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Multiobjective programming approaches in bilevel programming problems

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Dedication

- To my wife **Leonelle PIEUME** and my son **Nathan PIEUME**.

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List of abbreviations

BPP: **B**ilevel **P**rogramming **P**roblem (s)

B0P: **B**ilevel **O**ptimization **P**roblem (s)

BLPP: **B**ilevel **L**inear **P**rogramming **P**roblem (s)

MOPP: **M**ultiple **O**bjective **P**rogramming **P**roblem (s)

MOLPP: **M**ultiple **O**bjective **L**inear **P**rogramming **P**roblem (s)

BMPP: **B**ilevel **M**ultiobjective **P**rogramming **P**roblem (s)

BMLPP: **B**ilevel **M**ultiobjective **L**inear **P**rogramming **P**roblem (s)

DM: **D**ecision **M**aker (s)

GP: **G**oal **P**rogramming

Abstract

This thesis addresses two important classes of optimization: multiobjective optimization and bilevel optimization. The investigation concerns their solution methods, applications, and possible links between them.

First of all, we develop a procedure for solving Multiple Objective Linear Programming Problems (MOLPP). The method is based on a new characterization of efficient faces. It exploits the connectedness property of the set of ideal tableaux associated to degenerated points in the case of degeneracy. We also develop an approach for solving Bilevel Linear Programming Problems (BLPP). It is based on the result that an optimal solution of the BLPP is reachable at an extreme point of the underlying region. Consequently, we develop a pivoting technique to find the global optimal solution on an expanded tableau that represents the data of the BLPP. The solutions obtained by our algorithm on some problems available in the literature show that these problems were until now wrongly solved.

Some applications of these two areas of optimization problems are explored. An application of multicriteria optimization techniques for finding an optimal planning for the distribution of electrical energy in Cameroon is provided. Similarly, a bilevel optimization model that could permit to protect any economic sector where local initiatives are threatened is proposed.

Finally, the relationship between the two classes of optimization is investigated. We first look at the conditions that guarantee that the optimal solution of a given BPP is Pareto optimal for both upper and lower level objective functions. We then introduce a new relation that establishes a link between MOLPP and BLPP. Moreover, we show that, to solve a BPP, it is possible to solve two artificial M0PPs. In addition, we explore Bilevel Multiobjective Programming Problem (BMPP), a case of BPP where each decision maker (DM) has more than one objective function. Given a BMPP, we show how to construct two artificial M0PPs such that any point that is efficient for both problems is also efficient for the BMPP. For the linear case specially, we introduce an artificial MOLPP such that its resolution can permit to generate the whole feasible set of the leader DM. Based on this result and depending on whether the leader can evaluate or not his preferences for his different objective functions, two approaches for obtaining efficient solutions are presented.

Keywords: Multicriteria optimization, Bilevel optimization, Efficient point, Pareto-optimal point.

Resumé

Contexte et résultats obtenus

La convergence de divers facteurs tels que la concurrence accrue dans une économie mondiale, l'accès en temps réel à une foule d'informations transparentes et l'apparition d'une génération de consommateurs avertis et pragmatiques sont entrain de modifier le rôle et la nature de l'établissement des prises de décision. Bien que dans le passé, les modèles d'aides à la décision aient principalement fait l'objet de recherches universitaires pures, ces modèles constituent aujourd'hui des outils financiers et opérationnels fondamentaux au sein d'importants secteurs industriels et se rangent maintenant parmi les principaux facteurs de rentabilité. C'est dans cette optique qu'il est observé aussi bien au sein du secteur des affaires que dans la communauté des chercheurs en mathématiques appliquées une volonté croissante de développer des approches de prises de décision optimales.

Les problèmes de décision se modélisent en général par des problèmes d'optimisation sous contraintes. L'optimisation est l'un des domaines des mathématiques appliquées en plein expansion qui s'est imposée dès ses débuts dans les années 1940 comme un outil essentiel et incontournable dans la prise de décision. Le courant principal de recherche dans ce domaine s'est toutefois beaucoup plus préoccupé des modèles unicritères, c'est-à-dire, des modèles mathématiques bien définis, régis par le paradigme classique d'existence de solutions optimales, de caractérisations et des algorithmes convergeant vers celles-ci.

Même si l'utilisation de ces modèles unicritères s'est révélée particulièrement efficace dans de nombreuses situations, il est toutefois apparu qu'il existe une multitude d'applications pour lesquelles ne prendre en compte qu'un seul décideur ou un seul critère de décision ne répond en aucun cas à la réalité. C'est vers le milieu des années 1980 que la nécessité d'une aide multicritère à la décision se fit cruellement sentir et ce fut le mérite exceptionnel des scientifiques de renom tels que *B. Roy* pour l'école française et *A. Geofrion* pour l'école américaine. Depuis lors, l'aide multicritère à la décision connaît une expansion fulgurante avec le développement de deux principaux courants de recherches: l'optimisation multi-objectifs et l'optimisation à plusieurs niveaux.

Un problème d'optimisation à deux niveaux (BPP) est caractérisé par deux niveaux de prise de décision classés de manière hiérarchique et où chaque décideur essaie d'optimiser sa propre fonction objectif, sans tenir compte de l'objectif de l'autre partie, mais la décision de chaque partie affectant la valeur optimale de l'autre partie ainsi que l'espace de décision. Ces problèmes apparaissent dans de nombreuses situations réelles dont entre autres le contrôle optimal, l'optimisation des processus, le problème de transports.

L'optimisation multi-objectifs quant à elle, est une sorte de généralisation de l'optimisation classique dans laquelle plusieurs fonctions objectif conflictuelles sont maximisées (ou minimisées) simultanément. La plupart des problèmes de décision réalistes nécessitent l'optimisation simultanée de plus d'une fonction objectif.

Ces deux classes d'optimisation ont fait l'objet de nombreuses publications depuis une trentaine d'années. Cependant, très peu d'études ont porté sur les liens possibles entre les deux classes d'optimisation. La présente thèse porte principalement sur l'application des concepts et techniques d'optimisation multicritère en optimisation à deux niveaux. Les méthodes de résolution et les applications de chacune de ces deux classes d'optimisation sont aussi abordées. Dans le cadre des travaux réalisés, plusieurs apports ont été éffectués, notamment:

- Le développement d'une approche de résolution des problèmes d'optimisation linéaire multicritère;
- Le développement d'une approche de résolution des problèmes d'optimisation linéaire à deux niveaux;
- La proposition des solutions exactes à certains problèmes d'optimisation linéaire à deux niveaux jusque là mal résolus dans la littérature;
- L'établissement de deux nouvelles relations entre l'optimisation multicritère et l'optimisation à deux niveaux;
- La proposition de conditions de Pareto-optimalité pour les solutions des problèmes d'optimisation à deux niveaux;
- La proposition d'une nouvelle caractérisation de l'ensemble de solutions admissibles du problème du leader lors de la résolution de certaines classes de problème d'optimisation multicritère à deux niveaux;
- La proposition d'une approche de détermination des solutions efficaces lors de la résolution des problèmes d'optimisation multicritère à deux niveaux;

- La proposition de deux approches de résolutions des problèmes d'optimisation linéaire multicritère à deux niveaux.

Ce qui suit est une présentation du condensé de la thèse, chapitre par chapitre à partir du chapitre 2, le chapitre 1 tenant lieu de préliminaire.

L'optimisation multicritère

Le chapitre 2 traite de l'optimisation multiobjectifs. Un problème d'optimisation multicritère avec n variables de décisions, $m = m_1 + m_2$ contraintes et p fonctions objectif peut être formulé comme suit:

$$\text{" min "} f(x) = (f_1(x), f_2(x), ..., f_p(x))$$
$$\text{sujet à} \begin{cases} g_i(x) \leq 0, \ i \in I = \{1, .., m_1\} \\ \\ h_j(x) = 0, \ j \in J = \{1, .., m_2\} \\ x \geqq 0 \end{cases} \quad \text{(MOPP)}$$

La notation " min " évite ici l'obstacle de la mauvaise formulation du problème (MOPP) car la fonction de minimisation ne peut pas s'appliquer à un sous ensemble de $\mathbb{R}^n$ avec $n > 1$. Aussi, il n'existe pas en général une solution admissible qui optimise simultanément toutes les fonctions objectif. Résoudre le problème (MOPP) revient à présenter au décideur un ensemble de bonnes solutions possibles et éventuellement, avec les préférences du décideur, une solution de compromis (qui satisfasse au mieux le décideur). De telles solutions sont appelées solutions efficaces.

Après un parcours des principales caractérisations et propriétés des solutions efficaces, nous faisons une revue des différentes méthodes de résolutions des problèmes (MOPP). Il ressort qu'il existe trois familles d'approches, selon que le décideur interviene avant, pendant ou après le processus de sélection de la solution (efficace). Si les préférences du décideur sont connues préalablement à la procédure de resolution, on parle d'articulation a priori des préférences. Si par contre, le choix du décideur s'exprime après que la procédure de resolution lui fournisse l'ensemble des solutions efficaces, on parle d'une articulation a posteriori. Si enfin, ses préférences sont exprimées progressivement, au fur et à mesure de l'avancement de la procédure de resolution, on parle de méthode interactive.

Nous présentons ensuite une approche a priori de résolution des problèmes linéaires d'optimisation multicritère. L'approche est une généralisation de la méthode *d'Ecker et Kouada* qui est inefficace dans le cas de problèmes dégénérés car: (i) leurs méthodes ne guarantit pas le parcours de l'ensemble des sommets extrêmes efficaces et (ii) l'ensemble des solutions obtenues peut contenir des solutions non-efficaces.

Notre approche surmonte le problème (i) en débutant avec un sommet extrême efficace initial associé à un tableau idéal et en procédant ensuite au parcours des différents sommets extrêmes par la technique de pivotage lexicographique. Pour cela, nous exploitons le résultat de *P. Armand* selon lequel l'ensemble des tableaux idéaux associés aux sommets extrêmes efficaces dégénérés est connexe. Le problème (ii) est resolu par l'utilisation d'une nouvelle caractérisation des faces efficaces que nous avons développée. Ainsi, la méthode parcours les sommets extrêmes efficaces pour générer l'ensemble des faces efficaces dont l'union est l'ensemble des solutions efficaces.

Nous clôturons le chapitre par la présentation d'une application de l'optimisation multicritère au problème de planification de la distribution de l'énergie électrique au Cameroun.

Optimisation à deux niveaux

Le chapitre 3 est consacré à l'optimisation à deux niveaux. La formulation générale d'un problème d'optimisation à deux niveaux peut être donnée par :

$$\begin{aligned} &\min_{x\in X}\ F(x,y) \\ &\text{sujet à} \begin{cases} G(x,y) \leq 0 \\ \\ \text{où y résoud} \begin{cases} \min_{y\in Y} f(x,y) \\ \text{sujet à} \\ g(x,y) \leq 0 \end{cases} \end{cases} \end{aligned} \quad \text{(BPP)}$$

Nous débutons par une brève revue de littérature sur cette classe de problème d'optimisation. Il ressort que la résolution du problème (BPP) dépend significativement du cardinal de l'ensemble de réactions du suiveur pour un choix quelconque du leader. Si son cardinal est nul, alors le problème (BPP) n'a pas de solution. Si cet ensemble est de cardinal un, alors (BPP) peut être transformé en un problème d'optimisation unicritère classique. Si par contre, le cardinal de cet ensemble est supérieur ou égal à deux, alors le leader se trouve face à un dilemme en ce sens que même s'il connaît sa région réalisable, il ne sait pas quelle sera la décision du suiveur. Dans ce cas, deux situations sont envisageables : l'approche optimiste et l'approche pessimiste. Avec l'approche optimiste, le leader suppose que le suiveur sélectionnera le choix qui contribuera à l'optimisation de sa fonction objectif. Ce raisonnement n'est valable que si la collaboration entre le suiveur et le leader est permise. Au cas où elle n'est pas possible ou lorsque le leader n'est pas capable d'influencer le choix du suiveur, le leader limite les dégâts pouvant résulter d'un choix indésirable du suiveur. Dans ce cas, il raisonne comme si le suiveur prendra la décision qui détériore le plus sa fonction objectif, c'est l'approche pessimiste.

Nous présentons ensuite quelques méthodes développées dans la littérature pour résoudre des classes particulières de (BPP). Cette présentation est suivie par le développement d'une méthode de résolution des problèmes linéaires d'optimisation à deux niveaux (BLPP), où nous supposons que la réaction du suiveur est unique à chaque choix du leader. Nous formulons et démontrons le résultat selon lequel une solution optimale du problème (BLPP) se trouve parmi les sommets extrêmes de l'ensemble des solutions réalisable. Nous proposons par la suite une approche de parcours des sommets extrêmes à partir d'un tableau simplexe initial qui permet de passer d'un sommet extrême vers le sommet extrême adjacent avec la valeur optimale la moins dégradée. Nous exploitons ce résultat pour développer un algorithme de détermination d'une solution optimale du problème (BLPP). Nous montrons alors que toute solution obtenue de cet algorithme est effectivement une solution optimale du problème d'optimisation linéaire à deux niveaux résolu. A partir de l'implémentation de l'approche développée, nous proposons les solutions exactes de deux problèmes d'optimisation linéaire à deux niveaux jusque la mal résolus dans la littérature.

Nous achevons le chapitre par la présentation d'une application de l'optimisation à deux niveaux dans un contexte de mondialisation où les entreprises nationales ne sont pas encore capables de supporter la concurrence des firmes internationales et où il est indispensable d'introduire une taxe pour réguler cette inégale concurrence. Nous montrons que la modélisation du problème de détermination du taux de taxe qu'il faudra appliquer conduit à un problème d'optimisation à deux niveaux.

Les techniques d'optimisation multicritère en optimisation à deux niveaux

Le chapitre quatre est une étude de l'application des techniques d'optimisation multicritère en optimisation à deux niveaux. Le problème d'optimisation bicritère (BOP) associé au problème d'optimisation à deux niveaux (BPP) présenté dans le chapitre précédent est donné par :

$$\min_{x,y} \; T(x,y) = (F(x,y),\; f(x,y))$$
$$\text{sujet à} \begin{cases} G(x,y) \leq 0 \\ \\ g(x,y) \leq 0 \end{cases} \qquad \text{(BOP)}$$

Le chapitre débute par l'étude de possibilité qu'une solution optimale du problème d'optimisation à deux niveaux (BPP) soit solution Pareto-optimale du problème bicritère associé (BOP). Nous montrons qu'en général, cela n'est pas possible, sauf sous certaines conditions particulières.

Nous poursuivons par une analyse post-optimale qui consiste à montrer que s'il y a collaboration entre les deux décideurs, il est possible de déterminer une solution Pareto-optimale de BOP procurant à chacun une valeur optimale meilleure que la solution optimale obtenue en résolvant le problème d'optimisation à deux niveaux (BPP).

Par la suite, nous présentons une généralisation de l'approche de [Fulop 1993] qui établit une relation entre les problèmes linéaires à deux niveaux et une certaine classe de problèmes linéaires multi-objectif. Nous montrons que ce résultat peut être valide pour certaines classes de problèmes d'optimisation non linéaire à deux niveaux. En effet, si la fonction objectif du suiveur est continue pour chaque choix du leader, et si le domaine admissible du suiveur est compact, alors le résultat reste aussi valide.

Nous achevons le chapitre par la présentation d'une nouvelle relation entre certaines classes de problèmes d'optimisation à deux niveaux et les problèmes d'optimisation multicritère. Nous montrons que la solution optimale d'un problème d'optimisation à deux niveaux peut s'obtenir par la résolution de deux problèmes d'optimisation multicritère. Une solution optimale étant la solution Pareto-optimale correspondant au point non-dominé appartenant à l'intersection des deux ensembles non-dominés. Nous clôturons ce chapitre par la présentation de quatre conditions sous lesquelles ce dernier résultat établi ne peut être implémenté.

Modélisation multicritère en optimisation à deux niveaux

Le chapitre cinq étudie l'introduction de la modélisation multicritère en optimisation à deux niveaux. L'introduction de la modélisation multicritère en optimisation à deux niveaux conduit à une nouvelle classe de problème d'optimisation appelée optimisation multicritère à deux niveaux. C'est un cas particulier de problème à deux niveaux, où chaque décideur possède plusieurs fonctions objectif conflictuelles et désire les optimiser simultanément. Ce problème peut être formulé comme suit :

$$\min_{x\in X} F(x,y) = (F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y))$$
$$\text{sujet à} \begin{cases} G(x) \leq 0 \\ \text{où y résoud} \begin{cases} \min_{y\in Y} f(x,y) = (f_1(x,y), f_2(x,y), ..., f_{m_2}(x,y)) \\ \text{sujet à } \; g(x,y) \leq 0 \end{cases} \end{cases} \quad \text{(BMPP)}$$

Après une brève revue de la littérature de cette classe de problème d'optimisation, nous enchaînons par quelques caractérisations de la formulation optimiste.

Nous développons ensuite une approche de détermination des solutions efficaces lors de la résolution du problème (BMPP). A partir de deux cônes artificiels, nous construisons deux problèmes d'optimisation multicritère dont tout point appartenant à l'intersection des deux ensembles efficaces est aussi efficace pour le problème (BMPP). Nous présentons ensuite un algorithme générique permettant d'illustrer comment ce résultat peut être implémenté. Mais avant cela, nous développons quelque résultats théoriques permettant de définir sous quelles conditions utiliser l'algorithme. Nous exploitons par la suite l'approche générique pour construire une méthode spécifique aux problèmes linéaires. La méthode est illustrée à travers un exemple concret.

Nous exploitons enfin la caractérisation du domaine admissible du problème du leadeur proposée par in [Eichfelder 2008], pour développer une nouvelle caractérisation du domaine admissible du problème du leader plus conviviale et facile à implémenter. En se basant sur ce résultat, nous développons deux approches pouvant être utilisées lors de la résolution la forme linéaire du problème (BMPP). En effet, en supposant que le leader est capable de quantifier ses préférences pour l'ensemble des fonctions objectif, nous montrons qu'il est possible que le problème linéaire découlant de (BMPP) soit transformé en un problème d'optimisation d'une fonction linéaire avec comme domaine admissible l'ensemble des solutions Pareto-optimales d'un problème artificiel d'optimisation linéaire multicritère. Un algorithme basé sur ce résultat est présenté. La deuxième approche est une méthode permettant de générer un sous ensemble représentatif de l'ensemble des solutions Pareto-optimales.

Mots clés: *Optimisation, Optimisation à deux niveaux, Optimisation multicritère, Solution non dominée, Solution Pareto-optimale, Solution efficace.*

GENERAL INTRODUCTION

The convergence of various factors, such as increased competition in a global economy, the real-time access to a mass of information and the emergence of a generation of pragmatic consumers, are changing the role and nature of decision-making. Although in the past decision aid models have mostly been pure academic research, these models are now financial and basic operational tools in major industrial sectors. They rank among the key factors for profitability. For example, an increase of one percentage point in the price of a product can lead to increases in operating income more important than similar increase in variable costs, fixed costs or volumes. Conversely, poor decisions on price setting can affect the profit of an organization. Therefore, there is, within the business sector and particularly in the research community, a growing willingness to develop methods for optimal decision making. Optimization is the field of Applied Mathematics that has emerged since the early 1940s as the indispensable tool in decision making.

An optimization problem is to choose among a set of "alternatives" an "optimal one", where optimality refers to certain criteria, according to which the quality of the alternatives is measured. The mainstream research axis in this field delved almost exclusively into unicriteria models *i.e* mathematical models well defined, governed by the classical paradigm of the existence of an optimal solution and of algorithms converging towards it. Even if the use of these unicriteria models has proved to be particularly effective in many situations, it has appeared that there is a multitude of applications where only one criterion or (and) one decision maker would be not sufficient to describe the reality. The present work addresses two alternatives that have been developed in the literature since the mid-1970s in order to palliate these weaknesses: the bilevel optimization and the multicriteria optimization.

A bilevel optimization problem (BPP) is characterized by two levels of hierarchical decision making where each decision maker tries to optimize his own objective function without considering the objective of the other party, but the decision of each party affects the objective value of the other party as well as the decision space. These problems appear in many practical solving tasks, including optimal control, process optimization, game-playing strategy development, transportation problem and others [Bard 1988; 1998, Dempe 2002, Fortuny-Amat and Carl 1981, Migdalas 1995]. Multiobjective optimization in the other hand is a kind of generalization of classical optimization in which one wishes to optimize multiple functions at once. Most realistic optimization problems, particularly those in design, require the simultaneous optimization of more than one objective function. Some examples in [Brown et al. 1997, Martel June 1999, Philip 1972, Truong 2008, Vicente December 1997, Collete and Siarry 2003] are:

- In bridge construction, a good design is characterized by low total mass and high stiffness;
- Aircraft design requires simultaneous optimization of fuel efficiency, payload, and weight;
- In chemical plant design, or in design of a groundwater remediation facility, objectives to be considered include total investment and net operating costs;
- A good sunroof design in a car could aim at minimizing the noise the driver hears and maximizing the ventilation;
- The traditional portfolio optimization problem attempts to simultaneously minimize the risk and maximize the fiscal return.

The main focus of this thesis is on the use of multiobjective optimization results and notions in bilevel optimization problems. In addition, Solution approaches and applications of each of these classes of optimization problems are investigated. **More precisely, the thesis has the following three objectives:**

- (i) *Development of Solution approaches for each of these two classes of optimization, especially for the linear case:*
  Even if there are several methods proposed in the literature for solving problems related to these two areas of optimization, especially for the linear case, the majority of these methods turn out to be not convergent, incorrect or convergent towards a local non global optimum. For instance, Chenggen et al in [Chenggen et al. 2005] showed that the Kth-best method [Dempe 2002, Bialas and Karwan 1982, Candler and Townsley 1982], one of the most popular and workable methods for the linear bilevel programming problem, could badly deal with a linear bilevel programming problem when the constraint functions at the upper-level have an arbitrary linear form. More recently, C. Audet et al [Audet et al. 2006] showed that the definition of the linear bilevel programming problem and consequently the methods proposed by Lu, Shi, and Zhang [Lu et al. 2005a;b] do not solve a wider class of problems, but rather relax the feasible region, allowing infeasible points to be considered as feasible. In the same vein, P. Armand [Armand 1991; 1993] showed recently by a counter-example that the method developed by Ecker et al [Ecker and Kouada 1978, Ecker et al. 1980] for linear

multiobjective programming problem was incomplete in case of degeneracy. In the light of the foregoing, it becomes indispensable to develop new approaches with solid foundations.

- (ii) *Exploration and establishment of new relations between the two classes of optimization:* Although several authors have attempted to establish a link between bicriteria optimization and bilevel optimization [Colson et al. 2007, Fulop 1993, Haurie et al. 1990], none has succeeded thus far in proposing conditions that guarantee that the optimal solution of a given bilevel program is Pareto-optimal for both upper and lower level objective functions. Recently, several authors have started to study the possibility to exploit multicriteria approaches for solving bilevel programming problems [Ivanenko and Plyasunov 2008, Fliege and L.N.Vicente 2006, Dempe 2002, Bard 1983b, Winston 1994, Haurie et al. 1990, Clarke and Westerberg 1988, Bard 1984, Ünlü 1987]. Unfortunately, apart from relationship reported in [Fulop 1993] that has been used in the literature to develop an algorithm for solving bilevel linear programming problem, none of the other propositions has been implemented for solving BPP, due possibly to the fact that the proposed multicriteria optimization problems were defined by complicated relations [Fliege and L.N.Vicente 2006, Ivanenko and Plyasunov 2008] or the propositions were wrong [Bard 1983b, Ünlü 1987].

- (iii) *Application of these two classes of optimization for modeling and solving real problems in the Cameroonian context.*

For further guidance, here follows an outline of the general organization of this dissertation.

From the outset, *Chapter One* is a brief review of certain basic tools of mathematical analysis and optimization theory widely used in the dissertation. In particular, in section 1.2 some basic concepts and notations are introduced while in section 1.3, some well known results of classical optimization are provided.

In *Chapter Two*, after a survey on multiobjective programming problems provided in section 2.1, we devote section 2.2 to the presentation of a multiobjective linear programming method that we have developed. It can be regarded as a corrected form of the method of Ecker et al [Ecker and Kouada 1978, Ecker et al. 1980] shown to be incorrect by P. Armand [Armand 1993]. A numerical example is given to illustrate the proposed method. Section 2.3 presents an application of multicriteria optimization. We first describe the problem of electrical energy distribution planning generally faced by developing countries, inspired by the case of AES-SONEL company in Cameroon. We show that the mathematical formalization of the problem leads to a multicriteria programming problem. Two approaches for solving the obtained model are provided.

The focus of *Chapter Three* is bilevel programming. Mathematical formulation and classical solving methods of bilevel programming problem are discussed in section 3.1. In section 3.2, we present a new approach for solving linear bilevel programming problem. It is based on the result that an optimal solution to the BLPP is reachable at an extreme point of the underlying region. Consequently, we develop a pivoting technique to find the global optimal solution on an expanded tableau that represents the data of the BLPP. The last section is an application of bilevel programming. A strategy to protect national initiatives in the context of globalisation based on the resolution of a bilevel programming problem is presented.

In *Chapter Four*, relationships between bilevel and multicriteria optimization are investigated. In section 4.2, the issue of the Pareto-optimality of the optimal solution of a bilevel optimization problem is discussed. In section 4.3, we introduce a generalisation of the Fulop relation [Fulop 1993] that establishes a link between multiobjective linear programming and bilevel linear programming. We show that under the assumption that the follower admissible set is bounded and its constraint functions are continuous, the relation remains valid even if the objective function of the leader is non linear. We end in section 4.4 by presenting a new relation between bilevel programming and multicriteria optimization. We show that solving a certain class of bilevel programming problem can be equivalent to solving two independent multicriteria optimization problems.

The *last chapter*, Chapter Five, deals with bilevel multi-objective programming problems (BMPP). In the first section, we present the optimistic formulation and some related concepts. In section 5.2, we present an approach that we have developed in order to generate efficient solutions. Given a BMPP, we show how to construct two artificial multi-objective programming problems such that any point that is efficient for both problems is an efficient solution of the BMPP. A numerical example is provided to illustrate how the algorithm operates. In section 5.3, the focus is essentially on linear case. We introduce an artificial multiobjective linear programming problem such that its resolution can permit to generate the whole feasible set of the upper level decision maker. Based on this result, two approaches for obtaining efficient solutions are presented.

A general conclusion ends the thesis with the summary of all the work done and the discussion of new perspectives.

Chapter One

Elements of optimization theory

1.1 Introduction

Optimization problems can be broadly described as either continuous or discrete, but may be a mixture of both. Discrete optimization is concerned with the case where the variables may only take on discrete values. By contrast, the variables in continuous optimization problems are allowed to take any value permitted by the constraints. Throughout this dissertation, we shall be concerned with continuous optimization; more specifically, we shall study the class of mathematical model for which each function (objective or constraint) is described in terms of real valued mathematical function. This chapter is an overview of some basic tools of mathematical analysis and optimization widely used in formulations and proofs of the main results in the dissertation. Much of the material is based on the following books and courses [Canu December 2008, Ehrgott 2005, Horst et al. 1995, Wen and Hsu 1989] where the reader can find more details (proof of theorems and propositions), discussions, and references.

1.2 Basic concepts and notations

1.2.1 Notations

The following notations are extensively used throughout this thesis:

- The standard euclidean inner product, the norm and the distance defined in the euclidean space $\mathbb{R}^n$ are noted by $\langle ., \ . \rangle$, $\| \ . \ \|$ and $d(,)$ respectively.
- For any $x, y \in \mathbb{R}^n$, where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$, we have:

  (i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, ..., n$;

  (ii) $x \leqq y$ if and only if $x_i \leq y_i$ for all $i = 1, ..., n$;

  (iii) $x \leq y$ denotes $x \leqq y$ and $\exists j \in 1, ..., n$ such that $x_j < y_j$;

  (iv) $x < y$ if and only if $x_i < y_i$ for all $i = 1, ..., n$;
- $\mathbb{R}^n_+$ denotes the set of $x \in \mathbb{R}^n$ such that $x \geqq 0$;

- Let $S \subset \mathbb{R}^n$, S^c, $int(S)$; $ri(S)$; $bd(S)$; $cl(S)$ will denote the complementary, the interior ; the relative interior ; the boundary; the closure ($cl(S) = int(S) \cup bd(S)$) of S, respectively.

1.2.2 Elements from convex analysis

We start with the notion of *supporting hyperplane*. A set $H \subset \mathbb{R}^n$ is called a *hyperplane* if there exist $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$H = H(a, b) := \{x \in \mathbb{R}^n / \ \langle a,\ x\rangle = b\}.$$

In this case, the vector a is called a normal vector to H. Moreover, the sets

$$H(a, b)^+ := \{x \in \mathbb{R}^n / \langle a,\ x\rangle \geq b\} \text{ and } H(a, b)^- := \{x \in \mathbb{R}^n / \langle a,\ x\rangle \leq b\}$$

are called the positive and the negative halfspaces associated with H, respectively.

Definition 1.2.1. A *hyperplane* is said to support a set S in euclidean space $\mathbb{R}^n$ if it meets both of the following:

- S is entirely contained in one of the two closed half-spaces determined by the hyperplane;
- S has at least one point on the hyperplane.

Definition 1.2.2. Let $S \subseteq \mathbb{R}^n$, the set S is *convex* if $\forall x^1,\ x^2 \in S$ and $\lambda \in (0,\ 1)$, one has $\lambda x^1 + (1 - \lambda)x^2 \in S$.

The set $\{x \in \mathbb{R}^n / \ \|x\| \leq a\}$ is a convex set for every value of $a \in \mathbb{R}^n$. Let us remark that the empty set is convex.

Definition 1.2.3. The *affine hull* of a finite set $V = \{v^1, v^2, ..., v^m\} \subset \mathbb{R}^n$ is the set

$$affV = \{\lambda_1 v^1 + \lambda_2 v^2 + ... + \lambda_m v^m / \ \lambda_1, \lambda_2, ..., \lambda_m \in \mathbb{R};\ \sum_{i=1}^{n} \lambda_i = 1\}.$$

Definition 1.2.4. The convex hull of a finite set $V = \{v^1, v^2, ..., v^m\} \subset \mathbb{R}^n$ is the set

$$convV = \{\lambda_1 v^1 + \lambda_2 v^2 + ... + \lambda_m v^m / \ \lambda_1, \lambda_2, ..., \lambda_m \geq 0;\ \sum_{i=1}^{n} \lambda_i = 1\}.$$

The convex hull of $V \subset \mathbb{R}^n$ is the smallest convex set containing V. As illustration, for a given set $V = \{A,\ B\}$ where A and B are two given points, $AffV$ is the line through A and B and $convV$ is the segment through A and B.

Definition 1.2.5. A set $P \subset \mathbb{R}^n$ is a *polytope* if it is the convex hull of finitely many points of $\mathbb{R}^n$.

A cube and a tetrahedron are polytopes in $\mathbb{R}^3$.

Definition 1.2.6. A point v of a convex set P is called an *extreme point* if whenever $v = \lambda x^1 + (1-\lambda)x^2$, where $x^1,\ x^2 \in P$, and $\lambda \in (0,\ 1)$, then $v = x^1 = x^2$.

A triangle has three extreme points and a sphere has all its boundary points as extreme points. When a convex set has only a finite number of extreme points, these are often called vertices.

The following proposition presents a relation between polytope and extreme points.

Proposition 1.2.1. *Let P be the polytope $convV$, where $V = \{v^1, v^2, ..., v^m\} \subset \mathbb{R}^n$. Then P is equal to the convex hull of its extreme points.*

Definition 1.2.7. Let C be a convex set. A nonzero vector $d \in \mathbb{R}^n$ is said to be a *direction of recession* of C, if for every $x^0 \in \mathbb{R}^n$ the ray $\{\ x/; x = x^0 + \lambda d,\ \lambda \geq 0\}$ lies entirely in C.

At x^0 in P, a recession direction is a nonzero vector, h, for which the associated ray is contained in P. If h is a recession direction for some x in P, it is called a recession direction for P. It is clear from the definition 1.2.7 that, for a given convex set C, C is unbounded if and only if C has a direction. A direction d is called an *extreme direction* if it cannot be expressed as a positive combination of two other distinct directions d^1 and d^2 of P ie $\nexists \mu^1 > 0, \mu^2 > 0\ /d = \mu^1 d^1 + \mu^2 d^2$.

Definition 1.2.8. A subset P of $\mathbb{R}^n$ is a *polyhedron* if there exist a matrix $A \subset \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n / Ax \leq b\}$.

A convex polyhedron has a finite number of extreme points. The following proposition provides two equivalent definitions of extreme points with respect to a polyhedron P. Let $P = \{x \in \mathbb{R}^n / Ax \leq b\}$.

Proposition 1.2.2. *$x^0 \in P$ is an extreme point of the polyhedron P if:*

(i) There exists $c \in \mathbb{R}^n$ such that, for all $y \in P$, $y \neq x^0$, we have $c^t x^0 < c^t y$. That is, x^0 is the unique point in P where a certain linear function is minimized.

(ii) The point x^0 lies on n of the hyperplanes defining the polyhedron P, such that the rows of A associated with those hyperplanes are linearly independent. Thus, x^0 satisfies n linearly independent tight constraints.

Two extreme points are adjacent if they satisfy a common set of $n-1$ linearly independent tight constraints. One has also the following definition of a particular class of extreme points.

Definition 1.2.9. An extreme point $x^0 \in P$ is *degenerate* if it lies on more than n of the hyperplanes defining P.

Let a_i^T denote the i^{th} row of matrix A defining the polyhedron P. Further $x \in P$, let $I(x) := \{i \in \{1, ..., l\}\ /\ a_i^T x = b_i\}$ describe the inequalities which are saturated at x; and let $A_{I(x)}$ be the matrix with row $a_i^T, i \in I(x)$. Then, the vertex x of P is *nondegenerate* if $|I(x)| = n$.

Let P be an $n-$*dimensional* polyhedron and H be a supporting hyperplane, then, the intersection $F = P \cap H$ defines a *face* of P. An *edge* is a face of dimension 1. Edges are either line segments which connect adjacent extreme points or rays (respectively lines) parallel to an extreme point. A facet of an $n-$*dimensional* polyhedron is a face of dimension $n-1$.

Let us note that, every x of the polyhedron P can be represented as the sum of a convex combination of extreme points of P and a non negative linear combination of extreme directions of P:$x = \sum_{i=1}^{N} \alpha_i v^i + \sum_{j=1}^{M} \beta_j d^j, \ \ \sum_{i=1}^{N} \alpha_i = 1, \ \alpha_i \geq 0 (i = 1, ..., N), \ \beta_j \geq 0 \ (j = 1, ..., M)$, where $v^1, v^2, ..., v^N$ are extreme points of P, and $d^1, d^2, ..., d^M$ are extreme directions of P.

We end this subsection by introducing two topological notions that will be useful for some results in the text.

Definition 1.2.10. $Y \subset \mathbb{R}^p$ is called $\mathbb{R}^p_+ -$ *semicompact* if every open cover of Y of the form $\{(y^\alpha - \mathbb{R}^p_+)^c / \ y^\alpha \in Y, \ \alpha \in I\}$ has a finite subcover. This means: Whenever $Y \subset \cup_{\alpha \in I}(y^\alpha - \mathbb{R}^p_+)^c$ there is $m \in \mathbb{N}$ and $\{\alpha_1, \ ... \ \alpha_m\} \subset I$ such that $Y \subset \cup_{i=1}^{m}\{(y^{\alpha_i} - \mathbb{R}^p_+)^c$, where I is an index set.

Definition 1.2.11. $Y \subset \mathbb{R}^p$ is called $\mathbb{R}^p_+ -$ *compact* if for all $y \in Y$ the section $(y - \mathbb{R}^p_+) \bigcap Y$ is compact.

1.2.3 Orders and cones

1.2.3.1 Orders

Let $\mathbb{R}^n$ be a finite dimensional euclidean vector space, and $S \subset \mathbb{R}^n$. Every nonempty subset $R \subset S \times S$ defines a *binary relation* on Z. Below are some important properties of binary relation.

Definition 1.2.12. A binary relation R on S is called

- reflexive if $(a, a) \in R \ \ \forall a \in S$;
- symmetric if for all $a, b \in S$, $(a, b) \in R \Rightarrow (b, a) \in R$;
- asymmetric if for all $a, b \in S$, $((a, b) \in R \Longrightarrow (b, a) \notin R)$;
- antisymmetric if for all $a, b \in S$, $((a, b) \in R$ and $(b, a) \in R \Longrightarrow a = b)$;
- transitive if for all $a, b, c \in S$, $((a, b) \in R$ and $(b, c) \in R) \Rightarrow (a, c) \in R$;
- connected if for all $a, b \in S$, $a \neq b \ \Longrightarrow ((a, b) \in R$ or $(b, a) \in R)$.

In the context of orders, the relation R is usually written as $\precsim$. By convention, $(a, b) \in \precsim$ is written $a \precsim b$.

Definition 1.2.13. A binary relation $\precsim$ is called

- *a preorder* if it is reflexive and transitive.

- *a partial order* if it is reflexive, transitive and antisymmetric,
- *a strict partial order* if it is asymmetric and transitive.

Definition 1.2.14. A binary relation $\precsim$ on $\mathbb{R}^n$ is said to be compatible with
(i) addition if $x^1 \precsim x^2$ and $x^3 \precsim x^4 \Longrightarrow x^1 + x^2 \precsim x^3 + x^4, \ \forall x^1 x^2 x^3 x^4 \in \mathbb{R}^n$
(ii) scalar multiplication if $x^1 \precsim x^2 \Longrightarrow \lambda x^1 \precsim \lambda x^2, \ \forall x^1 x^2 \in \mathbb{R}^n$ and $\lambda > 0$.

1.2.3.2 Cones

Definition 1.2.15. A subset $K \subseteq \mathbb{R}^n$ is called cone, if $\lambda x \in K$ for all $x \in K$ and for all $\lambda \in \mathbb{R}, \ \lambda > 0$.

$K = \{x \in \mathbb{R}^2 / \ x_i \geq 0\}$ and $K = \{x \in \mathbb{R}^n / Ax \leq 0\}$ where $A \in \mathbb{R}^{n \times l}$ are examples of cones.

Definition 1.2.16. A cone K in $\mathbb{R}^n$ is called

- nontrivial if $K \neq \emptyset$ and $K \neq \mathbb{R}^n$
- Convex if $\alpha x^1 + (1-\alpha)x^2 \in K$ for all $x^1, x^2 \in K$ and for all $0 < \alpha < 1$
- Pointed if for $x \in K, x \neq 0$ the negative $-x \notin K$, ie $K \cap (-K) \subset \{0\}$.

It follows from the definition, that a cone K is convex if for all $x^1, x^2 \in K, \ x^1 + x^2 \in K$.

Proposition 1.2.3. *Let $Q = \{x \in \mathbb{R}^n / Ax \leq b\}$, P the convex hull of extreme points of Q, and $C := \{x \in \mathbb{R}^n / Ax \leq 0\}$. If $rankA = n$ then*

$$Q = P + C = \{x \in \mathbb{R}^n / x = u + v \textit{ for some } u \in P \textit{ and } v \in C\}$$

This last result is generally used to establish that there exists an optimal solution of linear optimization that can be found at an extreme point.

1.2.3.3 A relation between cone and order

Definition 1.2.17. Let K be a cone. The binary relation denoted by $\precsim_K$ and defined by

$x \precsim_K y \iff y - x \in K$ is called the cone relation on $\mathbb{R}^n$ induced by K.

Proposition 1.2.4. *Let K be a cone and $\precsim_K$ the induced relation on $\mathbb{R}^n$. Then $\precsim_K$ is compatible with addition and scalar multiplication and*

1. *is reflexive if and only if $0 \in K$*
2. *is transitive if and only if K is convex*
3. *is antisymmetric if and only if K is pointed*

For a given pointed convex cone K, it follows that $\precsim_K$ is a partial or a strict partial order if and only if $0 \in K$ or $0 \notin K$. The order relation induced by the set of nonnegative elements of $\mathbb{R}^n$ (considered as a cone) is extremely useful in multicriteria optimization.

1.2.4 Some particular classes of functions

Let $S \subset \mathbb{R}^n$ be a convex set.

Definition 1.2.18. A function $f : S \longrightarrow \mathbb{R}$ is convex if $\forall x^1,\ x^2 \in S$ and $\lambda \in (0,\ 1)$, one has $f(\lambda x^1 + (1-\lambda)x^2) \leq \lambda f(x^1) + (1-\lambda)f(x^2)$. The function f is *strictly convex* on S if the strict inequality $<$ holds for every $x^1 \neq x^2$.

Definition 1.2.19. A function $f : S \longrightarrow \mathbb{R}$ is *lower semi continuous* at $x^0 \in S$ if the value $f(x^0)$ is less than or equal to every limit of f as $x_k \longrightarrow x^0$. i.e $x_k \longrightarrow x^0 \Longrightarrow f(x^0) \leq \liminf_{k\longrightarrow\infty}\ f(x_k)$

Definition 1.2.20. A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is $\mathbb{R}^p_+ -$ *semicontinuous* if

$$f^{-1}(y - \mathbb{R}^p_+) = \{x \in \mathbb{R}^n /\ y - f(x) \in \mathbb{R}^p_+\} \text{ is closed } \forall y \in \mathbb{R}^p.$$

The following proposition gives a relation between $\mathbb{R}^p_+ -$ *semicontinuous* and lower semi continuous functions.

Proposition 1.2.5. *A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is $\mathbb{R}^p_+ -$ semicontinuous if and only if $f_i : \mathbb{R}^n \to \mathbb{R}$ are lower semicontinuous for all $i = 1, ..., p$.*

Definition 1.2.21. Let $K \subseteq \mathbb{R}^p$ be a cone, $Y \subseteq \mathbb{R}^p$ a set, and $s : Y \longrightarrow \mathbb{R}$ a function.
(i) The function s is *K-monotonically increasing* in Y if,

$$y \ \precsim_K\ y' \ \Rightarrow s(y)\ \leq\ s(y'),\ \forall y, y' \in Y$$

(ii) The function s is called *stricly K-monotonically increasing* in Y if,

$$y \ \precsim_K\ y' \text{ and } y \neq y' \ \Rightarrow\ s(y)\ <\ s(y'),\ \forall y, y' \in Y$$

We present below two results that permit to construct a *K-monotonically increasing function.* The importance of this class of function will be given in the next chapter .

Proposition 1.2.6. *$K \subseteq \mathbb{R}^p$ be a cone and $Y \subseteq \mathbb{R}^p$ be a set; let us consider the following notations:*

$$K^+ = \{w \in \mathbf{R^p} \setminus \{0\} /\ \langle w,\ k\rangle > 0\ \forall k \in K\} \tag{1.2.1}$$
$$K^* = \{w \in \mathbf{R^p} /\ \langle w,\ k\rangle \geq 0\ \forall k \in K\} \tag{1.2.2}$$

then
(i) $\forall w \in K^*$, the function $v : Y \to \mathbb{R}$ such that $v(x) = \langle w,\ x\rangle\ \forall x \in Y$ is K-monotone
(ii) $\forall w \in K^+$, the function $v : Y \to \mathbb{R}$ such that $v(x) = \langle w,\ x\rangle\ \forall x \in Y$ is strictly $K-$monotone.

Proposition 1.2.7. *Let $K \subseteq \mathbb{R}^p$ be a closed convex cone, let $Y \subseteq \mathbb{R}^p$ be a non-convex set, and L the subspace parallel to $aff(Y)$, the affine hull of Y. Let $Q \in \mathbb{R}^p \times \mathbb{R}^p$, be a symmetric positive semidefinite matrix, and define the function v by $v(y) = \langle Qy,\ y\rangle\ \ \forall y \in Y$,*

then v is K monotonically increasing on Y iff $QY \subseteq K^* + L^\perp$

1.3 Classical optimization

1.3.1 General formulation of optimization problems

A classical optimization(minimization) problem has the following form:

$$\begin{aligned} &\textit{minimize } f(x) \\ &\text{subject to } x \in S \end{aligned}$$

where S (feasible or admissible domain) is a subset of $\mathbb{R}^n$ and $f(x)$ (objective function) is a real valued function defined on S. In fact, the maximization problem is included in the minimization formulation because $\max\{f(x)/\ x \in S\} = \min\{-f(x),\ x \in S\}$. Solving an optimization problem consists in finding local and, if possible, global minimizers.

Definition 1.3.1. A feasible point x^* is a *local minimizer* of $f(x)$ if there is $\epsilon > 0$ such that for all $x \in S$ satisfying $\|x - x^*\| \leq \epsilon$, one has $f(x^*) \leq f(x)$. x^* is said to be a *global minimizer* if $f(x^*) \leq f(x)\ \forall x \in S$.

If f is a convex function and S is a convex set, then every local minimizer is a global minimizer. Moreover, if f is strictly convex, then a local minimizer is the unique global minimizer.

Proposition 1.3.1. *If S is a nonempty compact set in $\mathbb{R}^n$, and $f(x)$ is a lower semi continuous function on S, then $f(x)$ has at least one global minimizer in S.*

The simplest classes are unconstrained optimization and constrained optimization, corresponding to the case where $S = \mathbb{R}^n$ and $S \subset \mathbb{R}^n$, respectively.

1.3.2 Unconstrained optimization

We explore some well known optimal conditions and algorithms related to unconstrained optimization.

1.3.2.1 Optimality conditions for unconstrained optimization

We first consider what we might deduce if we were fortunate enough to have found a local minimizer of f(x). The following two results provide first- and second-order necessary optimality conditions (respectively).

Proposition 1.3.2. *Suppose that $f \in C^1$(differentiable) and x^* is a local minimizer of $f(x)$. Then* $\nabla_x f(x^*) = 0\ \ i.e\ \ \dfrac{\partial f(x^*)}{\partial x_i} = 0\ \ \forall i \in \{1, ..., n\}$.

Proposition 1.3.3. *Suppose that $f \in C^2$(twice differentiable) and x^* is a local minimizer of $f(x)$. Then $\nabla_x f(x^*) = 0$ i.e $\dfrac{\partial f(x^*)}{\partial x_i} = 0 \;\; \forall i \in \{1, ..., n\}$ and $H(x^*)$ is positive semi-definite. Where H is the Hessian matrix of f that is defined by $H(x) = \nabla_{xx} f(x)$.*

But what if we have found a point that satisfies the above conditions? Is it a local minimizer? Yes, an isolated one, provided the following second-order sufficient optimality conditions are satisfied.

Proposition 1.3.4. *Suppose $f \in C^2$(twice differentiable), x^* satisfies the condition $\nabla_x f(x^*) = 0$ and $H(x^*)$ is positive definite. Then x^* is a local minimizer of f.*

1.3.2.2 Algorithms for unconstrained optimization

One of the most used approaches is descent methods. The typical algorithm of *descent methods* is as follows:

Algorithm 1. 1

1: Starting point $x_0 \in \mathbb{R}^n$. Set $k = 0$
2: Search direction: $d_k \in \mathbb{R}^n$ such that $\langle d_k, \; \nabla_x f(x_k)\rangle < 0$ (if $\nabla_x f(x_k) \neq 0$).
3: Find the step length: $\alpha_k > 0$ such that $f(x_k + \alpha_k d_k) < f(x_k)$ holds.
4: Let $x_{k+1} = x_k + \alpha_k d_k$
5: Termination criterion: If fulfilled, stop. Otherwise, let $k := k + 1$, go to step 1.

At step 2, a *search direction* d_k is calculated from x_k. The direction is required to be a *descent direction* so that for small step along d_k, it is guaranteed that the objective function will be reduced (Taylor theorem). In step 3, it is stated that $\alpha_k > 0$ should be chosen to minimize $f(x_k + \alpha_k d_k)$. This is known as an exact linesearch. In most cases, exact linesearches prove to be very expensive. Modern linesearch methods prefer to use inexact linesearches, which are guaranteed to pick steps that are neither too long nor too short. The main contenders amongst the many possible inexact linesearches are the so-called «backtracking- Armijo» and the «Armijo-Wolfe» varieties [Canu December 2008]. The former are extremely easy to implement, and form the backbone of most Newton-like linesearch methods. The latter are particularly important when using secant quasi-Newton methods.

In the *Descent method*, one picks a descent direction d_k and then picks a *steplength* $\alpha_k > 0$ to "reduce" $f(x_k + \alpha_k d_k) < f(x_k)$ and finally sets $x_{k+1} = x_k + \alpha_k d_k$. More approaches for solving unconstrained optimization problem can be found in [Canu December 2008].

1.3.3 Constrained optimization

1.3.3.1 Optimality conditions for constrained optimization

When constraints are present, things get more complicated. The aim when dealing with constrained optimization is in general to

$$\begin{array}{c} \textit{minimize } f(x) \\ \text{subject to } x \in S \end{array}$$

where the admissible set has the following form:

$$S = \left\{ x \in \mathbb{R}^n / \quad \begin{array}{l} g_i(x) \leq 0,\ i \in I = \{1,..,m_1\},\ g_i:\ \mathbb{R}^n \longrightarrow \mathbb{R} \\ h_j(x) = 0,\ j \in J = \{1,..,m_2\},\ h_j:\ \mathbb{R}^n \longrightarrow \mathbb{R} \end{array} \right\}$$

where $g_i(.)$ $(i = 1,..,m_1$ are the functions of the inequality constraints and $h_j(.)$ $(1,..,m_2)$ are the functions of the equality constraints, and where m_1 and m_2 are the number of inequality and equality constraints, respectively. The *Karush Kuhn Tucker* conditions (or KKT conditions) are necessary for a solution in optimization problem to be optimal, provided that some regularity conditions are satisfied. The KKT approach to non-linear programming generalizes the method of Lagrange multipliers, which allowed only equality constraints. The following is the statement of the KKT optimal conditions.

Theorem 1.3.1. *(KKT-Optimality Conditions) If x^* is a local minimum that satisfies some regularity conditions, then there exist constants μ_i $(i = 1,..,m_1)$ and λ_j $(j = 1,..,m_2)$ such that*

$$\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m_1} \mu_i \nabla g_i(x^*) + \sum_{j=1}^{m_2} \lambda_j \nabla h_j(x^*) = 0 \\ \mu_i g_i(x^*) = 0 \\ \mu_i \geq 0\ (i = 1, ..., m_1)\ , \lambda_j \in \mathbb{R}\ (j = 1, ..., m_2) \end{cases}$$

The three conditions of the theorem above represent the KKT conditions.

In order for a minimum point x^* to satisfy the above KKT conditions, it should satisfy some regularity condition, the most used ones are listed below:

- *Linear independence constraint qualification (LICQ)*: the gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at x^*.

- *Mangasarian-Fromowitz constraint qualification (MFCQ)*: the gradients of the active inequality constraints and the gradients of the equality constraints are positive-linearly independent at x^*.

- *Constant rank constraint qualification (CRCQ)*: for each subset of the gradients of the active inequality constraints and the gradients of the equality constraints the rank at a vicinity of x^* is constant.

- *Constant positive linear dependence constraint qualification (CPLD)*: for each subset of the gradients of the active inequality constraints and the gradients of the equality constraints, if it is positive-linear dependent at x^* then it is positive-linear dependent at a vicinity of x^* . ($(v_1, v_2, ..., v_n)$ is positive-linear dependent if there exists $(a_1, a_2, ..., a_n)$ not all zero such that $a_1v_1 + ... + a_nv_n = 0$)

- *Slater condition*: for a convex problem, there exists a point x such that $h(x) = 0$ and $g_i(x) < 0$ for all i active in x^*.

- *Linearity constraints*: If f and g are affine functions, then no other condition is needed to assure that the minimum point is KKT.

It can be shown that $LICQ \Longrightarrow MFCQ$, $LICQ \Longrightarrow CRCQ \Longrightarrow CPLD$ (and the converses are not true). In practice weaker constraint qualifications are preferred since they provide stronger optimality conditions.

As stated in the following theorem, the necessary conditions can be sufficient for optimality in certain cases.

Theorem 1.3.2. *Assume the optimization problem is convex, that is , f as well a $g_i(i = 1, ..., m_1)$, are convex, and $h_j(i = 1, ..., m_2)$, are affine; all functions are in C^1 as well. Assume further that for $x^* \in S$, the KKT conditions are satisfied. Then x^* is a globally optimal solution to the optimization problem.*

There are many other optimality conditions developed in the literature, such as Lagrange theorem, Fritz-John condition etc.

1.3.3.2 Algorithms for constrained optimization

There are several methods developed in the literature to tackle constrained optimization problem. Most of them exploit unconstrained optimization algorithms. The concept of *merit function* is used in general. Given a parameter p, a composite function $\Phi(x, p)$ is a merit function if (some) minimizers of $\Phi(x, p)$ with respect to x approach those of $f(x)$ subject to the constraints as p approaches some set $\mathbb{P}$. In principle, what one has to do then is to choose the best unconstrained algorithm to solve the new problem obtained. Here are two merit functions in the case of equality constraints ($I = \emptyset$):

- $\Phi(x, \mu) = f(x) + \frac{1}{2\mu} \parallel h(x) \parallel_2$, $h(x) = (h_j(x))$, $j = 1, ..., m_2$. It is called the quadratic penalty function.

- $\Phi(x, u, \mu) = f(x) - u^T g(x) + \frac{1}{2\mu} \parallel h(x) \parallel_2^2$, where both u and μ are auxiliary parameters. It is called the *augmented Lagrangian function*.

For inequality constraints ($J = \emptyset$), the best known merit function is the *logarithmic barrier function* defined by $\Phi(x, \mu) = f(x) - \mu \sum_{i=1}^{m_1} log(-g(x)),\ g(x) = (g_i(x)),\ i = 1, ..., m_1$

Let us note that in practice, the methods developed relate to the nature of the optimization problem: linear problems, non linear problems, convex problems, quadratic problems etc. In particular, concerning linear optimization problems, the well known approach is the simplex method. The simplex method is an iterative process which exploits the fact that an optimal solution of linear optimization problem is an extreme point.

1.4 Conclusion

This chapter introduced notations with some familiar concepts and results of optimization theory such as optimal solution, optimality conditions, extreme point notion, order relations, cone, etc. As we have seen, classical optimization deals with problems having only one objective function. The next chapters focus on optimization problems having more than one objective function. They can be divided into two parts. The first part (Chapter Two and Chapter Three) is devoted to solution approaches and applications. The last part (Chapter Four and Chapter Five) focuses on the connections between bilevel optimization and multiobjective optimization.

Chapter Two

Multiobjective optimization

2.1 Introduction

Life is about decision making. Decisions usually involve several conflicting objectives. The observation that real world problems have to be solved optimally according to criteria, which prohibit an "ideal" solution (optimal for each decision-maker) has led to the development of multicriteria optimization. A multi-objective optimization (or programming), also known as multi-criteria or multi-attribute optimization, is the process of simultaneously optimizing two or more conflicting objectives, subject to certain constraints. It is especially during the last three decades that it has been much developed. Today, many decision support systems incorporate methods to deal with conflicting objectives. This chapter deals with multicriteria optimization. The main result of the chapter is a new approach for solving multiobjective linear programming problems. The chapter is organized as follows. In the next section 2.1, a brief survey on multiobjective programming is presented. A new a posteriori approach for solving linear multiobjective programming problems is provided in section 2.3. It is followed in section 2.4 by the presentation of a multicriteria programming optimization model that we have developed for the planning of the distribution of electrical energy in Cameroon. Finally, the chapter is concluded in section 2.5.

2.2 Multiobjective optimization problem

2.2.1 Formulation of multiobjective optimization problems

A multiobjective optimization problem with n decision variables, $m = m_1 + m_2$ constraints and p objectives is formulated as follows:

$$\text{"min"} f(x) = (f_1(x), f_2(x), ..., f_p(x)) \tag{2.2.1}$$

$$\text{subject to} \begin{cases} g_i(x) \leq 0,\ i \in I = \{1, .., m_1\} \\ h_j(x) = 0,\ j \in J = \{1, .., m_2\} \\ x \geqq 0 \end{cases} \tag{2.2.2}$$

where:

- $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$, each $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, each $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, and each $h_j : \mathbb{R}^n \longrightarrow \mathbb{R}$.
- Objective functions $(f_i, i = 1, ..., p)$ are at least partially conflicting, i.e they are conflicting in some regions of the search space.
- $X = \left\{ x \in \mathbb{R}^n_+ / \begin{array}{l} g_i(x) \leq 0,\ i \in I = \{1, .., m_1\} \\ h_j(x) = 0,\ j \in J = \{1, .., m_2\} \end{array} \right\}$ represents the set of the constraints (or the feasible set) of the multiobjective optimization problem.
- $Y := f(X) \subseteq \mathbb{R}^p$ is the set of all attainable outcomes or criterion vectors for all feasible solutions in the objective space.

The multiobjective optimization problem can then be modelled as follows:

$$\text{"}\min\text{"} f(x) = (f_1(x), f_2(x), ..., f_p(x)) \tag{2.2.3}$$

$$\text{s.t } x \in X \subseteq \mathbb{R}^n \tag{2.2.4}$$

When constraints and objectives are linear, one speaks about a *Linear Multiobjective Programming problem*. In such case, formulation is simply:

$$\text{"}\min\text{"} Cx \tag{2.2.5}$$

$$\text{s.t } x \in X \tag{2.2.6}$$

where C is a $p \times n$ matrix and $X = \{x \in \mathbb{R}^n_+, / \ Ax \leq b, x \geq 0\}$,
with $A = (a_{i,j})_{i=1..n, j=1..m}$ a $m \times n$ matrix, $b = (b_i)_{i=1..m}$ a column vector of $\mathbb{R}^m$ and n, p, m are integers.

2.2.2 Solutions of multiobjective programming problems

A feasible solution (decision) $x \in X$ is evaluated by the p objective functions producing the outcome $f(x)$. Since for $p \geq 2$, there is no canonical order in $\mathbb{R}^p$ like in $\mathbb{R}$, there does not exist a feasible solution which optimizes simultaneously all the objectives. Dealing with multiobjective programming problem then consists in finding compromise solutions. It seems natural to consider $x \in X$ as possible satisfying compromise, if there does not exist any other solution $y \in X$ which supplies values at least as good as x on every objective and better on at least one objective. Feasible points like x are called *Pareto-optimal solutions*. Solving a multiobjective problem is defined as finding a (or a subset of) Pareto-optimal solution(s) that satisfies the decision maker (DM), who knows or understands the problem better. It is important to deal only with Pareto-optimal points. In fact any x which is not Pareto-optimal cannot represent an optimal decision, because there exists at least one other point $x' \in X$, such that $f(x') \leq f(x)$, i.e x' is clearly better than x. Formally, one has the following definition of a Pareto-optimal point.

Definition 2.2.1.
A feasible point $x^* \in X$ is called *Pareto-optimal* if there does not exist $x \in X$ such that $f(x) \leq f(x^*)$. If x^* is Pareto-optimal then $f(x^*)$ is called *non-dominated point*.

Depending on the context, Pareto-optimal points and nondominated points are usually called Efficient points. Pareto-optimal points are then solutions that cannot be improved in one objective function without deteriorating their performance in at least one of the rest. The following proposition from [Ehrgott 2005] provides seven equivalent definitions of Pareto-optimality.

Proposition 2.2.1. *The following seven properties are equivalent:*

- *(i)* x^* *is Pareto-optimal;*
- *(ii) there is no* $x \in X$ *such that* $f_i(x) \leq f_i(x^*), i = 1, ..., p$ *and* $f_j(x) < f_j(x^*)$ *for some* $j \in \{1, ..., p\}$;
- *(iii) there is no* $x \in X$ *such that* $f(x) - f(x^*) \in -\mathbb{R}^p_+ \setminus \{0_p\}$;
- *(iv)* $f(x) - f(x^*) \in \mathbb{R}^p \setminus \{-\mathbb{R}^p_+ \setminus \{0_p\}\}$ *for all* $x \in X$;
- *(v)* $Y \cap (f(x^*) - \mathbb{R}^p_+) = \{f(x^*)\}$;
- *(vi)* $f(x) \leq f(x^*)$ *for some* $x \in X$ *implies* $f(x) = f(x^*)$;
- *(vii) there is no* $f(x) \in Y \setminus \{f(x^*)\}$ *such that* $f(x) \in f(x^*) - \mathbb{R}^p_+$.

We shall often refer to the one which is best suited to a given context.

There is a weaker concept of Pareto-optimality or efficiency, called *weak efficiency*. A feasible point $x^* \in X$ is *weak efficient* if there does not exist $x \in X$ such that $f(x) < f(x^*)$. Obviously, efficient points (Pareto-optimal points) are weakly efficient, but the contrary is not always true.

Let $K \subset \mathbb{R}^p$ be an arbitrary cone, the following is a more general definition of efficient points based on the cone concept.

Definition 2.2.2.
A point $y_0 \in Y$ is a *non-dominated point* with respect to the cone K if and only if there does not exist a point $y \in Y$ such that $y \precsim_K y_0$. If y^* is a non-dominated point with respect to the cone K, then $x^* \in X$ such that $y^* = f(x^*)$ is called a *Pareto-optimal point with respect to the cone K*.

Definition 2.2.1 is a particular case of *Definition* 2.2.2, where the cone used is $\mathbb{R}^p_+ \setminus \{0_p\}$. Throughout the rest of the text, the set of efficient(Pareto-optimal) points of a multi-objective optimization problem defined by a vector valued function f on a feasible set X with respect to a cone K is denoted by: $E(f,\ X,\ \precsim_K)$, while, non-dominated set is denoted by $ND(f(X), \precsim_K)$. If one speaks of efficient points without making a reference to a cone, it will be with respect to $Definition$ 2.2.1. In this case, the set of Pareto-optimal points will be denoted by X_{par} and the set of non-dominated points by Y_{eff}.

2.2.3 Pareto-optimality conditions

We present here some well known results for the characterization of Pareto-optimal points. New formulations and proofs are provided for most of them. We start with the following results related to non-dominated points.

Theorem 2.2.1. *Let $K \subseteq \mathbb{R}^p$ be a non-trivial cone, let $Y \subseteq \mathbb{R}^p$ a set, and $s : Y \longrightarrow \mathbb{R}$ a function.*
(i) If $argmin_{y\in Y}\ \{\ s(y)\ \} = \{a\}$ and s is K-monotone, then $a \in ND(Y, \precsim_K)$
(ii) If $a \in argmin_{y\in Y}\ \{\ s(y)\ \}$ and s is strictly K-monotone, then $a \in ND(Y, \precsim_K)$.

Proof:

(i) Suppose that $a \notin ND(Y, \precsim_K)$, then $\exists b \in Y\ (b \neq a)$ such that $b \precsim_K a$. From this result and since s is a K-monotonically function, we have $s(b) \leq s(a)$, which is equivalent to $s(b) < s(a)$ or $s(b) = s(a)$. But $a \in argmin_{y\in Y}\ \{\ s(y)\ \}$, so it cannot be possible to have $s(b) < s(a)$. Similarly, since a is the unique element which permits to have the minimum of the function s on Y, $s(b) = s(a)$ cannot be possible. Hence $a \in ND(Y, \precsim_K)$.

(ii) The proof of (ii) is similar. □

The following corollaries can be seen as a practical way to obtain Pareto-optimal or non-dominated points .

Corollary 2.2.1. *Let $K \subseteq \mathbb{R}^p$ be a convex cone and $Y \subseteq \mathbb{R}^p$.*
(i) If there exists a point $w \in K^+$ such that y^* is a solution to the problem

$$min_{y\in Y}\ \{\ \langle w,\ y\rangle\ \}\ \ then\ \ y^* \in ND(M, \precsim_K)$$

(ii)If there exists a point $w \in K^*$ such that y^* is the unique solution to the problem

$$min_{y\in Y}\ \{\ \langle w,\ y\rangle\ \},\quad then\ \ y^* \in ND(M, \precsim_K)$$

Proof: It follows from proposition 1.2.7 of chapter 1 and the preceding result. □

Proposition 2.2.2. *Suppose x^* is an optimal solution of the problem*

$$\min_{x\in X} \sum_{i=1}^{p} \lambda_i f_i(x)\ \ where\ \lambda \in \mathbb{R}^p$$

Then the following holds
(i) if $\lambda_i > 0,\ i = 1, ..., p$ and $\sum \lambda_i = 1$ (or $\lambda \in int\mathbb{R}^p_+$), then x^* is Pareto-optimal;
(ii) if $\lambda \geq 0$ and $\sum \lambda_i = 1$ (or $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$) and in addition, x^* is the unique solution of the optimization problem, then x^* is Pareto-optimal.

Proof

(i) Let x^* be an optimal solution of the problem with $\lambda_i > 0,\ i = 1, ..., p$ and $\sum \lambda_i = 1$, then one has $\sum_{i=1}^{p} \lambda_i f_i(x^*) \leq \sum_{i=1}^{p} \lambda_i f_i(x),\ \forall x \in X$.

Suppose that $x^* \notin X_{par}$, hence there must be $x^0 \in X$ *with* $f(x^0) \leq f(x^*)$. *i.e* $f_i(x^0) \leq f_i(x^*), i = 1, ..., p$ and $\exists j \in 1, ..., p/\ f_j(x^0) < f_j(x^*)$. Multiplying by the weights gives $\lambda_i f_i(x^0) \leq \lambda_i f_i(x^*), i = 1, ..., p$ and $\exists j \in 1, ..., p/\ \lambda_j f_j(x^0) < \lambda_j f_j(x^*)$, this is possible because all $\lambda_i > 0,\ i = 1, ..., p$ are strictly positive. Since the last inequality is strict, we then have $\sum_{i=1}^{p} \lambda_i f_i(x^0) < \sum_{i=1}^{p} \lambda_i f_i(x^*)$, contradicting the optimality of x^*.

(ii) The proof is a bit similar. In fact, let x^* be the unique optimal solution of the optimization problem with $\lambda \geq 0$, then one has $\sum_{i=1}^{p} \lambda_i f_i(x^*) \leq \sum_{i=1}^{p} \lambda_i f_i(x),\ \forall x \in X$.

Suppose that $x^* \notin X_{par}$, hence there must be $x^0 \in X,\ x^0 \neq x^*$ *with* $f(x^0) \leq f(x^*)$ *i.e* $f_i(x^0) \leq f_i(x^*), i = 1, ..., p$ and $\exists j \in 1, ..., p/\ f_j(x^0) < f_j(x^*)$. Multiplying by the weights leads to $\sum_{i=1}^{p} \lambda_i f_i(x^0) \leq \sum_{i=1}^{p} \lambda_i f_i(x^*)$. This implies two cases: $\sum_{i=1}^{p} \lambda_i f_i(x^0) < \sum_{i=1}^{p} \lambda_i f_i(x^*)$ or $\sum_{i=1}^{p} \lambda_i f_i(x^0) = \sum_{i=1}^{p} \lambda_i f_i(x^*)$. The first case is impossible because it contradicts the optimality of x^*. The second case implies that x^0 is also an optimal solution of the optimization problem, contradicting the uniqueness of x^*. □

This proposition can also be seen as a corollary of corollary 2.2.1 where the natural scalar product is considered.

Theorem 2.2.2. *Suppose x^*, an optimal solution of the problem*

$$\min_{x \in X} \parallel f(x) - y^0 \parallel$$

Where y^0 is a given point in $\mathbb{R}^p \setminus Y$. Then the following hold:
(i) If $\|\|$ is monotone and x^* is the unique solution, then x^* is Pareto-optimal;
(ii) If $\|\|$ is strictly monotone, then x^* is Pareto-optimal.

Proof

(i) Assume that x^* is a unique optimal solution and that x^* is not Pareto-optimal. Then there is some $x \in X$ such that $f_i(x) \leq f_i(x^*)$ for all $i \in 1, .., p$ and $f_j(x) < f_j(x^*)$ for some $j \in 1, .., p$. Therefore $0 \leq f_i(x) - y_i^0 \leq f_i(x^*) - y_i^0$ for $i = 1, .., p$ with one strict inequality. Because $\|\|$ is monotone, one has then $\parallel f(x) - y^0 \parallel < \parallel f(x^*) - y^0 \parallel$. From optimality of x^* equality must hold, which contradicts the uniqueness of x^*.

(ii) similar. □

Theorem 2.2.3. *let $x^* \in X$ and $y_i := f_i(x^*)$.*
Then x^* is Pareto-optimal if and only if $\cap_{i=1}^{p} L_{\leq}(y_i) = \cap_{i=1}^{p} L_{=}(y_i)$ where $\forall x^0 \in X$and $g : X \longrightarrow \mathbb{R}$,

$$L_{\leq}(g(x^0)) = \{\ x \in X/\ g(x) \leq g(x^0)\ \} \text{ and } L_{=}(g(x^0)) = \{\ x \in X/\ g(x) = g(x^0)\ \}$$

Proof

x^* is Pareto-optimal $\Leftrightarrow \nexists x \in X$ such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, ..., p$ and $f_j(x) < f_j(x^*)$ for some j. This is equivalent to say that there is no $x \in X$ such that both $x \in \cap_{i=1}^{p} L_{\leq}(y_i)$ and $x \in L_{<}(y_j)$ for some j, which leads to $\cap_{i=1}^{p} L_{\leq}(y_i) = \cap_{i=1}^{p} L_{=}(y_i)$ □

This theorem is illustrated by the following example taken from [Ehrgott 2005].

Example 2.2.1. *Let* $f_1(x) = x_1^2 - 4x_1 + x_2^2 - 6x_2 + 51$ *and* $f_2(x) = 7(x_1^2 - \frac{38}{7}x_1 + x_2^2 - \frac{44}{7}x_2) + 149$. *We want to check if* $A = (2, 2)$ *or* $B = (2, 3)$ *is Pareto-optimal.*

Let us start with $A = (2, 2)$. *One has* $f_1(2, 2) = 15$ *and* $f_2(2, 2) = 41$. *The computation of different sets of the theorem leads to:*

$$L_{=}(f_1(2, 2)) = \{x \in \mathbb{R}^2 / \ (x_1 - 2)^2 + (x_2 - 3)^2 = 1\}$$

and

$$L_{=}(f_2(2, 2)) = \{x \in \mathbb{R}^2 / \ (x_1 - \frac{19}{7})^2 + (x_2 - \frac{22}{7})^2 = \frac{89}{49}\}.$$

So $L_{=}(f_1(2, 2))$ *is a circle with center* $(2, 3)$ *and radius* 1 *and* $L_{=}(f_2(2, 2))$ *is a circle with center* $(\frac{19}{7}, \frac{22}{7})$ *and radius* $\sqrt{\frac{89}{7}}$.

One can check easily that $\cap_{i=1}^{2} L_{\leq}(f_i(2, 2)) \neq \cap_{i=1}^{2} L_{=}(f_i(2, 2))$, *so* $(2, 2)$ *is not Pareto-optimal.*

Now let us check the case of $B = (2, 3)$. *We have* $f_1(2, 3) = 12$ *and* $f_2(2, 3) = 32$. *Repeating the computations from above, one obtains* $f_1(x) = 12 \Longleftrightarrow (x_1 - 2)^2 + (x_2 - 3)^2 = 0$ *and* $f_2(x) = 32 \Longleftrightarrow (x_1 - \frac{19}{7})^2 + (x_2 - \frac{22}{7})^2 = \frac{26}{49}$. *Hence,*

$$L_{=}(f_1(2, 3)) = \{x \in \mathbb{R}^2 / \ (x_1 - 2)^2 + (x_2 - 3)^2 = 0\} = \{\ (2, 3)\ \}$$

and

$$L_{=}(f_2(2, 3)) = \{x \in \mathbb{R}^2 / \ (x_1 - \frac{19}{7})^2 + (x_2 - \frac{22}{7})^2 = \frac{26}{49}\}$$

$L_{=}(f_2(2, 3))$ *is then a circle around* $(\frac{19}{7}, \frac{22}{7})$ *with radius* $\frac{\sqrt{26}}{7}$.

One has now to check if $L_{=}(f_1(2, 3)) \cap L_{=}(f_2(2, 3))$ *is the same as* $L_{\leq}(f_1(2, 3)) \cap L_{\leq}(f_2(2, 3))$. *But* $L_{=}(f_1(2, 3)) = \{(2, 3)\}$, *i.e the level set consists of only one point, which is on the boundary of* $L_{\leq}(f_1(2, 3))$. *Thus* $(2, 3)$ *is Pareto-optimal.*

For a given $l \in \mathbb{R}^p$, now let us consider the following problem denoted by $P_k(l)$:

$$\begin{aligned} &\min_{x \in X} f_k(x) \\ &\text{subject to } f_j(x) \leq l_j \ ; j = 1, ..., p, \ j \neq k, \end{aligned}$$

The following result holds:

Theorem 2.2.4.

$$x^* \in X_{par} \ iff \ \exists l^* \in \mathbb{R}^p / \ x^* \ \textit{is an optimal solution of} \ P_k(l) \ \textit{for all} \ k = 1, .., p$$

Proof

$\Longrightarrow$

Let $l^* = f(x^*)$. Assume x^* is not optimal for $P_k(l^*)$ for some k. There must be $x \in X$ with $f_k(x) < f_k(x^*)$ and $f_j(x) \leq l_j^* = f_j(x^*)$ for all $j \neq k$, that is , $x^* \notin X_{par}$.

$\Longleftarrow$ Suppose x^* is not Pareto-optimal. Then there is an index $q \in 1, ..., p$ and a feasible point $x \in X$ such that, $f_q(x) < f_q(x^*)$ and $f_q(x) \leq f_q(x^*)$ when $j \neq q$. Since x^* is optimal for $P_q(l^*)$ by assumption, it must be feasible for $P_q(l^*)$. Therefore we have $f_j(x) \leq f_j(x^*) \leq l_j^*$ for all $j \neq k$. Thus x is feasible for $P_q(l^*)$ and $f_q(x) < f_q(x^*)$ contradicts the assumption. Hence x^* is Pareto-optimal. □

We end this section with the following Pareto-optimality conditions similar to the well known KKT conditions. It is taken from [Majumdar 1997].

Theorem 2.2.5. *Let consider the multiobjective optimization problem (2.2.1-2.2.2) presented in section 2.2.1. Suppose the following for some $x^* \in X$:*

- *(i) $f(x)$ is pseudoconvex at $x = x^*$;*
- *(ii) $g(x)$ and $h(x)$ are quasiconvex and differentiable at $x = x^*$;*
- *(iii) There exist $u^* \geq 0$, $v^* \geqq 0$, $w^* \geqq 0$ such that:*
  (a) $(\nabla f(x^*))^T u^* + (\nabla g(x^*))^T v^* + (\nabla h(x^*))^T w^* = 0$
  (b) $g(x^*) \leqq 0$
  (c) $h(x^*) = 0$

Then, x^* is a Pareto-optimal solution for problem (2.2.1-2.2.2).

Remark 2.2.1. This section was a brief review of some main results on Pareto-optimality conditions, without a reference to the structure of the optimization problem. In the literature, most of optimality conditions are developed with respect to the structure of the multicriteria optimization problems (linear problems, quadratic problems, convex problems etc.) More related studies can be found in [Cochrane and Zeleny 1973, Ehrgott 2005; 2000, Fotso April 1981, Jimenez and Novo 2002, Othmani May 1998].

2.2.4 Solution methods

More generally, three general approaches are normally taken: a priori optimization; interactive optimization and a posteriori optimization. This section is a brief survey on these methods. More detailed studies can be found in [Ehrgott 2005, Steuer 1986, Cochrane and Zeleny 1973, Truong 2008].

2.2.4.1 A priori methods

Here the DM is consulted before search and a mathematical model of his/her preferences is constructed to evaluate all solutions. The best solution found, according to the model, is returned and represents the outcome of the optimization process with no further input from the DM.

The simplest approach is qualified as *weighting method*. It reformulates the original multicriteria optimization problem as a convex linear combination of the individual objectives with weights $\lambda_k \geq 0$, such that $\sum_k \lambda_k = 1$. The task is then, to minimize this overall objective function, i.e solve:

$$\min_{x \in X} \sum_{i=1}^{p} \lambda_k f_k(x)$$

where each λ_k reflects (according to the point of view of the DM) the significance of the individual objective f_k. The obtained solution is Pareto-optimal if the solution of the optimization problem is unique or if λ provided by the DM is such that each $\lambda_k > 0$.
But, the most popular approach is the *Goal programming (GP)*. It can be seen as an extension of the *weighting method*. The mathematical formulation of GP model (see [Moussa 2007]) is as follows:

$$\min Z = \sum_{i=1}^{p} (w_i^+ \delta_i^+ + w_i^- \delta_i^-)$$

$$\text{subject to} \begin{cases} f_i(x) + \delta_i^- - \delta_i^+ = l_i \ (i = 1, ..., p) \\ x \in X \subset \mathbb{R}^n \\ \delta_i^+, \delta_i^- \geq 0 \ (i = 1, ..., p) \end{cases}$$

where

• δ_i^+ and δ_i^- indicate the positive and negative deviations between the achievement level $f_i(x)$ and the aspiration level l_i respectively.

• w_i^+ and w_i^- are the coefficients of relative importance given to the positive and negative deviations of the objective i respectively.

The solution obtained through this model represents the compromise that can be made between these objectives. However, there is no guarantee that the optimal solution is Pareto-optimal. The following is the most recent test proposed in the literature [Moussa 2007] in order to know if the obtained solution is Pareto-optimal and if not to provide a solution that is Pareto-optimal and dominates the later. (The test supposes that each f_i is continuous and X is compact.)

Step 1: Solve the GP optimization problem. Let x^* be the optimal solution.

Step 2: Solve the following optimization problem

$$\min_{x\in X} \sum_{k=1}^{p} f_k(x)$$
$$\text{subject to} f_i(x) \leq f_i(x^*) \; ; i = 1, ..., p$$

Let $\overline{x}$ be the solution of this problem.

Step 3: If $\sum_{k=1}^{p} f_k(x^*) = \sum_{k=1}^{p} f_k(\overline{x})$, then x^* is Pareto-optimal. x^* is then the optimal decision of the DM.

Else $\sum_{k=1}^{p} f_k(x^*) < \sum_{k=1}^{p} f_k(\overline{x})$, then x^* is not Pareto-optimal. But the decision $\overline{x}$ is Pareto-optimal and dominates x^*. The DM can adopt it as an optimal decision.

Remark 2.2.2. When solving the GP optimization problem, if each deviation has a strictly positive value, then the optimal solution obtained is Pareto-optimal.

The main difficulty of a posteriori method is that it is very hard for the DM to give adequate models determining which solutions he/she prefers, without knowing or having any idea what it is possible to attain, and how much one objective may have to be sacrificed with respect to others.

2.2.4.2 Interactive approaches

In this approach, the DM works together with the analyst or an interactive computer program. The analyst tries to determine the preference structure of the DM in an interactive way. The basic step of interactive algorithm is as follows:

Step 1: Find an initial feasible solution

Step 2: Interact with the DM and

Step 3: Obtain a new solution (or a set of new solutions). If the new solution satisfies the DM stop. Else go to step 2.

The following provides an example of interactive methods (see [Steuer 1986]).

Step 1: Initialization $(m = 0)$

Find for each $k = 1, ..., p$ the optimal solution x^{*k} of the following problem

$$\min f_k(x) \tag{2.2.7}$$
$$\text{s.t } \; x \in X \subseteq \mathbb{R}^n \tag{2.2.8}$$

Let $M_k = f_k(x^{*k})$ and $M = (M_1, M_2, ..., M_p)$; $z_k = M_k - \epsilon_k$; $k = 1, ..., p$, where ϵ_k are arbitrary positive fixed values.

Let $\Pi = \{\pi_k(k = 1, ..., p), \; /\pi_k \geq 0, \sum_{k=1}^{p} \pi_k = 1\}$ be the set of initial weight of each objective function provided by the DM.

Let p be the number of compromise solutions required by the DM at each iteration

Let $r(0 < r < 1)$ be a constant that will permit to define a convergence criteria

Step 2: General iteration $(m \geq 1)$

α) Let Π^m be the set of updated weights at iteration m

• If $m = 1$ then $\Pi^m = \Pi$;

• If $m > 1$ then let Π^{m-1} the set of weights to determine the compromise solution x_*^{m-1} retained by the DM at the precedent iteration (iteration $m-1$). The set Π^m is determined as follows:

$$\Pi^m = \{\pi_k(k = 1, ..., p),\ /l_k^m \leq \pi_k \leq u_k^m, \sum_{k=1}^{p} \pi_k = 1\}$$

where the interval $[l_k^m\ u_k^m]$ is defined as follows:

$$[l_k^m,\ \ u_k^m] = \begin{cases} [0,\ \ r^{m-1}] & \text{if } \overline{\pi_k^{m-1}} \leq \frac{r^{m-1}}{2} \\ [1 - r^{m-1},\ \ 1] & \text{if } \overline{\pi_k^{m-1}} \geq 1 - \frac{r^{m-1}}{2} \\ \left[\overline{\pi_k^{m-1}} - \frac{r^{m-1}}{2},\ \ \overline{\pi_k^{m-1}} + \frac{r^{m-1}}{2}\right] & \text{otherwise} \end{cases}$$

The length of different possible values of weight π_k at iteration m is then equal to r^m. So it will lead to zero by the way the approach iterates.

β) Choose p representative subsets of weights $\{\pi_k,\ \ k = 1, ..., p\} \in \Pi^m$. For each of these subsets, solve the following problem:

$$\min M\delta - \sum_{k=1}^{p} \epsilon_k \tag{2.2.9}$$

$$\text{s.t} \begin{cases} \pi_k(f_k(x) - z_k) \leq \delta - \epsilon_k,\ \ k = 1, ..., p \\ x \in X \\ \delta \geqq 0 \\ \epsilon_k,\ \ k = 1, ..., p \end{cases} \tag{2.2.10}$$

One obtains p optimization problems that lead to p Pareto-optimal solutions.

Step 3: Information phase and solution

The set of p attainable outcomes for all the p Pareto-optimal solution is presented to the DM. The DM is asked to provide the most preferred one. The corresponding Pareto-optimal solution is then considered as the compromise solution x_*^m. The weight to obtain the compromise solution is retained and represents $\overline{\pi^m}$. If the DM is satisfied, the algorithm stops. Else go to step 2.

Interactive approaches have some problems, mainly related to the preference information that the DM has to provide during the search process. For example, the DM can be asked to rank a set of solutions, to estimate weights or to adjust a set of aspiration levels for each objective. None of these tasks is trivial.

2.2.4.3 A posteriori approaches

First, search is conducted to find the Pareto-optimal set and the DM will then choose between these alternatives by inspection (with or without using some mathematical decision-making aid). Unfortunately, it is not easy to obtain an exact description of the Pareto-optimal set, that typically includes a very large or infinite number of points.

In practice, it is a finite subset of the efficient set that is found. W is a good representation of the efficient set $E(f,\ X,\ \precsim_K)$ if the following three conditions are fulfilled: W is finite and contains a reasonable number of points; non-dominated points corresponding to W do not miss a large portion of $ND(Y,\ \precsim_K)$ (coverage criterion); and these points do not include points that are very close to each other (uniformity criterion).

The coverage error is mathematically defined by:

$$\epsilon = \max_{x \in E(f,\ X,\ \precsim_K)} \min_{y \in W} d(f(x), f(y))$$

where $d(.,.)$ is a given distance defined in the decision space. This measure can be seen as the error associated with the worst representation of an element of $E(f,\ X,\ \precsim_K)$ in W. The uniformity of a representation is mathematically defined by:

$$\mu = \min_{y,z \in W\ y \neq z} d(f(y), f(z))$$

It measures the distance between a pair of closed elements of non dominated points corresponding to W. A smaller number of points, a lower coverage error and a more uniform level are desirable in order to have a good representation of the efficient set [Mena 2000, Messac and Mattson 2002, Saying 1996; 2003].

However, concerning the particular case of linear multiobjective programming problems, there exist approaches developed in the literature that generate the whole set of efficient points for small instances [Armand 1991; 1993, Armand and Malivert 1991, Ecker and Kouada 1978, Ecker et al. 1980, Iserman 1977, Yu and Zeleny 1975]. They are in general enumerative approaches based on the simplex method. We present in the next section an a posteriori method that we have developed and that can be seen as a corrected form of the method developed by [Ecker and Kouada 1978, Ecker et al. 1980] that has been shown by [Armand 1991; 1993] to be inefficient in case of degeneracy.

2.3 An a posteriori approach for solving linear multiobjective optimization problems

Most works were mainly conducted for identifying efficient solutions [Fotso April 1981, Gal 1977, Hong et al. 2005, Yu and Zeleny 1975, Ehrgott 2005]. There are two main approaches in the literature: by solving multiple parametric programming or by using multiple criteria methods to examine the adjacent extreme points. [Yu and Zeleny 1975] used a global view method and the 'top-down' search strategy, while [Philip 1972] used a local view method to obtain the efficient face incident to a given efficient extreme point. [Iserman 1977] and [Ecker and Kouada 1978, Ecker et al. 1980] combined these two view methods. Later on, [Ecker et al. 1980] and [Armand 1991; 1993] applied a 'bottom-up' search strategy to develop an algorithm. But the main difficulty for these methods arises especially when dealing with degenerate problems. Recently, [Armand 1991; 1993] showed by a counter-example that the approach developed in [Ecker and Kouada 1978, Ecker et al. 1980] was incomplete in case of degenerate problems.

In this section, we present a complete procedure for solving Multiple Objective Linear Programming Problems, which can be regarded as a corrected form and an alternative to the method of [Ecker et al. 1980]. In fact, we develop a new characterization of efficient faces similarly to the one developed in [Ecker and Kouada 1978, Ecker et al. 1980], but with an added property that a face incident to a given vertex depends not only on the vertex but also on the associated basis (tableau). In addition, the connectedness property of the set of ideal tableaux associated with a degenerate point is used to handle degenerate cases. The approach generates the whole efficient set and all extreme efficient points.

Throughout the section, we consider the following problem:

- Pb 1: $Max\ \{Cx \ \ s.t \ \ x \in X\}$ where $X = \{x/x \geq 0, Ax = b\} \subset \mathbb{R}^n$, C is a $k \times n$ matrix , A is a $m \times n$ matrix, $b \in \mathbb{R}^m$, $x \in R^n$.

To the problem, *Pb1*, we associate with respect to a given vector $\lambda \geq 0$, the following problem:

- $P\lambda$: $Max\ \{\lambda^t Cx \ \ s.t \ \ x \in X\}$.

We denote by E the whole set of efficient points of (Pb 1), $X(\lambda)$ the set of optimal solutions of $(P\lambda)$ and $e = (1, 1, ..., 1)$ an element of $\mathbb{R}^k$.

The presentation is organized as follows. In the next section, basic notations and definitions are presented. We also present a characterization of an efficient face incident to a point (vertex) with respect to an associated tableau. We then show how to construct an efficient face using the notion of $\lambda - efficient$ set. In section 2.3.2, an algorithm for finding all efficient faces incident to an extreme efficient point are provided. In section 2.3.3, we give an algorithm for finding the set of efficient points. We illustrate these algorithms with a numerical example in section 2.3.4.

2.3.1 Efficient face

2.3.1.1 Notations and definitions

At each iteration of the algorithm, the current efficient extreme point x is associated with a tableau T (not necessarily unique) called efficient extreme tableau which after column rearrangement can have the form:

$$(T): \quad \begin{array}{c|c} & x_N \quad x_B \\ \hline \text{d} & -C \quad 0 \\ \hline \text{b} & \text{A} \quad \text{I} \end{array}$$

The corresponding efficient extreme point can be put in the form $x = (x_N, x_B)$ where x_N and x_B represent basic variables and nonbasic variables, respectively. We denote by N_T, the nonbasic set which is the set of indices of nonbasic variables.

Tableau T or the corresponding efficient extreme point is degenerate if the left hand side b has at least one null component. A vertex is nondegenerate if and only if there is only one tableau that can be associated with it. Given a degenerate tableau, one cannot always generate all the efficient edges incident to the associated vertex.

Definition 2.3.1. [Armand 1993]: T is called $l-feasible$, if for any vertex x_1 of X, it is possible, to find a tableau T_1 associated with x_1 and a finite sequence of lexicographic pivots linking T and T_1.

This can be possible by using the finiteness property of the lexicographic rule ([Armand 1991]). A nondegenerate tableau is l-feasible. However, with a degenerate vertex, only a subset (nonempty) of its associated tableaux is l-feasible.

Definition 2.3.2. : Let C be the reduced cost matrix of T and let $\lambda > 0$, T is $\lambda - dual - efficient$ if $-\lambda^t C \geq 0$.

A pivot element on the jth column of T is $\lambda - dual - degenerate$ if T is $\lambda - dual - efficient$ and if the jth column of C satisfies $\lambda^t C^j = 0$.

On such a column, a pivot generates either an efficient edge of X, or a new dual-efficient tableau associated with the same efficient vertex.

Let JT be the set of nonbasic indices j leading to an efficient edge by a lexicographic pivot on column j of T. JT is called set of efficient indices incident to x with respect to T. If T is $\lambda - dual - efficient$, then $JT \subseteq \{j \in N_T / \lambda^t C^j = 0\}$. Thus, from JT we can characterize all or part of efficient edges (faces) incident to x. T is $\lambda - ideal$, with $\lambda > 0$, if it is $l - feasible$ and $\lambda - dual - efficient$.

Definition 2.3.3. : Two $\lambda - ideal$ tableaux are ideally-adjacent if it is possible to go from one to another with a lexicographic pivot on a column j of C such that $\lambda^t C^j = 0$.

This adjacency relation induces an undirected graph[Armand and Malivert 1991]. Any efficient extreme vertex can be associated with at least an ideal tableau [Armand 1991; 1993, Armand and Malivert 1991]. Using these results and given an efficient extreme point x characterized by a $\lambda - ideal$ tableau T, it follows that: If x is nondegenerate then T is unique and all the efficient edges incident to x will be generated.

In this case, $JT = \{j \in N_T / \lambda^t C^j = 0\}$. But if x is degenerate then T is not unique, only information about a subset of efficient edges (faces) incident to x will be given. Then $JT \subseteq \{j \in N_T / \lambda^t C^j = 0\}$.

Let $\alpha \subset N_T$. Consider the following faces with respect to T:

$f(T, \alpha) = \{x \in X / x_j = 0, j \in N_T \setminus \alpha\}$.

G(α)={ $(v, w) \geq 0 /\ C^t v + w = -C^t e$,$w_j = 0 \forall j \in \alpha\}$.

The face represented by $f(T, \alpha)$ for $\alpha \subset N_T$ is *efficient* if $f(T, \alpha) \subset E$.

If in addition there is no $\alpha^* \subset N_T / f(T, \alpha^*) \subset E$ with $\alpha \subset \alpha^*$ then the face is said *maximal efficient*.

A set $F \subset N_T$ is *E-maximal* (with respect to T) if and only if $G(F) \neq \emptyset$ and there is no set: $F^* \subset N_T / F \subset F^*, G(F^*) \neq \emptyset$.

In the following section, we consider a given extreme efficient point x characterized by an $l - feasible$ tableau T such that $JT \neq \emptyset$.

2.3.1.2 Characterization of an efficient face

The characterization of an efficient face presented in [Ecker et al. 1980] is incorrect if the problem is degenerate. Here, we introduce an extension which works even when the problem is degenerate. We show that, under some hypothesis, a face $f(T, F)$ is efficient if and only if $G(F) \neq \emptyset$. We first present some results necessary to establish our characterization.

Lemma 2.3.1. *[Ecker et al. 1980]: If $\lambda \geq e$, then $X(\lambda) \subset E$; conversely, given any maximal efficient face H, there is a vector $\lambda \geq e$ such that $H = X(\lambda)$*

We use this lemma to prove the following theorem.

Theorem 2.3.1. *: If $F \subseteq \{j \in N_T / \lambda^t C^j = 0\}$ and $-\lambda^t C \geq 0$ then $f(T, F) \subseteq X(\lambda)$.*

Proof: If $x^0 \in f(T, F)$ then: $x_j^0 = 0$ if $j \in N_T - F$, therefore: x^0 and x give the same value of the objective function for $(P\lambda)$. Since $-\lambda^t C \geq 0$, it follows that: $x_T \in X(\lambda)$ and then : $x_0 \in X(\lambda)$; we then have $f(T, F) \subset X(\lambda)$.

□

Lemma 2.3.2. *[Ecker et al. 1980]: If $F = \{j \in N_T / \lambda^t C^j = 0\}$ and $-\lambda^t C \geq 0$ then $f(T, F) = X(\lambda)$.*

Remark 2.3.1. : If $f(T, F) \subset X(\lambda)$ then $-\lambda^t C \geq 0$. In fact, $f(T, F) \subset X(\lambda)$ implies that $x \in X(\lambda)$, and thus $-\lambda^t C \geq 0$.

With respect to previous results, we can now give the main theorem of this section.

Theorem 2.3.2. *(*Characterization theorem*) : Let $F \subseteq JT$; The face $f(T, F)$ is efficient if and only if $G(F) \neq \emptyset$.*

Proof: $\Longrightarrow$ (necessary condition): If $f(T, F)$ is efficient, then it is contained in some maximal efficient face which, by lemma 2.3.1, is $X(\lambda)$ for some $\lambda \geq e$. From remark 2.3.1, $-\lambda^t C \geq 0$, so that T is $\lambda - ideal$ and $JT \subseteq \{j \in N_T / \lambda^t C^j = 0\}$. But $F \subseteq JT$, so $F \subseteq \{j \in N_T / \lambda^t C^j = 0\}$. Consequently, $(v, w) = (\lambda - e, -\lambda^t C) \in G(F)$. Therefore $G(F) \neq \emptyset$.

$\Longleftarrow$ (sufficient condition): Conversely, if $(v, w) \in G(F)$, then for $\lambda = v + e$, one has that $\lambda \geq e, -\lambda^t C \geq 0$ and $F \subseteq \{j \in N_T / \lambda^t C^j = 0\}$. Then according to theorem 2.3.1, $f(T, F) \subseteq X(\lambda)$ and from lemma 2.3.1, $X(\lambda) \subseteq E$. Consequently, $f(T, F) \subseteq E$. This completes the proof.

□

2.3.1.3 Construction of an efficient face

In this subsection, we present how to construct an efficient face incident to a given efficient extreme point. We have the following results.

Theorem 2.3.3. *: Let $f(T, F)$ be an efficient face incident to x. If $(v, w) \in G(F)$, then for $\lambda = v + e$, one has that $f(T, F) \subseteq X(\lambda)$.*

Proof: By theorem 2.3.1, it is sufficient to prove that $F \subseteq \{j \in N_T / \lambda^t C^j = 0\}$ and $-\lambda^t C \geq 0$. From theorem 2.3.2, $G(F) \neq \emptyset$. Let $(v, w) \in G(F)$ and $\lambda = v + e$. Then we have: $-\lambda^t C = -(v+e)^t C$ $= (-C^t v - C^t e)^t = w^t$ and $w \geq 0 \Rightarrow -\lambda^t C \geq 0$. Since $f(T, F)$ is efficient, $F \subseteq JT$; and $-\lambda^t C \geq 0$, then T is $\lambda - ideal$ so that $JT \subseteq \{j \in N_T / \lambda^t C^j = 0\}$; thus $F \subseteq \{j \in N_T / \lambda^t C^j = 0\}$. These results and theorem 2.3.1 imply that : $f(T, F) \subseteq X(\lambda)$.

□

From theorem 2.3.3, the following corollary can be deduced.

Corollary 2.3.1. *: Let $f(T, F)$ be an efficient face incident to x_T; $(v, w) \in G(F)$ and $\lambda = v + e$. If $X'(\lambda) = \{x \in X(\lambda)/x_i = 0 \ \forall i \in N_T - F\}$ then $f(T, F) = X'(\lambda)$.*

Proof: (i) First, we show that $f(T, F) \subseteq X(\lambda)$: Let $x \in f(T, F)$, then from theorem 2.3.3, $x \in X(\lambda)$ and $x_j = 0$, $\forall j \in N_T - F$. It follows that $x \in X(\lambda)$ and $x_j = 0 \ \forall j \in N_T - F$ which implies that $x \in X'(\lambda)$.
(ii) We now show that $f(T, F) \subseteq X(\lambda)$: Let $x \in X'(\lambda)$ then $x \in X(\lambda)$ and $x_i = 0 \forall \ i \in N_T - F$ or $X(\lambda) \subset X$. It follows that $x \in X$ and $x_i = 0 \ \forall i \in N_T - F$, *i.e* $x \in f(T, F)$.

From (i) and (ii) one can conclude that $X'(\lambda) = f(T, F)$.

□

Definition 2.3.4. : The vector λ constructed in theorem 2.3.3 is called the characteristic vector of the face $f(T, F)$.

Definition 2.3.5. : We call, $\lambda - efficient$ set incident to x_T, the set $X'(\lambda)$ constructed from $X(\lambda)$ in the corollary 2.3.1.

Remark 2.3.2. : The corollary 2.3.1 provides a method for finding the face $f(T, F)$ incident to x_T by finding the $\lambda - efficient$ set incident to x_T.

Remark 2.3.3. : Let $f(T, F)$ be an efficient face incident to x_T. If $(v, w) \in G(F)$, then for $\lambda = v + e$, if $(\forall j \in N_T, w_j = 0 \Rightarrow j \in F)$ then $X'(\lambda) = X(\lambda)$. In fact, from $F = \{j \in N_T/\lambda^t C^j = 0\}$ *and* $-\lambda^t C \geq 0$, lemma 2.3.2 implies that $f(T, F) = X(\lambda)$. Since $X'(\lambda) = f(T, F)$, then $X'(\lambda) = X(\lambda)$.

2.3.2 Finding all efficient faces incident to an extreme efficient point

The objective of this section is to use the results presented in the previous sections to develop an algorithm to find all efficient faces incident to a given extreme efficient point.

We assume that the tableaux associated with the current efficient extreme point x are ideal. Let T be such a tableau and N_T the corresponding nonbasic set. We first give a subroutine CALCUL1 that takes as input T and N_T, and returns all efficient points S2, all efficient extreme points S1, and all ideal tableaux S3 incident to x with respect to T. CALCUL1 is called by subroutine CALCUL2 to compute all efficient points and all efficient extreme points incident to the given extreme efficient point. CALCUL1 in step 1 finds two characteristic sets JT and KT where $KT = NT - JT$. In step 2, it finds for every subset of JT the characteristic vectors λ of an efficient face incident to x with respect to T. Then, for each vector λ the set $X'(\lambda)$ is found and saved in $S2$. Finally, step 3 finds for each element of KT an ideal tableau incident to x with respect to T and saves it in $S3$.

Algorithm 2. 1

Function $CALCUL1[T, N_T] = [S1, S2, S3]$

{T is an ideal tableau associated with the extreme efficient point x and N_T is the nonbasic set. }

LT:=∅; { LT is an intermediate set}

for all $j \in N_T$ **do**

Find optimal value m_j of P_j where

Problem P_j: $Min\ w_j$

such that

$C^t v + w = -C^t e\ v, w \geq 0$

if $m_j = 0$ **then** $LT := LT \cup \{j\}$ **endif**

endfor

for all $j \in LT$ **do**

Let p be the lexicographic pivot element.

if p is not primal degenerate **then** $JT := JT \cup \{j\}$

else $KT := KT \cup \{j\}$ **endif**

endfor

for all $j \in JT$ **do**

Let t be the pivot that leads to a new efficient extreme point,

T the ideal tableau obtained by pivoting on t and N_T the nonbasic set.

if $(T, N_T) \notin S1$ **then** $S1 := S1 \cup \{(T, N_T)\}$ **endif**

endfor

for all $\alpha \subseteq JT$ such that $G(\alpha) \neq \phi$ **do**

Find $(v, w) \in G(\alpha)$; $\lambda := v + e$; Find $X(\lambda)$; Find $X'(\lambda)$;

$S2 := S2 \cup \{X'(\lambda) - X'(\lambda) \cap S2\}$

endfor

for all $j \in KT$ **do**

Let t be the lexicographic pivot element;

Let T^* be the ideal tableau obtained by pivoting on t;

Let N_{T^*} be the nonbasic set;

if $(T^*, N_{T^*}) \notin S3$ **then** $S3 = S3 \cup \{(T^*, N_{T^*})\}$ **endif**

endfor

endfunction

Remark 2.3.4. : JT contains all j such that $\lambda^t C^j = 0$ and such that a pivot on such columns generates an efficient edge of X. KT contains all j such that $\lambda^t C^j = 0$ and such that a pivot on such columns generates a new ideal tableau associated with the same extreme efficient point. Given JT, we use theorem 2.3.3 and its corollary 2.3.1 to construct efficient face incident to x with respect to T (a subset of efficient faces incident to the current extreme point). $S1$ contains efficient extreme points adjacent to the current efficient extreme point with respect to T. Elements of $S1$ are characterized by an ideal tableau associated with the corresponding nonbasic set. $S2$ (possibly empty) contains efficient faces incident to the current efficient extreme point with respect to T. $S3$ contains ideal tableaux incident to the current efficient extreme point with respect to T.

Note that CALCUL1 considers only one ideal tableau incident to the current extreme efficient point. If there are more than one ideal tableau, it is necessary to go through these ideal tableaux and for each, generate the sets S1, S2 and S3. If the current extreme efficient point is nondegenerate, there is just one ideal tableau associated with it and CALCUL1 generates all incident efficient points and efficient extreme points. The variant algorithm (CALCUL2) uses the results of CALCUL1 and of the following P. Armand connectness theorem.

Theorem 2.3.4. *[Armand 1993]: Let x^0 be an efficient extreme point; and let $\Re(x^0)$ be (the subgraph spanned by) the set of ideal tableaux associated with x^0. Then $\Re(x^0)$ is a connected graph. Moreover, any efficient edge incident to x^0 can be generated from at least one tableau of $\Re(x^0)$.*

Since the subgraph of ideal tableau incident to an efficient extreme point x is connected, CALCUL2 is simply an iteration through different ideal tableaux incident to the current efficient extreme point x. For every unexplored tableau T, it calls CALCUL1 to find efficient points and extreme efficient points adjacent to x with respect to T.

Algorithm 2. 2

Function $CALCUL2[T, N_T] = [E1, E2]$
$[S1, S2, S3] := CALCUL1[T, N_T];$
$E1 := S1;\ E2 := S2;\ H1 := \{(T, N_T)\};$
for all $(T', NT') \in S3$ **do**
$S3 := S3 - \{(T', NT')\};$
$H1 := H1 \cup \{(T', NT')\};$
$[Z1, Z2, Z3] := CALCUL1(T', NT');$
$E1 := E1 \cup \{Z1 - Z1 \cap E1\};$
$E2 := E2 \cup \{Z2 - Z2 \cap E2\};$
$S3 := S3 \cup \{Z3 - Z3 \cap H1\};$
endfor
endfunction

Remark 2.3.5. : For every ideal tableau incident to the current efficient extreme point, $H1$ contains all ideal tableaux already dealt with. At the end of $CALCUL2$, $H1$ contains all ideal tableaux incident to the current extreme efficient point, $E1$ contains all efficient extreme points adjacent to the current efficient extreme point. An element of $E1$ is characterized by an ideal tableau and the corresponding nonbasic set (N_T). $E2$ contains all efficient points adjacent to the current efficient extreme point.

2.3.3 Finding all efficient points

Since efficient extreme points form a connected graph, the main algorithm EFFICACE iterates through these different efficient extreme points. For every unexplored point x characterized by an ideal tableau T and the corresponding non basic set NT, the main algorithm calls CALCUL2 to compute all the efficient points and all the efficient extreme points adjacent to x and then saves them in E and H respectively.

Algorithm 2. 3

Function $EFFICACE[T, N_T] = [E, H]$
$[S1, S2] := CALCUL2[T, N_T];$
$T1 := S1;\ E := S2;\ H := \{(T, N_T)\};$
for all $(T', NT') \in T1$ **do**
$T1 := T1 - \{(T', NT')\};$
$H := H \cup \{(T', NT')\};$
$[Z1, Z2] := CALCUL2(T, NT');$
$T1 := T1 \cup \{Z1 - Z1 \cap H\};$
$E := E \cup \{Z2 - Z2 \cap E\};$
endfor
endfunction

Remark 2.3.6. : Here E is the union of efficient faces. But, as in [Armand and Malivert 1991, Ecker and Kouada 1978, Ecker et al. 1980], the construction of the characteristic vectors can be done in such a way that E is exactly the union of maximal efficient faces.

2.3.4 Numerical illustration

The method developed in [Ecker and Kouada 1978, Ecker et al. 1980] assumes nondegeneracy. The example below proposed in [Armand 1993] shows that the method in [Ecker et al. 1980] cannot give all maximal efficient faces in the degenerate case. This example is used to illustrate how our approach gives all maximal efficient faces even for degenerate problems.

$$\max x_1 + x_2 - 3x_3$$

$$\max x_3$$

$$\text{subject to} \begin{cases} x_1 + x_2 \leq 4 \\ -x_1 - x_2 + 2x_3 \leq 0 \\ x_1 - 3x_2 - 4x_3 \leq 0 \\ x_1 + x_2 - 3x_3 \leq 0 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

2.3.4.1 Running CALCUL1

$x_0 = (0, 0, 0)$ is an efficient extreme point and the corresponding ideal tableau T_0 is:

0	-0.5	-0.5	1.5	0	0	0	0
0	0.5	0.5	-0.5	0	0	0	0
4	1	1	0	1	0	0	0
0	-0.5	-0.5	0.5	0	1	0	0
0	-1	5	2	0	0	1	0
0	-0.5	-0.5	1.5	0	0	0	1

with the nonbasic set $N_{T_0} = \{1, 2, 5\}$. $LT_0 = \{2, 5\}$, $JT_0 = \{2\}$ and $KT_0 = \{5\}$. From $JT_0 = \{2\}$, $(v, w) = (0, 0, 0, 0, 1) \in G(2)$. Thus $\lambda = (1, 1)$. Since $1 \notin \{2\}$, the first efficient face of the problem is $X'(\lambda_1)$ that is put in S1. $x_1 = (0, 4, 2)$ is an efficient extreme point adjacent to $x_0 = (0, 0, 0)$ with the corresponding ideal tableau T_1:

2	0	0.5	1.5	0	0	0	0
-2	0	-0.5	-0.5	0	0	0	0
4	1	1	0	1	0	0	0
2	0	0.5	0.5	0	1	0	0
-20	4	5	2	0	0	1	0
2	0	0.5	1.5	0	0	0	1

and the nonbasic set $N_{T_1} = \{1, 2, 5\}$. T_1 and N_{T_1} are put in S2. From $KT_0 = \{5\}$, only one ideal tableau T_0' is incident to $x_0 = (0, 0, 0)$ with respect to T_0. T_0' is:

0	1	1	-3	0	0	0	0
0	0	0	1	0	0	0	0
4	1	1	0	1	0	0	0
0	-1	-1	2	0	1	0	0
0	1	-3	-4	0	0	1	0
0	1	1	-3	0	0	0	1

and the nonbasic set is $N_{T_0'} = \{1,2,3\}$. Since $(T_0', N_{T_0'}) \notin S3$, we add it to $S3$ to have $S3 = \{(T_0', N_{T_0'})\}$. Therefore: (i) The set of efficient points adjacent to x_0 with respect to T_0 is $X'(\lambda_1), \lambda = (1,1)$. (ii) The set of extreme efficient points adjacent to x_0 with respect to T_0 is $x_1 = (0,4,2)$ characterized by (T_1, N_{T_1}). (iii) The set of ideal basis incident to x_0 with respect to T_0 is characterized by: $(T_0', N_{T_0'})$.

Remark 2.3.7. : Since the extreme efficient point considered is degenerate, it is possible that CALCUL1 gives only a subset of efficient points, efficient extreme points and ideal tableaux incident to $x_0 = (0,0,0)$.

2.3.4.2 Running CALCUL2

After calling CALCUL1 with T_0 and NT_0, we have: $S1 = \{(T_1, NT_1)\}$; $S2 = \{X'(\lambda_1)\}$ where $\lambda_1 = (1,1)$ and $S3 = \{(T_0', NT_0')\}$ where T_0' is:

0	1	1	-3	0	0	0	0
0	0	0	1	0	0	0	0
4	1	1	0	1	0	0	0
0	-1	-1	2	0	1	0	0
0	1	-3	-4	0	0	1	0
0	1	1	-3	0	0	0	1

and the nonbasic set $N_{T_0'} = \{1,2,3\}$.

We thus obtain; $E1 = \{(T_1, NT_1)\}; E2 = \{X'(\lambda_1)\}; H1 = \{(T_0, N_{T_0})\}$. The iteration begins with (T_0', NT_0'). (T_0', NT_0') is subtracted from S3 and added to H1 to give $S3 = \phi$ and $H1 = \{(T_0, N_{T_0}), (T_0', N_{T_0'})\}$. CALCUL1 with (T_0', NT_0') is called to give $Z1 = \phi$; $Z2 = \phi$ and $Z3 = \{(T_0, N_{T_0})\}$. In this process, $E1 = \{(T_1, NT_1)\}$, $E2 = \{X'(\lambda_1)\}$ and since $\{Z3 - Z3 \cap H1\} = \phi$, $S3 = \phi$. Since $S3 = \phi$ the iteration terminates with:

Set of efficient extreme points adjacent to $x_0 = (0,0,0)$, the set $E1 = \{(T_1, NT_1)\}$;

Set of efficient points adjacent to $x_0 = (0,0,0)$, the set $E2 = \{X'(\lambda_1)\}$.

2.3.4.3 Running EFFICACE

After calling CALCUL2 with the pair (T_0, N_{T_0}), $S2 = \{X'(\lambda_1)\}$ where $\lambda_1 = (1,1)$. $S1 = \{(T_1, N_{T_1})\}$. We found $T1 = \{(T_1, N_{T_1})\}$, $E = \{X'(\lambda_1)\}$, $H = \{(T_0, N_{T_0})\}$.

The iteration begins with (T_1, N_{T_1}).

Then, $T1 = \phi$, $H = \{(T_0, N_{T_0}), (T_1, N_{T_1})\}$.

After the call of CALCUL2 with (T_1, N_{T_1}), $Z1 = \{(T_2, N_{T_2})\}$, $Z2 = \{X'(\lambda_2)\}$ where the ideal tableau T_2 is given by:

2	0.5	0	1.5	0	0	0	0
-2	-0.5	0	-0.5	0	0	0	0
4	1	1	0	1	0	0	0
2	0.5	0	0.5	0	1	0	0
4	1	6	2	0	0	1	0
2	0.5	0	1.5	0	0	0	1

with the nonbasic set $N_{T_2} = \{2, 4, 5\}$ and the characteristic vector $\lambda_2 = (1,1)$. $T1 = \{(T_2, NT_2)\}$ and $E = \{X'(\lambda_1)\}$ because $X'(\lambda_1) = X'(\lambda_2)$. The first iteration finishes here. The second iteration starts with (T_2, N_{T_2}). Then $T1 = \phi$, $H = \{(T_0, N_{T_0}), (T_1, N_{T_1}), (T_2, N_{T_2})\}$. After CALCUL2 is called with (T_2, N_{T_2}), $Z1 = \{(T_0, N_{T_0})\}$, $Z2 = \{X'(\lambda_3)\}$ where $\lambda_3 = (1,1)$. At this point $T1 = \phi$ and E is still equal to $\{X'(\lambda_1)\}$ because $X'(\lambda_1) = X'(\lambda_3)$. Since $T1 = \phi$, the algorithm terminates with three efficient extreme points characterized by: (T_0, N_{T_0}), (T_1, N_{T_1}) and (T_2, N_{T_2}) respectively. The set of efficient points is $E = \{X'(\lambda_1), \quad \lambda_1 = (1,1)\}$.

2.4 Multicriteria optimization applications: planning of the distribution of electrical energy in Cameroon

2.4.1 Statement of the problem

The applications of multicriteria optimization are numerous and varied [Ehrgott 2005; 2000, Fotso April 1981]. For example, applications of multicriteria optimization in environmental management and in the management of wild animal reserves can be found in [Mbountcha June 2001]. Multicriteria optimization was also used by Glover and Martinson to solve a problem connected to regional development [Crama et al. 1997]. We present here a multicriteria optimization model for the planning of the distribution of electrical energy generally met in developing countries, inspired by the case of the AES-SONEL company of Cameroon.

In developing countries and particularly in Cameroon, the planning of distribution of electrical energy is a serious problem. Cameroon has only one single company, AES-SONEL, that produces and distributes electrical energy. The company possesses hydroelectric power plants scattered throughout the country. The electricity distribution system of AES-SONEL has two interconnected networks [Mattson et al. 2004]: the NIN (North Interconnected Network) which feeds the northern part of the country and the SIN (South Interconnected Network) which feeds the southern part except the East province. The production capacity of different hydroelectric power plants depends on the rainfall of the region where they are located. Until now, the policies adopted by AES-SONEL consist in assigning to a given region the closest hydroelectric power plant. One consequence of this strategy is that while certain regions have at their disposal a surplus of energy, others are in demand (due to low rainfalls). During the season of low rainfall in a given region, the company generally uses two policies:

- A load shedding strategy, which consists in depriving per given periods certain regions or localities of electrical energy.
- The policy of penalization of high consumers, which consists in increasing the unit cost of consumption when the total quantity of electrical energy consumed per month is higher than a fixed maximum level.

Both policies have disadvantages. The application of the first policy does not allow for example the company to make an optimal profit. The second policy obliges high electrical energy consuming companies not to work in their full capacities and consequently not to achieve their optimal production. Thus strategies for the distribution of electrical energy adopted so far by AES-SONEL slow down considerably Cameroonian economy. There is a necessity to develop a new policy for planning distribution of electrical energy in order to reduce this economical loss. The presentation is organized in two parts. First, a description of the problem of the distribution of electrical energy in Cameroon is provided. This is followed by the presentation of the mathematical formalism which leads to a multicriteria programming model. We conclude with the presentation of two approaches developed in order to solve the obtained model. The first approach is an interactive algorithm based on Geoffrion's method and analytic hierarchy process that can be used if an a priori knowledge about the decision problems is not available. The second approach is an aggregation method. A complete procedure written with the GAMS language [GAMS-Development October 2003 (www.gams.com/] is described, which can be directly tested if the software GAMS and real data are provided.

2.4.2 Multicriteria model for planning the distribution of electrical energy

2.4.2.1 Description of the problem

The problem concerns a company having a set of factories producers and distributors of electrical energy which has to satisfy the energy needs of a set of given regions. The search for an optimal planning is necessary at least for two reasons:

- The variability of the production capacities of the various hydroelectric power plants due to the fact that dams depend on the water level which in turn depends on the rainfall in the region. The price of petroleum in the market (which allows to turn the turbines of dams) is also variable.
- The quantities of electrical energy demanded by consumers are variable and must be satisfied.

The planning must try to reconcile at best the following often contradictory objectives:

1. Ensure the maximum profit to the company.
2. Ensure a distribution at the least possible cost.
3. Ensure a minimal pollution.
4. Ensure that each customer has the quantity of energy that he/she wishes at any moment.

These objectives are rather vague and sometimes contradictory; therefore, to remedy it, one has to impose that required needs are covered with the electrical energy production capacities of the various hydroelectric power plants and that certain objectives are translated into constraints. This inevitably involves a maximization of the quantity of energy produced by the company within the limits of its capabilities. We thus obtain a multicriteria optimization problem under the constraints of capacity and demand.

2.4.2.2 Mathematical formalization of the problem

Let us consider that the company has N factories of production of electrical energy and must satisfy the demands of P regions in electrical energy during a given period. Decision variables here can be represented by the set of x_{ij}, the quantity of energy transported by the factory i ($i = 1..N$) to the region j ($j = 1...P$). Although the main mission is the search for profit, the company has obligations vis- à-vis the country where it is established. It is supposed that two main clauses

are to satisfy the energy needs of the country and to respect current legislation related to the protection or pollution of the environment (destruction of the fauna and of its vegetation due to electricity transportation lines, atmospheric pollution due to gases emitted by turbines allowing the functioning of dams...).
The company would thus like to reach the following objectives:

- maximize the profit
- minimize the pollution or destruction of environment
- maximize the production of electrical energy
- satisfy incompressible demands (that of hospitals, security services...)

The following hypotheses are introduced:
• [H1)] *Determinism*: Apart from the unknown x_{ij} which is the quantity of energy transported by the factory i $(i = 1...N)$ to the region j $(j = 1...P)$, all other parameters of the model are known constants.
• [H2)]*Divisibility*: Levels of activities x_{ij} are not necessarily integers.
•[H3)]*Additivity*: By considering the level of activity x_{ij} , $i = 1..N, j = 1..P$, the total quantity of every used resource is found by accumulating corresponding quantities used by various activities. Thus the total quantity of electrical energy supplied by the factory i is equal to $\sum_{j=1}^{P} x_{ij}$ and the quantity of energy received by the region j is $\sum_{i=1}^{N} x_{ij}$.
•[H4)] *Proportionalities*: Objective functions values are directly proportional to decision variables x_{ij} .

If for the considered period, the transport of a unity of energy from factory i to region j requires a cost estimated by c_{ij} , then the cost of transport of the quantity of energy $\sum_{i=1}^{N} x_{ij}$ towards the region j is $\sum_{i=1}^{N} c_{ij}x_{ij}$. So the total cost to distribute all energy produced in all the P regions will be :

$$\sum_{j=1}^{P}\sum_{i=1}^{N} c_{ij}x_{ij}$$

Let us suppose that the transport of a unity of energy from factory i towards region j causes a loss of energy of p_{ij} with $o \leq p_{ij} < 1$; by considering that this loss has an impact on environment estimated by d_{ij} , the total quantity of energy lost will then be :

$$\sum_{j=1}^{P}\sum_{i=1}^{N} p_{ij}x_{ij}$$

and the quantity of poured pollutant may be estimated by:

$$\sum_{j=1}^{P}\sum_{i=1}^{N} d_{ij}x_{ij}$$

Let t_j be the cost of sale of a unity of energy in the region J, then the selling price of the energy produced for the region j will be:

$$\sum_{i=1}^{N} t_j x_{ij}$$

and for the P regions:

$$\sum_{j=1}^{P}\sum_{i=1}^{N} t_j x_{ij}$$

The loss of p_{ij} quantity of energy during the transportation of a unity of energy from factory i towards region j causes then a loss of $p_{ij}t_j$ and thus, for the company a loss of:

$$\sum_{j=1}^{P}\sum_{i=1}^{N} t_j p_{ij} x_{ij}$$

The profit to be realized by the company with regard to the distribution of the energy is estimated by : $price\ of\ sale - cost\ of\ transport$ - $cost\ of\ energy\ lost$; it can be expressed by:

$$\sum_{j=1}^{P}\sum_{i=1}^{N} t_j x_{ij} - \sum_{j=1}^{P}\sum_{i=1}^{N} c_{ij}x_{ij} - \sum_{j=1}^{P}\sum_{i=1}^{N} t_j p_{ij} x_{ij}$$

So the objectives of the company are:

- Maximize the profit value:

$$Max \sum_{j=1}^{P}\sum_{i=1}^{N} t_j x_{ij} - \sum_{j=1}^{P}\sum_{i=1}^{N} c_{ij}x_{ij} - \sum_{j=1}^{P}\sum_{i=1}^{N} t_j p_{ij} x_{ij}$$

- Maximize the total production to satisfy needs in energy of the country:

$$Max \sum_{j=1}^{P}\sum_{i=1}^{N} x_{ij}$$

- Minimize the environmental pollution:

$$Min \sum_{j=1}^{P}\sum_{i=1}^{N} d_{ij}x_{ij}$$

The company has to reach the objectives within the limits of production capacity of all the factories. Let b_i be the production capacity of the factory i for the considered period, we must have

$$\sum_{j=1}^{P} x_{ij} \leq b_i$$

furthermore, the company has to insure the satisfaction of its incompressible demand denoted by l, one must have:

$$\sum_{j=1}^{P}\sum_{i=1}^{N} x_{ij} \geq l$$

The quantity of energy being always positive, it implies the constraint of non negativity on x_{ij}, i=1..N., $j = 1..P$.

One obtains the following multicriteria optimization problem:

$$\begin{cases} Max \sum_{j=1}^{P}\sum_{i=1}^{N} t_j x_{ij} - \sum_{j=1}^{P}\sum_{i=1}^{N} c_{ij}x_{ij} - \sum_{j=1}^{P}\sum_{i=1}^{N} t_j p_{ij} x_{ij} \\ Max \sum_{j=1}^{P}\sum_{i=1}^{N} x_{ij} \\ Min \sum_{j=1}^{P}\sum_{i=1}^{N} d_{ij}x_{ij} \\ \text{subject to} \begin{cases} \sum_{j=1}^{P}\sum_{i=1}^{N} x_{ij} \geq l \\ \sum_{j=1}^{P} x_{ij} \leq b_i, \ ; \ i = 1..N. \\ x_{ij} \geq 0, \ i = 1..N, j = 1..P \end{cases} \end{cases}$$

2.4.3 Solution approach for the multicriteria optimization problem obtained

We consider the following notations:
$f_1(x) = \sum_{j=1}^{P}\sum_{i=1}^{N} t_j x_{ij} - \sum_{j=1}^{P}\sum_{i=1}^{N} c_{ij}x_{ij} - \sum_{j=1}^{P}\sum_{i=1}^{N} t_j p_{ij} x_{ij}$
$f_2(x) = \sum_{j=1}^{P}\sum_{i=1}^{N} x_{ij}$ and $f_3(x) = -\sum_{j=1}^{P}\sum_{i=1}^{N} d_{ij}x_{ij}$.
$\Omega = \left\{ x = (x_{ij}) / \ \sum_{j=1}^{P}\sum_{i=1}^{N} x_{ij} \geq l, \sum_{j=1}^{P} x_{ij} \leq b_i, \ x_{ij} \geq 0, i = 1..N., j = 1..P \right\}$
Then the preceding problem becomes:

$$\begin{cases} max \ f_1(x) \\ max \ f_2(x) \\ max \ f_3(x) \\ \text{subject to} \ \ x \in \Omega \end{cases}$$

The resolution of the obtained problem normally depends on the company distributing the electrical energy, AES-SONEL in the case of Cameroon. Two techniques can normally be use: It can be solve by using interactive methods as presented in 2.2.4.2. The difficulty of this approach is that, one needs the presence of the Decision Makers (AES-SONEL) at different level of the approach. The second method can be to use A priori methods as presented is section 2.2.4.1. In this case, it suffice to the Decision Maker (AES-SONEL) to provide the weights $\lambda_1, \lambda_2, \lambda_3$ to quantify his preferences concerning the three objectives f_1, f_2, f_3 respectively. The solution of the multicriteria optimization problem above can then be found by optimizing the aggregate function $f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ over the decision space. The optimal solution will be a Pareto-optimal solution and a good compromise. The GAMS [GAMS-Development October 2003 (www.gams.com/] program model that implements the aggregate approach is as follows:

GAMS model for solving the problem

Sets
i put in slashes the name of different factories / factory1, factory2, factory3 /
j put in slashes the name of different regions / centre, nord, sud, est ouest, sudouest, nordouest, littoral, extremenord, adam/
Parameters
t(j) in the slash, put in front every name of region, the cost of the unity of energy there /centre ,nord ,sud ,est ,ouest ,sudouest ,nordouest ,littoral ,extremenord ,adam /
b(j) in the slashes, put in front every name of region, the market demand of energy there /centre ,nord ,sud ,est ,ouest ,sudouest ,nordouest ,littoral ,extremenord ,adam /
Table c(i,j) Put in any intersection (i,j) the cost of transport of an unity of energy of the factory i towards the region j

	centre	nord	sud	est	ouest	sudouest	nordouest	littoral	extremenord	adam
factory1										
factory2										
factory3										

Table p(i,j) Put in any intersection (i,j) the quantity (rate) of energy lost during the transport of an unity of energy of factory i towards region j

	centre	nord	sud	est	ouest	sudouest	nordouest	littoral	extremenord	adam
factory1										
factory2										
factory3										

Table d(i,j) Put in any intersection (i,j) the degre of pollution of environment caused by the transport of an unity of energy of factory i towards region j

	centre	nord	sud	est	ouest	sudouest	nordouest	littoral	extremenord	adam
factory1										
factory2										
factory3										

scalar in the slaches put the value collected for these scalars
λ_1 / /, λ_2 / /, λ_3 / /, l / /
variables
x(i,j), f
Positive variable x
Equations

$$\text{cost..}f = e = \lambda_1 * sum((j,i), t(j) * x(i,j) - c(i,j) * x(i,j) - t(j) * p(i,j) * x(i,j)) + \lambda_2 * sum((j,i), x(i,j)) + \lambda_3 * sum((j,i), d(i,j) * x(i,j))$$

supply(i,j)..sum((j,i), x(i,j))=**g**=l

```
demand(i)..sum(j,x(i,j))=l=b(i)
Model energy /all/
solve energy using lp minimizing -f, display x.l
```

Remark 2.4.1. With the real data collected (c_{ij}, b_i,t_{ij},l,d_{ij}, p_{ij}), one just has to submit a file containing the statements above as input to the GAMS program, the planning problem model will be formulated and solved.

2.5 Conclusion

In this chapter, the main issues related to multicriteria optimization were discussed first. New formulations and proofs of some well known results in the literature where shown. We then presented a new approach for solving linear multiobjective programming problems, which can be regarded as a corrected form and an alternative to the method of [Ecker et al. 1980]. It ended with the development of a model for planning the distribution of electric energy based on multicriteria optimization techniques. The optimization process mainly comprises two steps: modelling of the problem and elaboration of approaches that can permit to find the best compromise solution of the modelled problem. The complete procedure of the GAMS model that implements the aggregate approach was provided. It will be enough then for any company to introduce its own data: b_i (production capacity of the factory i during the period considered), c_{ij} (cost of transport of a unity of energy from factory i towards region j), d_{ij} (pollution of environment caused by the transport of a unity of energy from factory i towards region j), p_{ij} (loss of quantity of energy during the transport of a unity of energy from factory i towards region j), t_{ij} (selling price of a unity of energy from factory i to region j) and incompressible demand l. The model provided can be used for other types of distributions of resources such as water, petroleum by pipeline, etc.

Chapter Three

Bilevel optimization

3.1 Introduction

A mathematical programming problem is classified as a Bilevel Programming Problem (BPP) when one of the constraints is also an optimization problem. The bilevel programming problem is a hierarchical optimization problem where a subset of the variables is constrained to be a solution of a given optimization problem parametrized by the remaining variables. The hierarchical optimization structure appears naturally in diverse applications, such as economics, civil engineering, chemical engineering, transportation (taxation, network design, trip demand estimation), management (coordination of multidivisional firms, network facility location, credit allocation), planning (agricultural policies, electric utility), etc. The aim of this chapter is to introduce a new approach for solving bilevel linear programming problem. The chapter is organized as follows. In the next section, a review of bilevel optimization is done. We investigate on the formulation and solution approaches. In section 3.3, an enumerative approach for solving bilevel linear programming problems is provided. It is followed in section 3.4 by a bilevel programming model that we have developed for protecting national initiatives in the context of globalization. The chapter is concluded in section 3.5.

3.2 The bilevel optimization problem

3.2.1 Mathematical formulation and notations

A standard Bilevel Programming Problem (BPP) is formulated as follows:

$$\min_{x \in X} F(x,y)$$
$$\text{subject to} \begin{cases} G(x,y) \leq 0 \\ \\ \text{y solves} \begin{cases} \min_{y \in Y} f(x,y) \\ s.t \\ g(x,y) \leq 0 \end{cases} \end{cases} \tag{BPP}$$

where the variables of the BPP are divided into two classes, namely the upper (or outer or planner's or leader's) variables $x \in X \subset \mathbb{R}^{n_1}$ and the lower-level (or inner or behavioral or follower's) variables $y \in Y \subset \mathbb{R}^{n_2}$. x (resp y) are decision variables controlled by the leader (resp the follower). Similarly, the functions $F : X\times \text{ Y} \longrightarrow \mathbb{R}$ and $f : X\times \text{ Y} \longrightarrow \mathbb{R}$ are the upper-level and lower-level objective functions, respectively, while $G : X \times Y \longrightarrow \mathbb{R}^{m_1}$ and $g : X\times Y \longrightarrow \mathbb{R}^{m_2}$ are called upper-level and lower-level constraints, respectively. When the objective functions (F, f) and the constraints (G, g) of the upper-level and lower-level problems are all linear, the resulting problem is a bilevel linear programming problem (BLPP), also called Linear Stackelberg game. The mathematical model of a bilevel linear programming problem can be stated as follows:

$$\min_{x\in X} F(x,y) = c_1x + d_1y$$
$$\text{subject to}\begin{cases} A_1x + B_1y \leq b_1 \\ \\ \text{y solves}\begin{cases} \min_{y\in Y} f(x,y) = c_2x + d_2y \\ s.t \\ A_2x + B_2y \leq b_2 \end{cases}\end{cases} \tag{BLPP}$$

where $c_1, c_2 \in \mathbb{R}^{n_1}$, $d_1, d_2 \in \mathbb{R}^{n_2}$, $b_1 \in \mathbb{R}^{m_1}$, $b_2 \in \mathbb{R}^{m_2}$, $A_1 \in \mathbb{R}^{m_1\times n_1}$,$B_1 \in \mathbb{R}^{m_1\times n_2}$, $A_2 \in \mathbb{R}^{m_2\times n_1}$, $B_2 \in \mathbb{R}^{m_2\times n_2}$.

Of course, once the leader selects an x, the first term in the follower's objective function becomes a constant and can be removed from the problem. Principles of bilevel program problems are summarized as follows:

• The system has interacting decision making units within a hierarchical structure.

• The lower unit executes its policies after, and in view of, the decisions of the higher unit.

• Each level minimizes its objective function, compromise is not allowed (the leader is not allowed to force the follower to take any of his optimal solutions).

• The effect of the upper decision maker on the lower problem is reflected in both its objective function and set of feasible decisions.

The following concepts and notations are usually used when dealing with BPP:

- The *relaxed problem* associated with BPP is:

$$\min_{x\in X,\, y\in Y} F(x,y) \tag{3.2.1}$$

$$\text{subject to}\begin{cases} G(x,y) \leq 0 \\ g(x,y) \leq 0 \end{cases} \tag{3.2.2}$$

and its optimal value is a lower bound for the optimal value of (BPP). A solution of a (BPP) is any $(x, y) \in X \times Y$. A solution (x, y) is admissible if it belongs to the admissible set Ω defined by

$$\Omega = \{(x, y) \in X \times Y / \ G(x, y) \leq 0 \text{ and } g(x, y) \leq 0\}$$

.

- Projection of Ω onto the leader's decision space is:

$$\Omega(X) = \{x \in X, \ / \exists y \in Y / \ \ G(x, y) \leq 0 \text{ and } g(x, y) \leq 0 \ \ \}$$

- For a given (fixed) vector $x \in X$, the follower's admissible region is defined by:

$$\Omega(x) = \{y \in Y / \ \ g(x, y) \leq 0 \ \}$$

while the lower-level reaction set is

$$R(x) = argmin_y\{f(x, \ y) / \ y \in \Omega(x)\}.$$

Every $y^* \in R(x)$ is a rational response. For a given x, $R(x)$ is an implicitly defined multi-valued function of x that may be empty (or have more than one element) for some values of its argument. Finally,

$$R = \{(x, y) \in X \times Y / \ G(x, y) \leq 0 \text{ and } y \in R(x)\}$$

represents the feasible set of the leader. Any $(x, y) \in R$ is called a feasible solution.

BPP can then be reformulated as:

$$\min_{x,y}\{F(x, y) / \ \ \ (x, y) \in R\} \tag{3.2.3}$$

R is usually nonconvex and it can even be disconnected or empty in presence of upper-level constraints. The following problem is an illustration:

Example 3.2.1.

$$\begin{array}{l} \min \ x - 2y \\ \textit{subject to} \left\{ \begin{array}{l} -x + 3y - 4 \leq 0 \\ \\ \textit{where } y \textit{ solves} \left\{ \begin{array}{l} \min \ x + y \\ x - y \leq 0 \\ -x - y \leq 0 \end{array} \right. \end{array} \right. \end{array} \tag{Ex1}$$

For this problem $\Omega(x) = \{y/\ y \geq \mid x \mid\}$ *and* $R(x) = \mid x \mid$. *Thus, the induced region is given by:*

$$\begin{aligned} R &= \{(x,y)/\ -x+3y-4 \leq 0 \ and\ y \in R(x)\} \\ &= \{(x,y)/\ y=-x,\ -1 \leq x \leq 0\} \cap \{(x,y)/\ y = x,\ 0 \leq x \leq 2\} \end{aligned}$$

which is nonconvex but connected. If the upper level constraints were changed to $\begin{cases} -x+3y-4 \leq 0 \\ -y+\frac{1}{2} \leq 0 \end{cases}$ *then the induced region would become*

$$R = \{(x,y)/\ y=-x,\ -1 \leq x \leq \frac{-1}{2}\} \cap \{(x,y)/\ y=x,\ \frac{1}{2} \leq x \leq 2\}$$

which would be a disconnected set.

3.2.2 Solutions of bilevel optimization problems

The notion of solution in bilevel programming is not easy to define. It depends on the number of optimal solutions in the lower level problem for some parameter values. The following example is an illustration:

Example 3.2.2. *Let us consider the following problem:*

$$\begin{aligned} &\min_x \ x^2+y^2 \\ &subject\ to \begin{cases} 0 \leq x \leq 1 \\ \\ where\ y\ solves \begin{cases} \min_y \ -xy \\ 0 \leq y \leq 1 \end{cases} \end{cases} \end{aligned} \tag{Ex2}$$

Then, evaluating the lower level problem and inserting the optimal solution of this problem into the objective function of the upper level results in

$$R(x) = \begin{cases} \{0\},\ if\ x>0 \\ \{0\},\ if\ x<0 \\ [0\ ,\ 1],\ if\ x=0 \end{cases} \Longrightarrow \qquad F(x,y(x)) \begin{cases} =x^2,\ if\ x>0 \\ =1+x^2,\ if\ x<0 \\ \in [0\ ,\ 1],\ if\ x=0 \end{cases}$$

The value of the function $x \mapsto F(x,y(x))$ *at the point* $x=0$ *is unclear. The infimal function value of* $F(x,y(x))$ *is equal to zero but this value is reached only if* $F(0,y(0)) = 0$. *This situation is called the optimistic position in what follows. If this is not the case, then the bilevel problem has no solution.*

To overcome this ambiguity, at least two approaches have been suggested. In the case of *optimistic bilevel programming*, it is assumed that, whenever the reaction set $R(x)$ is not a singleton, the leader is allowed to select the element in $R(x)$ that suits him best. In this situation, a point

$(x^*, y^*) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is said to be a local *optimistic solution* for problem (BPP) if the following holds:

$$\begin{cases} x^* \in X \\ G(x^*, y^*) \leq 0 \\ y^* \in R(x^*) \\ F(x^*, y^*) \leq F(x^*, y) \text{ for all } y \in R(x^*) \end{cases}$$

and there exists an open neighbourhood $V(x^*,\ \delta)$ *of* x^* (with radius $\delta > 0$) such that

$$\Phi_o(x^*) \leq \Phi_o(x) \text{ for all } \ x \in V(x^*,\ \delta) \cap X$$

where $\Phi_o(x) = \min_y\{F(x, y)/\ y \in R(x)\}$. It is called a global optimistic solution if $\delta = \infty$ can be selected corresponding to $V(x^*,\ \delta) = X$. The leader can use this approach if he supposes that the follower is willing to support him. In the preceding example, in either case, the problem has two local optimistic minimizers $(-1, 1)$ and $(2, 2)$ and one global optimistic minimizer $(2, 2)$. The following statement from [Dempe 2002] is an existence result for the optimistic bilevel problem(with the assumption that the leader constraint does not depend on the follower variables).

Theorem 3.2.1. *If the set* $\{(x, y)/G(x) \leq 0,\ g(x, y) \leq 0\}$ *is non-empty and compact and, for each* x *with* $G(x) \leq 0$*, the (MFCQ) is satisfied, then the optimistic formulation of BPP has an optimal solution.*

When cooperation of the leader and the follower is not allowed, or if the leader is risk-averse and wishes to limit the "damage" resulting from an undesirable selection from the follower, then a point $(x^*, y^*) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is said to be a local *pessimistic solution* for problem (BPP) if the following holds:

$$\begin{cases} x^* \in X \\ G(x^*, y^*) \leq 0 \\ y^* \in R(x^*) \\ F(x^*, y^*) \geq F(x^*, y) \text{ for all } y \in R(x^*) \end{cases}$$

and there exists an open neighbourhood $V(x^*,\ \delta)$ of x^* (with radius $\delta > 0$) such that

$$\Phi_p(x^*) \leq \Phi_p(x) \text{ for all } \ x \in V(x^*,\ \delta) \cap X$$

where $\Phi_p(x) = \max_y\{F(x, y)/\ y \in R(x)\}$. It is called a global pessimistic solution if $\delta = \infty$ can be selected. The following statement from [Dempe 2002] is an existence result for the pessimistic bilevel problem (with the assumption that the leader constraint does not depend on the follower variables).

Theorem 3.2.2. *Let the point-to-set mapping $R(.)$ be lower semicontinuous at all points x with $G(x) \leq 0$ and suppose that $\{(x, y)/G(x) \leq 0,\ g(x, y) \leq 0\}$ is non-empty and compact. Then, the pessimistic formulation of BPP has an optimal solution.*

A more complete discussion of these issues may be found in [Bard 1988; 1998; 1991, Dempe 2002, Pieume et al. 2008b]. Throughout the text, it is the optimistic formulation that is considered.

3.2.3 Some solution methods for bilevel programming problems

3.2.3.1 Penalty function methods

They usually incorporate exact penalty functions and are limited to computing stationary and local optimal points. An initial step in this direction was achieved by [Shimizu et al. 1997]. Their approach consists in replacing the lower-level problem by the penalized problem

$$min_y \rho(x, y, r) = f(x, y) + r\phi(g(x, y)),$$

where r is a positive scalar, ϕ is a continuous penalty function that satisfies

$$\phi(g(x, y)) > 0\ if\ y \in intS(x)$$
$$\phi(g(x, y)) \longrightarrow +\infty\ if\ y \longrightarrow bdS(x)$$
$$where\ S(x) = \{y/\ g(x, y) \leq 0\}$$

Problem BPP is then transformed into

$$\min_{x \in X, y} F(x, y^*(x, r))$$
$$\text{subject to} \begin{cases} G(x, y^*(x, r)) \leq 0 \\ \\ \rho(x, y^*(x, r), r) = \min_y \rho(x, y, r) \end{cases} \qquad \text{(BPPA1)}$$

[Colson et al. 2007] proved that the sequence $\{\ (x^k, y^*(x^k, r^k))\ \}$ of optimal solutions to (BPPA1) converges to a solution of (BPP).

3.2.3.2 Descent methods

Here we follow the presentation given in [Colson et al. 2007]. Assuming that, for any x, the optimal solution of the lower level problem is unique and define y as an implicit function $y(x)$ of x, problem (BPP) may be viewed solely in terms of the upper level variables $x \in \mathbb{R}^{n_1}$. Given a feasible point x, an attempt is made to find a feasible (rational) direction $d \in \mathbb{R}^{n_1}$ along which the upper level objective decreases. A new point $x + \alpha d(\alpha > 0)$ is computed so as to ensure a reasonable decrease in F while maintaining feasibility for the bilevel problem. However, a major issue is the availability of the gradient (or a sub-gradient) of the upper-level objective, $\nabla_x F(x, y(x))$, at a feasible point. Applying the chain rule of differentiation, we have, whenever $\nabla_x y(x)$ is well defined:

$$\nabla_x F(x, y(x)) = \nabla_x F(x, y) + \nabla_y F(x, y) \nabla_x y(x)$$

where the functions are evaluated at the current iterate. [Kolstad and Lasdon 1990] proposed a method for approximating this gradient. Another way is that of [Colson et al. 2007], for problems where no upper level constraints are present and where the lower level constraints are rewritten as:

$$g_i(x, y) \leq 0 \;\; \forall i \in I$$
$$g_j(x, y) = 0 \;\; \forall j \in J$$

The authors first show that an upper-level descent direction at a given point x is a vector $d \in \mathbb{R}^{n_1}$ such that:

$$\nabla_x F(x, y^*) d + \nabla_y F(x, y^*) \omega(x, d) < 0,$$

where $y^* = y(x)$ and $\omega \in \mathbb{R}^{n_2}$ is a solution of the program

$$\min_{\omega} \; (d^T, \; \omega^T) \nabla^2_{xy} L(x, y^*, \lambda)(d, \omega)$$
$$\text{subject to} \begin{cases} \nabla_y g_i(x, y^*)\omega \leq -\nabla_x g_i(x, y^*) d, \;\; i \in I(x); \\ \\ \nabla_y g_i(x, y^*)\omega = -\nabla_x g_j(x, y^*) d, \;\; j \in J; \\ \nabla_y f(x, y^*)\omega = -\nabla_x f(x, y^*) d + \nabla_x L(x, y^*, \lambda) d; \end{cases} \qquad \text{(BPPA2)}$$

with $I(x) = \{i \in I / \; g_i(x, y^*) = 0\}$ and $L(x, y, \lambda) = f(x, y) + \sum_{i \in I(x) \cap J} \lambda_i g_i(x, y)$ is the Lagrangian of the lower-level problem with respect to the active constraints. The steepest descent then coincides with the optimal solution of the linear-quadratic bilevel program:

$$\min_{d} \;\; \nabla_x F(x, y^*) d + \nabla_y F(x, y^*) \omega(x, d)$$
$$\text{subject to} \begin{cases} \| \; d \; \| \leq 1 \\ w(x, d) \; solves \; problem \; (BPPA2) \end{cases}$$

for which exact algorithms exist, such as those by [Bard and Moore 1990; 1992].

3.2.3.3 Branch-and-Bound methods

Branch and bound methods are widely applied to convex bilevel programs. Several approaches exploit the complementarity between the multipliers and the slack variables that arise from the KKT-conditions of the lower level problem [Bard and Moore 1990, Bialas and Karwan 1984, Bialas et al. 1980, Bialas and Karwan 1982]. In fact, when the lower-level problem is convex and regular, it can be replaced by its Karush-Kuhn-Tucker (KKT) conditions, yielding the single-level reformulation of BPP:

$$\min_{x\in X, y\in Y, \lambda} F(x,y)$$

$$\text{subject to} \begin{cases} G(x,y) \leq 0 \\ g(x,y) \leq 0 \\ \lambda_i \geq 0,\ i = 1, ..., m_2 \\ \lambda_i g_i(x,y) = 0,\ i = 1, ..., m_2 \\ \nabla_y L(x,y,\lambda) = 0 \end{cases}$$

where $L(x,y,\lambda) = f(x,y) + \sum_{i=1}^{m_2} \lambda_i g_i(x,y)$ is the Lagrangian function associated with the lower-level problem.

In addition to these methods, one can cite also trust-region algorithm developed in [Chen and Florian 1995, Colson et al. 2007] for solving nonlinear bilevel programs where the function G solely depends on the upper-level vector x. Complete and detailed presentations on methods for solving bilevel programming problems can be found in [Bard and Moore 1990, Bialas and Karwan 1984, Bialas et al. 1980, Bialas and Karwan 1982]. The next section presents a method that we have developed. The approach is specially developed for linear problems.

3.3 An enumerative method for solving bilevel linear programming problems

There have been nearly two dozen algorithms proposed for solving the BLPP. A wide class is based on vertex enumeration, using the basic idea that the extreme points of the admissible set of a BLPP are extreme points of the leader feasible region, and the optimal solution is one of these vertices [Bialas and Karwan 1984, Chenggen et al. 2005, Falks 1973, Lu et al. 2005b, Onal 1992]. Other approaches are based on replacing the lower level problem by its primal-dual optimality conditions [Lu et al. 2005a, White and Anandalingam 1993]. This operation reduces the original problem to a single-level program involving disjunctive constraints. Branch and bound algorithms have also been developed by [Bard and Falk 1982] and [Fortuny-Amat and Carl 1981]. Combining branch

and bound, monotonicity principles and penalization, [Hansen et al. 1992] have developed a code capable of solving medium size instances of the BLPP. Alternatively, [Bialas et al. 1980, Bialas and Karwan 1982] and [Judice and Faustino 1992; 1988] based their approach on complementary pivot theory. The recent books by [Dempe 2002], [Floudas et al. 1999] and [Bard 1998] present results, applications and solution methods for general bilevel problems where the functions and the constraints are not necessarily linear. However, only theoretical convergence is shown for the majority of these methods. In addition, many algorithms solving bilevel programming problems turned out to be incorrect, or convergent towards a local non global optimum.
[Ben-Ayed 1988] showed that the Parametric Complementary Pivot algorithm developed by [Bialas and Karwan 1984, Bialas et al. 1980, Bialas and Karwan 1982] and the Grid Search Algorithm presented by [Bard 1983a] do not always find the optimal solution. [Campelo and Scheimberg 2000] provided a counter-example to show that the method proposed by [Othmani May 1998] may not find a global solution as it was claimed. [Manoel and Scheimberg 2000] identified some troubles in the algorithm proposed by [Anandalingam and White 1990]. They showed that the set of cuts used in the algorithm to discard local optima is not well defined. [Chenggen et al. 2005] showed that the Kth-best approach [Bialas and Karwan 1982, Candler and Townsley 1982], one of the most popular and workable approaches for the BLPP, could badly deal with a linear bilevel programming problem when the constraint functions at the upper-level have an arbitrary linear form. More recently, [Audet et al. 2006] showed that the methods proposed by [Lu et al. 2005a;b] do not solve a wider class of problems, but rather relax the feasible region, allowing infeasible points to be considered as feasible. Consequently, it is extremely desirable to develop a simple technique that can locate the global solution.

The aim of this section is to describe a simple algorithm for solving the BLPP. We formalize and prove the well known result that extreme points of the admissible set of the BLPP are extreme points of the leader feasible region, and the optimal solution is one of these vertices. Using this result, we develop a pivot technique to find the optimal solution on an expanded tableau similar to a simplex tableau that represents the data of the BLPP. The pivot technique allows to rank in increasing order the outer level objective function value until a value is reached with a corresponding extreme point feasible for the BLPP. Then this extreme point is the required global solution. The algorithm starts with the optimal simplex tableau of the linear programming problem obtained by removing from the BLPP the follower objective function. Then an iteration procedure using the pivot technique developed allows to find the global optimal solution. Solutions obtained through our algorithm to some problems available in the literature show that these problems were until now wrongly solved.

The presentation is organized as follows. In the next section, BLPP formulation and properties are provided. This is followed by the presentation of the mathematical basis and the principle of our algorithm. In the third part, we present the algorithm and the implementation on some problems available in the literature. An example for highlighting the different steps of the algorithm is presented. We also present some problems from the literature that were until now wrongly solved and for which our method provides the correct answer.

Throughout this section, we consider the formulation (BLPP) as presented in the first section. With the following hypothesis (i) $X = \{x \in R^{n_1} / x \geqq 0\}$, (ii) $Y = \{y \in R^{n_2} / y \geqq 0\}$ and (iii) Ω bounded and nonempty.

3.3.1 Foundation of the approach

In this section, basic mathematical concepts and theorems necessary for the development of the algorithm are discussed. Based on notations provided in section 3.2.1, an admissible point (x^-, y^-) of Ω is feasible for the BLPP if y^- solves the lower level problem with x fixed at x^-, i.e,

$$c_2 y^- = \min_y \{d_2 y / B_2 y \leq b_2 - A_2 x^-, y \geq 0\}. \tag{3.3.1}$$

A *global solution* of a BLPP is any feasible point for the BLPP which provides the best value to the leader objective function.

Let us now state the following theorem on the existence of a BLPP solution.

Theorem 3.3.1. *: With the following assumptions:*
(i) The leader objective function is bounded over the relaxed feasible region.
(ii) The rational set $R(x)$ is a point $-$ to $-$ point map for all permissible x.
(iii) $R(x)$ is bounded for all permissible x.
Then, the problem $(BLPP)$ admits at least one optimal solution.

Proof: The proof is done in two steps:

- We first show that the admissible set can be equivalently written as a piecewise linear equality constraint comprised of supporting hyperplanes of Ω.
- Then, the result will be deduced from the new form of the inducible region.

Using the notation of section 3.2, we have

$$R(x) = \{y^* / \, y^* \in arg \min_y \{c_2 x + d_2 y / y \in \Omega(x)\}\} \tag{3.3.2}$$

and the BLPP feasible set

$$R = \{(x, y)/(x, y) \in \Omega, y \in R(x)\}$$

which can be written as

$$R = \{(x, y)/(x, y) \in \Omega, d_2 y = \min_{z}\{d_2 z/B_2 z \leq b_2 - A_2 x, z \geq 0\}\} \tag{3.3.3}$$

Now, let us define

$$T(x) = min\{d_2 z/B_2 z \leq b_2 - A_2 x, z \geq 0\}. \tag{3.3.4}$$

For each value of $x \in \Omega(X)$,the resulting feasible region to (BLPP) is nonempty and compact. Thus $T(x)$, which is a linear program parameterized in x and from proposition 1.3.1, always has a solution. From duality theory, the same problem can be stated as:

$$max\{u(A_2 x - b_2)/uB_2 \geq -d_2, u \geq 0\} \tag{3.3.5}$$

which has the same optimal value as the primal problem (3.3.4) at the solution u^*. Let $u^1, ...u^s$ be a listing of all the extreme points of the constraint region of (3.3.5) given by $U = \{u/uB_2 \geq -d_2, u \geq 0\}$. Because we know that a solution of problem (3.3.5) occurs at an extreme point of U, we get the equivalent problem:

$$max\{u^j(A_2 x - b_2)/u^j \in \{u^1, ..., u^s\}\} \tag{3.3.6}$$

which demonstrates that $T(x)$ is a piecewise linear function. Rewriting R as

$$R = \{(x, y) \in \Omega/T(x) - d_2 y = 0\} \tag{3.3.7}$$

permit to conclude that, the admissible set can be equivalently written as a piecewise linear equality constraint comprised of supporting hyperplanes of Ω.

The problem (BLPP) is then equivalent to minimizing F over an admissible set comprised of a piecewise linear equality constraint.

The function $T(x)$ is convex and continuous. In general, because we are minimizing a linear function $F = c_1 x + d_1 y$, over R, and because F is bounded from below on Ω by, say $min\{c_1 x + d_1 y/(x, y) \in \Omega\}$, one can conclude that a solution to the linear (BLPP) occurs at an extreme point of R.

□

We present now how to check the uniqueness condition.

Theorem 3.3.2. *: A solution (x^*, y^*) to a BLPP is not unique if the solution (a, b) of the following problem:*

$$\min_{x,y}\{d_2y/(x,y) \in \Omega, c_1x + d_1y = c_1x^* + d_1y^*\} \tag{3.3.8}$$

produces an objective function value such that $d_2b < d_2y^*$.

Proof: In fact, if (a, b) satisfies (3.3.8), then:

$$(i) \quad c_1a + d_1b = c_1x^* + d_1y^* \quad \text{and} \quad (ii) \quad d_2b < d_2y^*$$

(i) implies that (a, b) gives the same value for the leader's objective than (x^*, y^*), and

(ii) stipulates that (a, b) gives a better objective value for the follower. In this case, there is no reason that players prefer $(x*, y*)$ to (a, b). □

The remaining question is where to find the BLPP solution. We study how to localize the BLPP solution in the admissible set. Using some results presented in [Bard 1998], we show in the next theorem that the solution vertex $(x*, y*)$ of R is also an extreme point of Ω.

Theorem 3.3.3. *:*
The solution $(x, y*)$ of BLPP occurs at an extreme point of Ω.*

Proof: Let (x^1, y^1),..., (x^r, y^r) be the distinct extreme points of Ω.

Since any point in Ω can be written as a convex combination of these vertices, let $(x*, y*) = \Sigma_{i=1}^{i=\overline{r}}\lambda_i(x^i, y^i)$, where $\Sigma_{i=1}^{i=r}\lambda_i = 1$, $\lambda_i \geq 0$, $i = 1, ..., \overline{r}$ and $\overline{r} \leq r$. We will prove that $\overline{r} = 1$.

To see this, let us write the constraints of (BLPP) at $(x*, y*)$ in their piecewise linear form (3.3.7). This gives: $T(x*) - d_2y* = 0$, which implies that $T(\Sigma_i\lambda_ix^i) - d_2(\Sigma_i\lambda_iy^i) = 0$.
With the convexity of $T(x)$, we deduce that $\Sigma_i\lambda_i(T(x^i) - d_2y^i) \geq 0$.

But, by definition, $T(x^i) = min_{y\in\Omega(x^i)}d_2y$, so that $T(x^i) \leq d_2y^i \quad \forall i$.
Therefore, $T(x^i) - d_2y^i \leq 0 \quad \forall i; \quad i = 1, ..., \overline{r}$. Noting that $\lambda_i > 0 \quad \forall i; \quad i = 1, ..., \overline{r}$, equality must hold or else a contradiction would result.

Consequently, $T(x^i) - d_2y^i = 0, \quad \forall i$. This implies that $(x^i, y^i) \in R, \quad i = 1, ..., \overline{r}$, and that $(x*, y*)$ can be written as a convex combination of points in R. Because $(x*, y*)$ is an extreme point of R, a contradiction results unless $\overline{r} = 1$. □

Remark 3.3.1. : It can be seen from the proof of theorem 3.3.3 that any vertex of R is also an extreme point of Ω. This allows us to say that any extreme point of R is an extreme point of Ω. In addition, this section allows us to say that the solution of the BLPP lies at a vertex of $\Omega = \{(x, y)/A_1x + B_1y \leq b_1, A_2x + B_2y \leq b_2, x \in X, y \in Y\}$.

Remark 3.3.2. :

As stipulated in theorem 3.3.3, we only need to investigate extreme point in order to find the solution of the BLPP. Since the size of the set of extreme points is usually very large, it would be an impossible task to evaluate all the extreme points and select the feasible points for the BLPP.

Now a computational scheme that selects, in an orderly fashion, a small subset of the possible solutions which converges to the global BLPP solution is required. We present in the following section such a scheme that we have developed to iterate on different extreme points.

3.3.1.1 An orderly technique to iterate on potential BLPP solutions

Our objective in this section is to derive two rules that will be used in our algorithm. The main objective of the rules is to show how to move from an extreme point to one with the value of the objective function greater and closer to the current value than any other extreme point, when dealing with a minimization problem. The search for the smallest deterioration is in fact a variant of the simplex algorithm, only in reverse.
Let us consider the following problem: $\min_{x \in X}\{cx/Ax = b\}$, where $b \geq 0$.
Where $c = (c_1, c_2, ..., c_n)$ is a row vector, $A = (a_{ij})$ a $n \times m$ matrix, $x = (x_1, x_2, ..., x_n)$ and $b = (b_1, b_2, ..., b_m)$ are column vectors. We assume that the linear programming problem is realisable, that every extreme point (solution) is nondegenerate, and that we are given a basic extreme point.

The linear programming problem can be reformulated as follows:

Min$\{z = cx/x_1P_1 + x_2P_2 + ... + x_nP_n = P_0, x \geq 0\}$.

Where P_j for $j = 1, 2, ..., n$ is the jth column of A and $P_0 = b$.

We suppose that a basic extreme point (solution) is given, say $x_0 = (x_{10}, x_{20}, ..., x_{m0})$ and let the associated set of linearly independent vectors be $\{P_1, P_2, ..., P_m\}$. We then have

$$x_{10}P_1 + x_{20}P_2 + ... + x_{m0}P_m = P_0 \tag{3.3.9}$$

$$x_{10}c_1 + x_{20}c_2 + ... + x_{m0}c_m = z_0 \tag{3.3.10}$$

where all $x_{i0} > 0$, the c_i are the cost coefficients of the objective function and z_0 is the corresponding value of the objective function for the given solution. Since the set $\{P_1, P_2, ..., P_m\}$ is linearly independent and thus forms a basis, we can express any vector from the set $\{P_1, P_2, ..., P_n\}$ in terms of $P_1, P_2, ..., P_m$. Let P_j be given by

$$x_{1j}P_1 + x_{2j}P_2 + ... + x_{mj}P_m = P_j,\ \ j = 1, ..., n \tag{3.3.11}$$

and define

$$x_{1j}c_1 + x_{2j}c_2 + ... + x_{mj}c_m = z_j, j = 1, ..., n \tag{3.3.12}$$

where c_i are the cost coefficients corresponding to the P_i.

We then have the following result:

Theorem 3.3.4. *If for j, the condition $z_j - c_j < 0$ holds, then a set of extreme points (solutions) can be constructed such that, $z > z_0$ for any member of the set, where the upper bound of z is either finite or infinite (z is the value of the objective function for a particular member of the set of admissible points (solutions).*

Proof: Multiplying (3.3.11) by some number θ and subtracting from (3.3.9), and similarly multiplying (3.3.12) by the same θ and subtracting from (3.3.10), for $j = 1, 2, ..., n$, we get

$$(x_{10} - \theta x_{1,j})P_1 + (x_{20} - \theta x_{2,j})P_2 + ... + (x_{m0} - \theta x_{m,j})P_m + \theta P_j = P_0 \tag{3.3.13}$$

$$(x_{10} - \theta x_{1,j})c_1 + (x_{20} - \theta x_{2,j})c_2 + ... + (x_{m0} - \theta x_{m,j})c_m + \theta c_j = z_0 - \theta(z_j - c_j) \tag{3.3.14}$$

where θc_j has been added to both sides of (3.3.14). If all the coefficients of the vectors $P_1, P_2, ..., P_m, P_j$ in (3.3.13) are non negative, then we have determined a new admissible point (solution) whose value of the objective function is, by (3.3.14), $z = z_0 - \theta(z_j - c_j)$. Since the variables $x_{10}, x_{20}, ..., x_{m0}$ in (3.3.13) are all positive, it is clear, from the previous result, that there is a value of $\theta > 0$ (either finite or infinite) for which the coefficients of the vectors in (3.3.13) remain positive. From the assumption that, for a fixed j, $z_j - c_j < 0$, one have:

$\theta(z_j - c_j) < 0 \Rightarrow z = z_0 - \theta(z_j - c_j) > z_0$ for $\theta > 0$.

It can be seen that in either event, a new admissible point (solution) can be obtained with the value of the objective function greater than the value of the objective function of the previous admissible solution. □

Now, let us suppose given a simplex tableau [Winston 1994, Fotso Jully 2001] from a minimization linear programming problem. From the above results and some well known results of simplex algorithm, one can deduce the following rules:

Rule 1:

All the extreme points (tableau) adjacent to the current simplex tableau (current extreme point) can be obtained by making pivot operations on any column with a negative reduced cost coefficient in the objective function. With the pivot line obtained using *minimum (lexicographic) ratio rule.*

Rule 2:

The pivot element which leads to the new extreme point with the smallest deterioration of the objective function is obtained as follows:

- How to find candidate pivot columns: To be a candidate pivot column, a column must have a negative reduced cost coefficient in the objective function.

- For each candidate pivot column, use the *minimum ratio rule* to find the corresponding pivot line.

- For each candidate pivot column j, let p_j be the product of its pivot element by its coefficient in the objective function.

- The selected pivot belongs to the pivot column with the smallest p_j.

3.3.1.2 Principle of the approach

In the preceding section, we show that a global optimal solution of a BLPP always occurs at an extreme point of the admissible set. We will iterate through these admissible points to locate an optimal solution. We start with the best value of the objective function of the leader. Then, we progressively deteriorate this best solution until a feasible point (optimal solution) is obtained.

- **A description of the algorithm**

In this subsection, the problem BLPP will be called $(PB1)$ and the relaxed will be called $(PB2)$. We use T to represent the current simplex tableau, V the set of potential tableaux to be considered in the next iteration. $(PB3)$ will represent problem (3.3.1) and (PB 4) problem (3.3.8).

To search for the global solution of the BLPP, the algorithm starts with the optimal simplex tableau of the linear programming problem $(PB2)$, obtained from $(PB1)$ by removing the follower's objective function. Using problem $(PB3)$ and *functiontest* given below, the algorithm tests the feasibility (as stated in definition 3.3.1) of the corresponding extreme point. If it is verified, then the algorithm terminates. Otherwise, using *functionnewtableau* as given below, the algorithm generates the new tableau (extreme point) with the smallest increase in the value of the objective function, the feasibility of the new solution is tested again, and the entire scheme is repeated, until a feasible extreme point solution is found. The algorithm calls problem $(PB4)$ to check the uniqueness condition, to be sure that the extreme point obtained is really the solution of the BLPP. If it is verified, the feasible point is the required global solution, else the BLPP has no solution and the algorithm stops. Since Ω is regular, it has a finite number of extreme points; so the algorithm is bound to terminate in a finite number of steps.

- **Subroutine for testing if a tableau provides a BLPP solution**

We first present a function called *Functiontest*, which tests if the extreme solution corresponding to a given simplex tableau is the solution of the BLPP.

We consider T as the current simplex tableau and the notations given in the previous sections.

Algorithm 3. 1

Z=Functiontest(d2,b2,A2,B2,T)

BEGIN

•**Step 0 (Initialization):** $Z \leftarrow \emptyset$

• **Step 1:** Find the set V of adjacent tableaux to T which provides the same objective value as T. Add T to the set V.

•**Step 2:** Take an element (say T) from V, suppress it from V.

•**Step 3:** Let $(x*, y*)$ be the solution read on the simplex tableau T;

Find the solution y of problem $(PB3)$

If $y = y*$ then $Z \leftarrow T$ else if $V \neq \emptyset$ go to step 2 else stop.

END

If Z is empty, the current tableau does not provide a feasible solution for the BLPP. Else, Z contains a tableau that provides a feasible solution for the BLPP.

Remark 3.3.3. It is possible to obtain more than one simplex tableau which provide the same objective value. In order to tackle this case, we use at step 1 the well known scheme in the simplex method to obtain all the other tableaux (extreme points) which give the same objective value.

- **Subroutine for finding the new tableau to be considered**

Now, if at iteration $i-1$ ($i \geq 2$), the tableau T is tested and does not provide a feasible solution for the BLPP, the new tableau that has to be considered at iteration i can be determined by the following subroutine.

Let us call the subroutine *Functionnewtableau* and suppose the following notations:

(Z1,Z2) is a couple of tableaux such that:

Z1: is the tableau that will be considered at the current iteration.

Z2: is the tableau from which a pivot operation led to the tableau Z1.

V: the set of couples of tableaux with the same configuration as (Z1,Z2), that contains the other potential candidates for the current iteration. The elements of V were obtained from the previous iterations.

J: the set of potential candidates (couples of tableaux) with the same configuration as (Z1,Z2) for the next iteration $(i+1)$.

T: the tableau tested at the preceding iteration $(i-1)$.

T1: the tableau from where T was derived after a pivot operation

Algorithm 3. 2

[(Z1, Z2), J]=Functionnewtableau (V, (T, T1))

BEGIN

• **Step1:** Apply *Rule2* on tableau T to obtain the tableau T2 adjacent to T with the smallest increase in the value of the objective function.

• **Step2:** Apply *Rule1* on tableau T1 to obtain all the adjacent tableaux to T1. Delete T from the obtained set. For each of these tableaux (let T3), put the couple (T3,T1) in the set K.

• **Step3:** Put the pairs (T2,T), all the pairs of tableaux of the set $K - K \cap V$, and elements of V in the set J.

• **Step4:** Find, among the elements of J, the pairs such that the first tableau has the smallest objective value and consider the corresponding pair as (Z1,Z2). Suppress it from J.

END

At the end, J contains all the potential candidates for the next iteration (i+1), Z1 the new tableau to be considered in the current iteration (i) and Z2 the tableau from where Z1 was obtained after a pivot operation. We can now state our algorithm.

3.3.2 An algorithm for solving bilevel linear programming problems

At any iteration of the proposed algorithm, T represents the current tableau that has to be tested, T1 the tableau from which a pivot operation led to the tableau T and V the set of couples of tableaux with the same configuration as (T,T1), containing the other potential candidates for the next iteration.

3.3.2.1 Presentation of the algorithm

Algorithm 3. 3

BEGIN

- **Step 01:** Find the optimal simplex tableau T_0 of the following problem $(PB2)$:

$$\begin{aligned} \min f_1(x, y) = c_1 x + d_1 y \\ A_1 x + B_1 y \leq b_1 \\ A_2 x + B_2 y \leq b_2 \\ x \in X, y \in Y \end{aligned}$$

- **Step 02:** Use *functiontest* to check if T_0 provides the solution of the BLPP.

If it is the case, then go to step 4.

- **Step 03:** Use *Rule 2* to find all the adjacent tableaux T_1 to T_0 with the smallest increase in the value of the objective function.
- **Step 04:** Using *functiontest* , if there is T_1 that gives the solution of the BLPP, then go to step 4, else continue to step 1.
- **Step 1 (initialization):** $V \leftarrow \emptyset$, $T \leftarrow T_1$, $T1 \leftarrow T_0$.
- **Step 2:** With V, T, T1 use *functionnewtableau* to find the couple $(Z1, Z2)$ and J . Update T, T1 and V as follows: $V \leftarrow J$, $T \leftarrow Z1$, $T1 \leftarrow Z2$.
- **Step 3:** Use *functiontest* to check if T provides the solution of the BLPP.

  If it is not the case, go to step 2.
- **Step 4:** Let $(x*, y*)$ be the solution read on the tableau T. Find the optimal solution of the following problem $(PB4)$:

$$min\{d_2y/(x, y) \in \Omega, c_1x + d_1y = c_1x * +d_1y*\}$$

- **Step 5:** If the optimal value of the problem $(PB3)$ obtained at step 3 is less than the optimal value of the problem $(PB4)$ obtained in step 4, then the solution of the problem does not exist, stop. Else, the solution of the problem is $(x*, y*)$, stop.

END

Step 01 provides the first tableau (which is the simplex optimal solution of $(PB2)$) from which the algorithm will iterate. Then, in step 02 , the feasibility of any extreme point of problem $(PB2)$ which provides the optimal objective value is checked. If there is no feasible point among these extreme points, we use rule 2 to find the next closer tableau (with the smallest increase in the value of the objective function) at step 03. This new solution is tested at step 04. Step 2 and step 3 are schemes to find and test, respectively, at any moment the next closer tableau (with the smallest increase in the value of the objective function) until the first feasible extreme point for the BLPP is found. Step 4 tests if the problem satisfies the uniqueness condition which is a condition for the existence of the BLPP solution.

Theorem 3.3.5. *: Under the uniqueness assumption associated with the rational reaction set $R(x)$ and the boundness of the leader function, the algorithm terminates with the global optimum of the*

BLPP (PB1).

Proof: The algorithm forces the satisfaction of conditions of definition 1 to be true for the extreme point of the current tableau. So the obtained point is a feasible solution of (PB1). The globality of the solution comes from the fact that we start our algorithm with the tableau that contains the best value that the leader can have. Then iterate with a decreasing process, and stop at the first extreme point which satisfies the definition 1. So if there are any other extreme points that satisfy definition 1, they will provide an objective function value of the leader worse than the one provided by the algorithm. □

3.3.2.2 Computational experience

Illustrative example We used Scilab language to implement our algorithm. Below is an illustrative example taken from [Bard 1998].

$$\begin{aligned}
&\min x + 3y \\
&1 \leq x \leq 6 \\
&\min -y \\
&x + y \leq 8 \\
&-x - 4y \leq -8 \\
&x + 2y \leq 13 \\
&-y \leq 0 \\
&-x \leq 0
\end{aligned}$$

It represents our problem (PB 1) and (PB 2) is the following linear programming problem:

$$\begin{aligned}
&\min x + 3y \\
&1 \leq x \leq 6 \\
&x + y \leq 8 \\
&-x - 4y \leq -8 \\
&x + 2y \leq 13 \\
&-y \leq 0 \\
&-x \leq 0
\end{aligned}$$

The best known solution is (2,6).

At step 01

The optimal simplex tableau T_0 corresponding to problem (PB1) is:

-6.25	0.	0.	0.	0.25	0.	0.75	0.
1.	1.	0.	0.	$-1.$	0.	0.	0.
5.	0.	0.	1.	1.	0.	0.	0
5.25	0.	0.	0.	0.75	1.	0.25	0.
8.5	0.	0.	0.	0.5	0.	0.5	1.
1.75	0.	1.	0.	0.25	0.	-0.25	0

At Step 02,

functiontest shows that T_0 does not provide the solution of the BLPP.

At Step 03,

Rule 2 allows to have as tableau adjacent to T_0 with the smallest increase in the value of the objective function, the following tableau T_1:

-7.5	0.	0.	-0.25	0.	0.	0.75	0.
6.	1.	0.	1.	0.	0.	0.	0.
5.	0.	0.	1.	1.	0.	0.	0.
1.5	0.	0.	-0.75	0.	1.	0.25	0
6.	0.	0.	-0.5	0.	0.	0.5	1.
0.5	0.	1.	-0.25	0.	0.	-0.25	0

At step 04,

functiontest shows that T_1 does not provide the solution of the BLPP.

At step 1,

V is initialized as an empty set, $T \leftarrow T_1$, $T1 \leftarrow T_0$.

At step 2,

with V, T, T1 *functionnewtableau* gives the different tableaux Z1, Z2 and the set J. T, T1 and V are updated.

The new tableau T to be considered is:

$-12.$	0.	0.	2.	0.	$-3.$	0.	0.
6.	1.	0.	1.	0.	0.	0.	0.
5.	0.	0.	1.	1.	0.	0.	0.
6.	0.	0.	$-3.$	0.	4.	1.	0.
3.	0.	0.	1.	0.	$-2.$	0.	1.
2.	0.	1.	$-1.$	0.	1.	0.	0.

The tableau T1 from where T was derived is given by:

-7.5	0.	0.	-0.25	0.	0.	0.75	0.
6.	1.	0.	1.	0.	0.	0.	0.
5.	0.	0.	1.	1.	0.	0.	0.
1.5	0.	0.	-0.75	0.	1.	0.25	0
6.	0.	0.	-0.5	0.	0.	0.5	1.
0.5	0.	1.	-0.25	0.	0.	-0.25	0

And the set of potential candidates for the next iteration is given by:

$V = \{(K1, K2)\}$

where the candidate tableau K1 for the next iteration is :

$-19.$	0.	0.	0.	-0.5	0.	0.	-1.5
1.	1.	0.	0.	$-1.$	0.	0.	0..
5.	0.	0.	1.	1.	0.	0.	0..
1.	0.	0.	0.	0.5	1.	0.	-0.5
17.	0.	0.	0.	1.	0.	1.	2..
6.	0.	1.	0.	0.5	0.	0.	0.5

And the tableau $K2$ from where $K1$ was derived is given by:

-6.25	0.	0.	0.	0.25	0.	0.75	0.
1.	1.	0.	0.	$-1.$	0.	0.	0.
5.	0.	0.	1.	1.	0.	0.	0.
5.25	0.	0.	0.	0.75	1.	0.25	0.
8.5	0.	0.	0.	0.5	0.	0.5	1.
1.75	0.	1.	0.	0.25	0.	-0.25	0.

At step 3,

functiontest shows that T provides a feasible solution of the BLPP.

Then the algorithm goes to step 4

At step 4,

$(x*, y*) = (6, 2)$ is the solution read on the simplex tableau T. The optimal solution of the following problem $(PB4)$:

$$min\{-y/(x, y) \in \Omega, x + 3y = 12\}$$

is $(0, 4)$ and the optimal value is -4. Then the algorithm moves to step 5.

At step 5,

The optimal value of the problem $(PB3)$ obtained at step 3 is 12 which is greater than -4, the solution of the problem $(PB4)$ obtained at step 4. The algorithm then stops and the solution of the problem is $(x*, y*) = (6, 2)$.

3.3.2.3 Incorrectness of some results in the literature

Here, we will just present the problem and the simplex tableau provided by our algorithm.

PROBLEM 1: Let us consider the following problem taken from chapter 5 of [Bard 1998].

$$\begin{aligned}
&\min x - 4y \\
&\quad \min y \\
&\quad -x - y \leq -3 \\
&\quad -2x + y \leq 0 \\
&2x + y \leq 12 \\
&-3x + 2y \leq -4 \\
&\quad -y \leq 0 \\
&\quad -x \leq 0
\end{aligned}$$

The solution provided is (4 , 4). But a simple check shows that this solution is not feasible for the bilevel linear programming problem. In fact for $x = 4$, $y = 4$ is not the optimal solution of the following follower problem:

$$\begin{aligned}
&\min y \\
&-y \leq 1 \\
&y \leq 8 \\
&y \leq 4 \\
&2y \leq 8 \\
&-y \leq 0
\end{aligned}$$

The following tableau is provided by our algorithm:

2.	0.	0.	−2.	0.	0.	1.
7.	0.	0.	1.4	0.	1.	0.2
3.	0.	0.	−0.2	1.	0.	−0.6
2.	1.	0.	−0.4	0.	0.	−0.2
1.	0.	1.	−0.6	0.	0.	0.2

The global optimal solution is then $x = 2$ and $y = 1$. It gives as outer objective function value -2 and inner objective function value 1.

PROBLEM 2: Let us consider the following problem, taken from [Bard 1998] and used in [Shimizu et al. 1997].

$$\min x_1 + 10y_1 - y_2$$
$$\min -y_1 - y_2$$
$$-y_1 \leq 0$$
$$-y_2 \leq 0$$
$$-x \leq 0$$
$$x_1 - y_1 \leq 1$$
$$x_1 + y_2 \leq 1$$
$$y_1 + y_2 \leq 1$$

The solution provided in these papers is $(1\,,\,0\,,\,0)$. But a simple check shows that this solution is not feasible for the bilevel linear programming problem. In fact, for $x = 1$, the resolution of the follower problem provided the following optimal simplex tableau:

1	0.	0.	0.	0.	1
1.	0.	1	1	0	1.
0	0	1	0.	1	0
1.	1	1.	0	0.	1

The optimal solution for the follower when $x = 1$ is then $(1,\ 0)$ and not $(0,\ 0)$ as suggested. The solving of the optimistic formulation of problem 2 provide $(0, 0, 1)$ as a global optimal solution.

3.4 Applications of bilevel programming optimization: protecting national initiative in the context of globalization

The following section illustrates an application of the bilevel programs that we have formulated [Pieume and Fotso December 2005]. More examples of bilevel optimization applications can be found in [Bard et al. 2000, Bard 1998, Dempe 2002].

3.4.1 Protecting national initiative

One fundamental problem for a state is how to reconcile the necessary liberal economic imperatives with the worry of consolidating or even reinforcing as much as possible national economic enterprises which are still fragile. There is a risk of the settlement of the "law of the jungle" in developing countries like Cameroon where the most powerful crushes the weakest. It is sometimes important that the State intervenes in various sectors of economy where national initiatives are threatened in order to preserve national interest and, by extension, national cohesion and social peace. The solution consists in general to impose a tax in all non local initiative in any economic sector where local initiatives are threatened. Given an economic sector where national initiatives (enterprises) are threatened, we present in the following section how the problem for determining the tax rate to impose to non local initiatives in order to assist local enterprises can be modelled. We show in fact that the problem of the determination of the tax rate can be modelled as a bilevel programming problem.

3.4.2 Determination of the tax rate

Let x_1 and x_2 be the quantity produced by all non local enterprises and local enterprises respectively. A higher productivity of non local enterprises implies a lower productivity of local enterprises, this is due to the fact that products of non local enterprises are generally best quality and cheap and then are much demanded on the market. This implies an increase (respectively, decrease) of profit of non local enterprises (respectively, local enterprises). Let $p_n(x_1)$, $p_l(x_1, x_2)$ denote the profit of non local enterprises and local enterprises, respectively, depending on their respective productivity effort x_1, and x_2. Here, the profit obtained by local enterprises strongly depends on the effort of non local enterprises and will decrease with increasing x_1. Without loss of generality, the functions $p_n(.)$ and $p_1(x_1, .)$ are assumed to be concave with $p_1(x_1, .)$ decreasing in the space of non-negative arguments. If both parties try to maximize their respective profits, the market will fall since the non local enterprises will make maximum production which will dominate the market and will destroy local initiative and competition. As we have assumed, the State has the obligation to protect national initiative, by imposing a tax on all products coming from non local enterprises.

Let us assume for simplicity that this tax linearly depends on x_1. Let r be the tax rate imposed by the State, the profit of non local enterprises will be $p_n(x_1) - rx_1$, and the non local enterprises will now maximize their profit depending on r. Let $x_1(r)$ denote the optimal effort of the non local enterprises. Larger r implies smaller $x_1(r)$ of productivity of non local enterprises. Therefore, the competitiveness of local enterprises will increase with increasing r. Consequently, the optimal effort of local enterprises $x_2(x_1(r))$ also depends on r.

The objective of the State is to determine the tax rate that will satisfy both parties. If the State measures the social impact of the action it wants to take and the utility of both groups of enterprises by $f(x_1, x_2, r)$ then, it can decide to choose r that maximizes $f(x_1(r), x_2(x_1(r)), r)$ where $x_1(r))$ and $x_2(x_1(r))$ are the optimal efforts of the local and non local enterprises induced by a fixed value of r respectively. In a very natural way, this leads to a formulation of a three-level programming problem:

$$\max_{r\geq 0} \ f(x_1, x_2, r)$$
$$s.t \begin{cases} x_1 \text{ solves } \max\limits_{x_1\geq 0} p_n(x_1) - rx_1 \\ \text{and } x_2 \text{ solves } \begin{cases} \max\limits_{x_2\geq 0} p_l(x_1, x_2) \\ s.t\ P_l(x_1, x_2) \geq 0 \end{cases} \end{cases}$$

This trilevel program formulation can be transformed into a bilevel program by suggesting that the State also imposes that the profit gained by the local enterprises must at least exceed a certain estimated preset level said, for example, P_{min}. The State can then decide to control the quantities that local enterprises must produce (x_2) and the tax rate (r). It will lead to solving the following problem:

$$\max_{r, x_2\geq 0} \ f(x_1, x_2, r)$$
$$s.t \begin{cases} p_l(x_1, x_2) \geq P_{min} \\ x_1 \text{ solves } \max\limits_{x_1\geq 0} p_n(x_1) - rx_1 \end{cases}$$

The optimal solution of this bilevel program will allow the State to fix the proper tax rate (and the quantities that local enterprises must at least produce) that will permit national initiatives to be competitive with concurrent non local enterprises.

3.5 Conclusion

After a review of some main results and notions of bilevel programming, we have presented a new approach for solving bilevel linear programming problems. By using the basic idea that extreme points of the accessible region of the BLPP are extreme points for the feasible space of the leader problem, and that the optimal solution is one of these vertices, we have developed pivot techniques to find the optimal solution on an expanded tableau (simplex tableau) that represents the data of the BLPP. An example has been provided to illustrate how the approach works. A bilevel optimization model that can permit to determine the tax rate to impose to non local enterprises in order to protect national initiatives in the context of globalization was also formulated as a bilevel programming problem.

Chapter Four

Multicriteria optimization concepts in bilevel optimization

4.1 Introduction

Although several authors have attempted to establish a link between bicriteria optimization and bilevel optimization [Fulop 1993, Bard 1984], none has succeeded thus far in proposing conditions which guarantee that the optimal solution of a given bilevel program is Pareto-optimal for both upper and lower level objective functions (counter-examples were reported in [Candler and Townsley 1982, Clarke and Westerberg 1988, Haurie et al. 1990, Marcotte and Savard 1991]). More recently, certain authors have started to study the possibility to exploit multicriteria approaches for solving bilevel programming problems [Fulop 1993, Bard 1983b, Campelo and Scheimberg 2000, Fliege and L.N.Vicente 2006, Haurie et al. 1990, Ivanenko and Plyasunov 2008, Manoel and Scheimberg 2000, Marcotte and Savard 1991, Ünlü 1987, Wen and Hsu 1989]. Unfortunately, apart from the relationship reported in [Fulop 1993] that has been used in the literature to develop an algorithm for solving bilevel linear programming problems, none of the other propositions has been implemented for solving BPP, due certainly to the facts that the proposed multicriteria optimization problems were defined by complicated relations [Fliege and L.N.Vicente 2006, Ivanenko and Plyasunov 2008] or the propositions were wrong [Bard 1983b, Campelo and Scheimberg 2000, Haurie et al. 1990]. This chapter introduces our contributions in this study: the use of multicriteria optimization approaches when dealing with bilevel optimization problems [Pieume et al. 2009].

The chapter is organized as follows. In the next section, we first show through an example that the optimal solution of a bilevel optimization problem is not necessarily Pareto-optimal. We then discuss conditions under which a solution of a bilevel optimization problem can be a Pareto-optimal solution of the corresponding bicriteria optimization problem. Since it could be not economically admissible for the two DM to use a non Pareto-optimal solution, we end the section by a strategy that can permit the two decision makers to find a Pareto-optimal solution better than the bilevel optimal solution. In section 4.3, we introduce a generalization of the Fulop relation [Fulop 1993] that establishes a link between multiobjective linear programming and bilevel linear programming. We show that it could be valid even if the leader objective function is not linear. In section 4.4, a new relation between bilevel programming problem and multicriteria optimization problem is introduced. We also discuss the practical implementation of the new relation. The chapter is finally concluded in section 4.5.

4.2 Pareto-optimality of the bilevel optimization solution

4.2.1 Non Pareto-optimality of the optimal solution

consider the following bilevel programming problem (BPP):

$$\min_{x\in X} F(x,y)$$

$$\text{subject to} \begin{cases} G(x,y) \leq 0 \\ \text{y solves} \begin{cases} \min_{y\in Y} f(x,y) \\ s.t \\ g(x,y) \leq 0 \end{cases} \end{cases} \tag{BPP}$$

with $X \subset \mathbb{R}^{n_1}$ and $Y \subset \mathbb{R}^{n_2}$. The BPP corresponding bicriteria optimization problem BOP is defined by:

$$\min_{x,y} T(x,y) = (F(x,y),\ f(x,y))$$

$$\text{subject to} \begin{cases} G(x,y) \leq 0 \\ g(x,y) \leq 0 \end{cases} \tag{BOP}$$

The admissible set $\Omega = \{(x,y) \in X \times Y /\ G(x,y) \leq 0 \text{ and } g(x,y) \leq 0\}$ of BPP is also the admissible set of BOP.

As illustrated by the following example from [Dempe 2002], the optimal solution of a bilevel programming problem need not be Pareto-optimal for the corresponding bicriteria optimization problem.

Example 4.2.1.

$$\min_{x\in X} x + 2y$$

$$subject\ to \begin{cases} x \geq 0 \\ y\ solves \begin{cases} \min_{y\in Y} -y \\ s.t \\ x + y \leq 2 \\ x \leq 1 \\ y \geq 0 \end{cases} \end{cases}$$

with $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$. The optimal solution of this BLPP is $(1,\ 1)$. The following tableau proposes some admissible points that improve either the upper or the lower level objective or both.

(x,y)	Nature of the point	f(x,y)	F(x,y)	Remark
(1,1)	Optimal point	-1	3	Solution of the BLPP
(0,1)	Pareto point	-1	2	Improves the leader objective
(0,3/2)	Pareto point	-1.5	3	Improves the follower objective
(0,5/4)	Pareto point	-1.25	2.5	Improves both objectives

Hence, in general, it is not possible to directly use solution techniques based on the replacement of the bilevel program by the bicriteria optimization problem. Previously, several researchers have tried to do so, but their proposed approaches have all been proved inadequate. Let us remark that despite this, there is always a point which is at the same time Pareto-optimal for the associated bicriteria linear optimization problem and feasible for the bilevel linear problem. However, as illustrated by the following example, the corresponding feasible point can be the worst choice for the leader.

Example 4.2.2.

$$\min_{x\in X}\ x+3y$$

$$\textit{subject to}\ \begin{cases} 1\le x\le 6 \\ y\ \textit{solves} \begin{cases} \min_{y\in Y} -y \\ s.t \\ x+y\le 8 \\ x+4y\ge 8 \\ x+2y\le 13 \end{cases} \end{cases}$$

with $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$. The following figure (figure 4.1) illustrates the set of feasible points and Pareto-optimal points.

One can easily check that:

- The union of segment $[A,\ E]$ and segment $[E,\ D]$ gives the set of feasible solutions to the upper level problem.
- The global solution of the problem is the point $D = (6, 2)$ with an optimal function value to the leader equal to 12.
- The segment $[A,\ B]$ constitutes the set of all Pareto-optimal points of the corresponding bicriteria problem. The only efficient point which is feasible for the bilevel programming

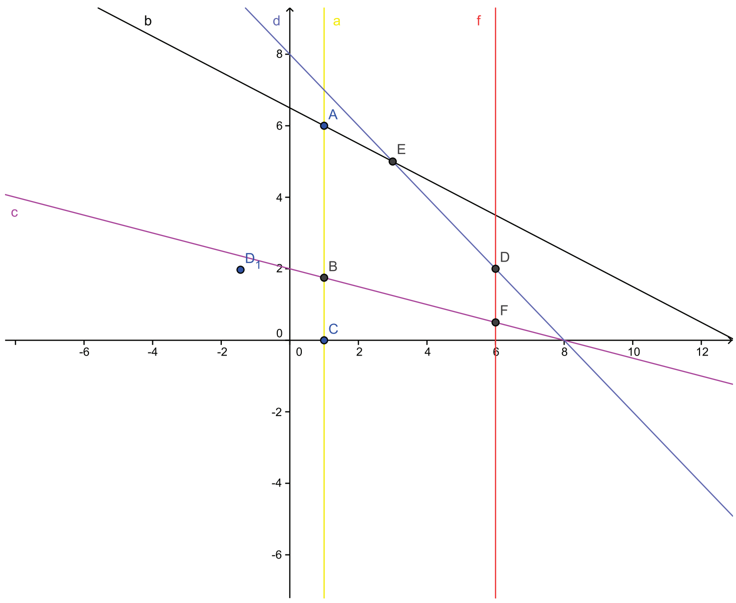

Figure 4.1: The linear bilevel programming problem

problem is the point $A = (1, 6)$, which has the worst possible function value for the upper level objective function.

4.2.2 Conditions for Pareto-optimality of the optimal solution

To our knowledge, there is only one result in the literature that permits to have sufficient conditions for the optimality of the bilevel programming problem. consider the following two assumptions:

Assumption 1

For any fixed element $x \in X$, $R(x) = Argmin_y \{f(x, y)/\ g(x, y) \leq 0\ \}$ has only one element $y(x)$.

Assumption 2

$f(x_1, y_1) < f(x_2, y_2) \Longrightarrow F(x_1, y_1) < F(x_2, y_2)\ \forall (x_1, y_1) \in \Omega\ , \forall (x_2, y_2) \in \Omega$

Theorem 4.2.1. *If assumption 1 and assumption 2 are satisfied, then the optimal solution* (x^*, y^*) *of* BPP *is a Pareto-optimal solution of* BOP*.*

Proof:

Let us suppose that, (x^*, y^*) is not an efficient solution of BOP, then there exists a point $(a, b) \in \Omega$ which dominates (x^*, y^*) i.e $\exists (a, b)$ such that $F(a, b) \leq F(x^*, y^*)$ (i)

and $f(a, b) \leq f(x^*, y^*)$ (ii) with at least one strict inequality. There are two cases:

Case 1: (a, b) is a feasible point of BPP, then from the inequality (i) and due to the fact that (x^*, y^*) is an optimal solution of BPP, we necessarily have $F(a, b) = F(x^*, y^*)$ (iii). From (ii), one have then $f(a,b) < f(x^*,y^*)$. Assumption 2 allow then to deduct that $F(a,b) < F(x^*,y^*)$, which contradicts the fact that (x^*,y^*) is an optimal solution of BPP. So (x^*,y^*) is an efficient point.

Case 2: If (a,b) is not a feasible point of BPP, then $f(a,y(a)) < f(a,b)$. From assumption 2, we have then $F(a,y(a)) < F(a,b)$. From relation (i) we have then $F(a,y(a)) < F(x^*,y^*)$ contradiction, because (x^*,y^*) is an optimal solution of BPP. □

We also have the following result:

Theorem 4.2.2. *If assumption 1 and assumption 2 are satisfied, then a Pareto-optimal point (x^*,y^*) of problem (BOP) such that (x^*,y^*) is feasible for BPP is an optimal solution of BPP.*

Proof:

Let us suppose that (x^*,y^*) is not an optimal solution of BPP, then there is (x,y) feasible for BPP such that $F(x,y) < F(x^*,y^*)$ $(***)$. Since (x^*,y^*) is a Pareto-optimal of BOP, we have $f(x^*,y^*) \leq f(x,y)$. This implies, due to assumption 2, that $F(x^*,y^*) < F(x,y)$; which contradicts the relation $(***)$. □

To our knowledge there do not exist other conditions in the literature that guarantee Pareto-optimality (with respect to BOP) of the optimal solution of BPP.

4.2.3 A post Pareto-optimality analysis

When the optimal solution of BPP is not Pareto-optimal with respect to BOP, it might be interesting to develop a post analysis approach that could permit to generate a Pareto-optimal point that improves both objective functions of the two DM. We present here an approach developed for the linear case that is a minor variant of the one developed in [Wen and Hsu 1991]. Consider the linear case (BLPP) defined as follows:

$$min_{x\in X} = c_1x + d_1y$$

$$where \text{ y } solves \begin{cases} min_{y\in Y} f(x,y) = c_2x + d_2y \\ s.t \\ A_1x + B_1y \leq 0 \end{cases}$$

If the optimal solution (x^*,y^*) of BLPP is not Pareto-optimal for the corresponding bicriteria linear programming problem (BOLP), we would like to develop a strategy that would permit to find a Pareto-optimal point better (with respect to the two DM) than (x^*,y^*). Let us introduce the following two results.

Lemma 4.2.1. *If Ω(the feasible set) is a compact and convex set, then the optimal solution (x^*,y^*) of BLPP is Pareto-optimal if and only if $int\Omega' = \emptyset$.*
where $\Omega' = \{(x,y) \in \Omega /\ F(x,y) \leq F(x^*,y^*) \text{ and } f(x,y) \leq f(x^*,y^*)\}$

The proof is very natural, because, if $int\Omega' \neq \emptyset$ then $\exists(x,y) \in \Omega'$ such that $F(x,y) < F(x^*,y^*)$ and $f(x,y) < f(x^*,y^*)$, which implies that (x^*,y^*) is not Pareto-optimal.

Lemma 4.2.2. *An efficient point to $min_{x,y}(F,f)$ over Ω' is also efficient to $min_{x,y}(F,f)$ over Ω.*

Proof:

Let $(a,b) \in \Omega'$ be an efficient point to $min_{x,y}(F,f)$ over Ω' and suppose that (a,b) is not an efficient point to $min_{x,y}(F,f)$ over Ω. Then there exists a point $(x',y') \in \Omega$ such that $F(x',y') \leq F(a,b)$ and $f(x',y') \leq f(a,b)$ (i) at least one inequality. Since $(a,b) \in \Omega'$, we have then $F(x',y') \leq F(x^*,y^*)$ and $f(x',y') \leq f(x^*,y^*)$ (ii). By the definition of Ω' it follows from (ii) that $(x',y') \in \Omega'$. From (i), it follows that the element (x',y') of Ω' dominates (a,b). This contradicts the fact that (a,b) is an efficient point to $min_{x,y}(F,f)$ over Ω' . □

Let us suppose that after having found that the optimal solution of BLPP is not Pareto-optimal, the two DM decide to make an agreement to adjust their decision policies by replacing the constraint set Ω by Ω' and preserving the hierarchical characteristics. The problem to solve will be then the following bilevel linear programming problem (BLPP'):

$$min_{x\in X} = c_1x + d_1y$$

$$where \text{ y } solves \begin{cases} min_{y\in Y} f(x,y) = c_2x + d_2y \\ s.t \\ (x,y) \in \Omega' \end{cases}$$

Where $\Omega' = \{(x,y) \in \Omega/;/F(x,y) \leq F(x^*,y^*) \text{ and } f(x,y) \leq f(x^*,y^*)\}$

The following result holds.

Theorem 4.2.3. *The optimal solution, (x_f,y_f), of minimizing f over Ω' is an optimal solution of $BLPP'$.*

Proof:

Let (x_f,y_f) be the optimal solution of minimizing f over Ω' (i). Let us suppose that (x_f,y_f) is not an optimal solution of $BLPP'$ (ii). Then (i) implies that (x_f,y_f) is a feasible point to BLPP'. (ii) implies that, there is another feasible point (a,b) of BLPP' which give better value to the leader than (x_f,y_f). This implies that $F(a,b) < F(x_f,y_f)$. Then (a,b) is a feasible point of BLPP. This contradicts the fact that (x^*,y^*) is the optimal solution of BLPP, since $F(a,b) < F(x_f,y_f) \leq F(x^*,y^*)$. It follows that (x_f,y_f) is an optimal solution to BLPP'. □

The following result can be naturally deducted.

Lemma 4.2.3. *The optimal solution (x_f,y_f) of $BLPP'$ (the one which is the optimal solution of minimizing f over Ω'), is a Pareto-optimal solution of $min_{x,y}(F,f)$ over Ω .*

Remark 4.2.1. The preceding results suggest that after solving the linear bilevel programming problem (BLPP), and after the non Pareto-optimality of the optimal solution identified, it suffices (the simple way) to retain the optimal solution of minimizing f over Ω' as an efficient solution of BLPP. In the case of uniqueness of the optimal solution and from the definition of Ω', this solution will be always better than the optimal optimal solution, of BLPP. In fact: If the the optimal solution of minimizing f over Ω' is unique, one has
$F(x_f, y_f) \leq F(x^*, y^*)$ and $f(x_f, y_f) < f(x^*, y^*) \Longrightarrow F(x_f, y_f) + f(x_f, y_f) < F(x^*, y^*) + f(x^*, y^*)$
Else, one has
$F(x_f, y_f) \leq F(x^*, y^*)$ and $f(x_f, y_f) \leq f(x^*, y^*) \Longrightarrow F(x_f, y_f) + f(x_f, y_f) \leq F(x^*, y^*) + f(x^*, y^*)$

4.3 A generalization of the Fulop approach

In[Fulop 1993], the author shows that, there exist an equivalence between linear bilevel programming and linear optimization over the efficient set. We show in this section that, under certain conditions, the same relation remains valid for some particulars non linear problems. The following class of BPP (call BPP') is considered:

$$\min_{x\in X} \{F(x,y)/\ G(x) \leq 0 \text{ and } y \in R(x)\ \}$$

where

$$R(x) = Argmin_{y\in Y} \left\{c^t y/\ g(x,y) \leq 0\ \right\}$$

with $c \in \mathbb{R}^{n_y}$, $g = (g_1, g_2, ..., g_q)$ and $g_i : \mathbb{R}^{n_x+n_y} \longrightarrow \mathbb{R}$ are continuous functions. Let $K = \mathbb{R}_+^{n_x+n_y} \setminus \{0\}$. Define a function T by: $T(z) = Cz\ \forall z \in \Omega$ with

$$C = \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix}$$

where e is a vector having each entry equals to 1, I is an n_x by n_x identity matrix.
Let us denote $Y = T(\Omega)$. We suppose that, for any fixed element $x \in X$, the set $\{(x,y) \in \Omega/\ g(x,y) \leq 0\ \}$ is bounded. We have the following result.

Theorem 4.3.1. *$z \in \Omega$ is feasible for BPP' if and only if $T(z) \in ND(Y, \precsim_K)$*

Proof:
$\Longrightarrow$ Suppose that $z = (x_0, y_0)$ is feasible for BPP' but that $T(z)$ is not in $ND(Y, \precsim_K)$. Then there

must be a point $z_1 \in \Omega$ such that $T(z_1) \precsim_K T(z)$, which means that: $T(z) - T(z_1) \in K$, this implies that $Cz - Cz_1 \in K$. From the definition of K, we have then $Cz - Cz_1 \geq 0$ and $Cz \neq Cz_1$. which is equivalent to: $Cz_1 \leq Cz$ and $Cz \neq Cz_1$. Using the structure of the matrix C and the fact that $z = (x_0, y_0)$ and $z_1 = (x, y)$, we obtain:

$$\begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cy \\ -x \\ e^T x \end{pmatrix} \leq \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} cy_0 \\ -x_0 \\ e^t x_0 \end{pmatrix}$$

and then

$$\begin{pmatrix} cy \\ -x \\ e^t x \end{pmatrix} \neq \begin{pmatrix} cy_0 \\ -x_0 \\ e^t x_0 \end{pmatrix}$$

This implies that: $x - x_0 \geq 0$ and $e^t(x - x_0) \leq 0$ which means that $x = x_0$ and also that $e^t x = e^t x_0$. Since $Cz \neq Cz_1$, it follows that $cy < cy_0$, contradicting the fact that $y_0 \in R(x_0)$.

$\Longleftarrow$ We suppose that there is a point $z = (x_0,\ y_0) \in \Omega$ with $T(z) \in ND(Y, \precsim_K)$ that is not feasible for BPP. Then, the fact that $z = (x_0,\ y_0)$ is not feasible for BPP' implies that y_0 does not solve $min_y \{cy/\ g(x_0, y) \leq 0\ \}$. Thus, there exists $\overline{y} \neq y_0$ (using the assumptions that the admissible set of the follower is bounded and that g is a continuous function) such that:

(i) $\overline{y} \neq y_0$ (ii) $g(x_0, \overline{y}) \leq 0$ (iii) $c\overline{y} < cy_0$ and (iv) $c\overline{y} < cy\ \forall y\ /\ g(x_0, y) \leq 0$ We can then say that $z^* = (x_0, \overline{y})$ is feasible for the BPP'. Let us remark that:

$$Cz^* = \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ \overline{y} \end{pmatrix} = \begin{pmatrix} c\overline{y} \\ -x_0 \\ e^t x_0 \end{pmatrix} \text{ and } Cz = \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} cy_0 \\ -x_0 \\ e^t x_0 \end{pmatrix}$$

Since, from (iv), one has $c\overline{y} < cy_0$,

$$C \begin{pmatrix} x_0 \\ \overline{y} \end{pmatrix} \leq C \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ and } C \begin{pmatrix} x_0 \\ \overline{y} \end{pmatrix} \neq C \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

So $T(x_0\ , \overline{y}) \precsim_K T(x_0,\ y_0)$, which contradicts the fact $T(x_0, y_0) \in ND(Y, \precsim_K)$. □

The following result which gives a way to find the optimal solution of a bilevel programming problem is a corollary of *Theorem* 4.3.1.

Corollary 4.3.1. *Solving the bilevel programming problem BPP' is equivalent to minimizing the upper level objective function $F(x, y)$ over the Pareto-optimal set corresponding to the non dominated set $ND(Y, \precsim_K)$.*

4.4 A multicriteria approach for bilevel optimization

4.4.1 New relation between unconstrained bilevel optimization and multicriteria optimization

We suppose here that our goal is to solve the following bilevel programming problem (that will be called BPP):

$$min_{x\in\mathbb{R}^{n_x}} \quad \{\ F(x,y)\ \}$$
$$s.t \qquad min_{y\in\mathbb{R}^{n_y}} \quad \{\ f(x,y)\ \}$$

In [Fliege and L.N.Vicente 2006], the authors define an order that captures exactly the optimality properties of the bilevel problem, in such a way that all solutions to the BPP are non-dominated with respect to the order introduced. In order to achieve this goal, the following order relation is first introduced:

$$z_1=(x_1,y_1)\preceq z_2=(x_2,y_2)\Longleftrightarrow\begin{cases}(x_1=x_2 \text{ and } \ f(z_1)<f(z_2))\\ \text{or}\\ (\|\nabla_y f(z_1)\|=0 \text{ and } F(z_1)<F(z_2))\end{cases}$$

And the following result is derived.

Lemma 4.4.1. *If $z=(a,b)\in\mathbb{R}^{n_x+n_y}$ is non-dominated with respect to $\preceq$, then z solves the bilevel problem (BPP).*

In order to compute the set of non-dominated points,the following function and notations are introduced.

Let H be the function

$$H : Z=(x,y)\longrightarrow(x,F(x,y),f(x,y),\|\nabla_y f(x,y)\|)$$

Let M be the image space of H, and define on M the cone K by:

$$K=\{(x,\alpha,\beta,\gamma)\in M/\ (x=0 \text{ and } \beta>0) \text{ or } (\alpha>0 \text{ and } \gamma\geq 0)\ \}\ (a)$$

Then the following result is a scheme to compute the set of non-dominated points with respect to $\preceq$.

Lemma 4.4.2. *If $H(z)$ is non-dominated with respect to $\precsim_K$ for some $z\in\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}$, then z is non-dominated with respect to $\preceq$.*

Proof:

If $H(z)$ is non-dominated with respect to $\precsim_K$ for some $z = (z_1,\ z_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$, then there is no $w = (a, b) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that $H(w)$ dominates $H(z)$. It means that, there is no w such that $H(w) \precsim_K H(z)$, which is equivalent to say that, there is no w such that $H(z) - H(w) \in K$ (i).

$$(i) \Longleftrightarrow \nexists w = (a, b)/ \begin{cases} (x = a \text{ and } \ f(z) < f(w)) \\ \text{or} \\ (\|\nabla_y f(w)\| \leq \|\nabla_y f(z)\| \text{ and } F(w) < F(z)) \end{cases}$$

This in turn implies that $\|\nabla_y f(z)\| = 0$ because otherwise, there is a point $w* = (a*, b*)$ such that $a* = z_1$ and $f(w*) < f(z)$. Contradicting what we have just stated above. Thus,

$$\nexists w = (a, b)/ \begin{cases} (z_1 = a \text{ and } \ f(z) < f(w)) \\ \text{or} \\ (\|\nabla_y f(w)\| = 0 \text{ and } F(w) < F(z)) \end{cases}$$

which proves that z is non-dominated with respect to $\preceq$. $\square$

The following corollary is a simple consequence of the preceding two lemmas.

Corollary 4.4.1. *If* $H(z) \in ND(M, \precsim_K)$ *for some* $z = (x, y) \in \mathbb{R}^{nx+ny}$ *then* z *is an optimal solution of BPP.*

Remark 4.4.1. Note that the same result was developed for constraint case, the difference was just that M was different. As this corollary states, in order to solve the problem BPP, one can find an element of the set $ND(M, \precsim_K)$ and the corresponding Pareto-optimal point will then be an optimal solution to the BPP. But, the main difficulty of this result is effectively to find the set (or an element of) $ND(M, \precsim_K)$. In fact, K is not convex, and most results developed in the literature for finding efficient points suppose that the cone used must be convex.

We now present how to construct a more practical relation. It can be seen as an improvement of the preceding results.

Note that K can be written as the union of two convex cones:

$$K = K_1 \cup K_2 \text{ where } \begin{cases} K_1 = \{(x, y, z, t) \in \mathbb{R}^r / \ (x = 0 \text{ and } z > 0)\ \} & (b) \\ K_2 = \{(x, y, z, t) \in \mathbb{R}^r / \ (y > 0 \text{ and } t \geq 0)\ \} & (c) \end{cases}$$

We exploit this representation and introduce the following definition.

Definition 4.4.1. Consider the binary relations $\precsim_{K_1}$ and $\precsim_{K_2}$ defined on M. We can consider their union to be a binary relation defined on M by:

$$x \precsim_{K_1} \cup \precsim_{K_2} y \Longleftrightarrow (x \precsim_{K_1} y) \ \text{ or } \ (x \precsim_{K_2} y)$$

Theorem 4.4.1. $ND(M, \precsim_{K_1} \cup \precsim_{K_2}) = ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$

Proof:

First, assume $y \in ND(M, \precsim_{K_1} \cup \precsim_{K_2})$, but $y \notin ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$. There are two possibilities. Case 1: $y \notin ND(M, \precsim_{K_1})$ and $y \in ND(M, \precsim_{K_2})$. Then there is $y' \in M$ such that $y' \precsim_{K_1} y$. Therefore $y' \precsim_{K_1} \cup \precsim_{K_2} y$, this implies $y \notin ND(M, \precsim_{K_1} \cup \precsim_{K_2})$, contradicting the assumption. Hence $y \in ND(M, \precsim_{K_1})$ and then $y \in ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$. Case 2: $y \in ND(M, \precsim_{K_1})$ and $y \notin ND(M, \precsim_{K_2})$. A similar proof leads to the same conclusion. So, $ND(M, \precsim_{K_1} \cup \precsim_{K_2}) \subseteq ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$.

Second, assume $y \in ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$ and $y \notin ND(M, \precsim_{K_1} \cup \precsim_{K_2})$. Then there is $y' \in M$ such that $y' \precsim_{K_1} \cup \precsim_{K_2} y$, this means that $y' \precsim_{K_1} y$ or $y' \precsim_{K_2} y$. If $y' \precsim_{K_1} y$, then $y \notin ND(M, \precsim_{K_1})$, which implies that $y \notin ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$, contradicting the assumption. If $y' \precsim_{K_2} y$, then $y \notin ND(M, \precsim_{K_2})$, which implies that $y \notin ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$, contradicting the assumption. So, $ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2}) \subseteq ND(M, \precsim_{K_1} \cup \precsim_{K_2})$. □

Theorem 4.4.2. *The binary relations $\precsim_K$ and $\precsim_{K_1} \cup \precsim_{K_2}$ are equivalent (on M) i.e :*

$$\forall x, y \in M;\ x \precsim_K y \Longleftrightarrow x \precsim_{K_1} \cup \precsim_{K_2} y$$

Proof:

$$\begin{aligned} \forall x, y \in M, x \precsim_K y &\Longleftrightarrow y - x \in K \\ &\Longleftrightarrow y - x \in K_1 \text{ or } y - x \in K_2 \\ &\Longleftrightarrow x \precsim_{K_1} y \text{ or } x \precsim_{K_2} y \\ &\Longleftrightarrow x \precsim_{K_1} \cup \precsim_{K_2} y \end{aligned}$$

□

We can then deduct the following two corollaries which are the fundamental results of this section:

Corollary 4.4.2. $ND(M, \precsim_K) = ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$

Proof: It follows immediately from theorem 4.4.1 and theorem 4.4.2 □

Corollary 4.4.3. *If $H(z) \in ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2})$ for some $z = (x, y) \in \mathbb{R}^{n_x + n_y}$, then z is an optimal solution of BPP.*

Proof: It follows immediately from corollary 4.4.1 and corollary 4.4.2 □

4.4.2 Practical implementation

The corollary 4.4.3 gives a simple way for solving BPP using multicriteria optimization techniques. In order to solve the BPP, one has to solve two multicriteria problems, for any point belonging to the intersection of the two non-dominated sets, the corresponding Pareto-optimal solution is an optimal solution to the bilevel programming problem BPP. The advantage of this approach is that K_1 and K_2 are convex sets, so results of corollary 2.1 can be easily used for finding the set $ND(M, \precsim_{K_1})$ and $ND(M, \precsim_{K_2})$.

Let us remark that, if the set $ND(M, \precsim_K)$ is empty, it does not mean that the bilevel problem (BPP) does not have a solution, but it means that the relationship given by corollary 4.4.3 is not applicable to solve BPP. So the corollary 4.4.3 is applicable if and only if $ND(M, \precsim_K) \neq \emptyset$. We present below three conditions for which the relation given by corollary 4.4.3 cannot be applied for solving BPP ($ND(M, \precsim_K) = \emptyset$). But we will first present some results necessary for establishing these conditions.

Proposition 4.4.1. *Let $C \subseteq \mathbb{R}^q_+$ a cone, and $Y \subseteq \mathbb{R}^q_+$ a set.*
(ii) If C is convex, then $ND(Y, \precsim_C) \subseteq ND(Y + C, \precsim_C)$;
(i) If $0 \in C$, then $ND(Y + C, \precsim_C) \subseteq ND(Y, \precsim_C)$.

Proof:

(i) Assume $y \in ND(Y, \precsim_C)$ but $y \notin ND(Y + C, \precsim_C)$. Then there exists a $y' \in Y + C$ such that $y' \precsim_C y$, which is equivalent to say that: $\exists y' \in Y + C \ / \ y' - y \in C$. $y' \in Y + C \ \Leftrightarrow \ \exists y'' \in Y$ and $d'' \in C$ such that $y' = y'' + d''$, let $d' = y - y'$, then $y = y' + d' = y'' + d' + d''$. Let $d = d' + d''$, since d', $d'' \in C$ and C is a convex cone, then $d \in C$. Therefore, $y = y'' + d \Rightarrow y - y'' \in C$, which is equivalent to $y'' \precsim_C y$. This implies that $y \notin ND(Y, \precsim_C)$, contradicting the assumption. Hence $ND(Y, \precsim_C) \subseteq ND(Y + C, \precsim_C)$.

(ii) Assume $y \in ND(Y + C, \precsim_C)$ but $y \notin ND(Y, \precsim_C)$, there are two possibilities:
Case 1: If $y \notin Y$, there exist $y' \in Y$ and $d \in C$ such that $y = y' + d$, which implies that $y - y' \in C$ $(*)$. Since $y' = y' + 0 \in Y + C$, $(*)$ implies that $y \notin ND(Y + C, \precsim_C)$, contradiction.
Case 2: If $y \in Y$, then there is $y' \in Y$ such that $y' \precsim_C y$. This implies that $y - y' \in C$. Let $d = y - y'$, we have $d \in C$, therefore $y = y' + d \in Y + C$ and $y \notin ND(Y + C, \precsim_C)$. Again contradicting the assumption. In either case, $y \in ND(Y, \precsim_C)$. □

Proposition 4.4.2. *Let $C \subseteq \mathbb{R}^q_+$ a cone, and $Y \subseteq \mathbb{R}^q_+$ a set, then: $ND(Y, \precsim_C) \subseteq bd(Y)$ where $bd(Y)$ denotes the boundary of Y.*

Proof:

Let $y \in ND(Y, \precsim_C)$ and suppose $y \notin bd(Y)$, therefore $y \in int(Y)$(where $int(M)$ is the interior of Y). Then, there exists an $\epsilon - neighbourhood$ $B(y, \epsilon)$ of y such that $B(y, \epsilon) \subseteq Y$. But $B(y, \epsilon) = y + B(0, \epsilon)$(where $B(0, \epsilon)$ is an open ball with radius centered at the origin). Let $d \in B(0, \epsilon)$, one can choose some $\lambda > 0$ such that $-\lambda d \in B(0, \epsilon)$, this implies that $y - \lambda d \in Y$, and there is $t \in Y$ such that $y - \lambda d = t$. This implies that $y - t = \lambda d$. Since C is a cone, $\lambda d \in C$, we have then $y - t \in C$, which is equivalent to $t \precsim_C y$. This implies that $y \notin ND(Y, \precsim_C)$. Contradicting the assumption. □

Theorem 4.4.3. *Consider M, K, K_1, K_2 as defined in (a), (b) and (c) in the preceding pages .*

(i) An approach for solving the bilevel programming problem (BPP) based on Corollary 4.4.3 is not applicable if M is closed-open (a set that is at the same time closed and open).

(ii) An approach for solving the bilevel programming problem (BPP) based on Corollary 4.4.3 is not applicable if $M + K_1$ is closed-open.

(iii) An approach for solving the bilevel programming problem (BPP) based on Corollary 4.4.3 is not applicable if $M + K_2$ is closed-open.

Proof:

(i) If M is closed-open, then $bd(M) = \emptyset$. From proposition 4.4.2, one has $ND(M, \precsim_K) \subseteq bd(M)$. This implies that $ND(M, \precsim_K) = \emptyset$.

(ii) If $M + K_1$ is closed-open, then $bd(M + K_1) = \emptyset$. From proposition 4.4.2, $ND(M + K_1, \precsim_{K_1}) = \emptyset$ (1) . Since K_1 is convex, then from proposition 4.4.1, we have : $ND(M, \precsim_{K_1}) \subseteq ND(M + K_1, \precsim_{K_1})$. The relation (1) permits to deduct that $ND(M, \precsim_{K_1}) = \emptyset$. From Corollary 4.4.2, one has:

$ND(M, \precsim_K) = ND(M, \precsim_{K_1}) \cap ND(M, \precsim_{K_2}) = \emptyset \cap ND(M, \precsim_{K_2}) = \emptyset$.

Hence $ND(M, \precsim_K) = \emptyset$.

(iii) Since K_2 is also convex as K_1, the proof of (iii) is similar to the one of (ii).

4.5 Conclusion

This chapter has presented a necessary and sufficient condition for the optimal solution of a bilevel programming problem to be Pareto-optimal. A strategy to find a Pareto-optimal solution better than the bilevel optimal solution was also developed. It has been shown that, in order to solve a particular class of bilevel programming problems, it suffices to solve two artificial multicriteria problems. Then for any point belonging to the intersection of the two non-dominated sets, the corresponding Pareto-optimal solution is an optimal solution to the bilevel programming problem. We have derived three necessary conditions by which the relation obtained is not applicable for solving bilevel programming problems.

Chapter Five

Multicriteria optimization modeling in bilevel optimization: the bilevel multiobjective optimization

5.1 Introduction

Standard bilevel programming problems where each DM has only one objective function have been extensively studied in the literature. However, despite their multiple applications, the special case of bilevel programming problems where each DM has more than one objective function has not yet received a broad attention in the literature. We have found only some few articles related to this class of problems in [Yaman and Chin-Hui 2008, Tuy et al. 1993, Shi and Xia 1997, Shimizu et al. 1997, Omar and Blair 1990, Kalyanmoy and Ankur 2008]. This situation is possibly due to the difficulty of searching and defining optimal solutions. Contrary to the standard two levels programming problems, where it is usually assumed that the set of rational responses of the follower is a singleton, in the bilevel multi-objective problem, the lower level optimization problem has a number of trade-off optimal solutions and the task of the upper level is to focus its search on multiple trade-off solutions, which are members of optimal trade-off solutions of lower level optimization problem. Bilevel Multi-objective Programming Problem is therefore computationally more complex than the conventional Multi-Objective Programming Problem or a Bilevel Programming Problem. The investigation of the present chapter is on Bilevel Multi-objective Programming Problem (BMPP).

The chapter is organized as follows: In the next section, we present the optimistic formulation of BMPP. Some concepts and notations are recalled. Section 5.3 presents an approach for generating efficient points when dealing with BMPP. We show how to construct two artificial multi-objective programming problems such that any point that is efficient for both problems is also efficient for the corresponding $BMPP$. Based on this result, a general algorithm for generating efficient solutions of $BMPP$ is provided. In section 5.4, the particular case of linear multi-objective programming problem ($LMPP$) is addressed. The optimistic formulation of $BMPP$ is considered. We introduce an artificial $LMPP$ such that its resolution permits to generate the whole set of admissible points of the upper level decision of the BMPP. Based on this result and depending on whether the leader can evaluate or not his preferences for his different objective functions, two approaches for obtaining Pareto-optimal solutions can be used.

5.2 Formulation of bilevel multiobjective optimization problems

Let us recall that standard Bilevel Programming Problems (BPP) are generally formulated as follows:

$$\min_{x\in X} \; F(x,y)$$
$$\text{subject to} \begin{cases} G(x,y) \leq 0 \\ \text{y solves} \begin{cases} \min_{y\in Y} f(x,y) \\ s.t \\ g(x,y) \leq 0 \end{cases} \end{cases} \quad \text{(BPP)}$$

with $X \subset \mathbb{R}^{n_1}$ and $Y \subset \mathbb{R}^{n_2}$.

If F and f are vector value functions ($F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1}$ and $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_2}$), then one speaks of bilevel multiobjective programming problems ($BMPP$). The formulation of a $BMPP$ can be given as follows:

$$\min_{x\in X} \; F(x,y) = (F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y))$$
$$\text{subject to} \begin{cases} G(x) \leq 0 \\ \text{y solves} \begin{cases} \min_{y\in Y} f(x,y) = (f_1(x,y), f_2(x,y), ..., f_{m_2}(x,y)) \\ s.t \;\; g(x,y) \leq 0 \end{cases} \end{cases} \quad \text{(BMPP)}$$

Let us denote by $R(x)$, the set of rational responses of the follower for each decision x of the leader, then it is defined as the set of Pareto-optimal points of the following problem:

$$\min_{y\in Y} f(x,y) = (f_1(x,y), f_2(x,y), ..., f_{m_2}(x,y))$$
$$s.t \;\; g(x,y) \leq 0$$

With this notation, one has the following formulation of $BMPP$:

$$\min_{x \in X} F(x,y) = (F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y))$$

$$\text{subject to} \begin{cases} G(x) \leq 0 \\ y \in R(x) \end{cases}$$

Let us denote by Ω the feasible space of $BMPP$ defined by:

$$\Omega = \{ \ (x,y) \in X \times Y \ / \ G(x) \leq 0 \text{ and } y \in R(x) \ \}$$

The optimistic formulation of $BMPP$ is given by:

$$\min_{x,y} F(x,y) = (F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y))$$

$$s.t \ \ (x,y) \in \Omega \tag{BMPP'}$$

Consider the following definition.

Definition 5.2.1.
(x^*,y^*) is an efficient solution of $BMPP'$ if and only if $(x^*,y^*) \in \Omega$ and $\nexists (x,y) \in \Omega$ such that $(F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y)) \leq (F_1(x^*,y^*), F_2(x^*,y^*), ..., F_{m_1}(x^*,y^*))$ and $(F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y)) \neq (F_1(x^*,y^*), F_2(x^*,y^*), ..., F_{m_1}(x^*,y^*))$.

Next sections are devoted to the development of approaches for finding efficient solutions of BMPP'. Throughout the rest of the chapter, the following notations are considered:
$K_1 = \mathbb{R}^{m_1}_+ \setminus \{0_{m_1}\}, \ K_2 = \mathbb{R}^{m_2}_+ \setminus \{0_{m_2}\}, \ X = \mathbb{R}^{n_1}_+, \ Y = \mathbb{R}^{n_2}_+, \ Z = \{ \ (x,y) \in \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+ \ / \ G(x) \leq 0 \text{ and } g(x,y) \leq 0 \ \}$ and S denotes the whole set of efficient solutions of $BMPP'$.
We intensively use the notation $E(f, \ X, \ \precsim_K)$ as stated in chapter 2, to represent the set of efficient(Pareto-optimal) points of a multi-objective optimization problem defined by a vector valued function f on an admissible set X with respect to a cone K.

5.3 Generating efficient solutions of BMPP'

5.3.1 Characterizations of efficient solutions

Let us consider the following multi-objective programming problem, constructed from the data of $BMPP'$:

$$\min_{x,y} \widetilde{f}(x,y) = (f_1(x,y), f_2(x,y), ..., f_{m_2}(x,y), x)$$

$$s.t \ \ (x,y) \in Z \tag{MPP2}$$

Let $K_3 = \mathbb{R}^{m_2}_+ \setminus \{0_{m_2}\} \times \{0_{n_1}\}$ and Ω as defined above. The following theorem provides a way to capture the feasible set of $BMPP'$.

Theorem 5.3.1. $\Omega = E(\widetilde{f},\ Z,\ \precsim_{K_3})$

Proof: This theorem is just a slight modified version of Theorem 4.1 given in [Eichfelder 2008]. □

Solving $BMPP'$ is then equivalent to solving the problem:

$$\begin{array}{ll} \min_{x\in X} & F(x,y) = (F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y)) \\ s.t & (x,y) \in E(\widetilde{f},\ Z,\ \precsim_{K_3}) \end{array} \tag{5.3.1}$$

This leads to the following corollary:

Corollary 5.3.1. $S = E(F,\ E(\widetilde{f},\ Z,\ \precsim_{K_3}),\ \precsim_{K_1})$.

Finding $E(\widetilde{f},\ Z,\ \precsim_{K_3})$ is not an easy task because it is difficult to generate the whole efficient set $E(\widetilde{f},\ Z,\ \precsim_{K_3})$ since it can be infinite and methods found in the literature are usually for cones defined in the form $\mathbb{R}^n_+ \setminus \{0_n\},\ n \in \mathbb{N}$.

Let $K_4 = \mathbb{R}^{m_2+n_1}_+ \setminus \{0_{m_2+n_1}\}$. The following result holds.

Theorem 5.3.2. $E(\widetilde{f},\ Z,\ \precsim_{K_4}) \subset E(\widetilde{f},\ Z,\ \precsim_{K_3})$

Proof:

Let $(x,y) \in E(\widetilde{f},\ Z,\ \precsim_{K_4})$ then there does not exist $(x',y') \in Z$ such that $\widetilde{f}(x',y') - \widetilde{f}(x,y) \in \mathbb{R}^{m_2+n_1}_+ \setminus \{0_{m_2+n_1}\}$ $(**)$. Since $\mathbb{R}^{m_2}_+ \setminus \{0_{m_2}\} \times \{0_{n_1}\} \subset \mathbb{R}^{m_2+n_1}_+ \setminus \{0_{m_2+n_1}\}$, $(**) \Rightarrow$ There does not exist $(x',y') \in Z$ such that $\widetilde{f}(x',y') - \widetilde{f}(x,y) \in \mathbb{R}^{m_2}_+ \setminus \{0_{m_2}\} \times \{0_{n_1}\}$. Consequently, $(x,y) \in E(\widetilde{f},\ Z,\ \precsim_{K_3})$. Therefore, one can conclude that $E(\widetilde{f},\ Z,\ \precsim_{K_4}) \subset E(\widetilde{f},\ Z,\ \precsim_{K_3})$. □

Theorem 5.3.2 suggests to capture a subset of S by solving the following problem:

$$\begin{array}{ll} \min_{x\in X} & F(x,y) = (F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y)) \\ s.t & (x,y) \in E(\widetilde{f},\ Z,\ \leq_{K_4}) \end{array} \tag{BMPP"}$$

This formulation and Corollary 5.3.1 lead obviously to the following result:

Corollary 5.3.2. $E(F,\ E(\widetilde{f},\ Z,\ \precsim_{K_4}),\ \precsim_{K_1}) \subseteq S$.

Even if the cone used in corollary 5.3.2 has the desired representation ($\mathbb{R}^n_+ \setminus \{0_n\}\ n \in \mathbb{N}$), optimizing multi-objective functions over an efficient set might not be an easy task. We introduce a new problem such that its resolution is easier and permits to capture a subset of efficient solutions of $BMPP''$.

Let us consider the following multi-objective programming problem:

$$\begin{array}{ll} \min\limits_{x,y} & F(x,y) = (F_1(x,y), F_2(x,y), ..., F_{m_1}(x,y)) \\ s.t & (x,y) \in Z \end{array} \tag{MPP1}$$

Theorem 5.3.3. $E(\widetilde{f},\ Z,\ \precsim_{K_4}) \cap E(F,\ Z,\ \precsim_{K_1}) \subseteq E(F,\ E(\widetilde{f},\ Z,\ \precsim_{K_4}),\ \precsim_{K_1})$

Proof:

Let $(x,y) \in E(\widetilde{f},\ Z,\ \precsim_{K_4}) \cap E(F,\ Z,\ \precsim_{K_1})$, then $(x,y) \in E(\widetilde{f},\ Z,\ \precsim_{K_4})$ and $(x,y) \in E(F,\ Z,\ \precsim_{K_1})$. Let us suppose that $(x,y) \notin E(F,\ E(\widetilde{f},\ Z,\ \precsim_{K_4}),\ \precsim_{K_1})$. Then there exists (x',y') that dominates (x,y) with respect to the problem $MPP1$ and the cone K_1, which means that $F(x',y') \precsim_{K_1} F(x,y)$. This implies that: $\exists (x',y') \in E(\widetilde{f},\ Z,\ \precsim_{K_4})$ such that $F(x',y') \leq F(x,y)$ and $F(x',y') \neq F(x,y)$. Since $E(\widetilde{f},\ Z,\ \precsim_{K_4}) \subseteq Z$, $\exists (x',y') \in Z$ such that $F(x',y') \leq F(x,y)$ and $F(x',y') \neq F(x,y)$. Consequently $(x,y) \notin E(F,\ Z,\ \precsim_{K_1})$. This contradicts the fact that $(x,y) \in E(\widetilde{f},\ Z,\ \precsim_{K_4}) \cap E(F,\ Z,\ \precsim_{K_1})$. □.

The following result is deduced from Theorem 5.3.3 and Corollary 5.3.2:

Corollary 5.3.3. $E(\widetilde{f},\ Z,\ \precsim_{K_4}) \cap E(F,\ Z,\ \precsim_{K_1}) \subseteq S$.

Corollary 5.3.3 stipulates that, in order to find an efficient point of $BMPP'$, one can solve the problem $MPP1$ (with respect to $\precsim_{K_1}$) and the problem $MPP2$ (with respect to $\precsim_{K_4}$). Then, any point belonging to the intersection of the efficient solution set of $MPP1$ and the efficient solution set of $MPP2$ can be retained as efficient point of $BMPP'$.

Corollary 5.3.3 can be use to generate efficient solutions to $BMPP'$ if and only if $E(\widetilde{f},\ Z,\ \precsim_{K_4}) \cap E(F,\ Z,\ \precsim_{K_1}) \neq \emptyset$. A necessary condition is that none of the two sets, $E(\widetilde{f},\ Z,\ \precsim_{K_4})$ and $E(F,\ Z,\ \precsim_{K_1})$ is empty. Theorem 5.3.4 gives sufficient conditions for this.

Theorem 5.3.4. *If the following three conditions hold:*
(i) Z is nonempty and compact set;
(ii) $\forall i \in \{1, ..., m_1\}$, F_i is lower semicontinuous ;
(iii) $\forall j \in \{1, ..., m_2\}$, f_j is lower semicontinuous ;
then $E(F,\ Z,\ \precsim_{K_1}) \neq \emptyset$ and $E(\widetilde{f},\ Z,\ \precsim_{K_4}) \neq \emptyset$.

Proof:

If (ii) holds, then F is $\mathbb{R}^{m_1}_+ - semicontinuous$ (i.e. the pre-image of the translated negative orthant is always closed). Since from (i) Z is nonempty and compact, $F(Z)$ is $\mathbb{R}^{m_1}_+ - semicompact$ and non-empty. This result and Theorem 2.8 of [Ehrgott 2005] imply that the set of non-dominated points is non-empty. Thus, the set of Pareto-points is non-empty i.e $E(F,\ Z,\ \precsim_{K_1}) \neq \emptyset$. Similarly, $E(\widetilde{f},\ Z,\ \precsim_{K_4}) \neq \emptyset$. □

Theorem 5.3.5 gives a sufficient condition for the implementation of an algorithm based on Corollary 5.3.3.

Theorem 5.3.5. *If the following two conditions hold:*
(i) $Z \subseteq \mathbb{R}_+^{n_1+n_2}$ *is a non-empty and compact set.*
(ii) $\exists i_0 \in \{1, ..., m_1\}, \exists j_0 \in \{1, ..., m_2\}, \exists \alpha > 0$ *such that*

- F_{i_0} *and* f_{j_0} *are lower semicontinuous functions;*
- $\forall x, y, \ \ f_{j_0}(x) = f_{j_0}(y) \Longrightarrow x = y;$
- $F_{i_0} = \alpha f_{j_0};$

Then $E(\widetilde{f}, \ Z, \ \precsim_{K_4}) \cap E(F, \ Z, \ \precsim_{K_1}) \neq \emptyset$.

Proof:

Let us suppose that (i) and (ii) hold. Consider the following optimization problem:

$$min\{f_{j_0}(z), \ z \in Z\} \ \ (pb1).$$

Since f_{j_0} is a lower semicontinuous function and Z is a nonempty compact set, $pb1$ has at least one optimal solution, let z_0 be such a solution (in fact z_0 is unique). We claim that $z_0 \in E(\widetilde{f}, \ Z, \ \precsim_{K_4}) \cap E(F, \ Z, \ \precsim_{K_1})$.

(a) Let us first show that $z_0 \in E(\widetilde{f}, \ Z, \ \precsim_{K_4})$. Suppose that $z_0 \notin E(\widetilde{f}, \ Z, \ \precsim_{K_4})$, then there exists $z' \in Z$, $z' \neq z_0$ such that $\widetilde{f}(z') \leq \widetilde{f}(z_0)$ and $\widetilde{f}(z') \neq \widetilde{f}(z_0)$. This implies that $f_{j_0}(z') \leq f_{j_0}(z_0)$. Due to the fact that z_0 is an optimal solution of $pb1$, one has $f_{j_0}(z') = f_{j_0}(z_0)$. The second condition of hypothesis (ii) implies that $z' = z_0$. This is a contradiction. Consequently $z_0 \in E(\widetilde{f}, \ Z, \ \precsim_{K_4})$.

(b) Since $F_{i_0} = \alpha f_{j_0}$ and $\alpha > 0$, z_0 is also an optimal solution of $min\{F_{i_0}(z), \ z \in Z\}$. With a similar proof as in (a), one obtains that $z_0 \in E(F, \ Z, \ \precsim_{K_1})$. Combining (a) and (b) leads to $z_0 \in E(\widetilde{f}, \ Z, \ \precsim_{K_4}) \cap E(F, \ Z, \ \precsim_{K_1})$, and hence $E(\widetilde{f}, \ Z, \ \precsim_{K_4}) \cap E(F, \ Z, \ \precsim_{K_1}) \neq \emptyset$ □

If conditions (i) and (ii) of Theorem 5.3.5 are fulfilled, then one can think of implementing an algorithm to generate efficient points of $BMPP'$ based on *Corollary* 5.3.3. At least two ideas can be used.

The first idea could be to generate the whole set of efficient points of $MPP1$ and then iterate on this set in order to retain points that are also Pareto-optimal for the second problem $MPP2$. But it would be a difficult task to generate the whole efficient set of $MPP1$ [Gal 1977, Iserman 1977, Pieume et al. 2008b, Saaty 2005].

The second idea could be to progressively generate efficient points for $MPP1$ and simultaneously test their efficiency for $MPP2$ before moving to another efficient point of $MPP1$. This seems to be more practical. An approach for generating efficient solutions for linear bilevel multi-objective programming problems based on this idea is presented in Algorithm 5.1.

5.3.2 Generating efficient solutions for bilevel linear multi-objective programming problems

5.3.2.1 Problem and implementation

We consider the optimistic formulation $(BLMPP')$ of the following problem:

$$\min_{x\in\mathbb{R}_+^{n_1}} \; F(x,y) = (C_1(x,y), C_2(x,y), ..., C_{m_1}(x,y))$$

$$\text{subject to} \begin{cases} A_1x \leq b_1 \\ \\ \text{y solves} \begin{cases} \min_{y\in\mathbb{R}_+^{n_2}} f(x,y) = (c_1(x,y), c_2(x,y), ..., c_{m_2}(x,y)) \\ s.t \;\; A_2x + A_3y \leq b_2 \end{cases} \end{cases} \tag{BLMPP}$$

The two linear multi-objective programming problems to be used are:

$$\min_{x,y} \; F(x,y) = (C_1(x,y), C_2(x,y), ..., C_{m_1}(x,y))$$

$$s.t \begin{cases} A_1x \leq b_1 \\ A_2x + A_3y \leq b_2 \\ x \geq 0, y \geq 0 \end{cases} \tag{LMPP1}$$

and

$$\min_{x,y} \widetilde{f}(x,y) = (c_1(x,y), c_2(x,y), ..., c_{m_2}(x,y), x)$$

$$s.t \begin{cases} A_1x \leq b_1 \\ A_2x + A_3y \leq b_2 \\ x \geq 0, y \geq 0 \end{cases} \tag{LMPP2}$$

Given $x \in \mathbb{R}_+^{n_1}$ and $y \in \mathbb{R}_+^{n_2}$, we introduce the vector $z \in \mathbb{R}_+^{n_1+n_2}$ whose first n_1 components are $x_1, x_2, ..., x_{n_1}$ and last n_2 components are $y_1, y_2, ..., y_{n_2}$.

We call C (resp c) the matrix such that $F(z) = (C_1z, C_2z, ..., C_{m_1}z) = Cz$ (resp $\widetilde{f}(z) = (c_1z, c_2z, ..., c_{m_2}z) = cz$). In order to be more concise, the notations and results that follow now are with respect to $LMPP1$, but all are also valid for $LMPP2$.

The data of $LMPP1$ can be represented by the following tableau:

(T) :

	z^N	z^B
d	C	0
b	A	I

Where $A = \begin{pmatrix} 0 & A_1 \\ A_2 & A_3 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

The algorithm starts with an initial efficient extreme point and iterates through different efficient extreme points of $LMPP1$. At each iteration, for every unexplored efficient extreme point z of $LMPP1$, the efficiency with respect to $LMPP2$ is tested. Points that are Pareto-optimal for both problems are retained. The entire scheme is repeated, until all the efficient extreme solutions of $LMPP1$ are tested. Since efficient extreme points form a connected graph and $LMPP1$ has a finite number of extreme efficient points, the algorithm will terminate in a finite number of steps.

At each iteration of the algorithm, the current efficient (extreme) point z^* is always associated with a tableau T (as presented above). NT denotes the nonbasic set of the current efficient point z^* associated with the tableau T. S_1 represents the set of Pareto-optimal points of the problem $LMPP1$ whose efficiency with respect to $LMPP2$ must be tested. S_2 represents the set of points found to be Pareto-optimal to both problems.

The result below [Ecker and Kouada 1978, Ecker et al. 1980] is the scheme used to check if an admissible point is a Pareto-optimal solution.

Lemma 5.3.1. *A point z^0 in Z is Pareto-optimal for problem $LMPP1$ if and only if the solution $(\overline{z}, \overline{s})$ of the following linear program yields to a maximum value zero:*

$$\max_{z \in Z,\, s \geq 0} e^t s$$
$$s - t \ \ Cz + Is = Cz_0$$

If the maximum value is not zero, then $\overline{z}$ is Pareto-optimal (with respect to $LMPP1$).

The following result of [Ecker et al. 1980] can be used to determine incident efficient edges.

Lemma 5.3.2. *The edge incident to the current efficient extreme point obtained by increasing the nonbasic variable z_j^N is efficient if an only if the system*

$$-Cv + w^N = C^t e \text{ has a solution } (v, w^N) \geq 0 \text{ with } w_j^N \geq 0 \tag{5.3.2}$$

provided the pivot in the z_j^N column has a pivot row (say row i) with $b_i > 0$. If $b_i = 0$ and $a_{ij} < 0$ for some j, then the edge obtained by pivoting using the minimum ratio rule is efficient if and only if (5.3.2) holds.

Based on this lemma, [Ecker and Song 1994] showed that one can implement this test for each edge by solving the following linear program:

$$\begin{aligned} &\max w_j^N \\ &s.t \\ &-C^t v + I w^N = C^t e \\ &v, w^N > 0 \end{aligned} \tag{5.3.3}$$

If the optimal value is zero, then the corresponding edge is efficient.

The following algorithm (Algorithm 5.1) exploits lemma 5.3.1, lemma 5.3.2 and formulation 5.3.3 to find efficient solutions to $BLMPP'$.

Algorithm 5. 1

1: Read data of problem $BLMPP$. Construct problems $LMPP1$, $LMPP2$ and let $S_1 := \emptyset$, $S_2 := \emptyset$.

2: Find an arbitrary admissible element $z_0 \in Z$.

3: Construct and solve the following problem (in order to find an initial efficient point to $LMPP1$)

$$\begin{aligned} &\max e^t s \\ &s.t \\ &Cz + Is = Cz_0 \\ &z \in Z \\ &s \geq 0 \end{aligned} \tag{Test0}$$

Let $(\overline{z}, \overline{s})$ the optimal solution of $Test0$.

If the optimum value of $(Test0)$ is 0, then z_0 is efficient, $S_1 := \{z_0\}$.

Otherwise, $\overline{z}$ is efficient, $S_1 := \{\overline{z}\}$.

4: If $S_1 = \emptyset$, stop. Otherwise, take an element z^* from S_1, $S_1 := S_1 \setminus \{z^*\}$.

5: Test if z^* is efficient for $LMPP2$ by solving the following problem:

$$\begin{aligned} &\max e^t s \\ &s.t \\ &cz + Is = cz^* \\ &z \in Z \\ &s \geq 0 \end{aligned} \tag{Test1}$$

Let $(\overline{z}, \overline{s}) = (\overline{x, y}, \overline{s})$ the optimal solution of $Test1$.

If the optimal value of $(Test1)$ is 0 then z^* is efficient to $LMPP2$ and hence is a solution of $BLMPP'$: $S_2 := S_2 \cup \{z^*\}$.

6: Find efficient edges (with respect to $LMPP1$) incident to z^*, by solving for each $j \in NT$ the problem (5.3.3).

Let $JT \subset NT$ the set of j such that the optimal value of PB_j is zero. (Let us recall that the edge obtained by increasing the nonbasic variable z_j^N in T is efficient if and only if $j \in JT$.)

7: For each $j \in JT$, generate efficient points incident to z^* by making a pivot operation on the column $j+1$ of tableau T. Let ST the set of efficient points incident to the current efficient

point z^* that have not yet been tested.
$S_1 = S_1 \cup ST$.
go to step 4

Remark 5.3.1. This algorithm is an implementation for linear problems of the second idea for generating efficient points discussed at the end of section 5.3.1. Algorithm 5.1 is applicable if conditions of Theorem 5.3.5 are fulfilled. The algorithm is certain to generate at least one efficient point for $BLMPP'$, but cannot assure the generation of all the desired efficient points.

5.3.2.2 Illustration of Algorithm 5.1

Let us consider the problem of finding efficient solutions to the optimistic formulation ($BLMPP'$) of the following problem:

$$\begin{aligned}
&\min \{-x_1 + 2x_3,\ x_1 - x_3,\ -x_1 - 2x_2\}\\
&\text{subject to} \begin{cases} x_1 + x_2 \leq 1 \\ x_2 \leq 2 \\ x_1 \geq 0,\ x_2 \geq 0 \\ x_3 \text{ solves } \begin{cases} \min \{-\frac{1}{2}x_1 + x_3,\ 2x_1 + x_2 + 2x_3\} \\ x_1 - x_2 + x_3 \leq 4 \\ x_3 \geq 0 \end{cases} \end{cases}
\end{aligned} \tag{BLMPP}$$

At Step 1: The following two problems are constructed:

$$\begin{aligned}
&\min \{-x_1 - 2x_2,\ -x_1 + 2x_3,\ x_1 - x_3\}\\
&\text{subject to} \begin{cases} x_1 + x_2 \leq 1 \\ x_2 \leq 2 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{cases}
\end{aligned} \tag{LMPP1}$$

and

$$\begin{aligned}
&\min \{-\tfrac{1}{2}x_1 + x_3,\ 2x_1 + x_2 + 2x_3,\ x_1,\ x_2\}\\
&\text{subject to} \begin{cases} x_1 + x_2 \leq 1 \\ x_2 \leq 2 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{cases}
\end{aligned} \tag{LMPP2}$$

At step 2: We start with $z_0 = (0,0,0) \in Z$

At step 3: The following problem is constructed in order to find an initial efficient extreme point (with respect to $LMPP1$):

$$\max\ s_1 + s_2 + s_3$$
$$\text{subject to} \begin{cases} -x_1 - 2x_2 + s_1 = 0 \\ -x_1 + 2x_3 + s_2 = 0 \\ x_1 - x_3 + s_3 = 0 \\ x_1 + x_2 \leq 1 \\ x_2 \leq 2 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0 \end{cases} \tag{Test0}$$

The optimal simplex tableau of the problem is:

2	1	0	1	2	0	0	0	0	0
2	1	0	0	2	0	0	1	0	0
0	-1	0	2	0	0	0	0	1	0
0	1	0	-1	0	0	0	0	0	1
1	1	1	0	1	0	0	0	0	0
1	-1	0	0	-1	1	0	0	0	0
5	2	0	1	1	0	1	0	0	0

The solution of the problem is $(\overline{z}, \overline{s}) = (\overline{(0,1,0)}, \overline{(2,0,0)})$ and the optimal value of the problem is 2. Since the value is different from 0, $z_0 = (0,0,0)$ is not an efficient point to $LMPP1$. We deduce from this optimal tableau that $\overline{z} = (0,1,0)$ is an efficient extreme point to $LMPP1$ and the corresponding efficient tableau is given by T_1:

2	1	0	0	2	0	0
0	-1	0	2	0	0	0
0	1	0	-1	0	0	0
1	1	1	0	1	0	0
1	-1	0	0	-1	1	0
5	2	0	1	1	0	1

(T_1)

So $S_1 = \{(0,1,0)\}$.

At step 4: Since $S_1 \neq \emptyset$, we take $(0,1,0)$ and remove it from S_1. S_1 becomes empty.

At Step 5: We must test if $(0,1,0)$ is efficient to $LMPP2$ by solving the following problem:

$$\max\ s_1+s_2+s_3+s_4$$
$$\text{subject to} \begin{cases} -\frac{1}{2}x_1+x_3+s_1=0 \\ 2x_1+x_2+2x_3+s_2=1 \\ x_1+s_3=0 \\ x_2++s_4=1 \\ x_1+x_2\leq 1 \\ x_2\leq 2 \\ x_1-x_2+x_3\leq 4 \\ x_1\geq 0,\ x_2\geq 0,\ x_3\geq 0,\ s_1\geq 0,\ s_2\geq 0,\ s_3\geq 0,\ s_4\geq 0 \end{cases} \tag{Test1}$$

The optimal simplex tableau of $Test1$ is:

2	2.5	2	3	0	0	0	0	0	0	0
0	-0.5	0	1	0	0	0	1	0	0	0
1	2	1	2	0	0	0	0	1	0	0
0	1	0	0	0	0	0	0	0	1	0
1	0	1	0	0	0	0	0	0	0	1
2	0	1	0	0	1	0	0	0	0	0
4	1	- 1	1	0	0	1	0	0	0	0

Since the optimal value is 2, so different from zero, $z_0=(1,0,1)$ is not efficient to $LMPP2$. We continue to step 6.

At step 6: We use T_1 to find incident edges to $z_0=(1,0,1)$. Here the set of non-basis variables is given by $NT_1=\{1,3,4\}$. We solve for different j in NT_1 the problem (PB_j) as presented in (5.3.3). For $j=1$, PB_1 is:

$$\begin{aligned} &\max w_j^N \\ &s-t \\ &-v_1+v_2-v_3+w_1^N=1 \\ &-2v_1+w_3^N=2 \\ &-2v_2+v_3+w_4^N=1 \\ &v,w^N>0 \end{aligned} \tag{PB_1}$$

The optimal value is 0. Similarly, for $j=3$, the optimal value of PB_3 is 2; for $j=4$, the optimal value of PB_4 is 0. So $JT_1=\{1,4\}$. We go to step 7.

At step 7: Only $j = 1$ leads to a new efficient extreme point. The new efficient point obtained is $(1, 0, 0)$ with the corresponding tableau given by (T_2):

1	0	- 1	0	1	0	0
1	0	1	2	1	0	0
- 1	0	- 1	- 1	- 1	0	0
1	1	1	0	1	0	0
2	0	1	0	0	1	0
3	0	- 2	1	- 1	0	1

(T_2)

The set of nonbasic variables is given by $NT_2 = \{2, 3, 4\}$. One has then $ST_2 = \{(1, 0, 0)\}$ and hence $S_1 = \{(1, 0, 0)\}$. We go again to step 4.

At step 4: Since $S_1 \neq \emptyset$, we take $(1, 0, 0)$ and remove it from S_1. S_1 becomes empty. We go to step 5.

At Step 5: We must test if $(1, 0, 0)$ is efficient to $LMPP2$ by solving the following problem:

$$\max\ s_1 + s_2 + s_3 + s_4$$
$$\text{subject to} \begin{cases} -\frac{1}{2}x_1 + x_3 + s_1 = -\frac{1}{2} \\ 2x_1 + x_2 + 2x_3 + s_2 = 2 \\ x_1 + s_3 = 1 \\ x_2 + s_4 = 2 \\ x_1 + x_2 \leq 1 \\ x_2 \leq 2 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0,\ s_4 \geq 0 \end{cases} \qquad \text{(Test1)}$$

The optimal value of $Test1$ is 0, so $(1, 0, 0)$ is efficient to $LMPP2$ and hence is an efficient solution to $BLMPP'$: $S_2 = \{(1, 0, 0)\}$. We continue to step 6.

At step 6: We find efficient extreme points incident to our current efficient point $z^* = (1, 0, 0)$, using T_2 and NT_2 by solving for each j in NT_2 the problem PB_j.

For $j = 2$, one obtains 2 as optimal value of PB_2; for $j = 3$ the optimal value of PB_3 is 0; for $j = 4$, the optimal value of PB_4 is 2. So $JT_2 = \{3\}$. We go to step 7.

At step 7: $j = 3$ leads to a new efficient extreme point $(1, 0, 3)$ with the corresponding tableau

T_3:

1	0	- 1	0	1	0	0
- 5	0	5	0	3	0	- 2
2	0	- 3	0	- 2	0	1
1	1	1	0	1	0	0
2	0	1	0	0	1	0
3	0	- 2	1	- 1	0	1

(T_3)

The set of nonbasic variables is given by $NT_3 = \{2, 4, 6\}$. One has then $ST_3 = \{(1, 0, 3)\}$ and hence $S_1 = \{(1, 0, 3)\}$. We go again to step 4.

At step 4: Since $S_1 \neq \emptyset$, we take $(1, 0, 3)$ and remove it from S_1. S_1 becomes empty. We go to step 5.

At step 5: We must test if $(1, 0, 3)$ is efficient to $LMPP2$ by solving the following problem:

$$\max\ s_1 + s_2 + s_3 + s_4$$
$$\text{subject to} \begin{cases} -\frac{1}{2}x_1 + x_3 + s_1 = \frac{5}{2} \\ 2x_1 + x_2 + 2x_3 + s_2 = 8 \\ x_1 + s_3 = 1 \\ x_2 + +s_4 = 0 \\ x_1 + x_2 \leq 1 \\ x_2 \leq 2 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0,\ s_4 \geq 0 \end{cases} \quad \text{(Test1)}$$

The optimal value of $Test1$ is 11.5, so $(1, 0, 3)$ is not efficient for $LMPP2$ and hence is not a solution to $BLMPP'$. We continue to step 6.

At step 6: We find efficient extreme points incident to the current efficient point $z^* = (1, 0, 3)$ by solving for each j in NT_3 the problem (PB_j).

For $j = 2$, the optimal value of the problem PB_2 is 2. For $j = 4$ and $j = 6$, the optimal values of the obtained problems are 0 and so $JT_3 = \{4, 6\}$. We go to step 7.

At step 7: One finds that it is only $j = 4$ that leads to a new efficient extreme point $(0, 0, 4)$

with the corresponding tableau T_4:

0	- 1	- 2	0	0	0	0
- 8	- 3	2	0	0	0	- 2
4	2	- 1	0	0	0	1
1	1	1	0	1	0	0
2	0	1	0	0	1	0
4	1	- 1	1	0	0	1

(T_4)

The set of nonbasic variables is given by $NT_4 = \{1, 2, 6\}$. Thus $ST_4 = \{(0, 0, 4)\}$ and hence $S_1 = \{(0, 0, 4)\}$. We go again to step 4.

At step 4: Since $S_1 \neq \emptyset$, we take $(0, 0, 4)$ and remove it from S_1 (so $S_1 = \emptyset$). We go to step 5.

At step 5: We must test if $(0, 0, 4)$ is efficient to $LMPP2$ by solving the following problem:

$$\max \; s_1 + s_2 + s_3 + s_4$$
$$\text{subject to} \begin{cases} -\frac{1}{2}x_1 + x_3 + s_1 = 4 \\ 2x_1 + x_2 + 2x_3 + s_2 = 8 \\ x_1 + s_3 = 0 \\ x_2 + +s_4 = 0 \\ x_1 + x_2 \leq 1 \\ x_2 \leq 2 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \; s_1 \geq 0, \; s_2 \geq 0, \; s_3 \geq 0, \; s_4 \geq 0 \end{cases} \tag{Test1}$$

The optimal value of $Test1$ is 12, so $(0, 0, 4)$ is not efficient for $LMPP2$ and hence is not a solution to $BLMPP'$. We continue to step 6.

At step 6: We find efficient extreme points incident to the current efficient point $z^* = (0, 0, 4)$ by solving for each j in NT_4 the problem (PB_j).

For all j in NT_4, 0 is the optimal value of PB_j. So $JT_4 = \{1, 2, 6\}$. We go to step 7.

At step 7: One finds that only $j = 2$ leads to a new efficient extreme point $(0, 1, 5)$ with the corresponding tableau T_5:

2	1	0	0	2	0	0
- 10	- 5	0	0	- 2	0	- 2
5	3	0	0	1	0	1
1	1	1	0	1	0	0
1	- 1	0	0	- 1	1	0
5	2	0	1	1	0	1

(T_5)

The set of nonbasic variables is given by $NT_5 = \{1, 4, 6\}$. So $ST_5 = \{(0, 1, 5)\}$ and hence $S_1 = \{(0, 1, 5)\}$. We go to step 4.

At step 4: Since $S_1 \neq \emptyset$, we take $(0, 1, 5)$ and remove it from S_1 (so $S_1 = \emptyset$). We go to step 5.

At step 5: We test if $(0, 1, 5)$ is efficient to $LMPP2$ by solving the following problem:

$$\max\ s_1 + s_2 + s_3 + s_4$$
$$\text{subject to} \begin{cases} -\frac{1}{2}x_1 + x_3 + s_1 = 5 \\ 2x_1 + x_2 + 2x_3 + s_2 = 11 \\ x_1 + s_3 = 0 \\ x_2 + +s_4 = 1 \\ x_1 + x_2 \leq 1 \\ x_2 \leq 2 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0,\ s_4 \geq 0 \end{cases} \tag{Test1}$$

The optimal value of $Test1$ is 17. So $(0, 1, 5)$ is not efficient for $LMPP2$ and hence is not a solution to $BLMPP'$. We continue to step 6.

At step 6: We find efficient extreme points incident to the current efficient point $z^* = (0, 1, 5)$ by solving (PB_j) for each j in NT_5.

For $j = 1$, 1.5 is the optimal value of PB_1; for $j = 4$ the optimal value of PB_4 is 2; for $j = 6$, the optimal value of PB_6 is 0. So $JT_5 = \{6\}$. We go to step 7.

At step 7: $j = 6$ leads to an efficient extreme point already tested (that is $(0, 1, 0)$). We go to step 4.

At step 4: S_1 is empty and the algorithm stops.

Since $S_2 = \{(1, 0, 0)\}$, we deduce that $(1, 0, 0)$ is an efficient solution to $BLMPP'$.

5.4 Solving bilevel linear multiobjective programming problems

5.4.1 The problem considered

The focus here is on the linear formulation of BMPP given as follows:

$$\min_{x \in \mathbb{R}_+^{n_1}} \; F(x,y) = (C_1(x,y), C_2(x,y), ..., C_{m_1}(x,y))$$

$$\text{subject to} \begin{cases} A_1 x \leq b_1 \\ \text{y solves} \begin{cases} \min_{y \in \mathbb{R}_+^{n_2}} f(x,y) = (c_1 y, c_2 y, ..., c_{m_2} y) \\ s.t \\ A_2 x + A_3 y \leq b_2 \end{cases} \end{cases} \qquad \text{(BLMPP)}$$

where $C_i, i = 1, ...m_1$ are $n_1 + n_2$-dimensional constant row vectors; $c_i, i = 1, ...m_2$ are n_2-dimensional constant row vectors; b_1 is a p-dimensional constant column vector and b_2 is a q-dimensional constant column vector; A_1, A_2 and A_3 are respectively $p \times n_1$; $q \times n_1$ and $q \times n_2$ constant matrices.

Let us denote by $R(x)$, the set of rational responses of the follower for each decision x of the leader. It is defined as the Pareto-optimal point of the following problem:

$$\min_{y \in \mathbb{R}_+^{n_2}} f(x,y) = (c_1 y, c_2 y, ..., c_{m_2} y)$$

$$s.t \;\; A_3 y \leq b_2 - A_2 x$$

With this notation, one has the following formulation of BLMPP:

$$\min_{x \in X} \; F(x,y) = (C_1(x,y), C_2(x,y), ..., C_{m_1}(x,y))$$

$$\text{subject to} \begin{cases} A_1 x \leq b_1 \\ y \in R(x) \end{cases}$$

Using the following representation for the feasible space of BLMPP:

$$\Omega = \{ \; (x,y) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2} \; / \; A_1 x \leq b_1 \text{ and } y \in R(x) \; \}$$

One obtains the following optimistic formulation of BLMPP:

$$\min_{x,y} \; F(x,y) = (C_1(x,y), C_2(x,y), ..., C_{m_1}(x,y))$$

$$s.t \qquad \text{(BLMPP')}$$

$$(x,y) \in \Omega$$

The next section presents a new characterization of the feasible set of $BMPP'$. Throughout the rest of this chapter, Z will represent the set defined by: $Z = \{\ (x,y) \in \mathbb{R}_+^{n_1} \times \mathbb{R}_+^{n_2}\ /\ A_1x \leq b_1$ and $A_2x + A_3y \leq b_2\ \}$. It will be assumed that Z is a non-empty and bounded set over the convex polyhedron. As adopted in section 5.3, S will represent the efficient solutions set of the problem $BLMPP'$.

5.4.2 A characterization of the feasible set of BLMPP'

We introduce a multi-objective programming problem which is such that its efficient solutions set is equal to the feasible set of BLMPP'. A similar result was already developed in [Eichfelder 2008], but with a different multiobjective programming problem. The author considered the following multi-objective programming problem:

$$\begin{aligned} &\min_{x,y}\ \widetilde{f}(x,y) = (c_1y, c_2y, ..., c_{m_2}y, x) \\ &s.t \\ &A_1x \leq b_1 \\ &A_2x + A_3y \leq b_2 \\ &x \geq 0, y \geq 0 \end{aligned} \tag{MPP2}$$

By letting $K_1 = \mathbb{R}_+^{m_2} \setminus \{0_{m_2}\} \times \{0_{n_1}\}$ and Ω as defined in section 5.3, the author showed the following result:

Lemma 5.4.1. $\Omega = E(\widetilde{f},\ Z,\ \precsim_{K_1})$

The disadvantage of this result is that it is not easily applicable. In fact, there do not exist approaches developed in the literature for finding efficient points with respect to the cone $K_1 = \mathbb{R}_+^{m_2} \setminus \{0_{m_2}\} \times \{0_{n_1}\}$. Methods are usually for cones that have the form $\mathbb{R}_+^{n} \setminus \{0_{n}\},\ n \in \mathbb{N}$. It is the reason why in [Eichfelder 2008], the author approximated the efficient set of $MPP2$ by the weakly efficient set.

Here, we introduce a new relation that can be applied directly to find an optimal solution. Let us consider the following linear multi-objective programming problem:

$$\min_{x,y} H(x,y) = \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$s.t \qquad \text{(LMPP1)}$$
$$A_1 x \leq b_1$$
$$A_2 x + A_3 y \leq b_2$$
$$x \geq 0, y \geq 0$$

where c is a $m_2 \times n_2$ matrix; e is a vector having each entry equal to 1 and I is the $n_1 \times n_1$ identity matrix. Each c_i represents the row vector that defines the ith-objective function of the follower. Let $K_2 = \mathbb{R}_+^{m_2+n_1+1} \setminus \{0_{m_2+n_1+1}\}$, then the following result holds.

Theorem 5.4.1. $\Omega = E(H,\ Z,\ \precsim_{K_2})$

Proof:

$\Longleftarrow$ Let us show that $E(H,\ Z,\ \precsim_{K_2}) \subset \Omega$

Let $z = (x_0,\ y_0) \in E(H,\ Z,\ \precsim_{K_2})$, from the definition of $E(H,\ Z,\ \precsim_{K_2})$, one has $A_2x_0 + A_3y_0 \leq b_2$ and $A_1x_0 \leq b_1$. So, in order to show that $z \in \Omega$, it suffices to show that $y_0 \in R(x_0)$. Let us suppose the contrary. Then there exists $\overline{y}$ such that: (i) $A_2x_0 + A_3\overline{y} \leq b_2$ and (ii) $\overline{y}$ dominate y_0. Relation (ii) is equivalent to $(c_1\overline{y}, c_2\overline{y}, ..., c_{m_2}\overline{y}) \leq (c_1y_0, c_2y_0, ..., c_{m_2}y_0)$ with at least one $k \in \{1, ..., m_2\}$ such that $c_k\overline{y} < c_ky_0$.

Let us now consider the point $z^* = (x_0, \overline{y})$, we have:

$$Hz^* = \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ \overline{y} \end{pmatrix} = \begin{pmatrix} c\overline{y} \\ -x_0 \\ e^t x_0 \end{pmatrix} \text{ and } Hz = \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} cy_0 \\ -x_0 \\ e^t x_0 \end{pmatrix}$$

Due to relation (ii), one has $c\overline{y} \leq cy_0$ and $c\overline{y} \neq cy_0$, this implies that:

$$H\begin{pmatrix} x_0 \\ \overline{y} \end{pmatrix} \leq H\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ and } H\begin{pmatrix} x_0 \\ \overline{y} \end{pmatrix} \neq H\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

So $(x_0,\ \overline{y})$ dominates $(x_0,\ y_0)$ with respect to the cone $K_2 = \mathbb{R}_+^{m_2+n_1+1} \setminus \{0_{m_2+n_1+1}\}$, which contradicts the fact that (x_0, y_0) is a Pareto-optimal point with respect to the cone $K_2 = \mathbb{R}_+^{m_2+n_1+1} \setminus \{0_{m_2+n_1+1}\}$.

$\Longrightarrow$ Let us now show that $\Omega \subset E(H,\ Z,\ \precsim_{K_2})$

Suppose that there is $z = (x_0, y_0,) \in \Omega$ such that $z \notin E(H,\ Z,\ \precsim_{K_2})$. Then there must be a point

$z_1 = (x_1, y_1)$ such that Hz_1 dominates Hz. This implies that $Hz_1 \leq Hz$ and $Hz \neq Hz_1$. Using the structure of the matrix H, and the fact that $z = (x_0, y_0)$ and $z_1 = (x, y)$, one obtains:

$$\begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} cy_1 \\ -x_1 \\ e^T x_1 \end{pmatrix} \leq \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} cy_0 \\ -x_0 \\ e^t x_0 \end{pmatrix}$$

and then

$$\begin{pmatrix} cy_1 \\ -x_1 \\ e^t x_1 \end{pmatrix} \neq \begin{pmatrix} cy_0 \\ -x_0 \\ e^t x_0 \end{pmatrix}$$

This implies that: $x_1 - x_0 \geq 0$ and $e^t(x_1 - x_0) \leq 0$ which means that $x_1 = x_0$ and also that $e^t x_1 = e^t x_0$. It follows that $cy_1 \leq cy_0$ and $cy_1 \neq cy_0$. Thus y_1 dominates y_0. Contradicting the fact that $y_0 \in R(x_0)$. □

From theorem 5.4.1, one can deduce that solving the problem $(BLMPP')$ is equivalent to solving:

$$\begin{aligned} &\min_{x,y} \ F(x,y) = (C_1(x,y), C_2(x,y), ..., C_{m_1}(x,y)) \\ &s.t \\ &(x,y) \in E(H,\ Z,\ \precsim_{K_2}) \end{aligned} \qquad \text{(BLMPP'')}$$

The theorem 5.4.1 leads to the following corollary.

Corollary 5.4.1. *$S = E(F,\ E(H,\ Z,\ \precsim_{K_2}),\ \precsim_{K_1})$ where $K_1 = \mathbb{R}_+^{m_1} \setminus \{0_{m_1}\}$ and $K_2 = \mathbb{R}_+^{m_2+n_1+1} \setminus \{0_{m_1+m_2+1}\}$.*

We provide in the next section, some discussions on the use of this last result for solving BLMPP'.

5.4.3 Two approaches for solving BLMPP'

5.4.3.1 First approach

Suppose that the upper decision maker is fully knowledgeable of all his preferences. One could then aggregate the leader objective functions using the weights $\lambda_1, \lambda_2 ... \lambda_{m_1}$ that measure his preferences concerning different objective functions. Solving BLMPP' will then be equivalent to solving the following problem:

$$\min_{x,y} \sum_{i=1}^{m_1} \lambda_{m_i} C_i(x,y)$$
$$s.t \ \ (x,y) \in E(H,\ Z,\ \precsim_{K_2})$$

which is an optimization of a linear function over a Pareto-optimal set. There are several methods developed in the literature that are devoted to the optimization of a linear function over an efficient set (see [Ecker and Song 1994] or the survey presented by [Yin 2000]). Any of these approaches can be applied.

5.4.3.2 Second approach

The second approach carried out in algorithm 5.2 below, could be to generate a representative subset of $E(H,\ Z,\ \precsim_{K_2})$ using well known schemes in [Hong et al. 2005, Mattson et al. 2004, Shi and Xia 2001]. Then, compute the image of the obtained subset by the leader objective functions and select elements that lead to non-dominated points for the leader. Algorithm 5.2 below is an illustration of this idea.

Algorithm 5. 2

1: Construct the following linear multiobjective programming problem:

$$\min_{x,y} H(x,y) = \begin{pmatrix} 0 & c \\ -I & 0 \\ e^t & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$s.t \qquad \text{(LMPP1)}$$
$$A_1 x \le b_1$$
$$A_2 x + A_3 y \le b_2$$
$$x \ge 0, y \ge 0$$

2: Compute a representative subset (call S) of the efficient set of $LMPP1$ using for instance, any of the methods developed in ([Mattson et al. 2004, Messac and Mattson 2002, Saying 1996]).

3: Compute the image set Y of S given by $Y = F(S)$.

4: Find Non-dominated points of Y (Y_{eff}) with respect to F.

5: Find the set X_E of Pareto-optimal points corresponding to Y_eff.

6: X_E is a representative subset of the efficient set of $BMLPP'$, STOP.

Remark 5.4.1. The Pareto-filter approach presented in [Mattson et al. 2004] can be used in step 4 and step 5.

5.5 Conclusion

This chapter has presented the optimistic formulation of a bilevel multi-objective programming problem ($BMPP$). We derived two multi-objective programming problems such that any point that is efficient for both problems is an efficient solution of the optimistic formulation of the considered $BMPP$. We proposed an approach to generate efficient solutions of the optimistic formulation of $BMPP$ and applied it to the resolution of the linear case. We also provided a necessary and sufficient condition under which the proposed algorithm is applicable. In addition, we proved that the admissible set of a bilevel linear multiobjective programming problem is equal to the set of efficient points of an artificial linear multiobjective programming problem. Based on this result, we proposed two approaches for generating efficient solutions when solving the optimistic formulation of a linear bilevel multiobjective programming problem.

General conclusion

This thesis investigated on multiobjective optimization and bilevel optimization. The main results obtained fall into three major parts:

We first developed methods for solving linear bilevel and multiobjective optimization problems [Pieume et al. 2008a;b]. We presented a new approach for solving Multiobjective Linear Programming Problems. We first developed a new characterization of efficient faces incident to efficient extreme points and showed how it can be used to construct these efficient faces. Secondly, we proposed a procedure to find all efficient faces incident to an extreme efficient point. Since efficient extreme points form a connected graph, the main algorithm iterates through these efficient extreme points, and for every unexplored point, computes all the efficient points and all the efficient extreme points incident to it. We then showed, using the concept of ideal basis (ideal tableau) of [Armand 1991; 1993] and the concept of $\lambda - efficient$ set introduced in the thesis, that the entire set of efficient points of a multiobjective linear programming problem can be obtained by using a revised version of efficient face characterization given by [Ecker and Kouada 1978, Ecker et al. 1980].

Still in part one, we developed a method for finding a global solution of some classes of bilevel linear programming problem $(BLPP)$. The approach is based on the basic idea that extreme points of the admissible region of BLPP are extreme points for the feasible space of the problem, and the optimal solution is one of these vertices. We then developed pivot techniques to find the optimal solution on an expanded tableau (simplex tableau) that represents the data of the BLPP. Experimental results showed that on all problems considered, our algorithm always finds the global optimal solution. We applied our algorithm to two problems incorrectly solved in the literature to find correct solutions.

The second part of our contributions is related to the applications of these two classes of optimization problems [Pieume and Fotso December 2005, Pieume et al. 22-26 May 2005]. We showed that, the mathematical formalization of the problem of planning of the distribution of electrical energy in Cameroon is a multiobjective programming problem. We discussed approaches for solving the obtained model. The complete procedure of the GAMS model that implements an aggregate approach was provided. Similarly, with the hypothesis that it is an obligation for a

State to protect local initiatives, a strategy that leads to the resolution of a bilevel programming problem was developed and proposed for the protection of national initiatives in the context of globalization.

The third part of our work dealt with links between bilevel optimization and multiobjective optimization [Pieume et al. 2009; 2010a;b]. We started by studying the question of the Pareto optimality of the optimal solution of a bilevel programming problem. We carried out a post optimal analysis approach and presented a technique that permits to generate a Pareto optimal solution better than the optimal solution of a BPP. We introduced a generalization of the Fulop relation [Fulop 1993] that establishes a link between multiobjective linear programming and bilevel linear programming. We showed that, under the assumptions that the follower admissible set is bounded and its constraints functions are continuous, the relation remains valid for some specific non linear problems. In addition, we showed that, under certain conditions, solving a bilevel linear programming problem is equivalent to solving two artificial linear multiobjective programming problems. This third part continued with the focus on bilevel multi-objective programming problems ($BMPPs$). Given a $BMPP$, we showed how to construct two artificial multi-objective programming problems such that any point that is efficient for both problems is an efficient solution of the $BMPP$. Some necessary and sufficient conditions under which the obtained result is applicable were provided. Concerning the particular case of BLMPP, we introduced an artificial multi-objective linear programming problem whose resolution can permit to generate the whole set of feasible points of the upper level decisions. Based on this result and depending on whether the leader can evaluate his preferences for his different objective functions or not, two approaches for obtaining Pareto-optimal solutions were developed. The first approach aggregates the leader objective functions and suggests the use of a technique of optimization of linear function over an efficient set, in order to find an optimal solution. The second approach uses a Pareto-filter schemes to find an approximated discrete representation of the efficient set. The second approach has the advantage of keeping the multi-criteria concept of the upper DM, while the first one uses an aggregation process to eliminate the multi-criteria concept for the leader.

Even if all the developments presented here are applicable to small instances of problems, the work carried out in this thesis and the results obtained so far are a source of encouragement for further studies, that may include:

- Finding of less restrictive conditions than the ones proposed in theorems 5.3.5;

- Finding of conditions under which the approach developed in section 4.4 for generating bilevel optimization optimal solutions based on multiobjective techniques is applicable. In our research, we just proposed conditions under which it is not applicable;

- Finding of more conditions that can guarantee that the optimal solution to the bilevel optimization problem is Pareto optimal. The ones presented in section 4.2.2 are too restrictive and seem to be valid only in the optimistic case;

- The search of what the new relation presented in section 4.1.1 (corollary 4.4.3) between bilevel optimization and multiobjective programming can lead to, when it is applied to particular classes of problems (linear, quadratic, convex...);

- The improvement of the performance of algorithms presented and the study of the complexities of these algorithms. It might be interesting to compare algorithms presented in the text with those developed in the literature;

- The derivation of Pareto optimality conditions for bilevel multiobjective programming by applying KKT Pareto-optimality conditions , as the one developed in [Jimenez and Novo 2002, Majumdar 1997], to the lower DM multiobjective optimization problem;

- Revisiting of some classical bilevel optimization or multiobjective optimization problems to try to model them as bilevel multiobjective optimization problems. The obtained models could be more realistic. For example, the bilevel transportation problem [Yin 2000, Winston 1994] could be modelled as follows:

  1. first level: minimization of the costs and time on the road;
  2. second level: minimization of the costs and minimization of the degradation of perishable products.

  while the bilevel supply chain planning, could be modelled as follows:

  1. first level: the distribution company minimizes transportation costs and maximizes retailers profits;
  2. second level: manufacturing plants minimize their own operation costs.

This research also presented an open issue, the technique used to find the optimal solution in bilevel linear problem seems to be a good way to solve multilevel programming problems with more than two players. Hopefully, this study will spur the interest in handling bilevel and multiobjective optimization to other interested researchers and practitioners.

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