

Uniqueness for an ill-posed reaction-dispersion model. Application to organic pollution in stream-waters

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Abstract

We are concerned with the inverse problem of detecting sources in a coupled diffusion-reaction system. This problem arises from the Biochemical Oxygen Demand-Dissolved Oxygen model⁽¹⁾ governing the interaction between organic pollutants and the oxygen available in stream waters. The sources we consider are point-wise and simulate stationary or moving pollution sources. The ultimate objective is to obtain their discharge location and recover their output rate from accessible measurements of DO when BOD measurements are difficult and time consuming to obtain. It is, as a matter of fact, the most realistic configuration. The subject to address here is the identifiability of these sources, in other words to determine if the observations uniquely determine the sources. The key tool is the study of coupled parabolic systems derived after restricting the global model to regions at the exterior of the observations. The absence of any prescribed condition on the BOD density is compensated by data recorded on the DO which provide over-determined Cauchy boundary conditions. Now, the first step toward the identifiability of the sources is precisely to recover the BOD at the observation points (of DO). This may be achieved by handling and solving the coupled systems. Unsurprisingly, they turn out to be ill-posed. That issue is investigated first. Then, we state a uniqueness result owing to a suitable saddle-point variational framework and to Pazy's uniqueness Theorem. This uniqueness complemented by former identifiability results proved in [2011, Inverse problems] for scalar reaction-diffusion equations yields the desired identifiability for the global model.

KEYWORDS: Biochemical Oxygen Demand, Dissolved Oxygen, Stream Waters, Ill-posed Parabolic systems, non-symmetric Mixed problem, saddle point theory, Pazy's Uniqueness Theorem, point-wise sources detection, Identifiability.

1 Introduction

Two indicators (tracers) are currently used in the analysis and management of stream water quality. The *Biochemical Oxygen Demand* (BOD) is widely employed to measure the pollution extent due to organic agents and then to evaluate the water characteristics. Another important constituent to consider is the *Dissolved Oxygen* (DO), the oxygen absorbed by water and is therefore available to biological micro-organisms and in general to aquatic life. Measuring the BOD is a procedure that determines the amount of oxygen consumed by biological organisms for the oxidation of polluting

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¹The acronym BOD-DO model is currently used.

material contained in agricultural, domestic or industrial waste (see [24, 20, 4]). The increase in BOD (at the expense of DO) is interpreted directly as the oxygen uptake in waters and is informative on the level of the organic pollution. We examine the reconstruction of pollution sources by observing the depletion of the DO concentration caused by the elevation of the BOD density. Notice that the ultimate objective of engineers is to know the Biochemical Oxygen Demand from some given observations. The advantage of this coupled BOD-DO model is that Dissolved Oxygen observations are much easier to conduct. Obtaining exploitable observations of Biochemical Oxygen Demand requires several days which is too long a period in the identification of an accidental spill where time is crucial (the standard is five days for BOD observations) (see [24]).

As detailed in [7, 18, 8], well developed models for rivers lead to time-dependent parabolic boundary value systems (see also [21]). These models are in general one dimensional in the curvilinear abscissa of the river. The dispersion of the pollutant and of the deficit (with respect to the saturation level) of the oxygen concentrations, denoted respectively by $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$, are governed by the partial differential equations

$$\begin{aligned} \partial_t b - (D(x)b')' + R(x)b &= F(x, t) && \text{in } (0, L) \times (0, T) \\ \partial_t c - (D_*(x)c')' + R_*(x)c - R(x)b &= G(x, t) && \text{in } (0, L) \times (0, T) \\ b(0, t) = c(0, t) &= 0 && \text{in } (0, T) \\ D(L)b'(L, t) = D_*(L)c'(L, t) &= 0 && \text{in } (0, T) \\ b(x, 0) = c(x, 0) &= 0 && \text{in } (0, L). \end{aligned}$$

where $F(x, t)$ is a pollution source, $G(x, t)$ is an oxygen source or sink. D, D_* are the longitudinal dispersion parameters and R, R_* are the deoxygenation and reoxygenation coefficients. They are all dependent on the space variable x . The term $R(x)b$ in the second equation on the DO is the depletion of oxygen due to elevated BOD. The boundary condition at $x = 0$ indicates that the water is clean upstream and the downstream condition at $x = L$ results from the truncation of the domain away from the source locations. Initial conditions indicates that the media is not polluted initially, the Biochemical Oxygen Demand is zero and the Dissolved Oxygen is at the saturation level, the deficit to the saturation is then zero. Throughout, we do not include the transport to the model we intend to study in detail. This extension and some others are discussed in a separate section at the end of the paper.

Often, the model treated in the literature consider the same diffusivity for the BOD and the DO concentrations, that is $D_* = D$. Nevertheless, when we are concerned with the interaction between the DO and other oxidizable contaminants, such as phosphorus or nitrates, the dispersion parameters are most likely to be different. We choose to handle the general case, given that it arises some technical points that are worth to deal with.

Now, the direct BOD-DO problem that is when the source F and G are known beforehand, is well-posed. It is triangular and both equations are weakly coupled. They may be handled sequentially. Solve first the BOD equation to obtain $b(\cdot, \cdot)$. Then, cope with the DO equation where the

global source term $(G + Rb)$ is fully known. We are however concerned with the inverse problem of determining (F, G) from data only given on $c(\cdot, \cdot)$. As will be seen later on the inverse problem is strongly coupled and the uncoupling in the direct problem does not work anymore.

Given ζ_L, ζ_R two distinct monitoring points such that $0 < \zeta_L < \zeta_R < L$, we suppose that the following observations are available

$$B[F, G] = \left\{ (c(\zeta_L, \cdot), D_*(\zeta_L)c'(\zeta_L, \cdot)), (c(\zeta_R, \cdot), D_*(\zeta_R)c'(\zeta_R, \cdot)), 0 \leq t \leq T \right\}. \quad (1)$$

With the above observations alone, one cannot hope to uniquely determine any arbitrary source. Some *a priori* information on the structure of the sources must be accessible. We suppose that each source is spatially supported by a moving point contained within (ζ_L, ζ_R) and has a time-varying strength. The sources are then expressed as

$$F(x, t) = f(t) \delta(x - r(t)) \quad \text{in } (0, L) \times (0, T) \quad (2)$$

$$G(x, t) = g(t) \delta(x - s(t)) \quad \text{in } (0, L) \times (0, T). \quad (3)$$

The symbol δ is for the Dirac distribution. The observations recorded at the two monitoring points (ζ_L, ζ_R) , should be located one upstream and the other downstream of the sources that is $\zeta_L < r(t), s(t) < \zeta_R$ for all $t \in (0, T)$ and the (pollution, oxygen deficit) rates (f, g) have finite energies. The identifiability result shows that the trajectory of the pollution source and the varying rate of release can both be retrieved if the variations in DO concentration are observed at two points. Many practical applications may be listed. One is the discrimination of the polluting agent in a river. Moving sources would indicate the responsibility of a ship in the pollution while fixed sources likely suggest the contribution of a factory. Another is to provide responsible authorities with valuable information about the location and the extent of contamination to make enlightened decisions of removing accidental organic pollution and clean-up stream-waters. This prevents eutrophication processes that are real threats to flora and fauna.

To our knowledge only partial results on the identifiability have been obtained on the inverse problem. Previously, El Badia and Hamdi [13] and later Hamdi [16] considered the case where no source G is active ($G = 0$) and the source F is point-wise with a fixed position r and a time-dependent intensity, i.e. $F(x, t) = f(t)\delta(x - r)$. With the supplementary assumptions that the physical parameters (D, D_*) and (R, R_*) are constants, that the source becomes inactive after a given time $T^* < T$ and at least one of the observations is made at a strategic point, they proved an identifiability result. *The result we pursue in this contribution is to consider and extend the identifiability result to the most general situation. We first remove the source inactivity assumption as well as the requirement that one observation point must be strategic⁽²⁾. More importantly, we suppose that the source is moving and our analysis holds for a large class of variable coefficients, (D, D_*) and (R, R_*) . Lastly, we show that with the same observation set, we can determine the additional point-wise source, G .* These generalizations are fundamentally those that are made in

²In the practice, it is hard and even impossible to tell whether an observation point is strategic or not. Indeed, the set of non-strategic points is dense in $(0, L)$.

[2] regarding the simpler problem considered earlier in [12, 17]. Let us emphasize the fact that the difference between [13] and our work is also methodological. We mainly use functional analysis tools essentially coming from saddle point methodology while in [13], the authors made use of Fourier computations which is valid only for constant (dispersion, reaction) coefficients (D, R) and (D_*, R_*) .

Since the identifiability process is conducted in several steps, we initially consider the coupled problem set in both external cylindrical time-space sub-domains, that coincide with the portions exterior to the observation points. The first and most difficult task will be to recover the Biochemical Oxygen Demand concentration $b(\cdot, \cdot)$ at observation points ζ_L and ζ_R from known data in (1). We have then to cope with two ill-posed parabolic systems set respectively in $(0, \zeta_L) \times (0, T)$ and in $(\zeta_R, L) \times (0, T)$. To state the uniqueness we choose a constructive formulation that will be retained in the numerical implementation. The treatment of the boundary value problems, written in exterior time-space strips is the subject of Section 2. Applying tools from the semi-group theory requires to study the resolvent of the spacial operator involved in the coupled system. The approach we follows relies on a suitable mixed variational formulation. The analysis of the resulting problem is achieved owing to the non-symmetric saddle point theory developed in [5]. Then, the uniqueness result established for the parabolic system is based on a theorem by A. Pazy (see [22, Chapter 4, Theorem 1.2]) that fits to a large class of ill-posed time-dependent problems. This uniqueness result is announced in [3] and the proofs are provided here. Armed with this result, we prove in Section 3 the identifiability of the sources (2)-(3) given the observation data on the dissolved oxygen.

Notation. The Lebesgue space of functions square integrable over I is denoted by $L^2(I)$. The scale of Sobolev spaces $H^\sigma(I)$, with $\sigma \in \mathbb{R}$, are defined as in [1]. We use also the vector valued Sobolev spaces $L^2(0, T; H^\sigma(I))$ and $H^\sigma(0, T; L^2(I))$ whose precise definitions are given in [19]. The space $L^\infty(I)$ is the space of functions essentially bounded in I . We use the bold symbols $\mathbf{L}^2(I)$ or $\mathbf{H}^1(I)$ for vector valued spaces, we have for instance $\mathbf{L}^2(I) = L^2(I) \times L^2(I)$. Finally, we indicate by $\mathcal{C}(I)$ the space of continuous functions in I . $\mathcal{D}(I)$ is the space of indefinitely differentiable functions with a compact support in I and $\mathcal{D}'(I)$ stands for the dual of $\mathcal{D}(I)$, the Schwarz space of distributions. We refer to [1] for more details on these functional spaces.

2 An Ill-Posed Parabolic System

The milestone of the identification we have in mind is the investigation of two parabolic systems obtained by restricting the coupled BOD-DO model to the strip $(0, \zeta_L) \times (0, T)$ for the first and to $(\zeta_R, L) \times (0, T)$ for the other. Throughout, we conduct our study in the reference domain $I = (0, \pi)$ and we choose to rather consider lower case physical coefficients to make this section independent

and self-contained. The first coupled system, the one related to $(0, \zeta_L) \times (0, T)$, reads then as follows

$$\partial_t b - (d(x)b')' + r(x)b = 0 \quad \text{in } I \times (0, T) \quad (4)$$

$$\partial_t c - (d_*(x)c')' + r_*(x)c - r(x)b = 0 \quad \text{in } I \times (0, T) \quad (5)$$

$$b(0, t) = c(0, t) = 0 \quad \text{in } (0, T) \quad (6)$$

$$c(\pi, t) = \alpha(t) \quad \text{in } (0, T) \quad (7)$$

$$d_*(\pi)c'(\pi, t) = \beta(t) \quad \text{in } (0, T) \quad (8)$$

$$b(x, 0) = c(x, 0) = 0 \quad \text{in } I. \quad (9)$$

Both (d, d_*) and (r, r_*) belong to $L^\infty(I)$, are positive and are assumed to be piecewise continuous on \bar{I} . The dispersion parameters are also bounded away from zero. There exists a constant $\underline{d} > 0$ such that $d(x) > \underline{d}$ and $d_*(x) > \underline{d}$. The positivity assumption on (d, d_*) is necessary and the one on (r, r_*) is only for convenience. It is possible at the cost of a suitable change of variable to write down an equivalent time-dependent problem where $r(x) > \underline{r} > 0$ and $r_*(x) > \underline{r}$. Notice that from now on we won't indicate explicitly in the equations the dependence of the parameters on the abscissa x .

The particularity here is that no boundary conditions are provided on $b(\cdot, \cdot)$ at point $x = \pi$ while $c(\cdot, \cdot)$ enjoys both Dirichlet and Neumann conditions. This fact induces a strong coupling between the concentrations $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ and implies the necessity of a mixed formulation for the study, as will be seen below. Notice that without loss of generality, one can consider for simplicity either $\alpha = 0$ or $\beta = 0$. We do not fix a choice now because we may need to adapt it to the context. We begin by investigating the ill-posedness of this coupled system. Then, as currently done, we study the problem where the boundary data (α, β) are replaced by volume sources (f, g) , the main objective pursued being to state a uniqueness result.

Remark 2.1 *The other parabolic problem, corresponding to the restriction to $(\zeta_R, L) \times (0, T)$, can also be set and studied on the reference segment $(0, \pi)$. The boundary conditions (6), (7) and (8) have to be interchanged and modified. The new conditions read as*

$$d_*(\pi)b'(\pi, t) = d_*(\pi)c'(\pi, t) = 0 \quad \text{in } (0, T)$$

$$c(0, t) = \alpha_*(t) \quad \text{in } (0, T)$$

$$d_*(0)c'(0, t) = \beta_*(t) \quad \text{in } (0, T).$$

The new Cauchy data (α_, β_*) are those coming from the observations realized on the dissolved oxygen by the monitoring station located at $x = \zeta_R$. The analysis we undertake for the first system (4)-(9) can be extended as well to this one with some slight adaptations, particularly on the functional subspaces we use.*

2.1 About Ill-Posedness

We assume throughout our discussion that $\beta = 0$. A formulation well fitted to investigate the ill-posedness consists in treating for instance the normal derivative $\gamma(t) = (db')(\pi, t), \forall t$ as an unknown.

Let then $\gamma \in L^2(0, T)$ be given and define b_γ the unique solution of the scalar reaction-diffusion boundary value problem

$$\begin{aligned} \partial_t b_\gamma - (db'_\gamma)' + rb_\gamma &= 0 & \text{in } I \times (0, T) \\ b_\gamma(0, t) &= 0 & \text{in } (0, T) \\ db'_\gamma(\pi, t) &= \gamma(t) & \text{in } (0, T) \\ b_\gamma(x, 0) &= 0 & \text{in } I. \end{aligned}$$

Then, we construct c_γ as the solution of the parabolic equation with rb_γ as a source term

$$\begin{aligned} \partial_t c_\gamma - (d_*c'_\gamma)' + r_*c_\gamma &= rb_\gamma & \text{in } I \times (0, T) \\ c_\gamma(0, t) &= 0 & \text{in } (0, T) \\ d_*c'_\gamma(\pi, t) &= 0 & \text{in } (0, T) \\ c_\gamma(x, 0) &= 0 & \text{in } I. \end{aligned}$$

With this notation, it is straight forward that the coupled system (4)-(9) may be reduced to the following: *Find $\gamma \in L^2(0, T)$ that satisfies the following equation*

$$S\gamma(t) = c_\gamma(\pi, t) = \alpha(t) \quad \text{in } (0, T). \quad (10)$$

It can be proved that the operator S is compact in $L^2(0, T)$. This results from the regularity of b_γ and c_γ . Indeed, S turns out to be bounded from $L^2(0, T)$ into $H^{3/2}(0, T)$. The compactness of S comes then from the compactness of the canonical embedding of $H^{3/2}(0, T)$ into $L^2(0, T)$. We refer to [10] for the technical details skipped over here. Let us stress the fact that the operator S is weakly regularizing given that we have no better regularity than $S\gamma \in H^{3/2}(0, T)$. As a result, the coupled problem is (mildly) ill-posed according to the classification by G. Wahba (see [25]) and existence and stability for (10) are expected to fail.

To have a better insight we put S as a kernel operator. In the case where (d, d_*) and (r, r_*) are constants, Fourier computations can be carried out. For simplicity, assume that $d = d_* = r = r_* = 1$. Formal calculations give (after setting $\lambda_k = (k + 1/2)^2 + 1$)

$$\begin{aligned} b_\gamma(x, t) &= \frac{2}{\pi} \sum_{k \in \mathbb{N}} \left((-1)^k \int_0^t \gamma(s) e^{-\lambda_k(t-s)} ds \right) \sin((k + 1/2)x) \\ c_\gamma(x, t) &= \frac{2}{\pi} \sum_{k \in \mathbb{N}} \left((-1)^k \int_0^t \gamma(s) (t-s) e^{-\lambda_k(t-s)} ds \right) \sin((k + 1/2)x). \end{aligned}$$

Plugging these formulas into the expression of S shows that it is a convolution operator. Indeed, we have that

$$(S\gamma)(t) = \int_0^t K(t-s)\gamma(s) ds \quad \forall t \in (0, T).$$

The convolution Kernel $K(\cdot)$ is given by

$$K(s) = \frac{2}{\pi} \sum_{k \in \mathbb{N}} s e^{-\lambda_k s} \quad \forall s \in (0, T).$$

Problem (10) is then a Volterra Equation of the first kind. It can be explicitly shown that S ranges from $L^2(0, T)$ into $H^{3/2}(0, T)$. The limited regularity of the kernel $K(\cdot)$ at the point $s = 0$, in particular the fact that $K'(0) = +\infty$, confirms the mild ill-posedness. We refer to [15] for further discussion on the ill-posedness of Volterra problems. Needless to say, the calculations that yield the Volterra equation are not economically reasonable for non-constant coefficients. Diagonalization of the Sturm-Liouville operator is in general avoided.

2.2 Abstract Formulation

To investigate the uniqueness issue, we need to put the coupled system (4)-(9) into an abstract form. Let f and g be given in $L^2(I \times (0, T))$, consider the time-dependent system

$$\partial_t \begin{pmatrix} b \\ c \end{pmatrix} + A \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad (11)$$

where A is an unbounded linear operator defined in $\mathbf{L}^2(I)$. The domain $D(A) \subset \mathbf{L}^2(I)$ is given by

$$D(A) = \left\{ (\varphi, \psi) \in \mathbf{H}^1(I), \quad ((d\varphi)', (d_*\psi)') \in \mathbf{L}^2(I) \right. \\ \left. \varphi(0) = \psi(0) = 0, \psi(\pi) = (d_*\psi)'(\pi) = 0 \right\},$$

and the operator A is defined to be

$$A \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} -(d\varphi)' + r\varphi \\ -(d_*\psi)' + r_*\psi - r\varphi \end{pmatrix}. \quad (12)$$

Remark 2.2 *Another possible and natural characterization of the (same) domain of A is as follows*

$$D(A) = \left\{ (\varphi, \psi) \in \mathbf{L}^2(I), \quad ((d\varphi)', (d_*\psi)') \in \mathbf{L}^2(I) \right. \\ \left. \varphi(0) = \psi(0) = 0, \psi(\pi) = (d_*\psi)'(\pi) = 0 \right\}.$$

The fact that it is embedded in $\mathbf{H}^1(I)$ may be derived by a hilbertian interpolation argument (see [10]).

It is well known that the properties of equation (11) are tightly connected with the resolvent $R(\lambda) = (\lambda + A)^{-1}$ (see [10]). We need then to state existence and boundedness results for $R(\lambda)$ in $\mathbf{L}^2(I)$, when $\lambda > 0$. Before proving such results, we begin by studying the operator A itself. Let $(f, g) \in \mathbf{L}^2(I)$, consider the existence and uniqueness of a solution of the problem

$$A \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \quad (13)$$

To solve (13), we write down a mixed formulation which allows to express the problem as a saddle point problem (see [6]).

2.3 The Stationary System

To construct a variational formulation of (13), we proceed formally and we attempt at last to make things mathematically rigorous with the accurate definition of the functional spaces we work in. We multiply the first equation of (13) by a test function, ψ with $\psi(0) = \psi(\pi) = 0$, and integrate over I to obtain

$$\int_I db' \psi' dx + \int_I rb \psi dx = \int_I f \psi dx. \quad (14)$$

Similarly, we multiply the second equation of (13) by another test function, φ with $\varphi(0) = 0$, and integrate which yields

$$\int_I d_* c' \varphi' dx + \int_I r_* c \varphi dx - \int_I rb \varphi dx = \int_I g \varphi dx. \quad (15)$$

Now, we need to render things more accurate and describe the spaces to which the unknowns (b, c) belong and in which the variational problem makes sense. We define the spaces $Q = H_0^1(I)$ and V as follows

$$V = \left\{ \varphi \in H^1(I) \quad \varphi(0) = 0 \right\}.$$

Let us also introduce the two bilinear forms that are involved in equations (14) and (15). We set then

$$\begin{aligned} a(\chi, \varphi) &= - \int_I r \chi \varphi dx \quad \forall (\chi, \varphi) \in V \times V \\ m_{(*)}(\psi, \varphi) &= \int_I d_{(*)} \psi' \varphi' dx + \int_I r_{(*)} \psi \varphi dx \quad \forall (\psi, \varphi) \in Q \times V. \end{aligned}$$

All these forms are continuous. With these new definitions, we can express equations (14) and (15) as follows *find* $(b, c) \in V \times Q$ *verifying*

$$m(b, \psi) = (f, \psi)_{L^2} \quad \forall \psi \in Q \quad (16)$$

$$m_*(\varphi, c) + a(b, \varphi) = (g, \varphi)_{L^2} \quad \forall \varphi \in V. \quad (17)$$

It is a non-symmetric mixed problem that does not fit into the saddle-point theory elaborated in [6]. The suitable framework for non-symmetry can be found in [5].

Before going further into the analysis we need to be sure that the (weak) solution of this mixed system satisfies also equation (13). This requires to check first that (b, c) lies in $D(A)$ and second that it fulfils the system (13). It holds that

Lemma 2.1 *Let $(b, c) \in V \times Q$ be the solution of the mixed problem (16)-(17). Then $(b, c) \in D(A)$ and satisfies equation (13).*

Proof: Assume for the moment that the problem in equation (16)-(17) admits a solution $(b, c) \in V \times Q$. We begin with the first equation of (16) which says that $b \in V$ satisfies

$$\int_I db' \psi' dx + \int_I rb \psi dx = \int_I f \psi dx \quad \forall \psi \in Q.$$

Because $b \in V$, we have immediately that $b(0) = 0$. Let $\psi \in \mathcal{D}(I)$ be arbitrarily chosen. Integrate by parts the first term in the above equation to find that

$$-(db')' + rb = f \quad \text{in } \mathcal{D}'(I).$$

The fact that $b \in L^2(I)$ yields that $(db')' \in L^2(I)$ and the above equality holds then in $L^2(I)$. Aggregating these results, we have that $b \in V$ satisfies

$$\begin{aligned} -(db')' + rb &= f & \text{in } I \\ b(0) &= 0. \end{aligned}$$

Next, we turn to the second equation (17). Having $c \in Q$ produces immediately that $c(0) = c(\pi) = 0$. Now, let $\varphi \in \mathcal{D}(I)$. Integrating by parts the first term in (15) provides that

$$-(d_*c')' + r_*c - rb = g \quad \text{in } \mathcal{D}'(I). \quad (18)$$

Proceeding as was done for b , we arrive at the fact that $(d_*c')' \in L^2(I)$ and that the above equality holds also in this space. What remains to verify is that c satisfies the third boundary condition, namely that $d_*c'(\pi) = 0$. Taking $\varphi \in V$ in (15) and integrating by parts yields

$$\int_I [-(d_*c')' + r_*c - rb - g]\varphi dx = -d_*c'(\pi)\varphi(\pi).$$

On account of equation (18) we deduce that $d_*c'(\pi) = 0$ when $\varphi(\pi) \neq 0$. Combining this with the above results, we have

$$\begin{aligned} -(d_*c')' + r_*c - rb &= g & \text{in } I \\ c(0) &= 0 \\ c(\pi) = d_*c'(\pi) &= 0. \end{aligned}$$

Any solution to (16)-(17) is hence a solution to (13). The proof is complete. \blacksquare

The next concern is to consider the issue of existence, uniqueness and stability issues for the mixed problem (16)-(17) by using the analysis tools of [5, 6] where some conditions are given for the existence and uniqueness of the (b, c) . The first one states that the bilinear form $a(\cdot, \cdot)$ must satisfy a couple of inf-sup conditions on the null-spaces of $m_{(*)}(\cdot, \cdot)$. The second is that each of $m(\cdot, \cdot)$ and $m_*(\cdot, \cdot)$ fulfills an inf-sup condition in $Q \times V$. Let us then denote by

$$\mathcal{N}_{(*)} = \left\{ \varphi \in V, \quad m_{(*)}(\varphi, \psi) = 0 \quad \forall \psi \in Q \right\}.$$

We have that

Lemma 2.2 *The bilinear form $a(\cdot, \cdot)$ satisfy the following inf-sup conditions, that is*

$$\inf_{\varphi \in \mathcal{N}} \sup_{\chi \in \mathcal{N}_*} \frac{a(\varphi, \chi)}{\|\varphi\|_{H^1} \|\chi\|_{H^1}} \geq \beta, \quad \inf_{\varphi \in \mathcal{N}_*} \sup_{\chi \in \mathcal{N}} \frac{a(\varphi, \chi)}{\|\varphi\|_{H^1} \|\chi\|_{H^1}} \geq \beta_*.$$

The constants β, β_ are positive.*

Proof: Let us first observe that $\dim \mathcal{N}_{(*)} = 1$. Both results are therefore achieved if we simply check that for a $\varphi \in \mathcal{N}$, $a(\varphi, \chi) = 0$ for all $\chi \in \mathcal{N}_*$, necessarily yields that $\varphi = 0$. To prove this, we need to introduce $\xi \in H^1(I)$, the solution of the problem

$$\begin{aligned} - (d\xi)' + r\xi &= 0 & \text{in } I \\ \xi(0) &= 0 \\ \xi(\pi) &= 1. \end{aligned}$$

Denote by $\xi_* \in H^1(I)$ its counterpart when the parameters are (d_*, r_*) . Invoking the maximum principle produces that $\xi(x) > 0$ for all $x \in I \setminus \{0\}$ (see [14]). The same result holds for ξ_* . Next, observe that any $(\varphi, \chi) \in \mathcal{N} \times \mathcal{N}_*$, may be obtained as $(\varphi, \chi) = (\varphi(\pi)\xi, \chi(\pi)\xi_*)$. Now assume that $a(\varphi, \chi) = 0$, for all $\chi \in \mathcal{N}_*$. Choosing $\chi = \xi_*$ necessarily yields that $\varphi(\pi) = 0$, given that $a(\xi, \xi_*) > 0$. This ends to $\varphi = 0$. Stating that for a $\chi \in \mathcal{N}_*$, $a(\varphi, \chi) = 0$, for all $\varphi \in \mathcal{N}$, concludes to $\chi = 0$ can be done similarly. The proof is complete. ■

Lemma 2.3 *The bilinear form $m(\cdot, \cdot)$ and $m_*(\cdot, \cdot)$ satisfy the inf-sup conditions in $V \times Q$,*

$$\inf_{\psi \in Q} \sup_{\varphi \in V} \frac{m(\varphi, \psi)}{\|\varphi\|_{H^1} \|\psi\|_{H^1}} \geq \alpha, \quad \inf_{\psi \in Q} \sup_{\varphi \in V} \frac{m_*(\varphi, \psi)}{\|\varphi\|_{H^1} \|\psi\|_{H^1}} \geq \alpha_*.$$

The constants α, α_ are positive.*

Proof: The inf-sup condition on $m(\cdot, \cdot)$ is derived easily. Because $Q \subset V$, we choose $\varphi = \psi$ from which we deduce that

$$\frac{m(\varphi, \psi)}{\|\varphi\|_{H^1} \|\psi\|_{H^1}} = \frac{m(\psi, \psi)}{\|\psi\|_{H^1}^2} \geq \alpha.$$

The desired result is therefore valid. The proof for $m_*(\cdot, \cdot)$ is achieved following the same lines. ■

All necessary tools for the well-posedness of the mixed problem (16)-(17) are available. We have then

Proposition 2.4 *The mixed problem (16)-(17) has a unique solution $(b, c) \in V \times Q$ such that*

$$\|b\|_{H^1} + \|c\|_{H^1} \leq C(\|f\|_{L^2} + \|g\|_{L^2}).$$

Proof: It is a direct consequence of the saddle point theory in [5] and Lemmas 2.2 and 2.3. ■

The next and most important point is to check the existence of the resolvent $R(\lambda)$, for $\lambda > 0$, and to derive its bound with respect to λ . Let us denote by $A_\lambda = \lambda + A$, then $R(\lambda) = (A_\lambda)^{-1}$. Let now $(f, g) \in \mathbf{L}^2(I)$ and consider the equation

$$A_\lambda \begin{pmatrix} b_\lambda \\ c_\lambda \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \quad (19)$$

The invertibility of A_λ can be proved following the same lines as for A . Define a mixed problem like in (16)-(17). The bilinear form $a(\cdot, \cdot)$ remains unchanged while the mixed form $m_\lambda(\cdot, \cdot)$ is obtained

by shifting $m(\cdot, \cdot)$ and $m_*(\cdot, \cdot)$ namely that $m_\lambda(\cdot, \cdot) = m(\cdot, \cdot) + \lambda(\cdot, \cdot)$ and $m_{*,\lambda}(\cdot, \cdot) = m_{*,\lambda}(\cdot, \cdot) + \lambda(\cdot, \cdot)$. The new mixed problem consists therefore in: *finding* $(b_\lambda, c_\lambda) \in V \times Q$ such that

$$m_\lambda(b_\lambda, \psi) = (f, \psi)_{L^2} \quad \forall \psi \in Q \quad (20)$$

$$m_{*,\lambda}(b_\lambda, \varphi) + a(b_\lambda, \varphi) = (g, \varphi)_{L^2} \quad \forall \varphi \in V. \quad (21)$$

Everything works just as well as for the problem (16)-(17). However, when it comes to the study of the unsteady problem, we need an accurate knowledge of the stability constants and how they grow with respect to λ . This has to do with the inf-sup constants of the bilinear form $a(\cdot, \cdot)$, when restricted to $\mathcal{N}_\lambda \times \mathcal{N}_{*,\lambda}$. The construction of \mathcal{N}_λ and $\mathcal{N}_{*,\lambda}$ follows the one of \mathcal{N} and \mathcal{N}_* where $m_\lambda(\cdot, \cdot)$ and $m_{*,\lambda}(\cdot, \cdot)$ are used instead of $m(\cdot, \cdot)$ and $m_*(\cdot, \cdot)$. The following holds

Lemma 2.5 *There exists $\underline{\lambda} > 0$ such that for all $\lambda \geq \underline{\lambda}$, the bilinear form $a(\cdot, \cdot)$ satisfy the inf-sup conditions,*

$$\inf_{\varphi \in \mathcal{N}_\lambda} \sup_{\chi \in \mathcal{N}_{*,\lambda}} \frac{a(\varphi, \chi)}{\|\varphi\|_{H^1} \|\chi\|_{H^1}} \geq \frac{\beta}{\lambda}, \quad \inf_{\varphi \in \mathcal{N}_{*,\lambda}} \sup_{\chi \in \mathcal{N}_\lambda} \frac{a(\varphi, \chi)}{\|\varphi\|_{H^1} \|\chi\|_{H^1}} \geq \frac{\beta_*}{\lambda}.$$

The constants β, β_* are positive and independent of λ .

Proof: The proof is based on tools from the asymptotic analysis. It is technical and postponed to the Appendix. \blacksquare

Remark 2.3 *We have also the following results*

$$\inf_{\varphi \in \mathcal{N}_\lambda} \sup_{\chi \in \mathcal{N}_{*,\lambda}} \frac{a(\varphi, \chi)}{\|\varphi\|_{L^2} \|\chi\|_{L^2}} \geq \beta, \quad \inf_{\varphi \in \mathcal{N}_{*,\lambda}} \sup_{\chi \in \mathcal{N}_\lambda} \frac{a(\varphi, \chi)}{\|\varphi\|_{L^2} \|\chi\|_{L^2}} \geq \beta_*,$$

for all $\lambda \geq \underline{\lambda}$. As will be seen later, these inf-sup conditions are the ones used to establish the desired stability of the resolvent $R(\lambda)$.

The following inf-sup conditions on $m_{(*),\lambda}(\cdot, \cdot)$ are also valid and their proofs can be conducted using the same arguments as for Lemma 2.3.

Lemma 2.6 *The bilinear form $m_\lambda(\cdot, \cdot)$ and $m_{*,\lambda}(\cdot, \cdot)$ satisfy the inf-sup conditions in $V \times Q$,*

$$\inf_{\psi \in Q} \sup_{\varphi \in V} \frac{m_\lambda(\varphi, \psi)}{\|\varphi\|_{H^1} \|\psi\|_{H^1}} \geq \alpha, \quad \inf_{\psi \in Q} \sup_{\varphi \in V} \frac{m_{*,\lambda}(\varphi, \psi)}{\|\varphi\|_{H^1} \|\psi\|_{H^1}} \geq \alpha_*.$$

The constants α, α_* are positive and independent of λ .

We are well equipped to derive the stability on $R(\lambda)$, needed for the uniqueness result of the unsteady coupled problem.

Proposition 2.7 *Let $(f, g) \in \mathbf{L}^2(I)$. The mixed problem (20)-(21) has a unique solution $(b_\lambda, c_\lambda) \in V \times Q$. The following estimates hold*

$$\|b_\lambda\|_{L^2} \leq C \left(\frac{1}{\lambda} \|f\|_{L^2} + \|g\|_{L^2} \right)$$

$$\|c_\lambda\|_{L^2} \leq \frac{C}{\lambda} \left(\frac{1}{\lambda} \|f\|_{L^2} + \|g\|_{L^2} \right).$$

The resolvent $R(\lambda)$ is then a bounded operator in $L^2(I)$ with a uniformly bounded norm that is

$$\|R(\lambda)\|_{(L^2(I) \rightarrow L^2(I))} \leq C' \quad \forall \lambda \geq \underline{\lambda}. \quad (22)$$

The constant C and C' are independent of λ .

Proof: Results by Lemmas 2.5 and 2.6 allow to derive the existence and uniqueness by applying the saddle point theory (see [5]). Let us emphasize on the fact that it is necessary here to have the inf-sup conditions of Lemma 2.5, written with the natural norms, those of H^1 . Now, that the existence of (b_λ, c_λ) are guaranteed, we aim to show the stabilities with respect to the L^2 -norm, for which the inf-sup condition provided in Remark 2.3 are better suited. To proceed with the stability in the space $L^2(I)$, we assume first that $f = 0$. Then, we handle the general case by linearity. On account of (20) (with $f = 0$), there comes out that $b_\lambda \in \mathcal{N}_\lambda$. Now, choosing $\psi \in \mathcal{N}_{*,\lambda}$ in (21), we can write that

$$a(b_\lambda, \psi) = (g, \psi)_{L^2}, \quad \forall \psi \in \mathcal{N}_{*,\lambda}.$$

Calling for the first inf-sup condition in Remark 2.3, we obtain the following bound

$$\|b_\lambda\|_{L^2} \leq C \|g\|_{L^2}.$$

Returning to equation (21), choose $\varphi = c_\lambda$, which lies in $Q \subset V$ and is hence admissible, we can state that

$$\lambda \|c_\lambda\|_{L^2}^2 \leq (b_\lambda, c_\lambda) + (g, c_\lambda).$$

Using the bound for b_λ , previously stated, yields that

$$\|c_\lambda\|_{L^2} \leq \frac{C}{\lambda} \|g\|_{L^2}.$$

Now, when the data f does not vanish, we argue by linearity. Thus, we split b_λ into $b_\lambda := b_\lambda + \underline{b}_\lambda$ ⁽³⁾ where $\underline{b}_\lambda \in H_0^1(I)$ is the unique solution of

$$\lambda \underline{b}_\lambda - (d\underline{b}_\lambda)' + r\underline{b}_\lambda = f \quad \text{in } I.$$

Notice that \underline{b}_λ fulfills Dirichlet condition at both extremities of I . Moreover, it is readily checked that

$$\|\underline{b}_\lambda\|_{L^2} \leq \frac{C}{\lambda} \|f\|_{L^2}.$$

Now, inserting the old b_λ in the coupled system (20)–(21), we come up with a new one where the unknowns are the new b_λ and c_λ . Besides, the data are changed to $f := 0$ and $g := g + r\underline{b}_\lambda$. The stability sought for is therefore established owing to the first part of the proof together with the stability on \underline{b}_λ . This completes the proof. \blacksquare

Remark 2.4 *Bounds on the norm of (b_λ, c_λ) in $H^1(I)$ and in $D(A)$ may be derived. Indeed, it is readily stated that*

$$\begin{aligned} \|b_\lambda\|_{H^1} + \lambda \|c_\lambda\|_{H^1} &\leq C\sqrt{\lambda} \left(\frac{1}{\lambda} \|f\|_{L^2} + \|g\|_{L^2} \right) \\ \|(db'_\lambda)'\|_{L^2} + \lambda \|(d_*c'_\lambda)'\|_{L^2} &\leq C\lambda \left(\frac{1}{\lambda} \|f\|_{L^2} + \|g\|_{L^2} \right). \end{aligned}$$

The constant C is independent of λ

³A notation abuse is made here.

2.4 A Uniqueness Result

The bound on the resolvent $R(\lambda)$ in Proposition 2.7 does not allow the application of the Hille-Yosida Theorem. Nevertheless, when we are only interested in the uniqueness, a result stated by A. Pazy (see [22, Chapter 4, Theorem 1.2]) turns out to be sufficient. Let us provide this theorem which ensures the uniqueness of the solution under a pretty weak assumption on the norm of the resolvent. Recall that (b, c) is a solution of problem (11) if

$$\begin{pmatrix} b \\ c \end{pmatrix} \in \mathcal{C}([0, T]; \mathbf{L}^2(I)) \cap \mathcal{C}(]0, T[; D(A)),$$

and satisfies equation (11) for all $t \in]0, T[$. We translate Pazy's Theorem into the context of the problem we are concerned with.

Theorem 2.8 (A. Pazy) *If $R(\lambda)$ exists for large real-numbers $\lambda(\geq \underline{\lambda})$ and*

$$\limsup_{\lambda \rightarrow +\infty} \frac{1}{\lambda} \log \|R(\lambda)\|_{(\mathbf{L}^2(I) \rightarrow \mathbf{L}^2(I))} = 0,$$

then the initial value problem (11) has at most one solution.

We are now in position to give and prove the key result of our work. The following proposition holds.

Theorem 2.9 *Problem (11) has at most one solution.*

Proof: Proposition 2.7 tells that $R(\lambda)$ is well defined for all $\lambda \geq \underline{\lambda}$. Moreover, owing to the resolvent estimate (22) we derive that

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \|R(\lambda)\|_{(\mathbf{L}^2(I) \rightarrow \mathbf{L}^2(I))} = 0.$$

Applying Pazy's Theorem 2.8 achieves the proof. ■

3 The BOD-DO System. Identifiability of point-wise sources

We intend to apply the previous results to the original pollution source identification problem of the Introduction. We show that the sources (2)-(3) are uniquely determined by the observation set (1). We demonstrate this for moving sources with $r(t), s(t), t \in (0, T)$ still satisfying $\zeta_L < r(t), s(t) < \zeta_R, \forall t$. The trajectories of both sources are assumed Lipschitz regular. We first consider two solutions $(b_i, c_i), i = 1, 2$ of the dispersion-reaction system

$$\begin{aligned} \partial_t b_i - (D(x)b_i')' + R(x)b_i &= F_i(t, x) && \text{in } (0, L) \times (0, T) \\ \partial_t c_i - (D_*(x)c_i')' + R_*(x)c_i - R(x)b_i &= G_i(t, x) && \text{in } (0, L) \times (0, T) \\ b_i(0, t) = c_i(0, t) &= 0 && \text{in } (0, T) \\ D(L)b_i'(L, t) = D_*(L)c_i'(L, t) &= 0 && \text{in } (0, T) \\ b_i(x, 0) = c_i(x, 0) &= 0 && \text{in } I. \end{aligned}$$

Following the notations in (2) and (3), we recall that $F_i(t, x) = f_i(t)\delta(x - r_i(t))$ and $G_i(t, x) = g_i(t)\delta(x - s_i(t))$. Without loss of generality, we suppose that $r_1 < r_2$ and that $s_1 < s_2$. The objective is to show that

$$(B[F_1, G_1] = B[F_2, G_2]) \text{ implies that } (b_1, c_1) = (b_2, c_2).$$

As a result, uniqueness of the sources will hold; namely that $(r_1, f_1) = (r_2, f_2)$ and $(s_1, g_1) = (s_2, g_2)$. Let us first introduce the notations $I_L = (0, \zeta_L)$ and $I_R = (\zeta_R, L)$.

Before starting our analysis we need to recall the regularity of the solution (b_i, c_i) . We need then to fix the assumptions on the diffusion and reaction parameters (D, D_*) and (R, R_*) . Suppose that $D, R \in L^\infty(I)$ and there exists \underline{D} and \underline{R} ⁽⁴⁾ two positive constants such that

$$D(x), D_*(x) \geq \underline{D}, \quad R(x), R_*(x) \geq \underline{R} \quad \forall x \in I.$$

The direct problem has a unique solution (see [10]),

$$\begin{pmatrix} b_i \\ c_i \end{pmatrix} \in L^2(0, T; \mathbf{H}^1(0, L)) \cap \mathcal{C}([0, T]; \mathbf{L}^2(0, L)). \quad (23)$$

In particular, neither $b(\cdot, \cdot)$ nor $c(\cdot, \cdot)$ has no jumps across the trajectories $(t, r(t))_{t \in [0, T]}$ and $(t, s(t))_{t \in [0, T]}$. They have well defined traces on those curves in the space-time domain. Now, denote by A_L (respectively A_R) the operator defined in (12) when restricted to the intervals I_L (respectively in I_R). Given that the sources $(s_i(t), r_i(t))_{t \in (0, T)}$ are permanently located in between the observation points ζ_L and ζ_R , and the initial conditions are zero we have the additional regularity

$$\begin{pmatrix} b_i \\ c_i \end{pmatrix} \in \mathcal{C}([0, T]; D(A_L)) \cap \mathcal{C}([0, T]; D(A_R)). \quad (24)$$

We begin by stating some preparatory lemmas. We have that

Lemma 3.1 *Suppose that $B[F_1, G_1] = B[F_2, G_2]$, then*

$$(b_1, c_1) = (b_2, c_2) \quad \text{in } (I_L \cup I_R) \times (0, T).$$

Proof: Assume that $B[F_1, G_1] = B[F_2, G_2]$. Let us focus on $I_L \times (0, T)$. The same argument applies as well to $I_R \times (0, T)$. Letting $\theta = (b_2 - b_1)$ and $\eta = (c_2 - c_1)$. According to (23) and (24) we have that

$$\begin{pmatrix} \theta \\ \eta \end{pmatrix} \in \mathcal{C}([0, T], \mathbf{L}^2(I)) \cap \mathcal{C}([0, T], D(A_L)).$$

Moreover, we can write the following system

$$\begin{aligned} \partial_t \theta - (D\theta)' + R\theta &= 0 & \text{in } I_L \times (0, T) \\ \partial_t \eta - (D_*\eta)' + R_*\eta - R\theta &= 0 & \text{in } I_L \times (0, T) \\ \theta(0, t) = \eta(0, t) &= 0 & \text{in } (0, T) \\ \eta(\zeta_L, t) &= 0 & \text{in } (0, T) \\ D_*\eta'(\zeta_L, t) &= 0 & \text{in } (0, T) \\ \theta(x, 0) = \eta(x, 0) &= 0 & \text{in } I_L. \end{aligned}$$

⁴The hypothesis on R and R_* is optional and can be removed.

On account of the uniqueness result established in Theorem 2.9, we derive immediately that $(\theta, \eta) = (0, 0)$ in $I_L \times (0, T)$. According to Remark 2.1, the same result holds in the right portion $I_R \times (0, T)$. The proof is complete. ■

Lemma 3.2 *Assume that $B[F_1, G_1] = B[F_2, G_2]$, then*

$$(r_1(t), f_1(t)) = (r_2(t), f_2(t)), \quad \text{in } (0, T).$$

Proof: Assume that the equality $B[F_1, G_1] = B[F_2, G_2]$ holds true. Set once again $\theta = (b_2 - b_1)$. We have that

$$\begin{aligned} \partial_t \theta - (D\theta')' + R\theta &= f_2(t)\delta(x - r_2(t)) - f_1(t)\delta(x - r_1(t)) && \text{in } I \times (0, T) \\ \theta(0, t) &= 0 && \text{in } (0, T) \\ D\theta'(L, t) &= 0 && \text{in } (0, T) \\ \theta(x, 0) &= 0 && \text{in } I. \end{aligned}$$

The proof is achieved step by step proceeding as in [2, Theorem 5]. To provide an idea about the tools used there while avoiding technical details, we limit ourselves to stationary sources. We assume then that (r_1, r_2) are time-independent with $r_1 < r_2$. We first look at the problem when restricted to $I_L \times (0, T)$. By Lemma 3.1, we derive that $\theta = 0$ in the cylinder $I_L \times (0, T)$. Then, applying the unique continuation theorem (see [23]) we deduce that $\theta = 0$ in $(0, r_1) \times (0, T)$. Similarly, we have that $\theta = 0$ in $(r_2, L) \times (0, T)$. Now rewriting the problem in $(r_1, r_2) \times (0, T)$, we obtain that

$$\begin{aligned} \partial_t \theta - (D\theta')' + R\theta &= 0 && \text{in } (r_1, r_2) \times (0, T) \\ \theta(r_1, t) = \theta(r_2, t) &= 0 && \text{in } (0, T) \\ \theta(x, 0) &= 0 && \text{in } (r_1, r_2). \end{aligned}$$

The traces at r_1 and r_2 are plainly justified by the regularity (23) of b_1 and b_2 . Then, we conclude that $\theta = 0$ in the whole $I \times (0, T)$ and as a consequence we obtain that (in a distributional sense)

$$f_1(t)\delta(x - r_1) - f_2(t)\delta(x - r_2) = 0 \quad \text{in } I \times (0, T).$$

This immediately implies that $(r_1, f_1(t)) = (r_2, f_2(t)), \forall t \in (0, T)$. The proof of the identifiability is complete for stationary sources (r, f) . It can be extended to moving sources at the cost of additional technical work. We refer to [2, Theorem 11] for details. ■

We now present the final result of the analysis, the identifiability of the sources for the full coupled system.

Theorem 3.3 *Assume that $B[F_1, G_1] = B[F_2, G_2]$, then*

$$\begin{aligned} (r_1(t), f_1(t)) &= (r_2(t), f_2(t)) && \text{in } (0, T) \\ (s_1(t), g_1(t)) &= (s_2(t), g_2(t)) && \text{in } (0, T). \end{aligned}$$

Proof: The first identifiability result is stated in Lemma 3.2. It remains to prove the second. Set again $\theta = (b_2 - b_1)$ and $\eta = (c_2 - c_1)$. We have already verified that $\theta = 0$. Additionally, Theorem 2.9 yields $\eta = 0$ in the external part of the space/time domain $(I_L \cup I_R) \times (0, T)$. The following equation holds for η ,

$$\begin{aligned} \partial_t \eta - (D_* \eta)' + R_* \eta &= g_2(t) \delta(x - s_2(t)) - g_1(t) \delta(x - s_1(t)) && \text{in } I \times (0, T) \\ \eta(0, t) &= 0 && \text{in } (0, T), \\ D_* \eta'(L, t) &= 0 && \text{in } (0, T) \\ \eta(x, 0) &= 0 && \text{in } I. \end{aligned}$$

Arguing as in Lemma 3.2, and [2, Theorem 11] we conclude with $\eta = 0$ in the whole $I \times (0, T)$ and then we obtain that

$$g_1(t) \delta(x - s_1(t)) - g_2(t) \delta(x - s_2(t)) = 0 \quad \text{in } I \times (0, T).$$

This yields that $(s_1(t), g_1(t)) = (s_2(t), g_2(t)), \forall t \in (0, T)$. The proof is complete. \blacksquare

4 Some extensions

We have already indicated that the BOD-DO model investigated here is sufficient to point out the difficulties arising in the analysis of the identifiability and to propose a way to solve them. Some generalizations may be however considered. Their study may be conducted following a similar methodology with presumably some more technical work.

Adding transport.— Another worthy extension is the addition of advective transport to be able to consider pollution in flowing stream environments. It makes sense to include this term to extend applicability to the wide range of flowing stream and river systems in which case the advective transport may be likely the most important. Pollutants are carried further downstream at faster rates and their impact is probably more extensive.

The addition of advective transport results in the following equations on b and c as follows

$$\begin{aligned} \partial_t c - (Dc)' + Vc + Rc &= f(t) \delta(x - r(t)) && \text{in } (0, L) \times (0, T), \\ \partial_t c - (D_* c)' + Vc + R_* c - Rb &= g(t) \delta(x - s(t)) && \text{in } (0, L) \times (0, T). \end{aligned}$$

where V is the stream velocity and we note that the spatial operator is not self-adjoint. A well known transformation permits one to remove the advection term in each equations, reducing them to a self-adjoint equations (the spatial operator is then said to be in a Sturm-Liouville form). The transformation with constant coefficients is given in [10, Remark 11, p. 87] and is demonstrated for the scalar BOD equation with coefficients in [2]. The extension of our results will not yield insurmountable difficulties, though some more technical tools are necessary.

Multiple point-wise sources.— Engineers may be facing the detection of more than one pollution source. BOD-DO equations on $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ could be enriched by multiple source terms

$$\begin{aligned}\partial_t b - (Db')' + Rb &= \sum_{1 \leq k \leq k_*} f_k(t) \delta(x - r_k(t)) && \text{in } (0, L) \times (0, T), \\ \partial_t c - (D_*c')' + R_*c - Rb &= \sum_{1 \leq k \leq k_*} g_k(t) \delta(x - s_k(t)) && \text{in } (0, L) \times (0, T).\end{aligned}$$

In [2], it is already noticed that for the scalar model (for BOD for instance), at least $2k_*$ observations are required to obtain the identifiability for k_* point-wise sources. They should be distributed in a particular way. Between each pair of neighboring sources one should place two distinct observation points. This requires of course some a-priori knowledge of the pollution source locations. The same rule has to be observed here to achieve identifiability. Between two consecutive pairs of sources (r, s) we must be able to access measurements on (c, D_*c') at two different points. The study is conducted following the same lines. The BOD-DO parabolic model to study in between two neighboring observation points is defined by two-sided Cauchy data on $c(\cdot, \cdot)$ while no boundary conditions are provided for $b(\cdot, \cdot)$. The variational framework written for one-sided Cauchy data extends and the saddle-point theory applies as well.

5 Conclusion

The identifiability of organic pollution sources by indirect observations of another interacting constituent is a challenging issue. A well established process is the detection of the BOD tracer from DO observations. The resulting model which is a coupled time-dependent system where the recovery of the missing observations on BOD is considered. The identifiability requires uniqueness results for two ill-posed parabolic systems. Such results may be stated by combining tools from the saddle point theory and some others coming from the semi-group theory. Specifically, we consider a mixed variational formulation to study the resolvents of the steady systems, then we call for Pazy's uniqueness theorem. They enable to show that the pollutant concentration tracer is unique with respect to the observations of the dissolved oxygen tracer. This turns out to be a key tool in order to show the identifiability of a moving source term on each constituent. Before closing, let us indicate that a careful bibliography makes us believe that this is the first work where the Pazy theorem is used to state uniqueness for parabolic systems where Hille-Yosida's theory fails.

6 Appendix : Inf-sup conditions on $a(\cdot, \cdot)$.

The statement of the necessary inf-sup conditions on $a(\cdot, \cdot)$, when restricted to $\mathcal{N}_\lambda \times \mathcal{N}_{*,\lambda}$ may be achieved following the proof of Lemma 2.2. However, the main point is not solved which is the knowledge of the behavior of the constants involved in those inf-sup conditions with respect to λ when it grows to infinity. This may be achieved through some constructive methodology which

requires some sharp a-priori estimates on the solution $\xi_\lambda(\lambda \geq 0)$ of the following problem

$$\begin{aligned}(\lambda + r)\xi_\lambda - (d\xi'_\lambda)' &= 0, & \text{in } I, \\ \xi_\lambda(0) &= 0, \\ \xi_\lambda(\pi) &= 1.\end{aligned}$$

We call $\xi_{*,\lambda}$ the unique solution of the same elliptic problem where the coefficients (d, r) are changed into (d_*, r_*) . The objective is to bound from below $a(\xi_\lambda, \xi_{*,\lambda})$, for large values of λ . Qualitatively, each of ξ_λ and $\xi_{*,\lambda}$ is well known to show a boundary layer along $x = \pi$ when λ grows to infinity. Zooming around $x = \pi$ and picking up tools from the asymptotic analysis enables to derive rigorously the desired result. Let us before recall the minoration (by $\underline{r} > 0$) of the reaction parameters (r, r_*) so as the piecewise continuity assumptions made on the dispersion coefficients (d, d_*) . A direct consequence is that (d, d_*) have finite limits at $x = \pi$. We will denote by $d_\pi, d_{*,\pi} \in \mathbb{R}$ those limits. Obviously, we have that $d_\pi, d_{*,\pi} \geq \underline{d}$. It holds that

Lemma 6.1 *There exists $\underline{\lambda}$ and a constant γ independent of λ such that*

$$a(\xi_\lambda, \xi_{*,\lambda}) \geq \frac{\gamma}{\sqrt{\lambda}}, \quad \forall \lambda \geq \underline{\lambda}.$$

Proof: The approach we use to state the lemma is the one currently employed in the asymptotic analysis for partial differential equations (see [11, 9] for example). We try to approximate the solutions ξ_λ and $\xi_{*,\lambda}$, in the boundary layers located at the vicinity of $x = \pi$. Let us focus on ξ_λ , the same work should be made for $\xi_{*,\lambda}$.

Observe that the maximum principle yields that $\xi_\lambda(x) > 0$ for all $x \in]0, \pi[$ (see [14]). Now, let us zoom on the narrow interval $[\pi - \epsilon, \pi]$, with $\epsilon = \frac{1}{\sqrt{\lambda}}$, for large λ . We introduce therefore the stretching coordinate $z \in [0, 1]$ so that $x = \pi - \epsilon z$, and for a function p in the variable x , we define \check{p} in the variable z so that $\check{p}(z) = p(x)$. With these notations, there comes out that $\check{\xi}_\lambda$ is such that

$$\begin{aligned}(1 + \epsilon^2 \check{r})\check{\xi}_\lambda - (\check{d}\check{\xi}'_\lambda)' &= 0, & \text{in } [0, 1], \\ \check{\xi}_\lambda(0) &= 1, \\ \check{\xi}_\lambda(1) &= \tau.\end{aligned}$$

Here $\tau \in]0, 1[$. Now, let us carefully examine $\check{d}(z) = d(\pi - \epsilon z)$. Due to the smoothness assumption on the diffusivity $d(\cdot)$ told of above, $\check{d}(z)$ converges toward $d_\pi > 0$ in $L^\infty(0, 1)$ for large λ . A straightforward consequence is that $(\check{\xi}_\lambda)_\lambda$ converges in $H^1(0, 1)$ towards

$$\xi(z) = \frac{\sinh(\omega_\pi(1 - z))}{\sinh(\omega_\pi)} + \tau \frac{\sinh(\omega_\pi z)}{\sinh(\omega_\pi)},$$

when λ tends to infinity. Here, we have set $d_\pi = (\omega_\pi)^2$. This, together with the minoration of the reaction parameter $\check{r}(z) \geq \underline{r} > 0$, yields the following estimate (valid for large values of λ)

$$\int_{[0,1]} \check{r}(z)\check{\xi}_\lambda(z)\check{\xi}_{*,\lambda}(z) dz \geq \gamma' \int_{[0,1]} \xi(z)\xi_*(z) dz = \gamma > 0,$$

where γ' is a positive constant that does not depend on λ . It is however dependent of \underline{r} and d_π . Returning to the primary functions, through a changing of variable, we obtain that

$$\int_{[\pi-\epsilon, \pi]} r(x)\xi_\lambda(x)\xi_{*,\lambda}(x) dx = \epsilon \int_{[0,1]} \check{r}(z)\check{\xi}_\lambda(z)\check{\xi}_{*,\lambda}(z) dz \geq \gamma\epsilon = \frac{\gamma}{\sqrt{\lambda}}.$$

Given the positivity of ξ_λ and $\xi_{*,\lambda}$, we derive finally that

$$a(\xi_\lambda, \xi_{*,\lambda}) = \int_I r(x)\xi_\lambda(x)\xi_{*,\lambda}(x) dx \geq \int_{[\pi-\epsilon, \pi]} r(x)\xi_\lambda(x)\xi_{*,\lambda}(x) dx \geq \frac{\gamma}{\sqrt{\lambda}}.$$

The proof is complete. \blacksquare

Remark 6.1 *The estimate by Lemma 6.1 is optimal and can not be improved. Analytical computations realized for constant coefficients (r, r_*) and (d, d_*) are liable to sweep away any doubt.*

Lemma 6.2 *There exists $\underline{\lambda}$ and a constant α independent of λ such that*

$$\lambda\|\xi_\lambda\|_{L^2(I)}^2 + \|\xi'_\lambda\|_{L^2(I)}^2 \leq \alpha\sqrt{\lambda}, \quad \forall \lambda \geq \underline{\lambda}.$$

Proof: The elliptic equation on ξ_λ being symmetric, it can be formulated as a minimization problem of the potential energy. The bound in the lemma is therefore directly issued from the formula

$$\lambda\|\xi_\lambda\|_{L^2(I)}^2 + \|\sqrt{r}\xi_\lambda\|_{L^2(I)}^2 + \|\sqrt{d}\xi'_\lambda\|_{L^2(I)}^2 \leq \lambda\|\zeta\|_{L^2(I)}^2 + \|\sqrt{r}\zeta\|_{L^2(I)}^2 + \|\sqrt{d}\zeta'\|_{L^2(I)}^2,$$

where ζ is selected as follows

$$\zeta(x) = \frac{\sinh(\sqrt{\lambda}x)}{\sinh(\sqrt{\lambda}\pi)}.$$

Straightforward calculations end to the bound $\lambda\|\zeta\|_{L^2(I)}^2 + \|\sqrt{d}\zeta'\|_{L^2(I)}^2 \leq \gamma''\sqrt{\lambda}$. The proof is therefore complete. \blacksquare

After these preparatory technical results, we are well equipped to establish the inf-sup condition of Lemma 2.5. The statement in Remark 2.3 is proved following the same lines.

Proof of Lemma 2.5: Let λ be sufficiently large and φ be given in \mathcal{N}_λ . There comes that $\varphi = \varphi(\pi)\xi_\lambda$. Choose $\psi = \varphi(\pi)\xi_{*,\lambda}$, it is readily checked that $\psi \in \mathcal{N}_{*,\lambda}$. Then, we have that

$$\frac{a(\varphi, \psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} = \frac{a(\xi_\lambda, \xi_{*,\lambda})}{\|\xi_\lambda\|_{H^1}\|\xi_{*,\lambda}\|_{H^1}}.$$

Considering the supremum on $\chi \in \mathcal{N}_{*,\lambda}$ and calling for Lemmas 6.1 and 6.2, we state that

$$\sup_{\chi \in \mathcal{N}_{*,\lambda}} \frac{a(\varphi, \chi)}{\|\varphi\|_{H^1}\|\chi\|_{H^1}} \geq \frac{a(\xi_\lambda, \xi_{*,\lambda})}{\|\xi_\lambda\|_{H^1}\|\xi_{*,\lambda}\|_{H^1}} \geq \frac{\beta}{\lambda}.$$

β is independent of λ . Switching to the infimum on $\varphi \in \mathcal{N}_\lambda$ achieves the proof. \blacksquare

Remark 6.2 *The inf-sup conditions in Remark 2.3 by means of the same arguments. The only modification is to switch from the H^1 -norm to the L^2 -norm. The lower bound comes from the estimate*

$$\frac{a(\xi_\lambda, \xi_{*,\lambda})}{\|\xi_\lambda\|_{L^2}\|\xi_{*,\lambda}\|_{L^2}} \geq \beta,$$

obtained owing to the L^2 bounds of ξ_λ and $\xi_{*,\lambda}$ by Lemma 6.2.

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