi(\frac{\kappa_d}{\kappa_\phi}s) \leq \frac{1}{s^2}$, $\forall s \geq S$. Let $\mathbb{K} \subseteq \mathbb{R}_+$ a compact set, for any $\eta \in \mathbb{K}$ and $s \geq S$

$$\left|s^2\left[1+\frac{g(s)}{\pi s}\right]\frac{f^\beta(s_\eta)}{s_\eta}\Phi_\gamma(s_\eta)\right| \leq \|f^\beta\|^{\mathbb{R}}_\infty\left[|h_1(s_\eta)| + \frac{|h_0(s_\eta)|}{2}\right]\phi\left(\frac{\kappa_d}{\kappa_\phi}s_\eta\right) \leq \frac{3}{2}\frac{\|f^\beta\|^{\mathbb{R}}_\infty}{s^2}.$$

Moreover, $(\eta, s) \longmapsto \left[|h_1(s_\eta)| + \frac{|h_0(s_\eta)|}{2}\right]\phi(s_\eta) \in \mathcal{C}^0(\mathbb{K}\times[-S,S])$, then there exists $C_{\mathbb{K},S} \geq 0$ such that $\left|s^2\left[1+\frac{g(s)}{\pi s}\right]\frac{f^\beta(s_\eta)}{s_\eta}\Phi_\gamma(s_\eta)\right| \leq \|f^\beta\|^{\mathbb{R}}_\infty\left[|h_1(s_\eta)| + \frac{|h_0(s_\eta)|}{2}\right]\phi(s_\eta) \leq C_{\mathbb{K},S}$, $\forall \eta \in \mathbb{K}$ and $s \in [-S,S]$. Then, $\eta \longmapsto H^F_{\phi,\eta,\eta} \in \mathcal{C}^0(\mathbb{R}_+)$, and the limit of Eq. $(\bigstar^F_{\phi,\eta,\eta})$ is well-defined

$$H^F_{\phi,\eta,\eta}(L,d) \xrightarrow[\eta\to 0^+]{} H^F_{\phi,0,0}(L,d) = H^F_\phi(L,d).$$

∎

B.3 Casimir energy

Here, we are interested in the computation of the standard Casimir effect using the formalism developed in appendix **B**. This consists in the determination of the contribution to the electromagnetic energy gievn by Eq. (2) due to the Casimir plates, *i.e.* the term depending on the function g, which is the correction to the density of states in vacuum

$$E^\phi_{\text{Casimir}}(L,d) = \frac{\pi}{2}\frac{L^2}{d^3}\int_{\mathbb{R}}\mathrm{d}s\, s^2\,(g\chi_{\mathbb{R}_+})(s)\phi\left(\frac{\kappa_d}{\kappa_\phi}s\right),$$

where we have directly identified $F(k) = \frac{k}{2}$, then deduce $f^\beta = 1$, $f_\gamma(s) = \frac{s^2}{2}$, $\beta = 0$ and $\gamma = 2$, and then apply theorem **2**. This is the action of the distribution $g\chi_{\mathbb{R}_+} \in \mathcal{S}'(\mathbb{R})$ on the test function $s \longmapsto s^2\phi(\frac{\kappa_d}{\kappa_\phi}s) \in \mathcal{S}(\mathbb{R})$. But, the action of its derivative is simpler, since $(g\chi_{\mathbb{R}_+})' = \pi\left[-\chi_{\mathbb{R}_+\setminus\mathbb{N}} + \sum_{p\in\mathbb{N}^*}\delta_p + \frac{\delta_0}{2}\right]$ in $\mathcal{S}'(\mathbb{R})$. Then, introducing the mapping Φ: $s \longmapsto \Phi(s) = \int_s^\infty \mathrm{d}t\, t^2\phi\left(\frac{\kappa_d}{\kappa_\phi}t\right) \in \mathcal{C}^\infty(\mathbb{R})$, which belongs also to $\mathcal{B}(\mathbb{R})$, we get

$$E^\phi_{\text{Casimir}}(L,d) = \frac{\pi}{2}\frac{L^2}{d^3}\int_{\mathbb{R}}\mathrm{d}s\,\Phi(s)(g\chi_{\mathbb{R}_+})'(s) = \frac{\pi^2}{2}\frac{L^2}{d^3}\left[\sum_{p\in\mathbb{N}^*}\Phi(p) + \frac{\Phi(0)}{2} - \int_{\mathbb{R}_+}\mathrm{d}s\,\Phi(s)\right].$$

We have that $\Phi'(0) = \Phi''(0) = 0$, $\Phi^{(3)}(0) = -2$ and $\Phi^{(k)}(0) = O\left(\frac{\kappa_d}{\kappa_\phi}\right)$, for any $k \geq 4$, and that for any $r \geq 3$

$$\left|\int_{\mathbb{R}_+} \mathrm{d}t\, \frac{\widetilde{B}_{2r}(t)}{(2r)!}\Phi^{(2r)}(t)\right| \leq \frac{|b_{2r}|}{(2r)!}\int_{\mathbb{R}_+} \mathrm{d}t \left|\frac{\mathrm{d}^{2r-1}}{\mathrm{d}t^{2r-1}} t^2 \phi\left(\frac{\kappa_d}{\kappa_\phi} t\right)\right| = \frac{2\zeta(2r)}{(2\pi)^{2r}}\left(\frac{\kappa_d}{\kappa_\phi}\right)^{2(r-2)} \int_{\mathbb{R}_+} \mathrm{d}s \left|\frac{\mathrm{d}^{2r-1}}{\mathrm{d}s^{2r-1}} s^2 \phi(s)\right|$$
$$= O\left[\left(\frac{\kappa_d}{\kappa_\phi}\right)^{2(r-2)}\right] \xrightarrow[\frac{\kappa_d}{\kappa_\phi}\to 0]{} 0.$$

From this result, we obtain the correct Casimir energy in the presence of the plates using the Euler-Maclaurin formula in the limit $\frac{\kappa_d}{\kappa_\phi} \to 0$, *i.e.*

$$E_{\text{Casimir}}(L, d) = -\frac{\pi^2}{720}\frac{L^2}{d^3}.$$

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