

## HESSIAN OF THE METRIC FORM ON TWISTOR SPACES

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**ABSTRACT.** We compute the hessian  $id'd''\mathbb{W}$  of the natural metric form  $\mathbb{W}$  on the twistor space  $\mathbb{T}(M, g)$  of a 4-dimensional Riemannian manifold  $(M, g)$ . We then adapt the computations to the case of the twistor space  $\mathbb{T}(M, g, D)$  of a hyperkähler manifold  $(M, g, D = (I, J, K))$ . We show a strong positivity property of the hessian  $id'd''\mathbb{W}$  on the twistor space  $\mathbb{T}(M, g, D)$  and prove, as an application, a convexity property of the component of the twistor lines in the cycle space of  $\mathbb{T}(M, g, D)$ .

### 1. INTRODUCTION

The Penrose twistor construction is a way of translating a problem on a 4-dimensional Riemannian manifold  $(M, g)$  into a problem on a hermitian (almost-)complex 3-dimensional manifold, its twistor space  $\mathbb{T}(M, g)$ . An analogous construction can be done, starting with a hyperkähler manifold  $(M, g, D = (I, J, K))$ . The main drawback is that the twistor spaces  $\mathbb{T}(M, g)$  or  $\mathbb{T}(M, g, D)$  are almost never of Kähler type, even under vanishing assumptions for the curvature of  $g$ . This defect can be quantified, at least for the natural metric on the twistor space  $\mathbb{T}(M, g, D)$ , by the Kodaira-Spencer class (see proposition 3.2).

Our first aim is to compute the hessian  $id'd''\mathbb{W}$  of the natural metric form  $\mathbb{W}$  on the twistor space  $\mathbb{T}(M, g)$  of a 4-dimensional Riemannian manifold  $(M, g)$  and extend the computations to the case of the twistor space  $\mathbb{T}(M, g, D)$  of a hyperkähler manifold  $(M, g, D = (I, J, K))$ . This is done by a study of commutation relations of various vector fields on the twistor spaces.

We then show, as a substitute to the Kähler property, a strong positivity property of the hessian  $id'd''\mathbb{W}$  of the natural metric form  $\mathbb{W}$  on the twistor space  $\mathbb{T}(M, g, D)$  of a hyperkähler manifold  $(M, g, D)$ .

Kobayashi conjectured that the canonical bundle of a projective manifold without non-constant entire curves would be ample. It would follow that every hyperkähler manifold contains non-constant entire curves, that is, in other words, is not Kobayashi hyperbolic. An approach to this problem has been initiated by Campana [5, 6]. It consists in deforming the rational twistor lines in  $\mathbb{T}(M, g, D)$  in order to produce an entire curve in  $(M, I)$ . We prove, as an application of the previous computations, a convexity property of the component of the twistor lines in the cycle space of  $\mathbb{T}(M, g, D)$ , that may be useful to control the deformations of twistor lines.

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## 2. TWISTOR SPACES OF 4-DIMENSIONAL RIEMANNIAN MANIFOLDS

**2.1. Constructions on  $\mathbb{R}^4$ .** An endomorphism  $A$  of the oriented euclidean real vector space  $\mathbb{R}^4$  is said to respect the orientation if for all vectors  $U, V \in \mathbb{R}^4$  the 4-tuple  $(U, AU, V, AV)$  is either linearly dependent or positively oriented. This will be denoted by  $A \gg 0$ . Examples are given by the following three orthogonal anti-involutive (hence anti-symmetric) endomorphisms

$$I = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The set  $F$  of complex structures on  $\mathbb{R}^4$  that respect the orientation and the euclidean product (called compatible complex structures) identifies with the sphere  $\{u = aI + bJ + cK / (a, b, c) \in \mathbb{S}^2\} \simeq \mathbb{S}^2$ . The standard metric  $g_0$  on the sphere reads on  $F$

$$g_0(V, W) = \frac{1}{2} \text{tr}(V^t W) = -\frac{1}{2} \text{tr}(VW), \quad \forall V, W \in T\mathbb{S}^2.$$

As the sphere  $\mathbb{S}^2$ , the set  $F$  inherits the complex structure of  $\mathbb{C}P^1$ . More precisely, at a point  $u \in F$ , the tangent space  $T_u F$  is  $\{V \in so(4) / Vu = -uV\}$  and the complex structure of  $F$  reads  $j \cdot V = uV$  as a matrix product.

This identification can be made intrinsic as follows. The euclidean product on  $\mathbb{R}^4$  gives an euclidean product on the exterior product  $\bigwedge^2 \mathbb{R}^4$ . The Hodge star operator decomposes  $\bigwedge^2 \mathbb{R}^4$  into  $\bigwedge^2 TM = \bigwedge^+ \oplus \bigwedge^-$ . An anti-symmetric endomorphism  $A$  of  $so(4)$  identifies with an element  $\phi(A)$  of  $\bigwedge^2 \mathbb{R}^4$  requiring

$$g(\phi(A), V \wedge W) = g(AV, W) \quad \forall V, W \in \mathbb{R}^4.$$

In particular, a compatible complex structure  $u$  identifies with an element  $\phi(u)$  precisely of the sphere of vectors of  $\bigwedge^+$  of norm  $\sqrt{2}$ . We will always identify  $\phi(u)$  and  $u$ , and  $\bigwedge^2 \mathbb{R}^4$  with  $so(4)$ .

**2.2. Constructions on a Riemannian 4-manifold.** Consider now a 4-dimensional oriented Riemannian manifold  $(M, g)$ . The twistor space of  $(M, g)$  is the fibre bundle  $\pi : \mathbb{T}(M, g) = \mathbb{T} \rightarrow M$  of vectors of  $\bigwedge^+ TM =: \bigwedge^+$  of norm  $\sqrt{2}$ , that fibre-wise identifies with the set of compatible complex structures on the tangent space of  $M$ . A natural Riemannian metric  $\mathbb{G}$  and a natural almost-complex structure  $\mathbb{J}$  are defined on the twistor space  $\mathbb{T}$  as follows. The bundle  $\mathcal{V}$  of vertical directions in  $T\mathbb{T}$  is the kernel of  $d\pi$ . Note that its structure group is  $SO(3) \subset PGL(2, \mathbb{C})$  so that fibres inherit complex structures and Riemannian metrics. The Levi-Civita connection  $\nabla^g$  of  $(M, g)$  provides us with a bundle  $\mathcal{H}$  of horizontal directions in  $T\mathbb{T}$  isomorphic via  $d\pi$  with  $TM$ , hence endowed with a complex structure and a Riemannian metric. The

decomposition  $T\mathbb{T} = \mathcal{H} \oplus \mathcal{V}$  is made  $\mathbb{G}$ -orthogonal and invariant by  $\mathbb{J}$ . The map  $\pi$  becomes a Riemannian submersion. We will use the notation  $\mathcal{H}(X) = \mathcal{H}_p(X)$  to denote the horizontal lift at  $p \in \pi^{-1}(m)$  of a vector  $X$  tangent to  $M$  at  $m$ , and likewise the notation  $\mathcal{H}(v)$  to denote the (orthogonal) projection of a vector  $v$  tangent to  $\mathbb{T}$  onto its horizontal part along the vertical direction.

Let  $\nabla^g$  be the Levi-Civita connection of  $(M, g)$ ,  $\eta$  its connection 1-form in a given frame with values in  $so(TM)$  and  $R$  its curvature tensor defined by  $R(X, Y)Z := [\nabla_Y^g, \nabla_X^g]Z + \nabla_{[X, Y]}^g Z$ . Recall that, with these conventions  $R(X, Y) = -(d\eta + \eta \wedge \eta)(X, Y)$ . As endomorphism of  $\Lambda^2 TM = \Lambda^+ \oplus \Lambda^-$  its decomposition is

$$R = \begin{bmatrix} W^+ + \frac{s}{12}Id & B \\ {}^t B & W^- + \frac{s}{12}Id \end{bmatrix} \text{ with } B : \begin{cases} \Lambda^+ \rightarrow \Lambda^- \\ \Lambda^- \rightarrow \Lambda^+ \end{cases}$$

The operator  $W = W^+ + W^-$  is called the Weyl operator and  $s$  is the scalar curvature of  $g$ .

By a fundamental theorem of [1] the complex structure  $\mathbb{J}$  is integrable if and only if the metric  $g$  on  $M$  is anti-self-dual, that is  $W^+ = 0$ . We will always assume this. By the works of Trudinger, Aubin, and Schoen on Yamabe problem, we will always choose a conformal representative of  $g$  with constant scalar curvature. This does not change the isomorphism class of  $(\mathbb{T}, \mathbb{J})$ .

**2.3. Properties of type decompositions.** For a complex tangent vector  $V$  on  $\mathbb{T}$ , we will denote by  $V^h, V^a \in T\mathbb{T}_{\mathbb{C}}$  its  $(1, 0)$  and  $(0, 1)$  parts :  $\mathbb{J}V^h = iV^h, \mathbb{J}V^a = -iV^a$ . Moreover, for a complex tangent vector  $X$  on  $M$ ,  $X^h$  will denote

$$X^h := \pi_*(\mathcal{H}(X)^h) = 1/2\pi_*(\mathcal{H}X - i\mathbb{J}\mathcal{H}X) = 1/2(X - iu(X))$$

and  $X^a = \pi_*(\mathcal{H}(X)^a)$ , omitting the dependence in  $p = (m, u) \in \pi^{-1}(m)$ . Note that by construction of  $\mathbb{J}$ , one has  $\mathcal{H}(X)^h = \mathcal{H}(X^h) =: \mathcal{H}X^h$  and  $\mathcal{H}(X)^a = \mathcal{H}(X^a) =: \mathcal{H}X^a$ .

**Lemma 2.1.** *Given a direct orthonormal frame  $(\theta_1, \dots, \theta_4)$  on a small open set  $\mathcal{U}$  of  $M$ , and  $(u, v)$  in  $\Lambda_{\mathbb{C}}^+ \times \Lambda_{\mathbb{C}}^-$ ,*

- (1) *the matrix bracket  $[u, v]$  (in fact  $[\phi^{-1}(u), \phi^{-1}(v)]$ ) vanishes.*
- (2)  *$\theta_i^h \wedge \theta_j^h \in \Lambda_{\mathbb{C}}^+$  and  $\theta_i^a \wedge \theta_j^a \in \Lambda_{\mathbb{C}}^+$*
- (3)  *$\theta_i^h \wedge \theta_j^a \in \Lambda_{\mathbb{C}}^- \oplus Vect(u)_{\mathbb{C}}$  (in fact  $\Lambda_{\mathbb{C}}^- \oplus Vect(\phi(u))_{\mathbb{C}}$ ).*

*Proof.* (1) Any  $u \in \phi^{-1}(\Lambda^+)$  coming from a bivector of norm  $\sqrt{2}$  can be described with a quaternion  $p$  as the quaternion product  $u(x) = p \cdot x$ , and likewise any  $v \in \phi^{-1}(\Lambda^-)$  coming from a bivector of norm  $\sqrt{2}$  can be described with a quaternion  $q$  as the quaternion product  $u(x) = x \cdot q$ . The result now follows from the associativity of the quaternion algebra. More explicitly note that the

$$\text{family } \begin{cases} \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4 \\ \theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4 \\ \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3 \end{cases} \text{ is a basis of } \Lambda^+ \text{ and that } \begin{cases} \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4 \\ \theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4 \\ \theta_1 \wedge \theta_4 - \theta_2 \wedge \theta_3 \end{cases} \text{ is a basis of } \Lambda^-.$$

(2) At a point  $p = (m, u)$ , expanding we get

$$\begin{aligned}\theta_i^h \wedge \theta_j^h &= \frac{1}{4}(\theta_i - iu\theta_i) \wedge (\theta_j - iu\theta_j) \\ &= \frac{1}{4}\left(\theta_i \wedge \theta_j - u\theta_i \wedge u\theta_j - i(\theta_i \wedge u\theta_j + u\theta_i \wedge \theta_j)\right) \\ &= \frac{1}{4}(Id - iu)(\theta_i \wedge \theta_j - u\theta_i \wedge u\theta_j) \in \Lambda_{\mathbb{C}}^+.\end{aligned}$$

The relation  $\theta_i^a \wedge \theta_j^a \in \Lambda_{\mathbb{C}}^+$  follows by conjugation.

(3) Expanding again, we get

$$\theta_i^h \wedge \theta_j^a = \frac{1}{4}\left(\theta_i \wedge \theta_j + u\theta_i \wedge u\theta_j + i(\theta_i \wedge u\theta_j - u\theta_i \wedge \theta_j)\right).$$

$$\text{We then check } \begin{cases} \theta_i \wedge \theta_j + u\theta_i \wedge u\theta_j \in \Lambda_{\mathbb{C}}^- \oplus Vect(u) \\ \theta_i \wedge u\theta_j - u\theta_i \wedge \theta_j \in \Lambda_{\mathbb{C}}^- \oplus Vect(u) \end{cases}$$

□

**2.4. Bracket computations.** The data of a direct orthonormal frame  $(\theta_1, \dots, \theta_4)$  on a small open set  $\mathcal{U}$  of  $M$  defines a trivialisation  $Z \supset \pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{S}^2$ . The local coordinates of a point  $p$  in  $Z$  will be denoted by  $(m, u)$ . Because the fibre of  $\pi$  over a point  $m \in M$  is  $\{u \in SO(T_m M)/u^2 = -Id \text{ and } u \gg 0\}$ , the vertical space  $\mathcal{V}_p$  at a point  $p = (m, u)$  is given by

$$\mathcal{V}_p = \{V \in so(T_m M)/Vu = -uV\}.$$

Let  $A : \mathcal{U} \rightarrow so(TM)$  be a section of the bundle of anti-symmetric endomorphisms. We define  $\widehat{A} : \pi^{-1}(\mathcal{U}) \rightarrow TZ$  to be the associated vertical vector field computed with matrix brackets

$$\widehat{A}(p) = [u, A(m)] \in \mathcal{V}_p.$$

Note that these special vectors generate the vertical directions.

**Remark 2.2.** *The first easy property of lemma 2.1 will hugely simplify the forthcoming computations. For example, if  $A : \mathcal{U} \rightarrow so(TM)$  is a section and  $A^+ : \mathcal{U} \rightarrow \Lambda^+$  its projection onto  $\Lambda^+$  then the associated vertical vector fields are equal  $\widehat{A} = \widehat{A}^+$ . In particular,  $R(\widehat{\theta_i^h \wedge \theta_j^h}) = (W^+ + \frac{s}{12}Id)(\theta_i^h \wedge \theta_j^h)$ . Similarly, for  $B$  maps  $\Lambda^+$  to  $\Lambda^-$ , the vertical vector field  $\widehat{B}(u)$  vanishes. Hence, for the vertical vector field  $B(\widehat{\theta_i^h \wedge \theta_j^a})$ , only the component in  $\Lambda_{\mathbb{C}}^-$  of the vector  $\theta_i^h \wedge \theta_j^a \in \Lambda_{\mathbb{C}}^- \oplus Vect(J)_{\mathbb{C}}$  is relevant.*

Let  $X : \mathcal{U} \rightarrow TM$  be a vector field on  $\mathcal{U}$ . Its horizontal lifting  $\mathcal{H}(X)$  is a basic vector field (i.e.  $\pi_* \mathcal{H}(X) = X$  everywhere on  $\pi^{-1}(\mathcal{U})$ ). In terms of the local trivialisation  $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{S}^2$  the vector field  $\mathcal{H}(X)$  reads

$$T\mathbb{T} \supset \mathcal{H}_p \ni \mathcal{H}(X) = X + \widehat{\eta(X)} \in TU \oplus TS^2.$$

The following bracket computations of basic vector fields will be used again and again.

**Lemma 2.3.** *Given a direct orthonormal frame  $(\theta_1, \dots, \theta_4)$  on a small open set  $\mathcal{U}$  of  $M$  and two sections  $A$  and  $B$  of  $\mathcal{U} \rightarrow so(TM)$ , the Lie brackets are computed by*

$$\begin{aligned} [\widehat{A}, \widehat{B}] &= \widehat{[A, B]} \\ [\mathcal{H}(\theta_i), \widehat{A}] &= \widehat{(\nabla_{\theta_i}^g A) + [\eta(\theta_i), A]} \\ [\mathcal{H}(\theta_i), \mathcal{H}(\theta_j)] &= \mathcal{H}[\theta_i, \theta_j] - \widehat{R(\theta_i \wedge \theta_j)}. \end{aligned}$$

*Proof.* In a point  $p = (m, u)$  of  $\mathbb{T}$ ,

- $[\widehat{A}, \widehat{B}] = [[u, A], [u, B]] = [[u, A], B] - [[u, B], A] = [u, [A, B]]$ . The map  $A \mapsto \widehat{A}$  is hence a morphism of Lie algebras.
- First note that,  $[\theta_i, \widehat{A}] = [\theta_i, [u, A]] = [u, \nabla_{\theta_i}^g A] = \widehat{\nabla_{\theta_i}^g A}$ . Hence,

$$[\mathcal{H}(\theta_i), \widehat{A}] = [\theta_i + \widehat{\eta(\theta_i)}, \widehat{A}] = \widehat{\nabla_{\theta_i}^g A} + [\widehat{\eta(\theta_i)}, \widehat{A}].$$

- The result derives from previous remarks

$$\begin{aligned} [\mathcal{H}(\theta_i), \mathcal{H}(\theta_j)] &= \widehat{[\theta_i + \eta(\theta_i), \theta_j + \eta(\theta_j)]} \\ &= [\theta_i, \theta_j] + \widehat{\nabla_{\theta_i}^g \eta(\theta_j)} - \widehat{\nabla_{\theta_j}^g \eta(\theta_i)} + [\widehat{\eta(\theta_i)}, \widehat{\eta(\theta_j)}] \\ &= [\theta_i, \theta_j] + d\widehat{\eta(\theta_i, \theta_j)} + \widehat{\eta([\theta_i, \theta_j])} + [\widehat{\eta(\theta_i)}, \widehat{\eta(\theta_j)}] \\ &= [\theta_i, \theta_j] + \widehat{\eta([\theta_i, \theta_j])} + (d\eta + \eta \wedge \eta)(\theta_i, \theta_j) \\ &= \mathcal{H}[\theta_i, \theta_j] - \widehat{R(\theta_i \wedge \theta_j)}. \end{aligned}$$

This relation shows that the curvature  $R$  accounts for the lack of integrability of the horizontal distribution  $\mathcal{H}$ . □

**Lemma 2.4.** *For every vertical vector field  $U$  on  $\mathbb{T}$ ,*

$$\begin{aligned} [\mathcal{H}\theta_i, \mathbb{J}U] &= \mathbb{J}[\mathcal{H}\theta_i, U] \quad \text{and} \quad \mathcal{V}[\mathbb{J}\mathcal{H}\theta_i, \mathbb{J}U] = \mathbb{J}\mathcal{V}[\mathbb{J}\mathcal{H}\theta_i, U] \\ [\mathcal{H}\theta_i, U^h_a] &= [\mathcal{H}\theta_i, U^h_a] \quad \text{and} \quad \mathcal{V}[\mathbb{J}\mathcal{H}\theta_i, U^h_a] = (\mathcal{V}[\mathbb{J}\mathcal{H}\theta_i, U])^h_a. \end{aligned}$$

*Proof.* The first formula follows from the fact that the parallel transport along horizontal directions respect the canonical metric and the orientation of the fibres, hence the vertical complex structures. The second follows from the fact that  $\mathcal{V}[\mathcal{H}\theta_i, \mathbb{J}U] = \mathbb{J}\mathcal{V}[\mathcal{H}\theta_i, U]$  is a tensor in  $\theta_i$ . The last two follow by linearity. □

**Lemma 2.5.** *For every vertical vector field  $U$  on  $\mathbb{T}$ ,*

$$\begin{aligned} \mathcal{H}[\mathcal{H}\theta_i, U] &= 0 \quad \text{and} \quad \mathcal{H}[\mathbb{J}\mathcal{H}\theta_i, U] = -U(\mathcal{H}\theta_i) \\ \mathcal{H}[\mathcal{H}\theta_i^h, U^h_a] &= \frac{i}{2}U^h_a(\mathcal{H}\theta_i^h) \quad \text{and} \quad \mathcal{H}[\mathcal{H}\theta_i^a, U^h_a] = -\frac{i}{2}U^h_a(\mathcal{H}\theta_i^a). \end{aligned}$$

*Proof.* We use the notations of the lemma 2.3. In order to get the first equality, simply note that for any smooth function  $f$ , using the previous lemma and the properties of

the Lie brackets  $[\mathcal{H}\theta_i, f\hat{A}]$  is vertical. The second computation hence reduces to

$$\begin{aligned}\mathcal{H}[\mathbb{J}\mathcal{H}\theta_i, U] &= \mathcal{H}[\mathcal{H}(\sum u_{ji}\theta_j), U] \\ &= -\sum (U \cdot u_{ji})\mathcal{H}\theta_j = -\sum U_{ji}\mathcal{H}\theta_j := -U(\mathcal{H}\theta_i).\end{aligned}$$

Use the bracket linearity to get  $\mathcal{H}[\mathcal{H}\theta_i^h, U] = \frac{i}{2}U(\mathcal{H}\theta_i) = -\mathcal{H}[\mathcal{H}\theta_i^a, U]$ . Note that this is a tensor in  $U$  so that in particular,

$$\mathcal{H}[\mathcal{H}\theta_i^h, U^a] = \frac{i}{2}U^a(\mathcal{H}\theta_i) = \frac{i}{2}\frac{Id + iJ}{2}U^a(\mathcal{H}\theta_i) = \frac{i}{2}U^a(\mathcal{H}\theta_i^a).$$

□

**Remark 2.6.** The formula  $\mathcal{H}[\mathbb{J}\mathcal{H}\theta_i, U] = -U(\mathcal{H}\theta_i)$  can be made more intrinsic by considering the map

$$\varphi : \pi^{-1}(m) \rightarrow \text{End}(T_m M)$$

that encodes the variation of complex structure on  $T_m M$ . We find

$$\pi_*[\mathbb{J}\mathcal{H}\theta_i, U] = -\varphi_*(U)(\theta_i).$$

**2.5. Computations of  $d\mathbb{W}$  and  $d'\mathbb{W}$ .** The results of this part are well known and can be found for example in [7]. Let  $\mathbb{W} = \mathbb{G}(\mathbb{J}\cdot, \cdot)$  be the metric form on the twistor space  $\mathbb{T}$ . Its exterior derivative is given by the following

**Proposition 2.7.** *The exterior derivative  $d\mathbb{W}$  of the metric form  $\mathbb{W}$  on the twistor space  $\mathbb{T}$  of an anti-self dual Riemannian 4-manifold  $(M, g)$  vanishes on pure directional (i.e. horizontal or vertical) vectors except when evaluated on two horizontal vectors and one vertical vector. More precisely then,*

$$\forall X, Y \in TM, \forall U \in \mathcal{V} \quad d\mathbb{W}(U, \mathcal{H}X, \mathcal{H}Y) = \mathbb{G}\left(\widehat{\left(\frac{1}{2}Id - R\right)}(X \wedge Y), \mathbb{J}U\right).$$

*Proof.* The usual formula for the exterior derivatives of a 2-form reduces here by orthogonality using the bracket computations of lemma 2.3 to

$$\begin{aligned}d\mathbb{W}(U, \mathcal{H}\theta_i, \mathcal{H}\theta_j) &= U \cdot \mathbb{W}(\mathcal{H}\theta_i, \mathcal{H}\theta_j) - \mathbb{W}([\mathcal{H}\theta_i, \mathcal{H}\theta_j], U) \\ &= U \cdot g(u(\theta_i), \theta_j) + \mathbb{G}([\mathcal{H}\theta_i, \mathcal{H}\theta_j], \mathbb{J}U) \\ &= -U_{ij} - \mathbb{G}(\widehat{R(\theta_i \wedge \theta_j)}, \mathbb{J}U).\end{aligned}$$

choosing vertical coordinates  $(u_{ij})$  such that  $u(\theta_j) = \sum u_{ij}\theta_i$ . Now set  $E = \theta_i \wedge \theta_j$ . From the definition of  $\mathbb{G}$ , and the property  $uU = -Uu$  of the vertical vector  $U$  one has

$$\begin{aligned}\mathbb{G}(\widehat{\theta_i \wedge \theta_j}, \mathbb{J}U) &= -1/2tr\left((uE - Eu)uU\right) = -1/2tr(uEuU) - 1/2tr(EU) \\ &= -tr(EU) = -2U_{ij}.\end{aligned}$$

It remains to check the vanishing of all the other pure directional components. As the fibres are of real dimension two, the 3-form  $d\mathbb{W}$  restricts to zero on fibres. Let

$A, B : M \rightarrow so(TM)$  be two sections. In normal coordinates at the centre of which the connection 1-form  $\eta$  vanishes

$$\begin{aligned} d\mathbb{W}(\mathcal{H}\theta_i, \widehat{A}, \widehat{B}) &= \mathcal{H}\theta_i \cdot \mathbb{W}(\widehat{A}, \widehat{B}) - \mathbb{W}([\mathcal{H}\theta_i, \widehat{A}], \widehat{B}) + \mathbb{W}([\mathcal{H}\theta_i, \widehat{B}], \widehat{A}) \\ &= \theta_i \cdot \mathbb{W}(\widehat{A}, \widehat{B}) - \mathbb{W}(\widehat{\nabla_{\theta_i}^g A}, \widehat{B}) + \mathbb{W}(\widehat{\nabla_{\theta_i}^g B}, \widehat{A}) \\ &= \mathbb{W}(\widehat{\nabla_{\theta_i}^g A}, \widehat{B}) + \mathbb{W}(\widehat{A}, \widehat{\nabla_{\theta_i}^g B}) - \mathbb{W}(\widehat{\nabla_{\theta_i}^g A}, \widehat{B}) + \mathbb{W}(\widehat{\nabla_{\theta_i}^g B}, \widehat{A}) \\ &= 0. \end{aligned}$$

Finally for a triple of horizontal lifts, still with normal coordinates,

$$\begin{aligned} d\mathbb{W}(\mathcal{H}\theta_i, \mathcal{H}\theta_j, \mathcal{H}\theta_k) &= \mathcal{H}\theta_i \cdot \mathbb{W}(\mathcal{H}\theta_j, \mathcal{H}\theta_k) - \mathcal{H}\theta_j \cdot \mathbb{W}(\mathcal{H}\theta_i, \mathcal{H}\theta_k) \\ &\quad + \mathcal{H}\theta_k \cdot \mathbb{W}(\mathcal{H}\theta_i, \mathcal{H}\theta_j) - \mathbb{W}([\mathcal{H}\theta_i, \mathcal{H}\theta_j], \mathcal{H}\theta_k) \\ &\quad + \mathbb{W}([\mathcal{H}\theta_i, \mathcal{H}\theta_k], \mathcal{H}\theta_j) - \mathbb{W}([\mathcal{H}\theta_j, \mathcal{H}\theta_k], \mathcal{H}\theta_i) \\ &= \theta_i \cdot g(u\theta_j, \theta_k) - \theta_j \cdot g(u\theta_i, \theta_k) + \theta_k \cdot g(u\theta_i, \theta_j) \\ &\quad - g(u[\theta_i, \theta_j], \theta_k) + g(u[\theta_i, \theta_k], \theta_j) - g(u[\theta_j, \theta_k], \theta_i) \\ &= 0 \end{aligned}$$

for  $\theta_i \cdot u = 0$  and for all the remaining quantities can be expressed in terms of  $\nabla_{\theta_a}^g \theta_b = 0$  only.  $\square$

This result gives an expression for the  $(2, 1)$ -part  $d'\mathbb{W}$  of  $d\mathbb{W}$ .

**Proposition 2.8.** *For all vertical vectors  $U$ , one has*

- i)  $d'\mathbb{W}(U^a, \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h) = (\frac{s}{6} - 1)U_{ij}^a$ .
- ii)  $d'\mathbb{W}(U^h, \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^a) = -i\mathbb{G}\left(\widehat{B(\theta_i^h \wedge \theta_j^a)}, U^h\right)$ .

*Proof.* i) Because  $\theta_i^h \wedge \theta_j^h = \frac{1}{4}(Id - iu)(\theta_i \wedge \theta_j - u\theta_i \wedge u\theta_j)$  belongs to  $\Lambda_{\mathbb{C}}^+$  we infer by lemma 2.1

$$\begin{aligned} d'\mathbb{W}(U^a, \widehat{\theta_i^h}, \widehat{\theta_j^h}) &= \frac{1}{4}\mathbb{G}\left(\left(\frac{1}{2}Id - R\right)(Id - iu)(\theta_i \wedge \theta_j - u\theta_i \wedge u\theta_j), \mathbb{J}U^a\right) \\ &= \frac{1}{4}\mathbb{G}\left(\left(\frac{1}{2} - \frac{s}{12}\right)(Id - iu)(\theta_i \wedge \theta_j - u\theta_i \wedge u\theta_j), \mathbb{J}U^a\right) \quad \text{for } W^+ = 0 \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{s}{12}\right)\mathbb{G}\left(\frac{Id - i\mathbb{J}}{2}(\theta_i \wedge \theta_j - u\theta_i \wedge u\theta_j), \mathbb{J}U^a\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{s}{12}\right)\mathbb{G}\left((\theta_i \wedge \theta_j - u\theta_i \wedge u\theta_j), \frac{Id + i\mathbb{J}}{2}\mathbb{J}U^a\right) \\ &= \frac{1}{4}\left(1 - \frac{s}{6}\right)\mathbb{G}\left((\theta_i \wedge \theta_j - u\theta_i \wedge u\theta_j), \mathbb{J}U^a\right). \end{aligned}$$

But we already found that  $\mathbb{G}(\widehat{\theta_i \wedge \theta_j}, \mathbb{J}U) = -2U_{ij}$ . Writing  $u\theta_i = \sum u_{ki}\theta_k$ , we find  $\mathbb{G}(u\theta_i \wedge u\theta_j, \mathbb{J}U) = -2u_{ki}u_{lj}U_{kl} = 2(uUu)_{ij} = 2U_{ij}$  that leads to  $d'\mathbb{W}(U^a, \widehat{\theta_i^h}, \widehat{\theta_j^h}) = (\frac{s}{6} - 1)U_{ij}^a$ .

ii) Because  $\theta_i^h \wedge \theta_j^a$  belongs to  $\widehat{\Lambda}_{\mathbb{C}} \oplus Vect(u)$ , we infer by lemma 2.1 on the one hand  $R(\widehat{\theta_i^h \wedge \theta_j^a}) = B(\widehat{\theta_i^h \wedge \theta_j^a})$  and on the other hand  $\widehat{\theta_i^h \wedge \theta_j^a} = 0$ . This gives

$$\begin{aligned} d'\mathbb{W}(U^h, \widehat{\theta_i^h}, \widehat{\theta_j^a}) &= \mathbb{G}\left(\left(\frac{1}{2}Id - R\right)(\widehat{\theta_i^h \wedge \theta_j^a}), JU^h\right) \\ &= -i\mathbb{G}\left(B(\widehat{\theta_i^h \wedge \theta_j^a}), U^h\right). \end{aligned}$$

□

**Corollary 2.9.** *The form  $\mathbb{W}$  is Kähler if and only if  $R|_{\Lambda^+} = \frac{1}{2}Id_{\Lambda^+}$ .*

*Proof.* The vanishing  $d\mathbb{W} = 0$  gives  $B = 0$  (cf *ii*) and  $s/12 = 1/2$  (cf. *i*). The converse is straightforward. □

**Remark 2.10.** *This is the case for the round sphere  $\mathbb{S}^4$  and the projective space  $\mathbb{C}P^2$  with a Fubini-Study metric. More generally, Hitchin [8] actually proved that these are the only Kähler twistor spaces.*

**2.6. Computation of  $id'd''\mathbb{W}$ .** In this section, we will compute the real 4-form  $id'd''\mathbb{W} = idd'\mathbb{W}$  of type  $(2, 2)$ . We will express its values on pure directional vectors.

**Theorem 1.** *The hessian  $id'd''\mathbb{W}$  of the metric form  $\mathbb{W}$  on the twistor space  $\mathbb{T}(M, g)$  of an anti-self dual Riemannian 4-manifold  $(M, g)$  is given on pure directions and pure types by the following formulae where  $\mathcal{H}\theta_i$  are basic horizontal lifts and  $U_i$  vertical vectors,*

$$(2.1) \quad id'd''\mathbb{W}(U_1^h, U_2^h, U_3^a, U_4^a) = 0$$

$$(2.2) \quad id'd''\mathbb{W}(U_1^h, U_2^h, U_3^a, \mathcal{H}\theta_i^a) = id'd''\mathbb{W}(\mathcal{H}\theta_i^h, U_3^h, U_1^a, U_2^a) = 0$$

$$(2.3) \quad id'd''\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U_1^a, U_2^a) = id'd''\mathbb{W}(U_1^h, U_2^h, \mathcal{H}\theta_i^a, \mathcal{H}\theta_j^a) = 0$$

$$(2.4) \quad \begin{aligned} id'd''\mathbb{W}(\mathcal{H}\theta_i^h, U_1^h, \mathcal{H}\theta_j^a, U_2^a) &= -U_2^a \cdot \mathbb{G}\left(B(\widehat{\theta_i^h \wedge \theta_j^a}), U_1^h\right) \\ &\quad - \frac{i}{2}U_2^a{}_{mj} \mathbb{G}\left(B(\widehat{\theta_m^a \wedge \theta_i^h}), U_1^h\right) \\ &\quad + \frac{1}{2}\left(\frac{s}{6} - 1\right)(U_2^a \cdot U_1^h)_{ij} \end{aligned}$$

$$(2.5) \quad id'd''\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U^a, \mathcal{H}\theta_k^a) = id'd''\mathbb{W}(U^h, \mathcal{H}\theta_k^h, \mathcal{H}\theta_i^a, \mathcal{H}\theta_j^a) = 0$$

$$(2.6) \quad \begin{aligned} id'd''\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, \mathcal{H}\theta_k^a, \mathcal{H}\theta_l^a) &= \mathbb{G}\left(B(\widehat{\theta_j^h \wedge \theta_i^a}), B(\widehat{\theta_i^h \wedge \theta_k^a})\right) \\ &\quad - \mathbb{G}\left(B(\widehat{\theta_i^h \wedge \theta_l^a}), B(\widehat{\theta_j^h \wedge \theta_k^a})\right) \\ &\quad - i\left(\frac{s}{6} - 1\right)\frac{s}{12}(\widehat{\theta_k^a \wedge \theta_l^a})_{ij}^a. \end{aligned}$$

*Proof.* (1) reflects the facts that the vertical distribution is integrable and that the metric on the fibres is Kähler.

(2) The non vanishing of  $d'\mathbb{W}$  requires two horizontal vectors. The integrability of the vertical distribution hence shows the results.

(3) By the usual formula for the exterior derivative, omitting zero terms, we get

$$\begin{aligned} & d''d'\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U_1^a, U_2^a) \\ &= U_1^a \cdot d'\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U_2^a) - U_2^a \cdot d'\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U_1^a) - d'\mathbb{W}([U_1^a, U_2^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h) \\ & \quad + d'\mathbb{W}([\mathcal{H}\theta_i^h, U_1^a], \mathcal{H}\theta_j^h, U_2^a) - d'\mathbb{W}([\mathcal{H}\theta_i^h, U_2^a], \mathcal{H}\theta_j^h, U_1^a) \\ & \quad + d'\mathbb{W}([\mathcal{H}\theta_j^h, U_1^a], \mathcal{H}\theta_i^h, U_2^a) - d'\mathbb{W}([\mathcal{H}\theta_j^h, U_2^a], \mathcal{H}\theta_i^h, U_1^a). \end{aligned}$$

From proposition 2.8 and lemma 2.5, we infer that the terms

$d'\mathbb{W}([\mathcal{H}\theta_i^h, U_1^a], \mathcal{H}\theta_j^h, U_2^a) = d'\mathbb{W}(\mathcal{H}[\mathcal{H}\theta_i^h, U_1^a]^h, \mathcal{H}\theta_j^h, U_2^a)$  in the last two lines vanishes for type reason. As the scalar curvature is constant, the proposition 2.8 leads to

$$\begin{aligned} & U_1^a \cdot d'\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U_2^a) - U_2^a \cdot d'\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U_1^a) - d'\mathbb{W}([U_1^a, U_2^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h) \\ &= \left(\frac{s}{6} - 1\right) \left( U_1^a \cdot (U_2^a)_{ij} - U_2^a \cdot (U_1^a)_{ij} - [U_1^a, U_2^a]_{ij} \right) = 0 \end{aligned}$$

The second equality follows by conjugation.

(4) By the usual formula for the exterior derivative, omitting zero terms, we get

$$\begin{aligned} & d''d'\mathbb{W}(\mathcal{H}\theta_i^h, U_1^h, \mathcal{H}\theta_j^a, U_2^a) \\ &= U_2^a \cdot d'\mathbb{W}(U_1^h, \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^a) - d'\mathbb{W}([\mathcal{H}\theta_i^h, U_2^a], U_1^h, \mathcal{H}\theta_j^a) \\ & \quad - d'\mathbb{W}([\mathcal{H}\theta_j^a, U_2^a], \mathcal{H}\theta_i^h, U_1^h) + d'\mathbb{W}([\mathcal{H}\theta_j^a, U_1^h], \mathcal{H}\theta_i^h, U_2^a) \\ &= -iU_2^a \cdot \mathbb{G}\left(B(\widehat{\theta_i^h \wedge \theta_j^a}), U_1^h\right) - d'\mathbb{W}(\mathcal{H}[\mathcal{H}\theta_i^h, U_2^a]^h, U_1^h, \mathcal{H}\theta_j^a) \\ & \quad - d'\mathbb{W}(\mathcal{H}[\mathcal{H}\theta_j^a, U_2^a]^a, \mathcal{H}\theta_i^h, U_1^h) + d'\mathbb{W}(\mathcal{H}[\mathcal{H}\theta_j^a, U_1^h]^h, \mathcal{H}\theta_i^h, U_2^a). \end{aligned}$$

From lemma 2.5, we infer that the second term vanishes for type reasons, and

that for the third term  $\mathcal{H}[\mathcal{H}\theta_j^a, U_2^a] = -\frac{i}{2}U_{2mj}^a \mathcal{H}\theta_m^a$ . Hence

$$-d'\mathbb{W}([\mathcal{H}\theta_j^a, U_2^a], \mathcal{H}\theta_i^h, U_1^h) = \frac{1}{2}U_{2mj}^a \mathbb{G}\left(B(\widehat{\theta_m^a \wedge \theta_i^h}), U_1^h\right).$$

From lemma 2.5, we infer that for the forth term  $\mathcal{H}[\mathcal{H}\theta_j^a, U_1^h] = -\frac{i}{2}U_{1mj}^h \mathcal{H}\theta_m^h$ .

Hence,

$$\begin{aligned} & d'\mathbb{W}([\mathcal{H}\theta_j^a, U_1^h], \mathcal{H}\theta_i^h, U_2^a) \\ &= -\frac{i}{2}d'\mathbb{W}(U_{1mj}^h \mathcal{H}\theta_m^h, \mathcal{H}\theta_i^h, U_2^a) \\ &= -\frac{i}{2}\left(\frac{s}{6} - 1\right)U_{1mj}^h U_{2mi}^a = \frac{i}{2}\left(\frac{s}{6} - 1\right)(U_2^a U_1^h)_{ij}. \end{aligned}$$

(5) Still from the formula of the exterior derivative

$$\begin{aligned}
& d'' d' \mathbb{W} \left( \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U^a, \mathcal{H}\theta_k^a \right) \\
&= -\mathcal{H}\theta_k^a \cdot d' \mathbb{W} \left( \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U^a \right) + d' \mathbb{W} \left( \mathcal{V}[\mathcal{H}\theta_k^a, U^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h \right) \\
&\quad + d' \mathbb{W} \left( \mathcal{V}[\mathcal{H}\theta_i^h, U^a], \mathcal{H}\theta_j^h, \mathcal{H}\theta_k^a \right) - d' \mathbb{W} \left( \mathcal{V}[\mathcal{H}\theta_j^h, U^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a \right) \\
&\quad + d' \mathbb{W} \left( \mathcal{H}[\mathcal{H}\theta_j^h, \mathcal{H}\theta_k^a], \mathcal{H}\theta_i^h, U^a \right) - d' \mathbb{W} \left( \mathcal{H}[\mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a], \mathcal{H}\theta_j^h, U^a \right).
\end{aligned}$$

From proposition 2.8 we can write

$$\mathcal{H}\theta_k^a \cdot d' \mathbb{W} \left( \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U^a \right) = \mathcal{H}\theta_k^a \cdot \left( \left( \frac{s}{6} - 1 \right) U_{ij}^a \right) = \mathcal{H}\theta_k^a \cdot \left( \left( \frac{s}{6} - 1 \right) U_{ij}^a \right) = 0$$

computed in normal coordinates centred at a point  $m$ . In such coordinates, we can choose  $U = \widehat{A}$  with furthermore  $\nabla_{\theta_i}^g A = 0$  at  $m$ . By lemma 2.4,  $\mathcal{V}[\mathcal{H}\theta_k^h, U^a] = (\mathcal{V}[\mathcal{H}\theta_k^h, U])^a = ((\widehat{\nabla_{\theta_k}^g A}) + [\eta(\theta_k), A])^a = 0$ . Now, the vanishing of the third and fourth terms, follows from lemma 2.4, after which  $\mathcal{V}[\mathcal{H}\theta_i^h, U^a]$  is of type  $(0, 1)$ . Still at the centre  $m$  of normal coordinates, we have  $\mathcal{H}[\widehat{\theta}_i^h, \widehat{\theta}_k^a] = 0$  because  $[\theta_i, \theta_j] = \nabla_{\theta_i}^g \theta_j - \nabla_{\theta_j}^g \theta_i = 0$ . As  $\mathcal{H}\theta_i \cdot u = 0$ , we conclude  $\mathcal{H}[\widehat{\theta}_j^h, \widehat{\theta}_k^a] = 0$ .

(6) Again from the formula of the exterior derivative and from 2.8

$$\begin{aligned}
& d'' d' \mathbb{W} \left( \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, \mathcal{H}\theta_k^a, \mathcal{H}\theta_l^a \right) \\
&= d' \mathbb{W} \left( [\mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a], \mathcal{H}\theta_j^h, \mathcal{H}\theta_l^a \right) + d' \mathbb{W} \left( [\mathcal{H}\theta_j^h, \mathcal{H}\theta_l^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a \right) \\
&\quad - d' \mathbb{W} \left( [\mathcal{H}\theta_j^h, \mathcal{H}\theta_k^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_l^a \right) - d' \mathbb{W} \left( [\mathcal{H}\theta_i^h, \mathcal{H}\theta_l^a], \mathcal{H}\theta_j^h, \mathcal{H}\theta_k^a \right) \\
&\quad - d' \mathbb{W} \left( [\mathcal{H}\theta_k^a, \mathcal{H}\theta_l^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h \right).
\end{aligned}$$

As  $\theta_i^h \wedge \theta_k^a$  belongs to  $\bigwedge_{\mathbb{C}}^- \oplus Vect(u)$  and  $\mathcal{V}[\mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a]$  is a tensor, we find  $[\mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a]_V = -R(\theta_i^h \wedge \theta_k^a) = -B(\widehat{\theta}_i^h \wedge \theta_k^a)$  using lemma 2.1. This leads to

$$\begin{aligned}
& d' \mathbb{W} \left( [\mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a], \mathcal{H}\theta_j^h, \mathcal{H}\theta_l^a \right) + d' \mathbb{W} \left( [\mathcal{H}\theta_j^h, \mathcal{H}\theta_l^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a \right) \\
&= -d' \mathbb{W} \left( B(\widehat{\theta}_i^h \wedge \theta_k^a)^h, \mathcal{H}\theta_j^h, \mathcal{H}\theta_l^a \right) - d' \mathbb{W} \left( B(\widehat{\theta}_j^h \wedge \theta_l^a)^h, \mathcal{H}\theta_i^h, \mathcal{H}\theta_k^a \right) \\
&= i\mathbb{G} \left( B(\widehat{\theta}_j^h \wedge \theta_l^a), B(\widehat{\theta}_i^h \wedge \theta_k^a)^h \right) + i\mathbb{G} \left( B(\widehat{\theta}_i^h \wedge \theta_k^a), B(\widehat{\theta}_j^h \wedge \theta_l^a)^h \right) \\
&= i\mathbb{G} \left( B(\widehat{\theta}_j^h \wedge \theta_l^a)^a, B(\widehat{\theta}_i^h \wedge \theta_k^a)^h \right) + i\mathbb{G} \left( B(\widehat{\theta}_i^h \wedge \theta_k^a)^a, B(\widehat{\theta}_j^h \wedge \theta_l^a)^h \right) \\
&= i\mathbb{G} \left( B(\widehat{\theta}_j^h \wedge \theta_l^a), B(\widehat{\theta}_i^h \wedge \theta_k^a) \right).
\end{aligned}$$

where we used the orthogonality of two  $(1, 0)$  vectors. For the last term, as  $\theta_k^a \wedge \theta_l^a$  belongs to  $\Lambda_{\mathbb{C}}^+$  we find  $\mathcal{V}[\mathcal{H}\theta_k^a, \mathcal{H}\theta_l^a] = -R(\widehat{\theta_k^a \wedge \theta_l^a}) = -\frac{s}{12}\widehat{\theta_k^a \wedge \theta_l^a}$ . Finally,

$$\begin{aligned} & -d'\mathbb{W}\left([\mathcal{H}\theta_k^a, \mathcal{H}\theta_l^a], \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h\right) \\ &= -d'\mathbb{W}\left(\mathcal{V}[\mathcal{H}\theta_k^a, \mathcal{H}\theta_l^a]^a, \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h\right) = \left(\frac{s}{6} - 1\right)\frac{s}{12}(\widehat{\theta_k^a \wedge \theta_l^a})_{ij}^a. \end{aligned}$$

□

### 3. TWISTOR SPACES OF HYPERKÄHLER MANIFOLDS

The previous sections have focused on the 4-dimensional case. We now briefly give a generalisation in higher dimension. Fix an integer  $n \geq 1$ .

A *quaternionic Kähler manifold* is an oriented  $4n$ -dimensional Riemannian manifold  $(M, g)$  whose holonomy group is contained in the product of quaternionic unitary groups  $Sp(1)Sp(n)$ . In other words, with the holonomy principle [4], such a manifold admits a rank 3 sub-bundle  $D \subset \text{End}(TM)$ , locally spanned by a triple  $(I, J, K = IJ = -JI)$  of  $g$ -orthogonal almost complex structures compatible with the orientation, that is stable by the Levi-Civita connection on  $TM$ . We will use the notation  $\nabla := \nabla^{(g,D)}$  for the restriction to  $D$  of the Levi-Civita connection, and subsequently  $R$  for the curvature of this restriction.

Let  $(M, g, D)$  be a quaternionic Kähler  $4n$ -manifold. One can define a scalar product on  $D$  by saying that a local admissible basis of  $D$  is orthonormal. A point  $u = (a, b, c)$  of the sphere  $\mathbb{S}^2$  gives a complex structure  $J_u = aI + bJ + cK$  on  $M$ . The corresponding complex manifold will be denoted by  $X_u$ . One can then define the *twistor space*  $\mathbb{T} = \mathbb{T}(M, g, D) \rightarrow M$  as the unit sphere bundle of  $D$ . This is a locally trivial bundle over  $M$  with fibre  $\mathbb{S}^2$  and structure group  $SO(3)$ . As before, the twistor space of a quaternionic Kähler  $4n$ -manifold can be endowed with a metric  $\mathbb{G}$  and an almost complex structure  $\mathbb{J}$  that is integrable ([12],[9]).

In particular, a *hyperkähler manifold* is an oriented  $4n$ -dimensional Riemannian manifold  $(M, g)$  whose holonomy group is contained in the quaternionic unitary group  $Sp(n)$ . In other words, with the holonomy principle, a hyperkähler manifold is an oriented  $4n$ -dimensional Riemannian manifold  $(M, g)$  endowed with three global  $g$ -orthogonal parallel (hence integrable Kähler) complex structures  $I, J$  and  $K$  compatible with the orientation such that  $IJ = -JI = K$ . The corresponding pencil of complex structures is called the Calabi family of  $(M, g, D = (I, J, K))$ . Note that for  $Sp(n) \subset SU(2n)$ , each of these complex structure is Ricci-flat.

For example, starting with a *holomorphic symplectic manifold* (i.e. a compact complex Kähler manifold  $X$  with a holomorphic symplectic 2-form  $\Omega$ , hence of vanishing first Chern class) and a Kähler class  $\kappa$ , the theorem of Yau [13] gives a unique Kähler metric  $g$  in the Kähler class  $\kappa$  with vanishing Ricci curvature. The form  $\Omega$  is  $g$ -parallel by the Bochner principle, showing that the holonomy of  $g$  is contained in the

quaternionic unitary group  $Sp(n)$ , and that  $g$  is a hyperkähler metric. The corresponding twistor space is called the Calabi family of  $(X, \Omega, \kappa)$ . The manifold  $X$  will be called *irreducible holomorphic symplectic* if furthermore  $X$  is simply connected and  $H^0(X, \Omega_X^2)$  is generated by the holomorphic symplectic 2-form  $\Omega$ .

In view of the applications, we will work in the rest of this section on a hyperkähler manifold  $(M, g, D = (I, J, K))$  and assume that the holonomy group is exactly  $Sp(n)$ . Note in this case, that the horizontal distribution on  $\mathbb{T} = \mathbb{T}(M, g, D)$  is integrable and given by a holomorphic map  $f$

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{f} & \mathbb{P}^1 \ni u \\ \downarrow \pi & & \\ m \in M & & \end{array}$$

and that each  $X_u := f^{-1}(u)$  is an irreducible holomorphic symplectic manifold [3].

**3.1. Bracket computations.** As before, a section of the bundle of anti-symmetric endomorphisms.  $A : \mathcal{U} \rightarrow so(D)$  gives rise to a special vertical vector field  $\widehat{A}$ . Because the complex structures  $I, J$  and  $K$  are  $g$ -parallel, the vector field  $R(\widehat{\theta_i \wedge \theta_j}) = [u, \nabla_{\theta_i}^{(g,D)} \nabla_{\theta_j}^{(g,D)} - \nabla_{\theta_j}^{(g,D)} \nabla_{\theta_i}^{(g,D)} - \nabla_{[\theta_i, \theta_j]}^{(g,D)}] = 0$ . The previous bracket computations reduce to

**Lemma 3.1.** *Given a direct orthonormal frame  $(\theta_1, \dots, \theta_{4n})$  on a small open set  $\mathcal{U}$  of  $M$  and two sections  $A$  and  $B$  of  $\mathcal{U} \rightarrow so(D)$ , the Lie brackets are computed by*

$$\begin{aligned} [\widehat{A}, \widehat{B}] &= \widehat{[A, B]} \\ [\mathcal{H}(\theta_i), \widehat{A}] &= (\nabla_{\theta_i}^{(g,D)} A) + [\eta(\theta_i), A] \\ [\mathcal{H}(\theta_i), \mathcal{H}(\theta_j)] &= \mathcal{H}[\theta_i, \theta_j]. \end{aligned}$$

For every vertical vector field  $U$  on  $\mathbb{T}$ ,

$$\begin{aligned} [\mathcal{H}\theta_i, U^h] &= [\mathcal{H}\theta_i, U]^h \quad \text{and} \quad \mathcal{V}[\mathbb{J}\mathcal{H}\theta_i, U^h] = (\mathcal{V}[\mathbb{J}\mathcal{H}\theta_i, U])^h. \\ \mathcal{H}[\mathcal{H}\theta_i^h, U^h] &= \frac{i}{2} U^h(\mathcal{H}\theta_i^h) \quad \text{and} \quad \mathcal{H}[\mathcal{H}\theta_i^a, U^h] = -\frac{i}{2} U^h(\mathcal{H}\theta_i^a). \end{aligned}$$

**3.2. Computations of  $d\mathbb{W}$  and  $d'\mathbb{W}$ .** Let  $(\theta_1, \dots, \theta_{4n})$  be a local orthonormal frame of  $TM$ .

**Proposition 3.2.** *The exterior derivative  $d\mathbb{W}$  of the metric form  $\mathbb{W}$  on the twistor space of a hyperkähler manifold  $(M, g, D)$  vanishes on pure directional vectors except when evaluated on two horizontal vectors and one vertical vector. More precisely then, for*

a vertical vector  $U \in \mathcal{V}_{(m,u)}$ ,

$$\begin{aligned} d\mathbb{W}(U, \mathcal{H}\theta_i, \mathcal{H}\theta_j) &= \frac{1}{2}\mathbb{G}\left(\widehat{\theta_i \wedge \theta_j}, \mathbb{J}U\right) = -U_{ij}. \\ d'\mathbb{W}(U^h, \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^a) &= 0 \\ d'\mathbb{W}(U^a, \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h) &= -U_{ij}^a \\ \text{equivalently } d''\mathbb{W}_{(m,u)}(U^h, \mathcal{H}\theta_i^a, \mathcal{H}\theta_j^a) &= -2\Omega_u(\kappa_U(\theta_i^a), \kappa_U(\theta_j^a)) \end{aligned}$$

where  $\Omega_u$  is the holomorphic symplectic  $(2,0)$ -form on  $X_u := f^{-1}(u)$  and  $\kappa_U$  is a closed  $(0,1)$ -form on  $X_u$  with values in  $TX_u$  that represents the Kodaira-Spencer class of the family  $f$  at  $u \in \mathbb{P}^1$  in the direction  $U$ .

*Proof.* The first item is proved along the same lines as in the previous section.

For the second,

$$\begin{aligned} d'\mathbb{W}(U^h, \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^a) &= \frac{1}{4}d'\mathbb{W}(U^h, \mathcal{H}\theta_i - iu_{mi}\mathcal{H}\theta_m, \mathcal{H}\theta_j + iu_{kj}\mathcal{H}\theta_k) \\ &= \frac{1}{4}\left(U^h - uU^h u + iuU^h + iU^h u\right)_{ij} = 0. \end{aligned}$$

For the third,

$$\begin{aligned} d'\mathbb{W}(U^a, \mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h) &= \frac{1}{4}d'\mathbb{W}\left(U^a, \mathcal{H}\theta_i - iu_{mi}\mathcal{H}\theta_m, \mathcal{H}\theta_j - iu_{kj}\mathcal{H}\theta_k\right) \\ &= -\frac{1}{4}(U_{ij}^a - u_{mi}u_{kj}U_{mk}^a - iu_{mi}U_{mj}^a - iu_{kj}U_{ik}^a) \\ &= -\frac{1}{4}(U^a + uU^a u + iuU^a - iU^a u)_{ij} = -U_{ij}^a. \end{aligned}$$

The previous formulae have a geometric content. We follow [10] (prop 25.7). Consider a path  $\gamma(t)$  in the base  $\mathbb{P}^1$  starting at  $u = I$  say, with derivative  $U \in T\mathbb{P}^1$ . Over every point  $m \in M$ , there is a vertical lift that we may write as  $u_m(t) = I_m + tU_m + t^2 \cdots$ . Note that  $U_m$  is the derivative in the direction  $U$ ,  $\varphi_*(U) \in \text{End}(T_m M)$ , of the map  $\varphi : \pi^{-1}(m) \rightarrow \text{End}(T_m M)$  that encodes the variation of complex structure on  $T_m M$ . For small  $t$ , we write  $T_m^{0,1}X_{u_m(t)}$  as the graph of a map  $K(t) = t\kappa_U + t^2 \cdots$  from  $T_m^{0,1}X_u$  to  $T_m^{1,0}X_u$ . Note that  $\kappa_U$  seen as a  $(0,1)$ -form on  $X_u$  with values on  $T^{1,0}X_u$  is closed by the integrability of  $\mathbb{T}$  and has, as cohomology class, the Kodaira-Spencer class  $\{\kappa_U\} \in H^1(TX_u)$ . For a vector  $v \in T_m^{0,1}X_u$ , we have the relation  $u_m(t)(v + K(t)(v)) = -i(v + K(t)(v))$  whose first order term gives  $I_m(\kappa_U(v)) + U_m(v) = i\kappa_U(v) + U_m(v) = -i\kappa_U(v)$ . This shows that  $\varphi_*(U)(v) = U_m(v) = -2i\kappa_U(v)$ . The formula  $d''\mathbb{W}(U^h, \mathcal{H}\theta_i^a, \mathcal{H}\theta_j^a) = -U_{ij}^h = U_{ji}^h$ , now reads

$$\begin{aligned} d''\mathbb{W}(U^h, \mathcal{H}\theta_i^a, \mathcal{H}\theta_j^a) &= g(\varphi_*(U)(\theta_i^a), \theta_j^a) = -2ig(\kappa_U(\theta_i^a), \theta_j^a) \\ &= -2\omega_u(\kappa_U(\theta_i^a), \theta_j^a) = -2\Omega_u(\kappa_U(\theta_i^a), \kappa_U(\theta_j^a)). \end{aligned}$$

The last equality is proved in [10].  $\square$

### 3.3. Computation of $id'd''\mathbb{W}$ .

**Theorem 2.** *The hessian  $id'd''\mathbb{W}$  of the metric form  $\mathbb{W}$  on the twistor space  $\mathbb{T}(M, g, D)$  of a hyperkähler manifold  $(M, g, D)$  vanishes on pure directions and pure types except in the following case where  $\mathcal{H}\theta_i$  are basic horizontal lifts and  $U_i$  vertical vectors,*

$$(3.1) \quad id'd''\mathbb{W}\left(\mathcal{H}\theta_i^h, U_1^h, \mathcal{H}\theta_j^a, U_2^a\right) = -\frac{1}{2}(U_2^a U_1^h)_{ij}.$$

*Proof.* We discuss each case as in the proof of theorem 1.

- (1) The vanishing  $id'd''\mathbb{W}(U_1^h, U_2^h, U_3^a, U_4^a) = 0$  reflects the facts that the vertical distribution is integrable and that the metric on the fibres is Kähler.
- (2) The non vanishing of  $d''\mathbb{W}$  requires two horizontal vectors. The integrability of the vertical distribution hence shows that  $id'd''\mathbb{W}(U_1^h, U_2^h, U_3^a, \mathcal{H}\theta_i^a) = id'd''\mathbb{W}(\mathcal{H}\theta_i^h, U_3^h, U_1^a, U_2^a) = 0$ .
- (3) Simply note that for types and directions reasons  $id'd''\mathbb{W}(U_1^h, U_2^h, \mathcal{H}\theta_i^a, \mathcal{H}\theta_j^a) = 0$ . By conjugation, we get the relation  $id'd''\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U_1^a, U_2^a) = 0$ .
- (4) Starting with the only non-zero term

$$\begin{aligned} & d''d''\mathbb{W}\left(\mathcal{H}\theta_i^h, U_1^h, \mathcal{H}\theta_j^a, U_2^a\right) \\ &= d''\mathbb{W}\left(\mathcal{H}[\mathcal{H}\theta_j^a, U_1^h], \mathcal{H}\theta_i^h, U_2^a\right) = d''\mathbb{W}\left(-\frac{i}{2}U_1^h(\mathcal{H}\theta_j^h), \mathcal{H}\theta_i^h, U_2^a\right) \\ &= -\frac{i}{2}U_{1mj}^h d''\mathbb{W}\left(\mathcal{H}\theta_m^h, \mathcal{H}\theta_i^h, U_2^a\right) = \frac{i}{2}U_{1mj}^h U_{2mi}^a = -\frac{i}{2}(U_2^a U_1^h)_{ij}. \end{aligned}$$

- (5) The term  $id'd''\mathbb{W}(U^h, \mathcal{H}\theta_k^h, \mathcal{H}\theta_i^a, \mathcal{H}\theta_j^a)$  vanishes for types and directions reasons. The vanishing of  $id'd''\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, U^a, \mathcal{H}\theta_k^a)$  then follows by conjugation,.
- (6) Finally,  $id'd''\mathbb{W}(\mathcal{H}\theta_i^h, \mathcal{H}\theta_j^h, \mathcal{H}\theta_k^a, \mathcal{H}\theta_l^a) = 0$  follows from the integrability of the horizontal directions. □

## 4. IRREDUCIBLE HOLOMORPHIC SYMPLECTIC MANIFOLDS AND HYPERBOLICITY

**4.1. Campana's result.** In [5, 6], Campana showed that every Calabi family of a hyperkähler manifold  $(M, g, D)$ , and in particular of a irreducible holomorphic symplectic manifold  $(X, \Omega, \kappa)$ , contains a non-hyperbolic member. His proof relies on the study of the component  $C_1(\mathbb{T})$  of the cycle space of the twistor space  $\mathbb{T} = \mathbb{T}(M, g, D)$  that contains the twistor lines. If  $C_1(\mathbb{T})$  were compact, then the evaluation map  $C_1(\mathbb{T})_x \rightarrow P(T_x \mathbb{T})$  at a twistor line  $x$  would be closed and open (by the very ampleness of the normal bundle to  $x$ ) hence surjective : there would be a one cycle with an horizontal tangent. If  $C_1(\mathbb{T})$  is not compact hence non equicontinuous, then there is a sequence of 1-cycles  $C_n$  of  $\mathbb{T}$  and points  $a_n \in C_n$  such that the tangent direction to  $C_n$  at  $a_n$  tends to an horizontal direction at a point  $t \in \mathbb{T}$ . In both cases, he constructed a Brody curve in  $f^{-1}(\mathcal{U}) \subset \mathbb{T}$  and consequently in a deformation  $X_u = f^{-1}(u)$  of  $(M, I)$ .

**4.2. Pseudo convexity of the cycle space.** We intend to go further in the range of ideas of Campana. Let  $(M, g, D = (I, J, K))$  be a hyperkähler manifold and  $\mathbb{T} = \mathbb{T}((M, g, D) \xrightarrow{f} \mathbb{P}^1)$  its Calabi family. Let  $C_1(\mathbb{T})$  be the component of the cycle space of  $\mathbb{T}$  containing the twistor lines.

For every  $s = [C_s] \in C_1(\mathbb{T})$ , we will identify a tangent vector  $\vec{n} \in T_s C_1(\mathbb{T})$  with a section  $n$  of the normal bundle  $N_{C_s/\mathbb{T}}$  of the 1-cycle  $C_s$  in the twistor space  $\mathbb{T}$ . If  $C_s$  happens to be a section of  $f$ , there is an horizontal lifting  $\tilde{n}$  of the normal section  $n$ , whose norm is simply denote by  $\|n\|$ .

**Theorem 3.** *The map  $\text{Vol} : C_1(\mathbb{T}) \rightarrow \mathbb{R}$  is pluri-sub-harmonic. More precisely,*

$$(4.1) \quad id' d''_{C_1(\mathbb{T})} \text{Vol}(C_s)(\vec{n}, J\vec{n}) \geq \int_{C'_s} 4\pi \|n'\|^2 d\text{vol} \geq 0$$

where  $C'_s$  is the irreducible component of the cycle  $C_s$  that maps onto  $\mathbb{P}^1$  by the pencil map  $f$ . In particular, the cycle space  $C_1(\mathbb{T})$  is pseudo-convex.

*Proof.* Choose a cycle  $C_0$ , a tangent vector  $\vec{n} \in T_0 C_1(\mathbb{T})$ , and a family

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Gamma} & \mathbb{T} \\ \downarrow \Pi & & \\ 0 \in S & & \end{array}$$

of cycles with this tangent vector  $\vec{n}$  at the origin. Then,

$$id' d''_{C_1(\mathbb{T})} \text{Vol}(C_s)(\vec{n}, J\vec{n}) = id' d'' \Pi_* \Gamma^* \mathbb{W}(\vec{n}, J\vec{n}) = \Pi_* \Gamma^* id' d'' \mathbb{W}(\vec{n}, J\vec{n}).$$

The intersection number with a fiber of  $f$  being constant, we infer that every member  $C_s$  of the relevant component of the cycle space contains, outside an irreducible section  $C'_s$  of the pencil  $f$ , a finite number  $\sum H_j$  of horizontal rational curves.

For the irreducible image  $C'_0$  of a section  $\sigma : \mathbb{P}^1 \rightarrow \mathbb{T}$  of the pencil  $f$ ,

$$\int_{C'_0} \Gamma^* id' d'' \mathbb{W}(\vec{n}, J\vec{n}) = \int_{\mathbb{P}^1} \sigma^* id' d'' \mathbb{W}(\tilde{n}, J\tilde{n}) = \int_{\mathcal{C}} id' d'' \mathbb{W}(\sigma_* \frac{\partial}{\partial \zeta}, \overline{\mathbb{J} \sigma_* \frac{\partial}{\partial \zeta}}, \tilde{n}, J\tilde{n}) d\lambda_{\mathcal{C}}(\zeta)$$

where  $\tilde{n}$  is any lifting of  $n$  under  $T\mathbb{T}|_{C_s} \rightarrow N_{C_s/\mathbb{T}}$ . For  $C'_0$  is a section of the map  $f$ , the composed map  $\mathcal{H}|_{C'_0} \hookrightarrow T\mathbb{T}|_{C'_0} \rightarrow N_{C'_0/\mathbb{T}}$  is an isomorphism and we can assume that the lifting  $\tilde{n}$  lies in  $\mathcal{H}|_{C'_0}$ . Hence, by theorem 2 only the vertical part  $\frac{\partial}{\partial \zeta}$  of  $\sigma_* \frac{\partial}{\partial \zeta} = \sum_{i=1}^{4n} \frac{\partial}{\partial \zeta} \sigma_i(\zeta) \frac{\partial}{\partial x_i} \oplus \frac{\partial}{\partial \zeta}$  contributes :

$$\int_{C'_0} \Gamma^* id' d'' \mathbb{W}(\vec{n}, J\vec{n}) = \int_{\mathcal{C}} id' d'' \mathbb{W}_{\sigma(\zeta)}(\frac{\partial}{\partial \zeta}, \overline{\mathbb{J} \frac{\partial}{\partial \zeta}}, \tilde{n}, J\tilde{n}) d\lambda_{\mathcal{C}}(\zeta).$$

Locally on  $C'_s$ , we may assume that  $C'_s$  is almost vertical so that there exists a coordinate chart  $\mathcal{U}$  on  $M$  containing  $\pi(C'_s)$  and with an orthonormal frame  $(\theta_1, \dots, \theta_{4n})$ . The

stereographic projection tell us that the complex structure  $\mathbb{J}$  on  $T\mathbb{T} \simeq \pi^*TM \oplus f^*T\mathbb{P}^1$  is given by

$$\mathbb{J}_{(m,\zeta)} = \left( \frac{1 - |\zeta|^2}{1 + |\zeta|^2} I_m + \frac{i(\zeta - \bar{\zeta})}{1 + |\zeta|^2} J_m + \frac{\zeta + \bar{\zeta}}{1 + |\zeta|^2} K_m, i \right).$$

Note that at the point  $\sigma(\zeta)$ , the vector  $\frac{\partial}{\partial \zeta}$  is of  $\mathbb{J}$ -type  $(1, 0)$  and reads

$$\frac{\partial}{\partial \zeta_{\sigma(\zeta)}} = \frac{1}{(1 + |\zeta|^2)^2} (-2\bar{\zeta} I_{\sigma(\zeta)} + i(1 + \bar{\zeta}^2) J_{\sigma(\zeta)} + (1 - \bar{\zeta}^2) K_{\sigma(\zeta)})$$

and then

$$\overline{\frac{\partial}{\partial \zeta_{\sigma(\zeta)}}} \frac{\partial}{\partial \zeta_{\sigma(\zeta)}} = -2 \frac{Id + i\mathbb{J}}{(1 + |\zeta|^2)^2}.$$

Theorem 2 gives

$$id'd''\mathbb{W}_{\sigma(\zeta)} \left( \frac{\partial}{\partial \zeta}, \mathbb{J} \overline{\frac{\partial}{\partial \zeta}}, \tilde{n}, \mathbb{J}\tilde{n} \right) d\lambda_{\mathbb{C}}(\zeta) = 2 \frac{\|n\|_{\mathcal{H}}^2}{(1 + |\zeta|^2)^2} d\lambda_{\mathbb{C}}(\zeta) = 4\pi \|n\|_{\mathcal{H}}^2 d\lambda_{\mathbb{P}^1}(\zeta).$$

As for the horizontal part  $H_j$ , using a parametrisation by  $\mathbb{P}^1$ , we get

$$\int_{H_j} \Gamma^* id'd'' \text{Vol}(\tilde{n}, \mathbb{J}\tilde{n}) = \int_{\mathbb{P}^1} id'd''\mathbb{W}(h, \mathbb{J}\tilde{h}, \tilde{n}, \mathbb{J}\tilde{n})$$

where  $h$  is horizontal and where  $\tilde{n}$  is any lifting of  $n$  under  $T\mathbb{T}|_{C_s} \rightarrow N_{C_s/\mathbb{T}}$ . By the Kähler property of the fibres of  $f$  or by theorem 2, only the vertical part of the lifting is relevant, and this contributes non-negatively to the hessian.

The map  $\text{Vol}$  being a continuous exhaustion [11], we infer from its pluri-sub-harmonicity, that the cycle space  $C_1(\mathbb{T})$  is pseudo-convex.  $\square$

**Remark 4.1.** *We could have used [2] (proposition 1) as a general argument to get the pluri-sub-harmonicity. The inequality (4.1) is however more precise and displays the fact that, because a non zero tangent vector  $\tilde{n} \in T_0 C_1(\mathbb{T})$  can have zero component  $n'$  on the slanted component  $C'_0$ , there can be compact families of horizontal 1-cycles, as for example, in the Hilbert scheme  $\text{Hilb}^n(S) \supset \text{Hilb}^n(C) = \mathbb{P}^n$  of a K3 surface that contains a smooth rational curve  $C$ .*

**4.3. The case of  $\mathbb{R}^4$ .** The twistor space  $\mathbb{T}(\mathbb{R}^4)$  of the flat euclidean  $\mathbb{R}^4$  is described as a complex manifold as the total space of the rank two vector bundle  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  on  $\mathbb{P}^1$ . From the product structure, the identification is done by the map

$$\begin{array}{ccc} \mathcal{O}(1) \oplus \mathcal{O}(1) & \xrightarrow{\phi} & \mathbb{C}P^1 \times \mathbb{C}^2 \\ (a\mu + b, c\mu + d) & \mapsto & (\mu, z_1, z_2) \end{array} \quad \text{with} \quad \begin{cases} z_1 = \frac{\bar{c}\bar{\mu} + \bar{d} + a|\mu|^2 + b\bar{\mu}}{1 + |\mu|^2} \\ z_2 = \frac{-\bar{a}\bar{\mu} - \bar{b} + c|\mu|^2 + d\bar{\mu}}{1 + |\mu|^2} \end{cases}$$

Twistor fibres are given by  $z_1 = \text{constant}$  and  $z_2 = \text{constant}$ , that is  $c = -\bar{b}$  and  $d = \bar{a}$ . The cycle space  $C_1(\mathbb{T}(\mathbb{R}^4))$  is simply the vector space  $H^0(\mathbb{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(1))$  of

holomorphic sections. Irreducible cycles are parametrised in the form  $(a\mu + b, c\mu + d)$ . The volume function that can be computed as

$$\text{Vol}_{FS}(\mathbb{C}P^1) \left( 1 + 2\pi|a - \bar{d}|^2 + 2\pi|\bar{b} + c|^2 \right)$$

achieves its minimum for twistor lines. To compute the Hessian of the metric form, we consider a point  $s \in C_1(\mathbb{T}(\mathbb{R}^4))$ , where  $C_s$  is parametrised by  $(a\mu + b, c\mu + d)$ . We consider a tangent vector  $n = (\alpha\mu + \beta, \gamma\mu + \delta) \in H^0(\mathbb{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \simeq T_s C_1(\mathbb{T}(\mathbb{R}^4))$ , whose norm is the flat Hermitian norm on  $\mathbb{C}^2$  of  $\phi_*(n)$ . Then theorem 3 gives :

$$id'd'' \text{Vol}_s(\bar{n}, \bar{n}) = 2i\pi \text{Vol}_{FS}(\mathbb{P}^1) \left( |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 \right)$$

which is coherent with the former expression.

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