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Analyse harmonique et équation de Schrödinger associées au laplacien de Dunkl trigonométrique

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Analyse harmonique et équation de Schrödinger associées au laplacien de Dunkl trigonométrique

Cette thèse est constituée de trois chapitres. Le premièr chapitre porte sur l'examen des conditions de validité du principe d'équipartition de l'énergie totale de la solution de l'équation des ondes associée au laplacien de Dunkl trigonométrique. Enfin, nous établissons le comportement asymptotique de l'équipartition dans le cas général. Les résultats de cette partie ont fait l'objet de la publication [8]. Le deuxième chapitre, publié avec J.Ph. Anker et M. Sifi [6], montre que les fonctions d'Opdam dans le cas de rang 1 satisfont à une formule produit. Cela nous a permis de définir une structure de convolution du genre hypergroupe. En particulier, on montre que cette convolution satisfait l'analogue du phénomène de Kunze-Stein. Le dernier chapitre est consacrée à l'étude des propriétés dispersives et estimations de Strichartz pour la solution de l'équation de Schrödinger associée au laplacien de Dunkl trigonométrique unidimensionnel [7]. Cette étude commence par des estimations optimales du noyau de la chaleur et de Schrödinger. À l'aide de ces résultats, ainsi que les outils d'analyse harmonique dévellopée dans le chapitre 2, on montre des éstimées de type Strichartz qui permettent de trouver des conditions d'admissibilité pour des équations de Schrödinger semi-linéaires.

Mots clés : Laplacien de Dunkl trigonométrique, formule produit, équation des ondes, équation de la chaleur, équation de Schrödinger, estimations de Strichartz.

Harmonic analysis and Schrödinger equation associated with the trigonometric Dunkl Laplacian

This thesis consists of three chapters. The first one is concerned with energy properties of the wave equation associated with the trigonometric Dunkl Laplacian. We establish the conservation of the total energy, the strict equipartition of energy under suitable assumptions and the asymptotic equipartition in the general case. These results were published in [8]. The second chapter, in collaboration with J.Ph. Anker and M. Sifi [6], shows that Opdam's functions in the rank one case satisfy a product formula. We then define and study a convolution structure related to Opdam's functions. In particular, we prove that this convolution fulfills a Kunze-Stein type phenomena. The last chapter deals with dispersive and Strichartz estimates for the linear Schrödinger equation associated with the one dimensional trigonometric Dunkl Laplacian [7]. We establish sharp estimates for the heat kernel in complex time, and therefore for the Schrödinger kernel. We then use these estimates together with tools from chapter 2 to deduce dispersive and Strichartz inequalities for the linear Schrödinger equation and apply them to well–posedness in the nonlinear case.

Keywords : Trigonometric Dunkl Laplacian, product formula, wave equation, heat equation, Schrödinger equation, Strichartz estiamtes.

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Chapitre 1

Introduction

Les équations aux dérivées partielles dispersives, dont les prototypes sont les équations des ondes et de Schrödinger, ont suscité l'intérêt de beaucoup de travaux. À partir des équations d'ondes et celle de Schrödinger euclidiennes on obtient les différentes équations d'ondes et de Schrödinger en considérant des perturbations et/ou des déformations. Ces perturbations et/ou déformations prennent différentes formes et modifient largement la nature des solutions.

Dans cette thèse, on s'intéresse principalement à l'étude de l'équation de Schrödinger et, dans une moindre mesure, à l'équation d'ondes associées au laplacien de Dunkl trigonométrique. Cet opérateur de second ordre est, schématiquement, une déformation du laplacien euclidien par des différentielles et d'opérateurs aux différences.

Cette thèse comporte deux parties indépendantes :

(I) La première partie porte sur l'examen des conditions de validité du principe d'équipartition de l'énergie totale de la solution de l'équation des ondes associée au laplacien de Dunkl trigonométrique ou laplacien de Dunkl–Cherednik. Cette partie est la continuation de mon mémoire de Master. Elle est constituée d'un seul chapitre qui reprend l'article suivant :

- *Equipartition of energy for the wave equation associated to the Dunkl – Cherednik Laplacian,* Journal of Lie Theory **18** (2008), 747–755.

(II) La deuxième partie, qui représente la partie principale de cette thèse, est consacrée à l'étude des propriétés dispersives et estimations de Strichartz pour la solution de l'équation de Schrödinger associée au laplacien de Dunkl trigonométrique unidimensionnel. L'étude de ces propriétés est fondamentale à plusieurs titre. Tout d'abord, elle est importante du point de vue physique dans l'étude des propriétés asymptotiques des solutions, par exemple en théorie du scattering. De plus, des estimations dispersives ont été utilisée avec succès dans beaucoup de problèmes non linéaires, en particulier dans la théorie moderne de l'existence de solutions locale et globale. Cette deuxième partie de la thèse est constituée de deux chapitres, chacun reprenant un article :

- (en collaboration avec J.–Ph. Anker & M. Sifi) *Opdam's hypergeometric functions : product formula and convolution structure in dimension 1*, accepté pour publication dans Advances in Pure and Applied Mathematics (2011).
- (en collaboration avec J.–Ph. Anker & M. Sifi) *Contributions to one–dimensional trigonometric Dunkl analysis* : *Abel transform, heat kernel estimates, Schrödinger equation*, prépublication.

1.1 Présentation du cadre de travail

En 1989 Dunkl a introduit dans [29] une famille remarquable d'opérateurs d'ordre un, mélange de dérivées partielles et d'opérateurs aux différences. Ces opérateurs de Dunkl font intervenir les systèmes de racines et ils sont à coefficients rationnels. Peu de temps auparavant Heckman et Opdam avaient entamé l'étude de fonctions hypergéométriques associées à tout système de racines, et introduit une notion nouvelle, les opérateurs de décalage [37, 38, 49, 50].

En 1991 Cherednik a introduit dans [22] une famille d'opérateurs différentiels aux différences mais à coefficients trigonométriques, généralisant ceux de Dunkl. Ces opérateurs de Dunkl trigonométriques, ou opérateurs de Dunkl–Cherednik, simplifièrent considérablement les travaux de Heckman et Opdam, notamment dans l'écriture des opérateurs de décalage.

L'invention des opérateurs de Dunkl trigonométriques a engendré une double évolution. D'une part un développement algébrique en lien avec la théorie des algèbres de Hecke [24]. D'autre part un développement analytique consistant à généraliser des problèmes d'analyse harmonique sur les espaces symétriques riemanniens. Le développement de cette branche est dû essentiellement à Heckman et Opdam, dont la principale motivation était de généraliser la théorie des fonctions sphériques de Harish–Chandra dans le but d'établir une théorie des fonctions hypergéométriques multivariées [39, 51, 52].

Commencons par quelques rappels concernant les opérateurs de Dunkl trigonométriques. On considère l'espace euclidien $\mathbb{R}^d$, $d \geq 1$, muni du produit scalaire usuel noté $\langle \cdot, \cdot \rangle$. Pour $\alpha \in \mathbb{R}^d$, on pose $\check{\alpha} := \frac{2}{\|\alpha\|^2}\alpha$, et on définit la réflexion orthogonale associée à α par

$$r_\alpha(x) = x - \langle \check{\alpha}, x \rangle \alpha.$$

Soit $\mathcal{R}$ un système de racines satisfaisant la condition cristallographique $\check{\alpha}(\mathcal{R}) \subset \mathbb{Z}$ pour tout $\alpha \in \mathcal{R}$. On choisit un système positif $\mathcal{R}^+$. On note W le groupe de Weyl associé. Pour tout $\xi \in \mathbb{R}^d$, l'opérateur de Dunkl trigonométrique est défini par

$$T_\xi(k)f(x) = \partial_\xi f(x) - \langle \rho, \xi \rangle f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{\langle \alpha, \xi \rangle}{1 - e^{-\langle \alpha, x \rangle}} \Big(f(x) - f(r_\alpha x) \Big), \qquad (1.1.1)$$

où $k = \{k_\alpha\}_{\alpha \in \mathcal{R}}$ est une fonction de multiplicité sur $\mathcal{R}$ (c'est–à–dire une fonction W–invariante, qu'on suppose dans toute la suite à valeurs positives ou nulles), et $\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha$. La propriété remarquable de ces opérateurs est le fait qu'ils commutent. De plus, ces opérateurs jouent le rôle des dérivées partielles pour le laplacien. Plus précisément, pour toute base orthonormale $\{\xi_1, \ldots, \xi_d\}$ de $\mathbb{R}^d$, le laplacien de Dunkl trigonométrique, ou laplacien de Dunkl–Cherednik, est défini par

$$\begin{aligned} \mathcal{L}_k f(x) &:= \sum_{i=1}^d T_{\xi_i}(k)^2 f(x) && (1.1.2) \\ &= \Delta f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \coth\left(\frac{\langle \alpha, x \rangle}{2} \right) \partial_\alpha f(x) + \|\rho\|^2 f(x) \\ &\quad - \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{4 \sinh^2 \frac{\langle \alpha, x \rangle}{2}} \Big(f(x) - f(r_\alpha x) \Big), \end{aligned}$$

où Δ désigne le laplacien euclidien. La théorie euclidienne classique correspond au cas $k \equiv 0$.

Exemple 1.1.1. (Le cas unidimensionnel) *Soit $\mathcal{R} = BC_1 = \{\pm\gamma, \pm 2\gamma\}$. On choisit la normalisation $\gamma = 2$. En utilisant les notations $\alpha = k_\gamma + k_{2\gamma} - \frac{1}{2}$ et $\beta = k_{2\gamma} - \frac{1}{2}$, l'opérateur de Dunkl trigonométrique s'écrit*

$$T^{(\alpha,\beta)} f(x) = f'(x) + \big\{ (2\alpha+1) \coth x + (2\beta+1) \tanh x \big\} \frac{f(x) - f(-x)}{2} - \rho f(-x), \qquad (1.1.3)$$

où $\rho = \alpha + \beta + 1$. Le laplacien de Dunkl trigonométrique, qu'on notera $\Delta_{\alpha,\beta}$, est donné par

$$\begin{aligned} \Delta_{\alpha,\beta} f(x) &= f''(x) + \{(2\alpha+1) \coth x + (2\beta+1) \tanh x\} f'(x) + \rho^2 f(x) \\ &\quad + \left\{ -\frac{2\alpha+1}{\sinh^2 x} + \frac{2\beta+1}{\cosh^2 x} \right\} \frac{f(x) - f(-x)}{2}. \qquad (1.1.4) \end{aligned}$$

La commutativité des opérateurs $T_\xi(k)$ amène naturellement à chercher, pour tout $\lambda \in \mathbb{C}^d$, les fonctions propres $G_\lambda^{(k)}$ du système

$$\begin{cases} T_\xi(k)\, G_\lambda^{(k)}(x) = i \,\langle \lambda, \xi \rangle\, G_\lambda^{(k)}(x) \quad \forall\, \xi \in \mathbb{R}^d, \\ G_\lambda^{(k)}(0) = 1. \end{cases}$$

Ce problème a été résolue par Opdam [51]. Lorsque $k \equiv 0$, $G_\lambda^{(k)}(x) = e^{i\langle\lambda,x\rangle}$. Un problème majeur de la théorie de Dunkl trigonométrique est l'absence de formule explicite pour $G_\lambda^{(k)}$ en général. Le seul cas connu est le cas unidimensionnel i.e. $\mathcal{R} = BC_1$. En utilisant les notations de l'Exemple 1.1.1, on a

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{(\alpha,\beta)}(x) + \frac{\rho + i\lambda}{4(\alpha+1)} \sinh 2x \, \varphi_\lambda^{(\alpha+1,\beta+1)}(x), \tag{1.1.5}$$

où $\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x\right)$ désigne la fonction de Jacobi.

Au moyen de $G_\lambda^{(k)}$, Opdam introduit dans [51] une transformation de Fourier définie pour f suffisament régulière par

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}^d} f(x) \, G_{-w_0\lambda}^{(k)}(w_0 x) \, d\mu(x),$$

où w_0 désigne le plus long élément du groupe de Weyl W, et μ est la mesure définie par

$$d\mu(x) = \prod_{\alpha \in \mathcal{R}^+} \left| 2 \sinh \frac{\langle \alpha, x \rangle}{2} \right|^{2k_\alpha} dx.$$

Cette transformation a été considérée ultérieurement par Cherednik [23] sous une forme légèrement différente. La transformation d'Opdam–Cherednik s'avère être un précieux outil pour généraliser l'analyse harmonique sur les espaces symétriques riemanniens. On dispose notamment d'une formule d'inversion et d'un théorème de Plancherel pour cette transformation de Fourier généralisée [51, 23].

Exemple 1.1.2. *Dans le cas unidimensionnel, la transformation $\mathcal{F}$ s'écrit*

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} f(x) \, G_\lambda^{(\alpha,\beta)}(-x) \, A_{\alpha,\beta}(|x|) \, dx \qquad \forall \, \lambda \in \mathbb{C}, \tag{1.1.6}$$

où $G_\lambda^{(\alpha,\beta)}$ est donnée par (1.1.5) *et*

$$A_{\alpha,\beta}(|x|) = |\sinh x|^{2\alpha+1} (\cosh x)^{2\beta+1}. \tag{1.1.7}$$

1.2 Principaux résultats obtenus

1.2.1 Première partie : Équation d'ondes et équipartition d'énergie

Cette partie m'a été proposée comme une continuation de mon mémoire de Master. Elle a fait l'objet de la publication [8]. Il s'agit d'étudier certaines propriétés d'énergie pour les solutions de l'équation des ondes suivante :

$$\begin{cases} \partial_t^2 u(x,t) - \mathcal{L}_k u(x,t) = 0, & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(x,0) = f(x),\, \partial_t u(x,0) = g(x), \end{cases} \tag{1.2.1}$$

où $\mathcal{L}_k$ désigne le laplacien de Dunkl trigonométrique (1.1.2) dans la variable d'espace $x \in \mathbb{R}^d$, $d \geq 1$. Nous nous sommes intéressés à la conservation de l'énergie totale et à son équipartition en énergie cinétique et en énergie potentielle.

L'examen des conditions de validité du principe d'équipartition dans le cas euclidien est dû à Brodsky [21]. Ensuite de nombreux auteurs ont étudié ce problème dans différents contextes. Citons les travaux de Branson, Ólafsson, Schlichtkrull, Helgason, et Pasquale pour l'équation des ondes sur les espaces symétriques riemanniens [17, 20, 40, 18]. Citons aussi [13, 12] pour l'équation des ondes associée au laplacien de Dunkl rationnel.

Décrivons maintenant plus en détail les résultats de cette première partie. Commencons par rappeler le théorème suivant concernant la propagation des ondes à vitesse finie pour (1.2.1).

Théorème 1.2.1. *(cf. [12]) Si les données initiales f et g ont leur support dans une boule $B(0,R)$, alors la solution u au temps T a son support dans la boule $B(0,R+|T|)$.*

On définit l'énergie totale au temps t de la solution u de (1.2.1) par

$$\mathcal{E}(u)(t) = \mathcal{K}(u)(t) + \mathcal{P}(u)(t),$$

où

$$\mathcal{K}(u)(t) := \frac{1}{2}\int_{\mathbb{R}^d} |\partial_t u(x,t)|^2 \, d\mu(x)$$

désigne l'énergie cinétique de u au temps t et

$$\mathcal{P}(u)(t) := -\frac{1}{2}\int_{\mathbb{R}^d} \mathcal{L}_k u(x,t)\,\overline{u(x,t)}\, d\mu(x)$$

son énergie potentielle. En utilisant le théorème de Plancherel pour la transformation $\mathcal{F}$ d'Opdam–Cherednik, on montre le résultat suivant :

Théorème 1.2.2. *L'énergie totale $\mathcal{E}(u)(t)$ est indépendante de t, i.e. $\mathcal{E}(u)(t) = \mathcal{E}(u)(0)$ pour tout $t \in \mathbb{R}$.*

Rappelons que la mesure de Plancherel a pour densité

$$\nu(\lambda) = \prod_{\alpha \in \mathcal{R}_0^+} \frac{\Gamma\Big(i\langle\lambda,\check{\alpha}\rangle + k_\alpha\Big)}{\Gamma\Big(i\langle\lambda,\check{\alpha}\rangle\Big)} \frac{\Gamma\Big(\frac{i\langle\lambda,\check{\alpha}\rangle + k_\alpha}{2} + k_{2\alpha}\Big)}{\Gamma\Big(\frac{i\langle\lambda,\check{\alpha}\rangle + k_\alpha}{2}\Big)} \frac{\Gamma\Big(-i\langle\lambda,\check{\alpha}\rangle + k_\alpha\Big)}{\Gamma\Big(-i\langle\lambda,\check{\alpha}\rangle + 1\Big)} \frac{\Gamma\Big(\frac{-i\langle\lambda,\check{\alpha}\rangle + k_\alpha}{2} + k_{2\alpha} + 1\Big)}{\Gamma\Big(\frac{-i\langle\lambda,\check{\alpha}\rangle + k_\alpha}{2}\Big)} \;\forall\, \lambda \in \mathbb{R}^d,$$

où $\mathcal{R}_0^+ = \{\alpha \in \mathcal{R}^+ \mid |\frac{\alpha}{2} \notin \mathcal{R}\}$ et $k_{2\alpha} = 0$ si $2\alpha \notin \mathcal{R}$. Soit $\gamma_0 \in]0, \infty]$ la largeur de la plus grande bande $\{z \in \mathbb{C} \mid |\operatorname{Im} z| < \gamma_0\}$ sur laquelle $z \mapsto \nu(z\omega)$ se prolonge en une fonction holomorphe, pour tout $\omega \in S^{d-1}$. Notons que $\gamma_0 = \infty$ si et seulement si $k_\alpha \in \mathbb{N}$ pour tout $\alpha \in \mathcal{R}$. Dans ce cas on a le principe suivant d'équipartition d'énergie.

Proposition 1.2.1. *Supposons que la dimension d est impaire, que la fonction de multiplicité k est à valeurs entières, et que les conditions initiales $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ ont leur support dans une boule $B(0, R)$. Alors*

$$\mathcal{K}(u)(t) = \mathcal{P}(u)(t) = \frac{1}{2}\,\mathcal{E}(u)(t) \qquad \text{si } |t| \geq R.$$

En fait on déduit ce résultat de l'estimation suivante : il existe une constante $C > 0$ dépendant des conditions intiales f, g telle que, pour tout $0 < \gamma \leq \gamma_0$ et pour tout $t \in \mathbb{R}$,

$$|\mathcal{K}(u)(t) - \mathcal{P}(u)(t)| \leq Ce^{2\gamma(R-|t|)}.$$

En général, on a le principe suivant d'équipartition asymptotique à vitesse exponentielle ou polynomiale.

Proposition 1.2.2. *(1) En dimension d impaire, pour tout $0 < \gamma < \gamma_0$, il existe une constante $C > 0$ dépendant de $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ telle que*

$$|\mathcal{K}(u)(t) - \mathcal{P}(u)(t)| \leq Ce^{-2\gamma|t|} \qquad \forall\, t \in \mathbb{R}.$$

(2) En dimension d paire, il existe une constante $C > 0$ dépendant de $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ telle que

$$|\mathcal{K}(u)(t) - \mathcal{P}(u)(t)| \leq C(1 + |t|)^{-d-|\mathcal{R}_0^+|} \qquad \forall\, t \in \mathbb{R}.$$

1.2.2 Deuxième partie : Analyse harmonique et équation de Schrödinger dans le cadre de la théorie de Dunkl trigonométrique unidimensionnelle

La deuxième partie de cette thèse se place dans le cadre de la théorie de Dunkl trigonométrique unidimensionnelle, qui fait l'objet de l'exemple 1.1.1 et dont nous reprenons les notations. Il s'agit d'une généralisation de la théorie des fonctions de Jacobi, qui fut développée dans les années 1970 par Flensted–Jensen et Koornwinder (cf. [47] pour une présentation d'ensemble) et sur laquelle nous nous basons.

Nous établissons tout d'abord plusieurs résultats fondamentaux d'analyse harmonique. Dans le chapitre 3, qui fait l'objet d'une publication commune [6] avec

J.–Ph. Anker et M. Sifi, nous obtenons notamment une formule produit explicite pour les fonctions d'Opdam (1.1.5), qui permet définir des translations généralisées, ainsi qu'un produit de convolution généralisé, dont nous démontrons des propriétés fonctionnelles. Dans le chapitre 4, qui fait l'objet d'une prépublication commune [7] avec J.–Ph. Anker et M. Sifi, nous commencons par étudier la transformation d'Abel et son inverse, pour lesquelles nous obtenons des expressions explicites. Nous utilisons ensuite ces outils d'analyse harmonique pour estimer le noyau de la chaleur en temps complexe et pour étudier l'équation de Schrödinger

$$\begin{cases} i\partial_t u(x,t) + \Delta_{\alpha,\beta} u(x,t) = F(x,t) & \forall\, (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = f(x) & \forall\, x \in \mathbb{R}, \end{cases} \tag{1.2.2}$$

où $\Delta_{\alpha,\beta}$ désigne le laplacien de Dunkl trigonométrique (1.1.4) opérant sur la variable d'espace $x \in \mathbb{R}$. Nous établissons notamment des inégalités dispersives pour $e^{it\Delta_{\alpha,\beta}}$ et des inégalités de Strichartz, que nous appliquons ensuite à l'existence (locale ou globale) de solutions pour l'équation de Schrödinger semi–linéaire

$$\begin{cases} i\partial_t u(x,t) + \Delta_{\alpha,\beta} u(x,t) = F_\gamma(u(x,t)) & \forall\, (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = f(x) & \forall\, x \in \mathbb{R}, \end{cases} \tag{1.2.3}$$

avec des non–linéarités de type polynomial

$$F_\gamma(u) \sim |u|^\gamma \qquad (\gamma > 1).$$

Décrivons maintenant nos résultats plus en détail. Rappelons que la fonction d'Opdam unidimensionnelle est donnée par

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{(\alpha,\beta)}(x) + \frac{\rho + i\lambda}{4(\alpha+1)} \sinh 2x\; \varphi_\lambda^{(\alpha+1,\beta+1)}(x), \tag{1.2.4}$$

où $\varphi_\lambda^{(\alpha,\beta)}$ désigne la fonction de Jacobi, $\alpha \geq \beta \geq -\frac{1}{2}$ avec $\alpha \neq -\frac{1}{2}$, et $\rho = \alpha+\beta+1$. En utilisant la formule produit des fonctions de Jacobi, due à Koornwinder [47], on démontre le théorème suivant :

Théorème 1.2.3. *Pour tout $x, y \in \mathbb{R}$ et pour tout $\lambda \in \mathbb{C}$, on a*

$$G_\lambda^{(\alpha,\beta)}(x)\, G_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(z)\, d\mu_{x,y}^{(\alpha,\beta)}(z)\,,$$

où $d\mu_{x,y}^{(\alpha,\beta)}$ est une mesure signée de masse totale 1, uniformément bornée en x et en y, et à support compact dans le double intervalle symétrique

$$I_{x,y} = \Big[-|x|-|y|, -\big||x|-|y|\big|\Big] \cup \Big[\big||x|-|y|\big|, |x|+|y|\Big].$$

Pour l'expression explicite de $\mu_{x,y}^{(\alpha,\beta)}$, nous renvoyons le lecteur au chapitre 3 de cette thèse. Notons que la mesure $\mu_{x,y}^{(\alpha,\beta)}$ n'est pas positive, en général. Signalons que notre résultat permet de retrouver comme cas limite la formule produit du noyau de Dunkl unidimensionnel obtenue par Rösler dans [56].

La formule produit nous permet de définir l'opérateur de translation généralisée

$$\tau_x^{(\alpha,\beta)} f(y) = \int_{\mathbb{R}} f(z)\, d\mu_{x,y}^{(\alpha,\beta)}(z) \tag{1.2.5}$$

dont on démontre facilement les propriétés de base. Citons par exemple :

$$\begin{aligned}\tau_x^{(\alpha,\beta)}\,\tau_y^{(\alpha,\beta)} &= \tau_y^{(\alpha,\beta)}\,\tau_x^{(\alpha,\beta)},\\ T^{(\alpha,\beta)}\,\tau_x^{(\alpha,\beta)} &= \tau_x^{(\alpha,\beta)}\,T^{(\alpha,\beta)},\\ \mathcal{F}(\tau_x^{(\alpha,\beta)} f)(\lambda) &= G_\lambda^{(\alpha,\beta)}(x)\,\mathcal{F}(f)(\lambda)\,,\end{aligned}$$

où $T^{(\alpha,\beta)}$ l'opérateur de Dunkl trigonométrique (1.1.3) et $\mathcal{F}$ désigne la transformation d'Opdam–Cherednik (1.1.6). Signalons toutefois qu'en général

$$\tau_x^{(\alpha,\beta)} \circ \tau_y^{(\alpha,\beta)} \neq \tau_{x+y}^{(\alpha,\beta)}\,.$$

Mentionnons finalement que les translations généralisées (1.2.5) ont été introduites différemment dans [48], au moyen des opérateurs de transmutation.

Pour tout $1 \le p \le \infty$, désignons par $L^p_{\alpha,\beta}(\mathbb{R})$ l'espace de $L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, où $A_{\alpha,\beta}$ désigne le poids (1.1.7). Nous démontrons le résultat fondamental (mais non trivial) que les translations généralisées (1.2.5) sont des opérateurs uniformément bornés sur $L^p_{\alpha,\beta}(\mathbb{R})$. Plus précisément :

Théorème 1.2.4. *Il existe une constante $c_{\alpha,\beta} > 0$, qui ne dépend que de α et de β, telle que*

$$\|\tau_x^{(\alpha,\beta)} f\|_p \le c_{\alpha,\beta}\, \|f\|_p\,,$$

pour tout $x \in \mathbb{R}$, pour tout $p \in [1,\infty]$ et pour tout $f \in L^p_{\alpha,\beta}(\mathbb{R})$.

La prochaine notion nous permettra plus tard d'écrire la solution du problème homogène associé à (1.2.2) comme produit de convolution d'un noyau avec la donnée initiale f. Au moyen des translations généralisées (1.2.5), on définit le produit de convolution généralisé suivant :

$$f *_{\alpha,\beta} g(x) = \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y)\, g(y)\, A_{\alpha,\beta}(|y|)\, dy\,, \tag{1.2.6}$$

où $A_{\alpha,\beta}$ désigne le poids (1.1.7). Parmi les propriétés attendues, citons par exemple :

$$\begin{aligned} f *_{\alpha,\beta} g &= g *_{\alpha,\beta} f,\\ \mathcal{F}(f *_{\alpha,\beta} g) &= \mathcal{F}(f)\,\mathcal{F}(g),\end{aligned}$$

où $\mathcal{F}$ désigne la transformation d'Opdam–Cherednik (1.1.6).

Le théorème suivant ou plutôt son corollaire est crucial pour obtenir au chapitre 4 les propriétés dispersives du propagateur de Schrödinger $e^{it\Delta_{\alpha,\beta}}$. Sa démonstration repose sur des estimations précises des fonctions d'Opdam (1.2.4), établies en toute dimension dans [57].

Théorème 1.2.5. *Pour tout* $1 \leq p < 2 < q \leq \infty$, *on a*

$$L^p_{\alpha,\beta}(\mathbb{R}) *_{\alpha,\beta} L^2_{\alpha,\beta}(\mathbb{R}) \subset L^2_{\alpha,\beta}(\mathbb{R}),$$

et

$$L^2_{\alpha,\beta}(\mathbb{R}) *_{\alpha,\beta} L^2_{\alpha,\beta}(\mathbb{R}) \subset L^q_{\alpha,\beta}(\mathbb{R}).$$

Il s'agit de l'analogue du célèbre phénomène de Kunze–Stein [45] :

$$L^p(G) * L^2(G) \subset L^2(G) \qquad \forall\, 1 \leq p < 2,$$

découvert initialement dans le cas particulier du groupe $G = \mathrm{SL}(2,\mathbb{R})$, muni de sa mesure de Haar, puis démontré par Cowling [27] dans le cas général d'un groupe de Lie semisimple G (connexe, non compact, de centre fini). On en déduit les inclusions suivantes :

Corollaire 1.2.6. *(1) Pour tout* $1 \leq p < q \leq 2$, *on a*

$$L^p_{\alpha,\beta}(\mathbb{R}) *_{\alpha,\beta} L^q_{\alpha,\beta}(\mathbb{R}) \subset L^q_{\alpha,\beta}(\mathbb{R}).$$

(2) Pour tout $1 < p < 2$ *et pour tout* $p < q \leq \frac{p}{2-p}$, *on a*

$$L^p_{\alpha,\beta}(\mathbb{R}) *_{\alpha,\beta} L^p_{\alpha,\beta}(\mathbb{R}) \subset L^q_{\alpha,\beta}(\mathbb{R}).$$

(3) Pour tout $2 < p, q < \infty$ *avec* $\frac{q}{2} \leq p < q$, *on a*

$$L^p_{\alpha,\beta}(\mathbb{R}) *_{\alpha,\beta} L^{q'}_{\alpha,\beta}(\mathbb{R}) \subset L^q_{\alpha,\beta}(\mathbb{R}).$$

La dernière partie du chapitre 3 est consacrée à la construction d'une base orthogonale de $L^2_{\alpha,\beta}(\mathbb{R})$. Etant donné $\delta > 0$, celle–ci est constituée des fonctions

$$\begin{cases} \mathrm{H}^\delta_{2n}(x) = (\cosh x)^{-\alpha-\beta-\delta-2}\, \mathrm{P}^{(\alpha,\delta)}_n(1-2\tanh^2 x), \\ \mathrm{H}^\delta_{2n+1}(x) = (\cosh x)^{-\alpha-\beta-\delta-2}\, \mathrm{P}^{(\alpha+1,\delta-1)}_n(1-2\tanh^2 x)\, \tanh x, \end{cases}$$

où $n \in \mathbb{N}$ et $\mathrm{P}^{(a,b)}_n$ désigne le polynôme de Jacobi de degré n. En utilisant le fait que $\mathrm{P}^{(a,b+\varepsilon^{-2})}_n(1-2\tanh^2 \varepsilon x)$ tend vers le polynôme de Laguerre $\mathrm{L}^a_n(x^2)$ lorsque $\epsilon \to 0$, on retrouve comme cas limite les fonctions d'Hermite construites par Rosenblum [55], qui forment une base orthogonale de $L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$. Nous exprimons ensuite leur transformée d'Opdam–Cherednik et établissons une formule de type Rodrigues au moyen des polynômes de Wilson

$$\begin{aligned} \mathrm{P}_n(t^2; a, b, c, d) &= (a+b)_n\,(a+c)_n\,(a+d)_n \\ &\quad \times {}_4\mathrm{F}_3\Big(\begin{matrix} -n, a+b+c+d+n-1, a+t, a-t \\ a+b, a+c, a+d \end{matrix}; 1\Big). \end{aligned}$$

Théorème 1.2.7. *Considérons les polynômes*

$$\begin{cases} \widetilde{\mathrm{P}}^{\delta}_{2n}(t) = \dfrac{(-1)^n}{n!\,(\frac{\alpha+\beta+\delta}{2}+1)_n\,(\frac{\alpha-\beta+\delta}{2}+1)_n}\, \mathrm{P}_n\big(\tfrac{t^2}{4}; \tfrac{\delta+1}{2}, \tfrac{\delta+1}{2}, \tfrac{\alpha+\beta+1}{2}, \tfrac{\alpha-\beta+1}{2}\big), \\ \widetilde{\mathrm{P}}^{\delta}_{2n+1}(t) = \dfrac{(-1)^n\,(\rho+t)}{2\,n!\,(\frac{\alpha+\beta+\delta}{2}+1)_{n+1}\,(\frac{\alpha-\beta+\delta}{2}+1)_n}\, \mathrm{P}_n\big(\tfrac{t^2}{4}; \tfrac{\delta+1}{2}, \tfrac{\delta-1}{2}, \tfrac{\alpha+\beta+3}{2}, \tfrac{\alpha-\beta+1}{2}\big). \end{cases}$$

Alors, pour tout $n \in \mathbb{N}$, *on a*

$$\mathcal{F}\big(\mathrm{H}^{\delta}_n\big)(\lambda) = \frac{\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{\Gamma(\frac{\alpha+\beta+\delta}{2}+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+1)}\, \widetilde{\mathrm{P}}^{\delta}_n(i\lambda)$$

et

$$\mathrm{H}^{\delta}_n(x) = \widetilde{\mathrm{P}}^{\delta}_n(\mathrm{T}^{(\alpha,\beta)}_x)(\cosh x)^{-\alpha-\beta-\delta-2}\,,$$

où $T^{(\alpha,\beta)}$ *désigne l'opérateur de Dunkl trigonométrique* (1.1.3).

Nous avons découvert récemment que cette dernière partie du chapitre 3 faisait l'objet de l'article [53].

Le chapitre 4 débute par l'étude de la transformation d'Abel $\mathcal{A}$ et de son inverse, qui servira à exprimer et à estimer le noyau de la chaleur en temps complexe. Cette transformation est définie sur $\mathcal{C}^\infty_c(\mathbb{R})$ par

$$\mathcal{A} = \mathcal{F}_0^{-1} \circ \mathcal{F},$$

où

$$\mathcal{F}_0(f)(\lambda) = \int_{\mathbb{R}} e^{i\lambda x} f(x)\, dx$$

est la transformation de Fourier classique sur $\mathbb{R}$ et $\mathcal{F}$ désigne la transformation d'Opdam–Cherednik unidimensionnelle (1.1.6). Le théorème suivant fournit des expressions de $\mathcal{A}$ et de son inverse $\mathcal{A}^{-1}$ en termes de transformations de Weyl fractionnaires (voir [47])

$$\mathcal{W}^{\tau}_{\mu} f(x) = \tfrac{1}{\Gamma(\mu+m)} \int_t^{+\infty} \big(-\tfrac{\partial}{\partial(\cosh s)}\big)^m f(s)\,(\cosh\tau s - \cosh\tau x)^{\mu+m-1} d(\cosh\tau s), \tag{1.2.7}$$

où $\tau > 0$, $\mu \in \mathbb{C}$ et $m \in \mathbb{N}$ avec $m > -\operatorname{Re}\mu$ (cette définition ne dépend pas de m).

Théorème 1.2.8. *On a*

$$\mathcal{A}f(x) = c_{\alpha,\beta}\Big[\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\big)f_{\text{paire}}(x) - \big(\tfrac{\partial}{\partial x}-\rho\big)\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\circ\mathcal{I}\big)f_{\text{impaire}}(x)\Big]$$

et

$$\mathcal{A}^{-1}f(x) = c^{-1}_{\alpha,\beta}\Big[\big(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\big)f_{\text{paire}}(x) + \big(\tfrac{\partial}{\partial x}-\rho\big)\big(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\circ\mathcal{I}\big)f_{\text{impaire}}(x)\Big], \tag{1.2.8}$$

où $c_{\alpha,\beta}$ est une constante >0, f_{paire} et f_{impaire} désignent les parties paire et impaire de f, et

$$\mathcal{I}f(x) = \int_{|x|}^{+\infty} f(y)\,dy.$$

A partir de l'expression (1.2.7) des transformations de Weyl, on peut obtenir une expression intégrale explicite de la transformation d'Abel :

$$\mathcal{A}f(x) = \int_{|y|>|x|} K^{(\alpha,\beta)}(x,y)\,f(y)\,dy \tag{1.2.9}$$

où le noyau $K^{(\alpha,\beta)}(x,y)$ s'exprime au moyen des fonctions hypergéométriques ${}_2F_1$ (voir chapitre 4, section 4). Signalons que Gallardo & Trimèche [32] ont obtenu récemment, au moyen des opérateurs de transmutation, une représentation équivalente à (1.2.9) pour la transformation d'Abel duale $\mathcal{A}^*$, au sens suivant :

$$\int_{\mathbb{R}} \mathcal{A}f(x)\,g(x)\,dx = \int_{\mathbb{R}} f(x)\,\mathcal{A}^*g(x)\,A_{\alpha,\beta}(|x|)\,dx\,.$$

Nous considérons ensuite l'équation de la chaleur

$$\begin{cases} \partial_t u(x,t) = \Delta_{\alpha,\beta}u(x,t) & \forall\, x\in\mathbb{R}, \forall\, t>0,\\ u(x,0) = f(x) & \forall\, x\in\mathbb{R}, \end{cases}$$

dont la solution est donnée par

$$u(x,t) = e^{t\Delta_{\alpha,\beta}}f(x) = \big[h_t *_{\alpha,\beta} f\big](x),$$

où

$$h_t(x) = \mathcal{F}^{-1}\big(e^{-t\lambda^2}\big)(x) = c^{-1}_{\alpha,\beta}\,\mathcal{A}^{-1}\Big(\tfrac{1}{\sqrt{4\pi t}}\,e^{-\frac{y^2}{4t}}\Big)(x) \tag{1.2.10}$$

est le noyau de la chaleur. A l'exception de quelques valeurs particulières de α et β, où (1.2.10) se réduit à une expression élémentaire, on ne dispose pas en général de formule plus explicite pour $h_t(x)$. A défaut, on se contente d'estimations précises. Notre principal résultat dans cette direction est la majoration globale suivante du noyau de la chaleur en temps complexe.

Théorème 1.2.9. *Il existe une constante $C>0$ telle que, pour tout $t\in\mathbb{C}^*$ avec* $\operatorname{Re} t\geq 0$ *et pour tout $x\in\mathbb{R}$,*

$$|h_t(x)|\leq C\begin{cases}|t|^{-\frac{3}{2}}\,(1+|x|)\,e^{-\rho|x|}\,e^{-\frac{x^2}{4}\operatorname{Re}(\frac{1}{t})} & si\ \ |t|\geq 1+|x|,\\ |t|^{-\alpha-1}\,(1+|x|)^{\alpha+\frac{1}{2}}\,e^{-\rho|x|}\,e^{-\frac{x^2}{4}\operatorname{Re}(\frac{1}{t})} & si\ \ |t|\leq 1+|x|.\end{cases} \tag{1.2.11}$$

Une telle estimation était connue dans le cas des espaces hyperboliques, et plus généralement des espaces de Damek–Ricci [28, 2, 35, 3, 4]. On reprend ici la même méthode, en partant de l'expression (1.2.10) du noyau de la chaleur au moyen de la transformation d'Abel inverse (1.2.8). L'intervention de deux transformations intégrales (en général) au lieu d'une (au plus) représente une difficulté technique nouvelle.

On peut montrer que l'estimation (1.2.11) est optimale pour $t>0$ et elle l'est vraisemblablement plus généralement (voir [35, 4]). Nous travaillons actuellement sur cette question.

Considérons finalement l'équation de Schrödinger et commencons par des rappels dans le cas euclidien, où l'équation (1.2.2) s'écrit

$$\begin{cases} i\partial_t u(x,t)+\Delta_x u(x,t)=F(x,t) & \forall\,(x,t)\in\mathbb{R}^d\times\mathbb{R},\\ u(x,0)=f(x) & \forall\,x\in\mathbb{R}^d.\end{cases} \tag{1.2.12}$$

Inspiré par des travaux de Tomas et de Segal [60, 61, 58], Strichartz établit dans [59] l'inégalité a priori suivante pour les solutions u de (1.2.12) :

$$\|u\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{d+1})}\lesssim\|f\|_{L^2(\mathbb{R}^d)}+\|F\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}^{d+1})}.$$

Ginibre & Velo [34] et Kato [43] ont ensuite généralisé cette inégalité comme suit, au moyen d'arguments abstraits d'analyse fonctionnelle :

$$\|u(x,t)\|_{L^p_tL^q_x}\lesssim\|f(x)\|_{L^2_x}+\|F(x,t)\|_{L^{\tilde{p}'}_tL^{\tilde{q}'}_x}.$$

Ici (p,q) et $(\tilde{p},\tilde{q})$ sont des couples admissibles, au sens où

$$\begin{cases} 2<p\leq\infty,\ 2\leq q<\infty \quad \text{avec} \quad \frac{1}{p}=\frac{d}{2}\big(\frac{1}{2}-\frac{1}{q}\big),\\ 2<\tilde{p}\leq\infty,\ 2\leq\tilde{q}<\infty \quad \text{avec} \quad \frac{1}{\tilde{p}}=\frac{d}{2}\big(\frac{1}{2}-\frac{1}{\tilde{q}}\big),\end{cases}$$

et on considère les normes mixtes en espace–temps

$$\begin{cases} \|u(x,t)\|_{L^p_tL^q_x}=\big\|\,t\mapsto\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}\big\|_{L^p(\mathbb{R})},\\ \|F(x,t)\|_{L^{\tilde{p}'}_tL^{\tilde{q}'}_x}=\big\|\,t\mapsto\|F(\cdot,t)\|_{L^{\tilde{q}'}(\mathbb{R}^d)}\big\|_{L^{\tilde{p}'}(\mathbb{R})}.\end{cases}$$

Keel & Tao [44] ont conclu cette famille d'inégalités en incluant les cas limites $(p,q)=(2,\frac{d-2}{2d})$ ou $(\tilde{p},\tilde{q})=(2,\frac{d-2}{2d})$ en dimension $d\geq 3$.

L'intérêt principal des estimations de type Strichartz réside dans leurs applications aux équations d'évolution, notamment au problème d'existence (locale ou globale) pour les solutions d'EDP non linéaires dispersives.

Durant la dernière décennie, ces estimations ont été étendues à différents contextes, par exemple aux variétés compactes [15, 16, 33], au groupe de Heisenberg [9], aux variétés coniques [36], aux espaces hyperboliques [10, 54, 11, 3, 41], aux espaces de Damek–Ricci [4], ... Sans parler de toute la litérature consacrée aux autres EDP dispersives, notamment les équations d'ondes.

Nous poursuivons dans cette voie, en adaptant la démarche de [3] à notre contexte. Considérons l'équation de Schrödinger

$$\begin{cases} i\partial_t u(x,t)+\Delta_{\alpha,\beta}u(x,t)=F(x,t) & \forall\ (x,t)\in\mathbb{R}\times\mathbb{R}, \\ u(x,0)=f(x) & \forall\ x\in\mathbb{R}, \end{cases} \tag{1.2.13}$$

dont la solution est donnée par la formule de Duhamel

$$u(x,t)=e^{it\Delta_{\alpha,\beta}}f(x)-i\int_0^t e^{i(t-s)\Delta_{\alpha,\beta}}F(x,s)\,ds \tag{1.2.14}$$

avec

$$e^{it\Delta_{\alpha,\beta}}f=h_{it}*_{\alpha,\beta}f.$$

On commence par établir l'inégalité dispersive suivante.

Théorème 1.2.10. *Pour tout $2<q,\tilde{q}\leq\infty$, il existe une constante $C>0$ telle que, pour tout $t\in\mathbb{R}^*$,*

$$\|\,e^{it\Delta_{\alpha,\beta}}\,\|_{L^{\tilde{q}'}_{\alpha,\beta}(\mathbb{R})\to L^{q}_{\alpha,\beta}(\mathbb{R})}\leq\ C\begin{cases}|t|^{-2(\alpha+1)\max\left\{\frac12-\frac1q,\frac12-\frac1{\tilde{q}}\right\}} & si\ \ 0<|t|<1,\\ |t|^{-\frac32} & si\ \ |t|\geq 1.\end{cases}$$

Lorsque $q=\tilde{q}=2$, on a bien entendu $\|\,e^{it\Delta_{\alpha,\beta}}\,\|_{L^2_{\alpha,\beta}(\mathbb{R})\to L^2_{\alpha,\beta}(\mathbb{R})}=1$.

La démonstration de ce théorème repose sur l' estimation (1.2.11) du noyau de Schrödinger $s_t=h_{it}$, qui s'écrit

$$|s_t(x)|\lesssim\begin{cases}|t|^{-\frac32}\,(1+|x|)\,e^{-\rho|x|} & \text{si}\ \ |t|\geq 1+|x|,\\ |t|^{-\alpha-1}\,(1+|x|)^{\alpha+\frac12}\,e^{-\rho|x|} & \text{si}\ \ |t|\leq 1+|x|.\end{cases}$$

On raisonne par interpolation complexe pour t petit et, pour t grand, on fait appel au phénomène de Kunze–Stein sous la forme du Corollaire 1.2.6(3). On en déduit les inégalités de Strichartz suivantes, pour les solutions (1.2.14) de (1.2.13), en utilisant des arguments devenus standards : méthode TT^* de Ginibre & Velo, lemme de Christ & Kiselev, interpolation bilinéaire de Keel & Tao, ... Posons

$$T_\alpha = \left\{ \left(\tfrac{1}{p}, \tfrac{1}{q}\right) \in \left(0, \tfrac{1}{2}\right] \times \left(0, \tfrac{1}{2}\right) \;\middle|\; \tfrac{1}{p} \geq (\alpha+1)\left(\tfrac{1}{2} - \tfrac{1}{q}\right) \right\} \cup \left\{ \left(0, \tfrac{1}{2}\right) \right\}.$$

Un couple (p, q) est dit admissible si $\left(\frac{1}{p}, \frac{1}{q}\right) \in T_\alpha$.

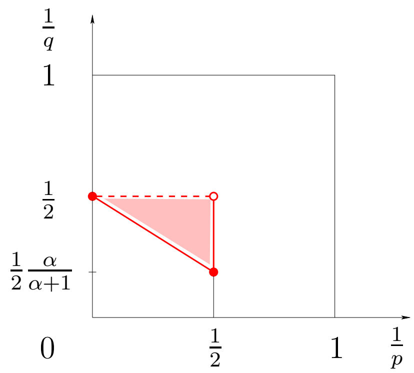

FIGURE 1.1 – L'ensemble T_α lorsque $\alpha > 0$

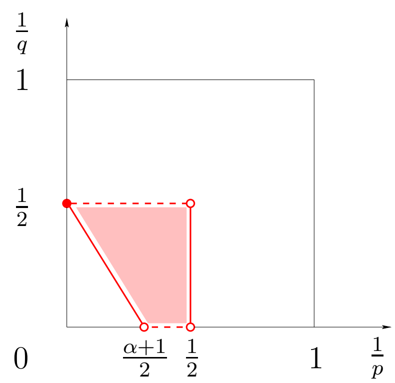

FIGURE 1.2 – L'ensemble T_α lorsque $-\frac{1}{2} \leq \alpha \leq 0$

Théorème 1.2.11. *Pour tous couples admissibles* (p, q) *et* $(\tilde{p}, \tilde{q})$*, on a l'inégalité a priori*

$$\|u\|_{L^p(\mathbb{R}, L^q_{\alpha,\beta}(\mathbb{R}))} \lesssim \|f\|_{L^2_{\alpha,\beta}(\mathbb{R})} + \|F\|_{L^{\tilde{p}'}(\mathbb{R}, L^{\tilde{q}'}_{\alpha,\beta}(\mathbb{R}))} \tag{1.2.15}$$

pour la solution (1.2.14) *de l'équation* (1.2.13).

Ces inégalités font intervenir les normes mixtes

$$\begin{cases} \|u\|_{L^p(\mathbb{R}, L^q_{\alpha,\beta}(\mathbb{R}))} = \left[\displaystyle\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(x,t)|^q \, A_{\alpha,\beta}(|x|) \, dx \right)^{\frac{p}{q}} dt \right]^{\frac{1}{p}}, \\ \|F\|_{L^{\tilde{p}'}(\mathbb{R}, L^{\tilde{q}'}_{\alpha,\beta}(\mathbb{R}))} = \left[\displaystyle\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(x,t)|^{\tilde{q}'} A_{\alpha,\beta}(|x|) \, dx \right)^{\frac{\tilde{p}'}{\tilde{q}'}} dt \right]^{\frac{1}{\tilde{p}'}}, \end{cases}$$

avec la modification usuelle lorsque $p = \infty$ ou $\tilde{p} = \infty$. Observons, comme dans le cas des espaces hyperboliques ou des espaces de Damek–Ricci [11, 3, 41], que les conditions d'admissibilités sont beaucoup larges que dans le cas euclidien. En conséquence, on obtient de meilleurs résultats d'existence pour les solutions de l'équation de Schrödinger semi–linéaire

$$\begin{cases} i\partial_t u(x,t) + \Delta_{\alpha,\beta} u(x,t) = F_\gamma(u(x,t)) & \forall\,(x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = f(x) & \forall\, x \in \mathbb{R}, \end{cases} \tag{1.2.16}$$

avec des non–linéarités F_γ vérifiant

$$|F_\gamma(u)| \lesssim |u|^\gamma \quad \text{et} \quad |F_\gamma(u) - F_\gamma(v)| \lesssim (|u|^{\gamma-1} + |v|^{\gamma-1})\,|u-v| \quad (\gamma > 1)\,.$$

Plus précisément, on établit le résultat suivant :

Théorème 1.2.12. *(i) Le problème* (1.2.16) *est localement bien-posé dans* $L^2_{\alpha,\beta}(\mathbb{R})$ *si* $1 < \gamma < 1 + \frac{2}{\alpha+1}$.

(ii) Il est globalement bien-posé dans $L^2_{\alpha,\beta}(\mathbb{R})$ *si* $1 < \gamma \leq 1 + \frac{2}{\alpha+1}$ *et si la donnée initiale* f *est suffisamment petite.*

Chapitre 2

Équation des ondes associée au laplacien de Dunkl trigonométrique et équipartition d'énergie

Ce chapitre reprend l'article [8], publié à *Journal of Lie Theory* **18** *(2008), 747–755.*

Résumé: Cet article porte sur l'étude des propriétés de l'énergie de l'équation des ondes associée au laplacien de Dunkl–Cherednik. Nous établissons, d'abord, la conservation de l'énergie totale, puis l'examen des conditions de son équipartition en énergie cinétique et en énergie potentielle, et enfin le comportement asymptotique de l'équipartition dans le cas général.

Abstract: This paper is concerned with energy properties of the wave equation associated to the Dunkl–Cherednik Laplacian. We establish the conservation of the total energy, the strict equipartition of energy under suitable assumptions and the asymptotic equipartition in the general case.

2.1 Introduction

We use [52] as a reference for the Dunkl–Cherednik theory. Let $\mathfrak{a}$ be a Euclidean vector space of dimension d equipped with an inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{R}$ be a crystallographic root system in $\mathfrak{a}$, $\mathcal{R}^+$ a positive subsystem and W the Weyl group generated by the reflections $r_\alpha(x)=x-2\frac{\langle\alpha,x\rangle}{\|\alpha\|^2}\alpha$ along the roots $\alpha\in\mathcal{R}$. We let $k:\mathcal{R}\to[0,+\infty)$ denote a multiplicity function on the root system $\mathcal{R}$, and $\rho=\frac{1}{2}\sum_{\alpha\in\mathcal{R}^+}k_\alpha\alpha$. We note that k is W-invariant. The Dunkl–Cherednik operators are the following differential–difference operators, which are deformations of partial derivatives and still commute pairwise:

$$T_\xi(k)f(x)=\partial_\xi f(x)-\langle\rho,\xi\rangle f(x)+\sum_{\alpha\in\mathcal{R}^+}k_\alpha\,\frac{\langle\alpha,\xi\rangle}{1\,-\,e^{-\langle\alpha,x\rangle}}\,\{f(x)-f(r_\alpha x)\}\,.$$

Given an orthonormal basis $\{\xi_1,\dots,\xi_d\}$ of $\mathfrak{a}$, the Dunkl–Cherednik Laplacian is defined by

$$\mathcal{L}_kf(x)=\sum_{j=1}^{d}T_{\xi_j}(k)^2f(x)\,.$$

More explicit formulas for $\mathcal{L}_k$ exist but they will not be used in this paper. The Laplacian $\mathcal{L}_k$ is selfadjoint with respect to the measure $\mu(x)dx$ where

$$\mu(x)=\prod_{\alpha\in\mathcal{R}^+}\Big|\,2\,\sinh\frac{\langle\alpha,x\rangle}{2}\,\Big|^{2k_\alpha}\,.$$

Consider the wave equation

$$\begin{cases}\partial_t^2u(t,x)=\mathcal{L}_ku(t,x)\,,\\ u(0,x)=f(x)\,,\;\partial_t\big|_{t=0}u(t,x)=g(x)\,,\end{cases}\tag{2.1.1}$$

with smooth and compactly supported initial data (f,g). Let us introduce:

- the *kinetic energy* $\mathcal{K}[u](t)=\frac{1}{2}\int_{\mathfrak{a}}\big|\partial_tu(t,x)\big|^2\mu(x)\,dx$,
- the *potential energy* $\mathcal{P}[u](t)=-\frac{1}{2}\int_{\mathfrak{a}}Lu(t,x)\,\overline{u(t,x)}\,\mu(x)\,dx$,
- the *total energy* $\mathcal{E}[u](t)=\mathcal{K}[u](t)+\mathcal{P}[u](t)$.

In this paper we prove

- the conservation of the total energy:

$$\mathcal{E}[u](t)=\text{constant},\tag{2.1.2}$$

- the strict equipartition of energy, under the assumptions that the dimension d is odd and that all the multiplicities k_α are integers :

$$\mathcal{K}[u](t) = \mathcal{P}[u](t) = \tfrac{1}{2}\,\mathcal{E}[u] \ \text{ for } |t| \text{ large}\,, \tag{2.1.3}$$

- the asymptotic equipartition of energy, for arbitrary d and positive real valued multiplicity function k :

$$\mathcal{K}[u](t) \to \tfrac{1}{2}\,\mathcal{E}[u] \ \text{ and } \ \mathcal{P}[u](t) \to \tfrac{1}{2}\,\mathcal{E}[u] \ \text{ as } \ |t| \text{ goes to } \infty. \tag{2.1.4}$$

The proofs follow [20] and use the Fourier transform in the Dunkl–Cherednik setting, which we will recall in next section. We mention that during the past twenty years, several works were devoted to Huygens' principle and equipartition of energy for wave equations on symmetric spaces and related settings. See for instance [17], [40, ch. V], [20], [42], [13], [18], [19], [12].

2.2 Generalized hypergeometric functions and Dunkl–Cherednik transform

Opdam [51] introduced the following special functions, which are deformations of exponential functions $e^{\langle\lambda,x\rangle}$, and the associated Fourier transform.

Theorem 2.2.1. *There exist a neighborhood U of 0 in $\mathfrak{a}$ and a unique holomorphic function $(\lambda,x)\mapsto G_\lambda(x)$ on $\mathfrak{a}_{\mathbb{C}}\times(\mathfrak{a}+i\,U)$ such that*

$$\begin{cases} T_\xi(k)\,G_\lambda^{(k)}(x) = \langle\lambda,\xi\rangle\,G_\lambda^{(k)}(x) \quad \forall\,\xi\in\mathfrak{a}\,,\\ G_\lambda^{(k)}(0) = 1\,. \end{cases}$$

Moreover, the following estimate holds on $\mathfrak{a}_{\mathbb{C}}\times\mathfrak{a}$:

$$|G_\lambda^{(k)}(x)| \le |W|^{\frac{1}{2}}\,e^{\|\mathrm{Re}\lambda\|\|x\|}\,.$$

Definition 2.2.2. *The Dunkl–Cherednik transform of a nice functions f on $\mathfrak{a}$, say $f\in\mathcal{C}_c^\infty(\mathfrak{a})$, is defined by*

$$\mathcal{F}f(\lambda) = \int_{\mathfrak{a}} f(x)\,G^{(k)}_{-i\,w_0\lambda}(w_0\,x)\,\mu(x)\,dx.$$

Here w_0 denotes the longest element in the Weyl group W and $\mathcal{C}_c^\infty(\mathfrak{a})$ the space of smooth functions on $\mathfrak{a}$ with compact support.

The involvement of w_0 in the definition of $\mathcal{F}$ is related to the below skew–adjointness property of the Dunkl–Cherednik operators with respect to the inner product

$$\langle f, g\rangle = \int_{\mathfrak{a}} f(x)\,\overline{g(x)}\,\mu(x)\,dx, \qquad f,g\in\mathcal{C}_c^\infty(\mathfrak{a}).$$

Lemma 2.2.3. *The adjoint of $T_\xi(k)$ is $-w_0\,T_{w_0\xi}(k)\,w_0$:*

$$\langle T_\xi(k)f, g\rangle = \langle f, -w_0\,T_{w_0\xi}(k)\,w_0\,g\rangle\,.$$

As an immediate consequence, we obtain:

Corollary 2.2.4. *For every $\xi,\lambda\in\mathfrak{a}$ and $f\in\mathcal{C}_c^\infty(\mathfrak{a})$, we have*

$$\mathcal{F}(T_\xi(k)f)(\lambda) = i\,\langle\lambda,\xi\rangle\,\mathcal{F}f(\lambda),$$

and therefore

$$\mathcal{F}(\mathcal{L}_k f)(\lambda) = -\,\|\lambda\|^2\,\mathcal{F}f(\lambda)\,.$$

Next we will recall from [51] the three main results about the Dunkl–Cherednik transform (see also [52]). $\mathcal{C}_R^\infty(\mathfrak{a})$ denotes the space of smooth functions on $\mathfrak{a}$ vanishing outside the ball $B_R=\{\,x\in\mathfrak{a}\mid\|x\|\le R\,\}$. We let $\mathcal{H}_R(\mathfrak{a}_{\mathbb{C}})$ is the space of holomorphic functions h on the complexification $\mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{a}$ such that, for every integer $N>0$,

$$\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}}(1+\|\lambda\|)^N\,e^{-R\,\|\mathrm{Im}\lambda\|}\,|h(\lambda)| < +\infty\,.$$

Theorem 2.2.5. (Paley-Wiener) *The transformation $\mathcal{F}$ is an isomorphism of $\mathcal{C}_R^\infty(\mathfrak{a})$ onto $\mathcal{H}_R(\mathfrak{a}_{\mathbb{C}})$, for every $R>0$.*

The Plancherel formula and the inversion formula of $\mathcal{F}$ involve the complex measure $\nu(\lambda)\,d\lambda$ with density

$$\nu(\lambda) \;=\; \prod_{\alpha\in\mathcal{R}_0^+}\frac{\Gamma\big(i\,\langle\lambda,\check\alpha\rangle+k_\alpha\big)}{\Gamma\big(i\,\langle\lambda,\check\alpha\rangle\big)}\,\frac{\Gamma\big(\frac{i\,\langle\lambda,\check\alpha\rangle+k_\alpha}{2}+k_{2\alpha}\big)}{\Gamma\big(\frac{i\,\langle\lambda,\check\alpha\rangle+k_\alpha}{2}\big)}\,\frac{\Gamma\big(-i\,\langle\lambda,\check\alpha\rangle+k_\alpha\big)}{\Gamma\big(-i\,\langle\lambda,\check\alpha\rangle+1\big)}\,\frac{\Gamma\big(\frac{-i\,\langle\lambda,\check\alpha\rangle+k_\alpha}{2}+k_{2\alpha}+1\big)}{\Gamma\big(\frac{-i\,\langle\lambda,\check\alpha\rangle+k_\alpha}{2}\big)}\,,$$

where $\mathcal{R}_0^+=\{\,\alpha\in\mathcal{R}^+\mid\frac{\alpha}{2}\notin\mathcal{R}\,\}$ is the set of positive indivisible roots, $\check\alpha=2\,\|\alpha\|^{-2}\,\alpha$ the coroot corresponding to α, and $k_{2\alpha}=0$ if $2\alpha\notin\mathcal{R}$. Notice that ν is an analytic function on $\mathfrak{a}$, which grows polynomially and which extends meromorphically to $\mathfrak{a}_{\mathbb{C}}$. It is actually a polynomial if the multiplicity function k is integer–valued and it has poles otherwise.

Theorem 2.2.6. (Inversion formula) *There is a constant $c_0>0$ such that, for every $f\in\mathcal{C}_c^\infty(\mathfrak{a})$,*

$$f(x) \;=\; c_0\int_{\mathfrak{a}}\mathcal{F}f(\lambda)\,G_{i\lambda}^{(k)}(x)\,\nu(\lambda)\,d\lambda\,.$$

Theorem 2.2.7. (Plancherel formula) *For every* $f, g \in \mathcal{C}_c^\infty(\mathfrak{a})$,

$$\int_{\mathfrak{a}} f(x)\,\overline{g(x)}\,\mu(x)\,dx \;=\; c_0 \int_{\mathfrak{a}} \mathcal{F}f(\lambda)\,\widetilde{\mathcal{F}}g(\lambda)\,\nu(\lambda)\,d\lambda\,,$$

where

$$\widetilde{\mathcal{F}}g(\lambda) := \overline{\mathcal{F}(w_0 g)(w_0\lambda)} = \int_{\mathfrak{a}} \overline{g(x)}\,G_{i\lambda}^{(k)}(x)\,\mu(x)\,dx.$$

2.3 Conservation of energy

This section is devoted to the proof of (2.1.2). Via the Dunkl–Cherednik transform, the wave equation (2.1.1) becomes

$$\begin{cases} \partial_t^2 \mathcal{F}u(t,\lambda) = -\|\lambda\|^2\,\mathcal{F}u(t,\lambda)\,, \\ \mathcal{F}u(0,\lambda) = \mathcal{F}f(\lambda)\,,\; \partial_t\big|_{t=0}\,\mathcal{F}u(t,\lambda) = \mathcal{F}g(\lambda)\,, \end{cases}$$

and its solution is given by

$$\mathcal{F}u(t,\lambda) = (\cos t\,\|\lambda\|)\,\mathcal{F}f(\lambda) + \tfrac{\sin t\,\|\lambda\|}{\|\lambda\|}\,\mathcal{F}g(\lambda)\,. \tag{2.3.1}$$

According to the Paley–Wiener theorem, (2.1.1) has thus a unique solution in $\mathcal{C}_c^\infty(\mathfrak{a})$, which satisfies moreover the following finite speed propagation property (see [12, p. 52-53]) :

> Assume that the initial data f and g belong to $\mathcal{C}_R^\infty(\mathfrak{a})$. Then the solution $u(t,x)$ belongs to $\mathcal{C}_{R+|t|}^\infty(\mathfrak{a})$ as a function of x.

Let us express the potential and kinetic energies defined in the introduction via the Dunkl–Cherednik transform. Using the Plancherel formula and Corollary 2.2.4, we have

$$\mathcal{P}[u](t) = \tfrac{c_0}{2}\int_{\mathfrak{a}} \|\lambda\|^2\,\mathcal{F}u(t,\lambda)\,\widetilde{\mathcal{F}}u(t,\lambda)\,\nu(\lambda)\,d\lambda\,. \tag{2.3.2}$$

Moreover, since the Dunkl–Cherednik Laplacian is $W-$invariant, it follows that $(w_0 u)(t,x) = u(t, w_0 x)$ is the solution to the wave equation (2.1.1) with the initial data $w_0 f$ and $w_0 g$. Thus

$$\widetilde{\mathcal{F}}u(t,\lambda) = (\cos t\,\|\lambda\|)\,\widetilde{\mathcal{F}}f(\lambda) + \tfrac{\sin t\,\|\lambda\|}{\|\lambda\|}\,\widetilde{\mathcal{F}}g(\lambda)\,. \tag{2.3.3}$$

Now, by substituting (2.3.1) and (2.3.3) in (2.3.2), we get

$$\begin{aligned} \mathcal{P}[u](t) = \;& \tfrac{c_0}{2}\int_{\mathfrak{a}} \|\lambda\|^2\,(\cos t\,\|\lambda\|)^2\,\mathcal{F}f(\lambda)\,\widetilde{\mathcal{F}}f(\lambda)\,\nu(\lambda)\,d\lambda \\ &+ \tfrac{c_0}{2}\int_{\mathfrak{a}} (\sin t\,\|\lambda\|)^2\,\mathcal{F}g(\lambda)\,\widetilde{\mathcal{F}}g(\lambda)\,\nu(\lambda)\,d\lambda \\ &+ \tfrac{c_0}{4}\int_{\mathfrak{a}} \|\lambda\|\,(\sin 2\,t\,\|\lambda\|)\,\big\{\,\mathcal{F}f(\lambda)\,\widetilde{\mathcal{F}}g(\lambda) + \mathcal{F}g(\lambda)\,\widetilde{\mathcal{F}}f(\lambda)\,\big\}\,\nu(\lambda)\,d\lambda\,. \end{aligned} \tag{2.3.4}$$

Similarly to $\mathcal{P}[u]$, we can rewrite the kinetic energy as

$$\mathcal{K}[u](t) = \tfrac{c_0}{2} \int_{\mathfrak{a}} \partial_t \, \mathcal{F}u(t,\lambda) \, \partial_t \, \widetilde{\mathcal{F}}u(t,\lambda) \, \nu(\lambda) \, d\lambda \, .$$

Using the following facts

$$\begin{cases} \partial_t \, \mathcal{F}u(t,\lambda) = - \, \|\lambda\| \, (\sin t \, \|\lambda\|) \, \mathcal{F}f(\lambda) + (\cos t \, \|\lambda\|) \, \mathcal{F}g(\lambda) \, , \\ \partial_t \, \widetilde{\mathcal{F}}u(t,\lambda) = - \, \|\lambda\| \, (\sin t \, \|\lambda\|) \, \widetilde{\mathcal{F}}f(\lambda) + (\cos t \, \|\lambda\|) \, \widetilde{\mathcal{F}}g(\lambda) \, , \end{cases}$$

we deduce that

$$\begin{aligned} \mathcal{K}[u](t) = \; & \tfrac{c_0}{2} \int_{\mathfrak{a}} \|\lambda\|^2 \, (\sin t \, \|\lambda\|)^2 \, \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) \, \nu(\lambda) \, d\lambda \\ & + \tfrac{c_0}{2} \int_{\mathfrak{a}} (\cos t \, \|\lambda\|)^2 \, \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) \, \nu(\lambda) \, d\lambda \\ & - \tfrac{c_0}{4} \int_{\mathfrak{a}} \|\lambda\| \, (\sin 2 \, t \, \|\lambda\|) \, \big\{ \, \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) + \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) \, \big\} \, \nu(\lambda) \, d\lambda \, . \end{aligned} \tag{2.3.5}$$

By suming up (2.3.4) and (2.3.5), we obtain the conservation of the total energy:

$$\mathcal{E}[u](t) \; = \; \tfrac{c_0}{2} \int_{\mathfrak{a}} \big\{ \, \|\lambda\|^2 \, \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) + \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) \, \big\} \, \nu(\lambda) \, d\lambda \; = \; \mathcal{E}[u](0) \, .$$

That is $\mathcal{E}[u](t)$ is independent of t.

2.4 Equipartition of energy

This section is devoted to the proof of (2.1.3) and (2.1.4). Using the classical trigonometric identities for double angles, we can rewrite the identities (2.3.4) and (2.3.5) respectively as

$$\begin{aligned} \mathcal{P}[u](t) = \; & \tfrac{c_0}{4} \int_{\mathfrak{a}} \big\{ \, \|\lambda\|^2 \, \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) + \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) \, \big\} \, \nu(\lambda) \, d\lambda \\ & + \tfrac{c_0}{4} \int_{\mathfrak{a}} (\cos 2 \, t \, \|\lambda\|) \, \big\{ \, \|\lambda\|^2 \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) - \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) \, \big\} \, \nu(\lambda) \, d\lambda \\ & + \tfrac{c_0}{4} \int_{\mathfrak{a}} \|\lambda\| \, (\sin 2 \, t \, \|\lambda\|) \, \big\{ \, \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) + \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) \, \big\} \, \nu(\lambda) \, d\lambda \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}[u](t) = \; & \tfrac{c_0}{4} \int_{\mathfrak{a}} \big\{ \, \|\lambda\|^2 \, \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) + \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) \, \big\} \, \nu(\lambda) \, d\lambda \\ & - \tfrac{c_0}{4} \int_{\mathfrak{a}} (\cos 2 \, t \, \|\lambda\|) \, \big\{ \, \|\lambda\|^2 \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) - \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) \, \big\} \, \nu(\lambda) \, d\lambda \\ & - \tfrac{c_0}{4} \int_{\mathfrak{a}} \|\lambda\| \, (\sin 2 \, t \, \|\lambda\|) \, \big\{ \, \mathcal{F}f(\lambda) \, \widetilde{\mathcal{F}}g(\lambda) + \mathcal{F}g(\lambda) \, \widetilde{\mathcal{F}}f(\lambda) \, \big\} \, \nu(\lambda) \, d\lambda \, . \end{aligned}$$

Hence

$$
\begin{aligned}
&\mathcal{P}[u](t)-\mathcal{K}[u](t) \;=\\
&=\; \tfrac{c_0}{2}\int_{\mathfrak{a}}(\cos 2\,t\,\|\lambda\|)\,\big\{\,\|\lambda\|^2\mathcal{F}f(\lambda)\,\widetilde{\mathcal{F}}f(\lambda)-\mathcal{F}g(\lambda)\,\widetilde{\mathcal{F}}g(\lambda)\,\big\}\,\nu(\lambda)\,d\lambda\\
&+\; \tfrac{c_0}{2}\int_{\mathfrak{a}}\|\lambda\|\,(\sin 2\,t\,\|\lambda\|)\,\big\{\,\mathcal{F}f(\lambda)\,\widetilde{\mathcal{F}}g(\lambda)+\mathcal{F}g(\lambda)\,\widetilde{\mathcal{F}}f(\lambda)\,\big\}\,\nu(\lambda)\,d\lambda\,.
\end{aligned}
\tag{2.4.1}
$$

Introducing polar coordinates in $\mathfrak{a}$, (2.4.1) becomes

$$
\mathcal{P}[u](t)-\mathcal{K}[u](t) \;=\; \tfrac{c_0}{2}\int_0^{+\infty}\big\{\cos(2\,t\,r)\,\Phi(r)+\sin(2\,t\,r)\,r\,\Psi(r)\,\big\}\,r^{\,d-1}\,dr\,, \tag{2.4.2}
$$

where

$$
\begin{aligned}
\Phi(r) \;&=\int_{S(\mathfrak{a})}\big\{\,r^2\,\mathcal{F}f(r\sigma)\,\widetilde{\mathcal{F}}f(r\sigma)-\mathcal{F}g(r\sigma)\,\widetilde{\mathcal{F}}g(r\sigma)\,\big\}\,\nu(r\sigma)\,d\sigma\,,\\
\Psi(r) \;&=\int_{S(\mathfrak{a})}\big\{\,\mathcal{F}f(r\sigma)\,\widetilde{\mathcal{F}}g(r\sigma)+\mathcal{F}g(r\sigma)\,\widetilde{\mathcal{F}}f(r\sigma)\,\big\}\,\nu(r\sigma)\,d\sigma\,,
\end{aligned}
$$

and $d\sigma$ denotes the surface measure on the unit sphere $S(\mathfrak{a})$ in $\mathfrak{a}$. Let $\gamma_0\in(0,+\infty\,]$ be the width of the largest horizontal strip $|\operatorname{Im} z|<\gamma_0$ in which $z\mapsto\nu(z\sigma)$ is holomorphic for all directions $\sigma\in S(\mathfrak{a})$.

Lemma 2.4.1. (i) *$\Phi(z)$ and $\Psi(z)$ extend to even holomorphic functions in the strip $|\operatorname{Im} z|<\gamma_0$.*
(ii) *If $\gamma_0<+\infty$, the following estimate holds in every substrip $|\operatorname{Im} z|\le\gamma$ with $\gamma<\gamma_0$: For every $N>0$, there is a constant $C>0$ (depending on $f,g\in\mathcal{C}_R^\infty(\mathfrak{a})$, N and γ) such that*

$$
|\Phi(z)|+|\Psi(z)|\le C\,|z|^{|\mathcal{R}_0^+|}\,(1+|z|)^{-N}\,e^{\,2\,R\,|\operatorname{Im} z|}\,.
$$

(iii) *If $\gamma_0=+\infty$, the previous estimate holds uniformly in $\mathbb{C}$.*

Proof. (i) follows from the definition of Φ and Ψ. Let us turn to the estimates (ii) and (iii). On one hand, according to the Paley–Wiener Theorem (Theorem 2.2.5), all transforms $\mathcal{F}f(z\sigma)$, $\widetilde{\mathcal{F}}f(z\sigma)$, $\mathcal{F}g(z\sigma)$, $\widetilde{\mathcal{F}}g(z\sigma)$ are $\mathrm{O}\big(\{1+|z|\}^{-N}\,e^{\,R\,|\operatorname{Im} z|}\big)$. On the other hand, let us discuss the behavior of the Plancherel measure. Consider first the case where all multiplicities are integers. Without loss of generality, we may assume that $k_\alpha\in\mathbb{N}^*$ and $k_{2\alpha}\in\mathbb{N}$ for every indivisible root α. Then

$$
\begin{aligned}
\nu(\lambda) \;=\; &\text{const.}\,\textstyle\prod_{\alpha\in\mathcal{R}_0^+}\langle\lambda,\check{\alpha}\rangle\,\big\{\langle\lambda,\check{\alpha}\rangle+i\,(k_\alpha+2\,k_{2\alpha})\big\}\\
&\times\,\textstyle\prod_{0<j<k_\alpha}\big\{\langle\lambda,\check{\alpha}\rangle^2+j^2\big\}\,\prod_{0\le\tilde{j}<k_{2\alpha}}\big\{\langle\lambda,\check{\alpha}\rangle^2+(k_\alpha+2\,\tilde{j})^2\big\}
\end{aligned}
$$

is a polynomial of degree $2\,|k| = 2\sum_{\alpha\in\mathcal{R}^+} k_\alpha$. In general,

$$\nu(\lambda) = \text{const.}\,\pi(\lambda)\,\widetilde{\nu}(\lambda)\,,$$

where

$$\pi(\lambda) = \prod\nolimits_{\alpha\in\mathcal{R}_0^+}\langle\lambda,\breve{\alpha}\rangle$$

is a homogeneous polynomial of degree $|\,\mathcal{R}_0^+|$ and

$$\widetilde{\nu}(\lambda) = \prod_{\alpha\in\mathcal{R}_0^+}\frac{\Gamma\left(i\,\langle\lambda,\breve{\alpha}\rangle+k_\alpha\right)}{\Gamma\left(i\,\langle\lambda,\breve{\alpha}\rangle+1\right)}\,\frac{\Gamma\left(\frac{i\,\langle\lambda,\breve{\alpha}\rangle+k_\alpha}{2}+k_{2\alpha}\right)}{\Gamma\left(\frac{i\,\langle\lambda,\breve{\alpha}\rangle+k_\alpha}{2}\right)}\,\frac{\Gamma\left(-i\,\langle\lambda,\breve{\alpha}\rangle+k_\alpha\right)}{\Gamma\left(-i\,\langle\lambda,\breve{\alpha}\rangle+1\right)}\,\frac{\Gamma\left(\frac{-i\,\langle\lambda,\breve{\alpha}\rangle+k_\alpha}{2}+k_{2\alpha}+1\right)}{\Gamma\left(\frac{-i\,\langle\lambda,\breve{\alpha}\rangle+k_\alpha}{2}\right)}$$

is an analytic function which never vanishes on $\mathfrak{a}$. Notice that $z\mapsto\nu(z\sigma)$ or $\widetilde{\nu}(z\sigma)$ has poles for generic directions $\sigma\in S(\mathfrak{a})$ as soon as some multiplicities are not integers. Using Stirling's formula

$$\Gamma(\xi)\sim\sqrt{2\,\pi}\,\xi^{\,\xi-\frac{1}{2}}\,e^{-\xi}\quad\text{as }|\xi|\to+\infty\text{ with }|\arg z|<\pi-\varepsilon\,,$$

we get the following estimate for the Plancherel density, in each strip $|\operatorname{Im} z|<\gamma$ with $0<\gamma<\gamma_0$:

$$|\nu(z\sigma)|\,\le\,C\,|z|^{|\mathcal{R}_0^+|}\,(1+|z|)^{2\,|k|-|\mathcal{R}_0^+|}\,.$$

The estimates (ii) and (iii) follow easily from these considerations. □

Proposition 2.4.2. *Assume that the dimension d is odd and that all multiplicities are integers. Then there exists a constant $C>0$ (depending on the initial data $f,g\in\mathcal{C}_R^\infty(\mathfrak{a})$) such that, for every $\gamma\ge0$ and $t\in\mathbb{R}$,*

$$|\mathcal{P}[u](t)-\mathcal{K}[u](t)|\le C\,e^{\,2\,\gamma\,(R-|t|)}\,.$$

Proof. Evenness allows us to rewrite (2.4.2) as follows:

$$\mathcal{P}[u](t)-\mathcal{K}[u](t) = \tfrac{c_0}{4}\int_{-\infty}^{+\infty}e^{\,i\,2\,t\,r}\left\{\,\Phi(r)-i\,r\,\Psi(r)\,\right\}r^{\,d-1}\,dr\,.$$

Let us shift the contour of integration from $\mathbb{R}$ to $\mathbb{R}\pm i\,\gamma$, according to the sign of t, and estimate the resulting integral, using Lemma 2.4.1.iii. As a result, the difference of energy

$$\begin{aligned}&\mathcal{P}[u](t)-\mathcal{K}[u](t) = \\ &= \tfrac{c_0}{4}\,e^{-2\,\gamma\,|t|}\int_{-\infty}^{+\infty}e^{\,i\,2\,t\,r}\left\{\,\Phi(r\pm i\,\gamma)-i\,(r\pm\gamma)\,\Psi(r\pm i\,\gamma)\,\right\}(r\pm i\,\gamma)^{\,d-1}\,dr\end{aligned}$$

is $\mathrm{O}\big((1+\gamma)^{-N}\,e^{\,2\,\gamma\,(R-|t|)}\big)$. □

As an immediate consequence of the above statement and in view of the fact that $\gamma_0 = \infty$ if k is integer valued, we deduce the strict equipartition of energy (2.1.3) for $|t| \geq R$, by letting $\gamma \to \infty$.

Henceforth, we will drop the above assumption on k. By resuming the proof of Proposition 2.4.2 and using Lemma 2.4.1. ii instead of Lemma 2.4.1. iii , we obtain the following result.

Proposition 2.4.3. *Assume that the dimension d is odd. Then, for every $0 < \gamma < \gamma_0$, there is a constant $C > 0$ (depending on the initial data $f, g \in \mathcal{C}_R^\infty(\mathfrak{a})$) such that*

$$|\mathcal{P}[u](t) - \mathcal{K}[u](t)| \leq C\, e^{-2\gamma|t|} \quad \forall\, t \in \mathbb{R}\,.$$

As a corollary, we obtain the asymptotic equipartition of energy (2.1.4) in the odd dimensional case, with an exponential rate of decay. In the even dimensional case, the expression (2.4.2) cannot be handled by complex analysis and we proceed differently.

Proposition 2.4.4. *Assume that the dimension d is even. Then there is a constant $C > 0$ (depending on the initial data $f, g \in \mathcal{C}_R^\infty(\mathfrak{a})$) such that*

$$|\mathcal{P}[u](t) - \mathcal{K}[u](t)| \leq C\,(1+|t|)^{-d-|\mathcal{R}_0^+|} \quad \forall\, t \in \mathbb{R}\,.$$

Proof. The problem lies in the decay at infinity. According to Lemma 2.4.1, $\Phi(r)$ and $\Psi(r)$ are divisible by r^D, where $D = |\mathcal{R}_0^+|$. Let us integrate (2.4.2) $d+D$ times by parts. This way

$$\int_0^{+\infty} \cos(2\,t\,r)\, r^{d-1} \underbrace{\Big\{ \int_{S(\mathfrak{a})} \mathcal{F}g(r\sigma)\, \widetilde{\mathcal{F}}g(r\sigma)\, \nu(r\sigma)\, d\sigma \Big\}}_{\widetilde{\Phi}(r)} dr$$

becomes

$$\pm\, \tfrac{1 \text{ or } 0}{(2\,t)^{d+D}}\, \tfrac{(d+D)!}{(D+1)!} \left(\tfrac{\partial}{\partial r}\right)^{D+1} \widetilde{\Phi}(r)\,\Big|_{r=0} \pm \int_0^{+\infty} \tfrac{\cos(2\,t\,r) \text{ or } \sin(2\,t\,r)}{(2\,t)^{d+D}} \left(\tfrac{\partial}{\partial r}\right)^{d+D} \big\{ r^{d-1}\, \widetilde{\Phi}(r) \big\}\, dr$$

which is $\mathrm{O}\big(|t|^{-d-D}\big)$. Similarly

$$\int_0^{+\infty} \cos(2\,t\,r)\, r^{d+1} \Big\{ \int_{S(\mathfrak{a})} \mathcal{F}f(r\sigma)\, \widetilde{\mathcal{F}}f(r\sigma)\, \nu(r\sigma)\, d\sigma \Big\}\, dr \,=\, \mathrm{O}\big(|t|^{-d-D-2}\big)$$

and

$$\int_0^{+\infty} \cos(2\,t\,r)\, r^d\, \Psi(r)\, dr \,=\, \mathrm{O}\big(|t|^{-d-D-1}\big)\,.$$

This concludes the proof of Proposition 2.4.4. ∎

As a corollary, we obtain the asymptotic equipartition of energy (2.1.4) in the even dimensional case, with a polynomial rate of decay.

Remark 2.4.5. *Our result may not be optimal. In the W–invariant case, one obtains indeed the rate of decay* $\mathrm{O}\big(\{1+|t|\}^{-d-2|\mathcal{R}_0^+|}\big)$ *as in [20].*

Chapitre 3

Fonctions hypergéométriques d'Opdam : formule produit et structure de convolution unidimensionnelle

Ce chapitre reprend l'article [6], à paraître dans *Advances in Pure and Applied Mathematics (2011).*

Résumé: Soit $\mathrm{G}_\lambda^{(\alpha,\beta)}$ la fonction propre normalisée de l'opérateur $\mathrm{T}^{(\alpha,\beta)}$ de Dunkl–Cherednik sur $\mathbb{R}$. Dans cet article, on exprime le produit $\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\mathrm{G}_\lambda^{(\alpha,\beta)}(y)$ comme une intégrale en terme de $\mathrm{G}_\lambda^{(\alpha,\beta)}(z)$ avec un noyau explicite. En général, le noyau n'est pas positif. De plus, notre résultat permet de retrouver comme cas limite la formule produit du noyau de Dunkl unidimensionnel obtenue par M. Rösler. En suite, on définit et en étudiant une structure de convolution associée à $\mathrm{G}_\lambda^{(\alpha,\beta)}$.

Abstract: Let $\mathrm{G}_\lambda^{(\alpha,\beta)}$ be the eigenfunction of the Dunkl–Cherednik operator $\mathrm{T}^{(\alpha,\beta)}$ on $\mathbb{R}$. In this paper we express the product $\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(y)$ as an integral in terms of $\mathrm{G}_\lambda^{(\alpha,\beta)}(z)$ with an explicit kernel. In general this kernel is not positive. Furthermore, by taking the so–called rational limit, we recover the product formula of M. Rösler for the Dunkl kernel. We then define and study a convolution structure associated to $\mathrm{G}_\lambda^{(\alpha,\beta)}$.

3.1 Introduction

The Opdam hypergeometric functions $\mathrm{G}_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}$ are normalized eigenfunctions

$$\begin{cases} \mathrm{T}^{(\alpha,\beta)}\,\mathrm{G}_\lambda^{(\alpha,\beta)}(x) = i\lambda\,\mathrm{G}_\lambda^{(\alpha,\beta)}(x) \\ \mathrm{G}_\lambda^{(\alpha,\beta)}(0) = 1 \end{cases} \tag{3.1.1}$$

of the differential–difference operator

$$\mathrm{T}^{(\alpha,\beta)} f(x) = f'(x) + \underbrace{\big\{(2\alpha{+}1)\coth x + (2\beta{+}1)\tanh x\big\}\,\tfrac{f(x)-f(-x)}{2}}_{\{(\alpha-\beta)\coth x\,+\,(2\beta+1)\coth 2x\}\,\{f(x)-f(-x)\}} - \rho\, f(-x)\,.$$

Here $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha > -\frac{1}{2}$, $\rho = \alpha{+}\beta{+}1$ and $\lambda \in \mathbb{C}$. Notice that, in Cherednik's notation, $\mathrm{T}^{(\alpha,\beta)}$ writes

$$\mathrm{T}(k_1,k_2) f(x) = f'(x) + \big\{\tfrac{2\,k_1}{1-e^{-2x}} + \tfrac{4\,k_2}{1-e^{-4x}}\big\}\,\big\{f(x) - f(-x)\big\} - (k_1{+}2k_2) f(x)\,,$$

with $\alpha{=}k_1{+}k_2{-}\frac{1}{2}$ and $\beta{=}k_2{-}\frac{1}{2}$. We use as main references the article [51] and the lecture notes [52] by Opdam.

The functions $\mathrm{G}_\lambda^{(\alpha,\beta)}$ are closely related to Jacobi or hypergeometric functions (see e.g. [51, p. 90], [52, Example 7.8], [32, Proposition 2.1]). Specifically,

$$\begin{aligned} \mathrm{G}_\lambda^{(\alpha,\beta)}(x) &= \varphi_\lambda^{(\alpha,\beta)}(x) - \frac{1}{\rho{-}i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x) \\ &= \varphi_\lambda^{(\alpha,\beta)}(x) + \frac{\rho{+}i\lambda}{4(\alpha{+}1)}\sinh 2x\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,, \end{aligned} \tag{3.1.2}$$

where $\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2\mathrm{F}_1\big(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}\,; \alpha{+}1\,; -\sinh^2 x\big)\,.$

This paper deals with harmonic analysis for the functions $\mathrm{G}_\lambda^{(\alpha,\beta)}$. We derive mainly a product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$, which is analogous to the corresponding result of Flensted-Jensen and Koornwinder [30] for Jacobi functions, and of Ben Salem and Ould Ahmed Salem [14] for Jacobi–Dunkl functions. The product formula is the key information needed in order to define an associated convolution structure on $\mathbb{R}$. More precisely, we deduce the product formula

$$\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} \mathrm{G}_\lambda^{(\alpha,\beta)}(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z) \quad \forall x,y \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}, \tag{3.1.3}$$

from the corresponding formula for $\varphi_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}^+$. Here $\mu_{x,y}^{(\alpha,\beta)}$ is an explicit real valued measure with compact support on $\mathbb{R}$, which may not be positive and which is uniformly bounded in $x,y \in \mathbb{R}$. We conclude the first part of the paper by recovering as a limit case the product formula for the Dunkl kernel obtained in [56].

In the second part of the paper, we use the product formula (3.1.3) to define and study the translation operators

$$\tau_x^{(\alpha,\beta)} f(y) := \int_{\mathbb{R}} f(z)\, d\mu_{x,y}^{(\alpha,\beta)}(z).$$

We next define the convolution product of suitable functions f and g by

$$f *_{\alpha,\beta} g(x) := \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y)\, g(y)\, A_{\alpha,\beta}(|y|)\, dy,$$

where $A_{\alpha,\beta}(y) = (\sinh y)^{2\alpha+1}(\cosh y)^{2\beta+1}$. We show in particular that $f *_{\alpha,\beta} g = g *_{\alpha,\beta} f$ and that $\mathcal{F}(f *_{\alpha,\beta} g) = \mathcal{F}(f)\,\mathcal{F}(g)$, where $\mathcal{F}$ is the so–called Opdam–Cherednik transform. Eventually we prove an analog of the Kunze–Stein phenomenon for the $*_{\alpha,\beta}$-convolution product of L^p-spaces.

In the last part of the paper, we construct an orthogonal basis of the Hilbert space $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, generalizing the corresponding result of Koornwinder [47] for $L^2(\mathbb{R}^+, A_{\alpha,\beta}(x)dx)$. As a limit case, we recover the Hermite functions constructed by Rosenblum [55] in $L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$.

Our paper is organised as follows. In Section 3.2, we recall some properties and formulas for Jacobi functions. In Section 3.3, we give the proof of the product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$. Section 3.4 is devoted to the translation operators and the associated convolution product. Section 3.5 contains a Kunze–Stein type phenomenon. In Section 3.6, we construct an orthogonal basis of $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$ and compute its Opdam–Cherednik transform.

3.2 Preliminaries

In this section we recall some properties of the Jacobi functions. See [30] and [31] for more details, as well as the survey [47].

Let $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$ and $\lambda \in \mathbb{C}$. The Jacobi function $\varphi_\lambda^{(\alpha,\beta)}$ is defined by

$$\begin{aligned}\varphi_\lambda^{(\alpha,\beta)}(x) &= {}_2\mathrm{F}_1\Big(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x\Big) \\ &= (\cosh x)^{-\rho-i\lambda}\, {}_2\mathrm{F}_1\Big(\frac{\rho+i\lambda}{2}, \frac{\alpha-\beta+1+i\lambda}{2}; \alpha+1; \tanh^2 x\Big) \quad \forall\, x \in \mathbb{R}\,,\end{aligned} \tag{3.2.1}$$

where $\rho = \alpha+\beta+1$ and ${}_2\mathrm{F}_1$ denotes the hypergeometric function.

Its asymptotic behavior is generically given by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = \mathrm{c}_{\alpha,\beta}(\lambda)\, \Phi_\lambda^{(\alpha,\beta)}(x) + \mathrm{c}_{\alpha,\beta}(-\lambda)\, \Phi_{-\lambda}^{(\alpha,\beta)}(x) \qquad \forall\, \lambda \in \mathbb{C} \smallsetminus i\mathbb{Z}\,,\ \forall\, x \in \mathbb{R}^*, \tag{3.2.2}$$

where

$$\mathrm{c}_{\alpha,\beta}(\lambda)=\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac12)}\,\frac{\Gamma(i\lambda)}{\Gamma(\alpha-\beta+i\lambda)}\,\frac{\Gamma(\frac{\alpha-\beta+i\lambda}{2})}{\Gamma(\frac{\rho+i\lambda}{2})}=\frac{\Gamma(\alpha+1)\,2^{\rho-i\lambda}\,\Gamma(i\lambda)}{\Gamma(\frac{\rho+i\lambda}{2})\,\Gamma(\frac{\alpha-\beta+1+i\lambda}{2})} \tag{3.2.3}$$

and

$$\Phi_\lambda^{(\alpha,\beta)}(x)=(2\cosh x)^{-\rho+i\lambda}\,{}_2\mathrm{F}_1\Big(\frac{\rho-i\lambda}{2},\frac{\alpha-\beta+1-i\lambda}{2};1-i\lambda;\cosh^{-2}x\Big). \tag{3.2.4}$$

In the limit case $\lambda=0$, we obtain

$$\varphi_0^{(\alpha,\beta)}(x)=\frac{2^{\rho+1}\,\Gamma(\alpha+1)}{\Gamma(\frac{\rho}{2})\,\Gamma(\frac{\alpha-\beta+1}{2})}\,|x|\,e^{-\rho\,|x|}+\mathcal{O}\big(e^{-\rho\,|x|}\big)\qquad\text{as }\ |x|\to+\infty\,, \tag{3.2.5}$$

after multiplying (3.2.2) by λ and applying $\frac{\partial}{\partial\lambda}\big|_{\lambda=0}$.

The Jacobi functions satisfy the following product formula, for $\alpha>\beta>-\frac12$ and $x,y\geq0$:

$$\varphi_\lambda^{(\alpha,\beta)}(x)\,\varphi_\lambda^{(\alpha,\beta)}(y)=\int_0^1\!\int_0^\pi\varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)\,dm_{\alpha,\beta}(r,\psi), \tag{3.2.6}$$

where

$$\gamma(x,y,r,\psi)=\cosh x\cosh y+\sinh x\sinh y\,re^{i\psi},$$

and

$$dm_{\alpha,\beta}(r,\psi)=2\,M_{\alpha,\beta}\,(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta}\,r\,dr\,d\psi \tag{3.2.7}$$

with

$$M_{\alpha,\beta}=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac12)}.$$

When $\alpha=\beta>-\frac12$, the product formula becomes

$$\varphi_\lambda^{(\alpha,\alpha)}(x)\,\varphi_\lambda^{(\alpha,\alpha)}(y)=M_{\alpha,\alpha}\int_0^\pi\varphi_\lambda^{(\alpha,\alpha)}(\arg\cosh|\gamma(x,y,1,\psi)|)\,(\sin\psi)^{2\alpha}\,d\psi, \tag{3.2.8}$$

where $M_{\alpha,\alpha}=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac12)}$. Notice that the limit cases $\alpha>\beta=-\frac12$ and $\alpha=\beta>-\frac12$ are connected by the quadratic transformation $\varphi_\lambda^{(\alpha,-\frac12)}(x)=\varphi_{2\lambda}^{(\alpha,\alpha)}\big(\frac{x}{2}\big)$.

For $\alpha>\beta>-\frac12$ and fixed $x,y>0$, we perform the change of variables $]0,1[\times]0,\pi[\,\ni(r,\psi)\longmapsto(z,\chi)\in\,]0,+\infty[\times]0,\pi[$ defined by

$$\cosh z\;e^{i\chi}=\gamma(x,y,r,\psi)\quad\Longleftrightarrow\quad\begin{cases}r\cos\psi=\frac{\cosh z\cos\chi-\cosh x\cosh y}{\sinh x\,\sinh y}\,,\\ r\sin\psi=\frac{\cosh z\,\sin\chi}{\sinh x\,\sinh y}\,.\end{cases} \tag{3.2.9}$$

This implies in particular that

$$\cosh(x-y) \leq \cosh(z) \leq \cosh(x+y),$$

and therefore x, y, z satisfy the triangular inequality

$$|x-y| \leq |z| \leq x+y.$$

Moreover, an easy computation gives

$$1 - r^2 = (\sinh x \sinh y)^{-2}\, g(x, y, z, \chi),$$

where

$$g(x, y, z, \chi) := 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2 \cosh x \cosh y \cosh z \cos \chi. \quad (3.2.10)$$

Furthermore, the measure $\sinh^2 x \sinh^2 y\, r\, dr d\psi$ becomes $\cosh z \sinh z\, dz\, d\chi$ and therefore the measure (3.2.7) becomes

$$dm_{\alpha,\beta}(r, \psi) = 2M_{\alpha,\beta}\, g(x, y, z, \chi)^{\alpha-\beta-1} \big(\sinh x \sinh y \sinh z\big)^{-2\alpha} (\sin \chi)^{2\beta} A_{\alpha,\beta}(z)\, dz d\chi,$$

where

$$A_{\alpha,\beta}(z) := (\sinh z)^{2\alpha+1} (\cosh z)^{2\beta+1}. \quad (3.2.11)$$

Hence, the product formula (3.2.6) reads

$$\varphi_\lambda^{(\alpha,\beta)}(x)\, \varphi_\lambda^{(\alpha,\beta)}(y) = \int_0^{+\infty} \varphi_\lambda^{(\alpha,\beta)}(z)\, W_{\alpha,\beta}(x, y, z)\, A_{\alpha,\beta}(z)\, dz, \qquad x, y > 0, \quad (3.2.12)$$

where

$$W_{\alpha,\beta}(x, y, z) := 2\, M_{\alpha,\beta}\, (\sinh x \sinh y \sinh z)^{-2\alpha} \int_0^\pi g(x, y, z, \chi)_+^{\alpha-\beta-1} (\sin \chi)^{2\beta}\, d\chi$$

if $x, y, z > 0$ satisfy $|x - y| < z < x + y$ and $W_{\alpha,\beta}(x, y, z) = 0$ otherwise. Here

$$g_+ = \begin{cases} g & \text{if } \; g > 0, \\ 0 & \text{if } \; g \leq 0. \end{cases}$$

We point out that the function $W_{\alpha,\beta}(x, y, z)$ is nonnegative, symmetric in the variables x, y, z and that

$$\int_0^{+\infty} W_{\alpha,\beta}(x, y, z)\, A_{\alpha,\beta}(z)\, dz = 1.$$

Furthermore, in [30, Formula (4.19)] the authors express $W_{\alpha,\beta}$ as follows in terms of the hypergeometric function ${}_2\mathrm{F}_1$: For every $x, y, z > 0$ satisfying the triangular inequality $|x-y| < z < x+y$,

$$\begin{aligned} W_{\alpha,\beta}(x,y,z) &= M_{\alpha,\alpha}\,(\cosh x\cosh y\cosh z)^{\alpha-\beta-1}\,(\sinh x\sinh y\sinh z)^{-2\alpha} \\ &\quad\times(1-B^2)^{\alpha-\frac{1}{2}}\,{}_2\mathrm{F}_1\Big(\alpha+\beta,\alpha-\beta;\alpha+\frac{1}{2};\frac{1-B}{2}\Big), \end{aligned} \tag{3.2.13}$$

where

$$B := \frac{\cosh^2 x+\cosh^2 y+\cosh^2 z-1}{2\cosh x\cosh y\cosh z}.$$

Notice that

$$1\pm B = \frac{[\,\cosh(x+y)\pm\cosh z\,]\,[\,\cosh z\pm\cosh(x-y)\,]}{2\cosh x\cosh y\cosh z},$$

hence

$$\begin{aligned} 1-B^2 &= \frac{[\,\cosh 2(x+y)-\cosh 2z\,]\,[\,\cosh 2z-\cosh 2(x-y)\,]}{16\cosh^2 x\cosh^2 y\cosh^2 z} \\ &= \frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)}{4\cosh^2 x\cosh^2 y\cosh^2 z}. \end{aligned} \tag{3.2.14}$$

In the case $\alpha=\beta>-\frac{1}{2}$, we use instead the change of variables

$$\cosh z = |\gamma(x,y,1,\psi)| = |\cosh x\cosh y+\sinh x\sinh y\,e^{i\psi}\,|\,,$$

and we obtain the same product formula (3.2.12), where $W_{\alpha,\alpha}$ is given by

$$\begin{aligned} &W_{\alpha,\alpha}(x,y,z) = 2^{4\alpha+1}M_{\alpha,\alpha}\,\big[\sinh 2x\sinh 2y\sinh 2z\big]^{-2\alpha} \\ &\times\big[\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)\big]^{\alpha-1/2}. \end{aligned} \tag{3.2.15}$$

In the case $\alpha>\beta=-\frac{1}{2}$, we use the quadratic transformation

$$\varphi_\lambda^{(\alpha,-\frac{1}{2})}(2x) = \varphi_{2\lambda}^{(\alpha,\alpha)}(x),$$

and we obtain again the product formula (3.2.12), with

$$W_{\alpha,-\frac{1}{2}}(x,y,z) = 2^{-2\alpha}\,W_{\alpha,\alpha}(\frac{x}{2},\frac{y}{2},\frac{z}{2}).$$

As noticed by Koornwinder [46] (see also [31]), the product formulas (3.2.6) and (3.2.12) are closely connected with the addition formula for the Jacobi functions, that we recall now for later use:

$$\begin{aligned} &\varphi_\lambda^{(\alpha,\beta)}(\operatorname{arg\,cosh}|\gamma(x,y,r,\psi)|) \\ &= \sum\nolimits_{0\le\ell\le k<\infty}\varphi_{\lambda,k,\ell}^{(\alpha,\beta)}(x)\,\varphi_{-\lambda,k,\ell}^{(\alpha,\beta)}(-y)\,\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi)\,\Pi_{k,\ell}^{(\alpha,\beta)}\,, \end{aligned} \tag{3.2.16}$$

where

$$\varphi_{\lambda,k,\ell}^{(\alpha,\beta)}(x) = \frac{\mathrm{c}_{\alpha,\beta}(-\lambda)}{\mathrm{c}_{\alpha+k+\ell,\beta+k-\ell}(-\lambda)}\,(2\sinh x)^{k-\ell}\,(2\cosh x)^{k+\ell}\,\varphi_\lambda^{(\alpha+k+\ell,\beta+k-\ell)}(x)$$

are modified Jacobi functions, the functions

$$\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi) = r^{k-\ell}\,\tfrac{\ell\,!}{(\alpha-\beta)_\ell}\,\mathrm{P}_\ell^{(\alpha-\beta-1,\beta+k-\ell)}(2r^2-1)\,\tfrac{(k-\ell)!}{(\beta+\frac12)_{k-\ell}}\,\mathrm{P}_{k-\ell}^{(\beta-\frac12,\beta-\frac12)}(\cos\psi)\,,$$

which are expressed in terms of Jacobi polynomials (see for instance [1])

$$\mathrm{P}_n^{(a,b)}(z) = \tfrac{(a+1)_n}{n!}\,{}_2\mathrm{F}_1\big(-n,a+b+n+1;a+1;\tfrac{1-z}{2}\big)\,, \tag{3.2.17}$$

are orthogonal with respect to the measure (3.2.7), and

$$\begin{aligned}\Pi_{k,\ell}^{(\alpha,\beta)} &= \Big(\int_0^1\int_0^\pi \chi_{k,\ell}^{(\alpha,\beta)}(r,\psi)^2\,dm_{\alpha,\beta}(r,\psi)\Big)^{-1}\\ &= \tfrac{(\alpha+k+\ell)\,(\beta+2k-2\ell)}{(\alpha+k)\,(2\beta+k-\ell)}\,\tfrac{(\alpha+1)_k\,(\alpha-\beta)_\ell\,(2\beta+1)_{k-\ell}}{(\beta+1)_k\,\ell!\,(k-\ell)!}\,.\end{aligned} \tag{3.2.18}$$

3.3 Product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$

For $x,y,z\in\mathbb{R}$ and $\chi\in[0,\pi]$, let

$$\sigma_{x,y,z}^\chi = \begin{cases} \frac{\cosh x\,\cosh y-\cosh z\,\cos\chi}{\sinh x\,\sinh y} & \text{if } \;xy\neq0\,,\\ \quad 0 & \text{if } \;xy=0\,.\end{cases} \tag{3.3.1}$$

Furthermore, if $\alpha>\beta>-\frac12$, let us define $\mathcal{K}_{\alpha,\beta}$ by

$$\begin{aligned}\mathcal{K}_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta}\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha}\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}&\\ \times\Big[1-\sigma_{x,y,z}^\chi+\sigma_{x,z,y}^\chi+\sigma_{z,y,x}^\chi+\frac{\rho}{\beta+\frac12}\coth x\coth y\coth z\,(\sin\chi)^2\Big]&\\ \times(\sin\chi)^{2\beta}\,d\chi&\end{aligned}$$

if $x,y,z\in\mathbb{R}^*$ satisfy the triangular inequality $||x|-|y||<|z|<|x|+|y|$, and $\mathcal{K}_{\alpha,\beta}(x,y,z)=0$ otherwise. Here $g(x,y,z,\chi)$ is as in (3.2.10).

Remark 3.3.1. *The following symmetry properties are easy to check :*

$$\begin{cases}\mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(y,x,z),\\ \mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(-z,y,-x),\\ \mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(x,-z,-y).\end{cases}$$

Recall the Opdam functions $\mathrm{G}_\lambda^{(\alpha,\beta)}$ defined in (3.1.2). This section is devoted to the proof of our main result, that we state first in the case $\alpha>\beta>-\frac{1}{2}$.

Theorem 3.3.2. *Assume* $\alpha>\beta>-\frac{1}{2}$. *Then* $\mathrm{G}_\lambda^{(\alpha,\beta)}$ *satisfies the following product formula*

$$\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\,\mathrm{G}_\lambda^{(\alpha,\beta)}(y)=\int_{-\infty}^{+\infty}\mathrm{G}_\lambda^{(\alpha,\beta)}(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z),$$

for $x,y\in\mathbb{R}$ *and* $\lambda\in\mathbb{C}$. *Here*

$$d\mu_{x,y}^{(\alpha,\beta)}(z)=\begin{cases}\mathcal{K}_{\alpha,\beta}(x,y,z)\,A_{\alpha,\beta}(|z|)\,dz & \text{if } xy\neq 0\\ d\delta_x(z) & \text{if } y=0\\ d\delta_y(z) & \text{if } x=0\end{cases}\tag{3.3.2}$$

and $A_{\alpha,\beta}$ *is as in* (3.2.11).

Let us split the Opdam function

$$\mathrm{G}_\lambda^{(\alpha,\beta)}=\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}+\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}$$

into its even part

$$\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(x)=\varphi_\lambda^{(\alpha,\beta)}(x)$$

and odd part

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(x)=-\frac{1}{\rho-i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x)=\frac{\rho+i\lambda}{4(\alpha+1)}\sinh 2x\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x).$$

For $x,y\in\mathbb{R}^*$, the product formula (3.2.12) for the Jacobi functions yields

$$\begin{aligned}\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(x)\,\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(y)&=\int_{||x|-|y||}^{|x|+|y|}\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(z)\,W_{\alpha,\beta}(|x|,|y|,z)\,A_{\alpha,\beta}(z)\,dz\\&=\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_\lambda^{(\alpha,\beta)}(z)\,W_{\alpha,\beta}(|x|,|y|,|z|)\,A_{\alpha,\beta}(|z|)\,dz,\end{aligned}$$

where

$$I_{x,y}:=[-|x|-|y|,-||x|-|y||]\cup[||x|-|y||,|x|+|y|]\,.\tag{3.3.3}$$

Next let us turn to the mixed products. The following statement amounts to Lemma 2.3 in [14].

Lemma 3.3.3. *For* $\alpha>\beta>-\frac{1}{2}$, $\lambda\in\mathbb{C}$ *and* $x,y\in\mathbb{R}^*$, *we have*

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(x)\,\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(y)=M_{\alpha,\beta}\int_{I_{x,y}}\mathrm{G}_{\lambda}^{(\alpha,\beta)}(z)\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha}$$
$$\times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\,\sigma_{x,z,y}^{\chi}\,(\sin\chi)^{2\beta}\,d\chi\Big\}\,A_{\alpha,\beta}(|z|)\,dz\,,$$

where $g(x,y,z,\chi)$ *is given by* (3.2.10), $\sigma_{x,z,y}^{\chi}$ *by* (3.3.1) *and* $I_{x,y}$ *by* (3.3.3).

We consider now purely odd products, which is the most difficult case.

Lemma 3.3.4. *For* $\alpha>\beta>-\frac{1}{2}$, $\lambda\in\mathbb{C}$ *and* $x,y\in\mathbb{R}^*$, *we have*

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(x)\,\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(y)=M_{\alpha,\beta}\int_{I_{x,y}}\mathrm{G}_{\lambda}^{(\alpha,\beta)}(z)\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha}$$
$$\times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\Big[-\sigma_{x,y,z}^{\chi}-\frac{\rho}{(\beta+\frac{1}{2})}\coth x\coth y\coth z\,(\sin\chi)^2\Big]$$
$$\times(\sin\chi)^{2\beta}d\chi\Big\}\,A_{\alpha,\beta}(|z|)\,dz.$$

Proof. For $x,y>0$, we have

$$\begin{aligned}\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(x)\,\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(y)&=\frac{(\rho+i\lambda)^2}{16\,(\alpha+1)^2}\,\sinh 2x\,\sinh 2y\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(y)\\&=\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y)+\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y),\end{aligned}\tag{3.3.4}$$

where

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y):=-\frac{\rho^2+\lambda^2}{16\,(\alpha+1)^2}\,\sinh 2x\,\sinh 2y\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(y),$$

and

$$\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y):=\frac{\rho\,(\rho+i\lambda)}{8\,(\alpha+1)^2}\,\sinh 2x\,\sinh 2y\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(y).\tag{3.3.5}$$

Consider first $\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}$. We deduce from the addition formula (3.2.16) that

$$\int_0^1\int_0^\pi\varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)\,\chi_{1,0}^{(\alpha,\beta)}(r,\psi)\,dm_{\alpha,\beta}(r,\psi)=\varphi_{\lambda,1,0}^{(\alpha,\beta)}(x)\,\varphi_{-\lambda,1,0}^{(\alpha,\beta)}(-y),$$

where $\chi_{1,0}^{(\alpha,\beta)}(r,\psi)=r\cos\psi$ and $\varphi_{\pm\lambda,1,0}^{(\alpha,\beta)}(x)=\frac{\rho m_2^- i\lambda}{4(\alpha+1)}\sinh 2x\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)$. Hence

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y)=\int_0^1\int_0^\pi\varphi_\lambda^{(\alpha,\beta)}(\arg\cosh|\gamma(x,y,r,\psi)|)\,r\cos\psi\,dm_{\alpha,\beta}(r,\psi).\tag{3.3.6}$$

By performing the change of variables (3.2.9) and arguing as in Section 3.2, (3.3.6) becomes

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = -\,2\,M_{\alpha,\beta}\int_{|x-y|}^{x+y} \mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\beta)}(z)\,(\sinh x\sinh y\sinh z)^{-2\alpha}$$
$$\times\Big\{\int_0^\pi \sigma_{x,y,z}^{\chi}\,g(x,y,z,\chi)_+^{\alpha-\beta-1}\,(\sin\chi)^{2\beta}\,d\chi\Big\}\,A_{\alpha,\beta}(z)\,dz.$$

By using the symmetries

$$\begin{cases} g(x,y,z,\chi) = \, g(|x|,|y|,|z|,\chi)\,, \\ \mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = \, \operatorname{sign}(xy)\,\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(|x|,|y|)\,, \\ \sigma_{x,y,z}^{\chi} = \, \operatorname{sign}(xy)\,\sigma_{|x|,|y|,|z|}^{\chi}\,, \end{cases}$$

we conclude, for all $x,y\in\mathbb{R}^*$, that

$$\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}(x,y) = -\,M_{\alpha,\beta}\int_{I_{x,y}} \mathrm{G}_{\lambda}^{(\alpha,\beta)}(z)\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha}$$
$$\times\Big\{\int_0^\pi \sigma_{x,y,z}^{\chi}\,g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta}\,d\chi\Big\}\,A_{\alpha,\beta}(|z|)\,dz.$$

Consider next $\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}$. By using this time the product formula (3.2.12) for $\varphi_\lambda^{(\alpha+1,\beta+1)}$, we obtain, for $x,y>0$,

$$\begin{aligned} \mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y) &= \frac{\rho\,(\rho+i\lambda)}{4\,(\alpha+1)^2}\,M_{\alpha+1,\beta+1}\sinh 2x\sinh 2y \\ &\quad\times\int_{|x-y|}^{x+y}\varphi_\lambda^{(\alpha+1,\beta+1)}(z)(\sinh x\sinh y\sinh z)^{-2\alpha-2} \\ &\quad\times\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta+2}d\chi\Big\}\,A_{\alpha+1,\beta+1}(z)\,dz \\ &= 2\,M_{\alpha,\beta}\int_{|x-y|}^{x+y}\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(z)\,(\sinh x\sinh y\sinh z)^{-2\alpha}\,\frac{\rho}{\beta+\frac12} \\ &\quad\times\coth x\coth y\coth z\,\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta+2}d\chi\Big\}\,A_{\alpha,\beta}(z)\,dz\,. \end{aligned}$$

By arguing again by evenness and oddness, we deduce, for all $x,y\in\mathbb{R}^*$,

$$\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x,y) = M_{\alpha,\beta}\int_{I_{x,y}}\mathrm{G}_{\lambda}^{(\alpha,\beta)}(z)\,\big|\sinh x\,\sinh y\,\sinh z\,\big|^{-2\alpha}$$
$$\times\,\frac{\rho}{\beta+\frac12}\,(\coth x\coth y\coth z)\,\Big\{\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta+2}d\chi\Big\}\,A_{\alpha,\beta}(|z|)\,dz\,.$$

This concludes the proof of Lemma 3.3.4 and hence the proof of Theorem 3.3.2. ☐

Next we turn our attention to the case $\alpha=\beta>-\frac{1}{2}$. For $x,y,z\in\mathbb{R}$, let

$$\sigma_{x,y,z}=\begin{cases}\frac{\cosh 2x\,\cosh 2y-\cosh 2z}{\sinh 2x\,\sinh 2y} & \text{if } \; xy\neq 0\,,\\ 0 & \text{if } \; xy=0\,.\end{cases} \tag{3.3.7}$$

Moreover, we define the kernel $\mathcal{K}_{\alpha,\alpha}$ by

$$\begin{aligned}\mathcal{K}_{\alpha,\alpha}(x,y,z)&=2^{4\alpha+2}M_{\alpha,\alpha}\,e^{x+y-z}\\ &\times\frac{\big[\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)\big]^{\alpha-1/2}}{\big|\sinh 2x\,\sinh 2y\,\sinh 2z\,\big|^{2\alpha}}\\ &\times\frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)}{\sinh 2x\,\sinh 2y\,\sinh 2z}\end{aligned} \tag{3.3.8}$$

if $||x|-|y||<|z|<|x|+|y|$, and $\mathcal{K}_{\alpha,\alpha}(x,y,z)=0$ otherwise. The symmetry properties of $\mathcal{K}_{\alpha,\beta}$ (see Remark 3.3.1) remain true for $\mathcal{K}_{\alpha,\alpha}$.

Theorem 3.3.5. *In the case $\alpha=\beta>-\frac{1}{2}$, the product formula reads*

$$\mathrm{G}_\lambda^{(\alpha,\alpha)}(x)\,\mathrm{G}_\lambda^{(\alpha,\alpha)}(y)=\int_{-\infty}^{+\infty}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,d\mu_{x,y}^{(\alpha,\alpha)}(z), \tag{3.3.9}$$

for $x,y\in\mathbb{R}$ and $\lambda\in\mathbb{C}$. Here

$$d\mu_{x,y}^{(\alpha,\alpha)}(z)=\begin{cases}\mathcal{K}_{\alpha,\alpha}(x,y,z)\,A_{\alpha,\alpha}(|z|)\,dz & \text{if } \; xy\neq 0\,,\\ d\delta_x(z) & \text{if } \; y=0\,,\\ d\delta_y(z) & \text{if } \; x=0\,.\end{cases} \tag{3.3.10}$$

Proof. The even product formula

$$\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(x)\,\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(y)=\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz$$

and the mixed product formulas

$$\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(x)\,\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(y)=\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,\sigma_{z,y,x}\,W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz\,,$$

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(x)\,\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(y)=\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_\lambda^{(\alpha,\alpha)}(z)\,\sigma_{x,z,y}\,W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz$$

are obtained as in the case $\alpha>\beta$. Here $W_{\alpha,\alpha}$ is given by (3.2.15), $I_{x,y}$ by (3.3.3) and $\sigma_{x,z,y}$ by (3.3.7). As far as they are concerned, odd products are splitted up as in (3.3.4):

$$\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(x)\,\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(y)=\mathcal{I}_{\lambda,1}^{(\alpha,\alpha)}(x,y)+\mathcal{I}_{\lambda,2}^{(\alpha,\alpha)}(x,y).$$

The first expression $\mathcal{I}_{\lambda,1}^{(\alpha,\alpha)}$ is handled as $\mathcal{I}_{\lambda,1}^{(\alpha,\beta)}$ in the case $\alpha > \beta$. We perform this time the change of variables $]0,\pi[\,\ni\psi \longmapsto z\in\,]0,+\infty[$ defined by $\cosh z = |\gamma(x,y,1,\psi)|$ and we obtain this way

$$\begin{aligned}\mathcal{I}_{\lambda,1}^{(\alpha,\alpha)}(x,y) &= -\,M_{\alpha,\alpha}\int_0^\pi \mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(\arg\cosh|\gamma(x,y,1,\psi)|)\,\cos\psi\,(\sin\psi)^{2\alpha}\,d\psi \\ &= -\int_{|x-y|}^{x+y}\mathrm{G}_{\lambda,\mathrm{e}}^{(\alpha,\alpha)}(z)\,\sigma_{x,y,z}\,W_{\alpha,\alpha}(x,y,z)\,A_{\alpha,\alpha}(z)\,dz\,,\end{aligned}$$

hence

$$\mathcal{I}_{\lambda,1}^{(\alpha,\alpha)}(x,y) = -\,\frac{1}{2}\int_{I_{x,y}}\mathrm{G}_{\lambda}^{(\alpha,\alpha)}(z)\,\sigma_{x,y,z}\,W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz\,,$$

first for $x,y>0$ and next for $x,y\in\mathbb{R}^*$. According to the product formula for $\varphi_\lambda^{(\alpha+1,\alpha+1)}$, the second expression $\mathcal{I}_{\lambda,2}^{(\alpha,\alpha)}$ becomes

$$\begin{aligned}\mathcal{I}_{\lambda,2}^{(\alpha,\alpha)}(x,y) = \frac{2\alpha+1}{2(\alpha+1)}&\int_{|x-y|}^{x+y}\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\alpha)}(z) \\ &\times\frac{\sinh 2x\,\sinh 2y}{\sinh 2z}\,W_{\alpha+1,\alpha+1}(x,y,z)\,A_{\alpha+1,\alpha+1}(z)\,dz\end{aligned}$$

for all $x,y>0$. By using

$$\begin{aligned}&W_{\alpha+1,\alpha+1}(x,y,z) = 16\,\frac{\alpha+1}{\alpha+1/2} \\ &\times\frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)}{\sinh^2 2x\,\sinh^2 2y\,\sinh^2 2z}\,W_{\alpha,\alpha}(x,y,z)\end{aligned}$$

and

$$A_{\alpha+1,\alpha+1}(z) = \frac{\sinh^2 2z}{4}\,A_{\alpha,\alpha}(z)\,,$$

we obtain

$$\begin{aligned}\mathcal{I}_{\lambda,2}^{(\alpha,\alpha)}(x,y) = 2&\int_{I_{x,y}}\mathrm{G}_{\lambda}^{(\alpha,\alpha)}(z) \\ &\times\frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\sinh(x+y-z)}{\sinh 2x\,\sinh 2y\,\sinh 2z} \\ &\times W_{\alpha,\alpha}(|x|,|y|,|z|)\,A_{\alpha,\alpha}(|z|)\,dz\,,\end{aligned}$$

first for $x,y>0$ and next for $x,y\in\mathbb{R}^*$. We conclude the proof of Theorem 3.3.5 by summing all partial product formulas and by using the remarkable identity

$$\begin{aligned}\varrho(x,y,z) &= 1-\sigma_{x,y,z}+\sigma_{z,y,x}+\sigma_{x,z,y} \\ &= 4\,\frac{\sinh(x+y+z)\sinh(-x+y+z)\sinh(x-y+z)\cosh(x+y-z)}{\sinh 2x\,\sinh 2y\,\sinh 2z}\,.\end{aligned}$$

□

Consider next the rational limit of the product formula (3.3.9). It is well known that the hypergeometric function ${}_2\mathrm{F}_1\big(a,b;c;z\big)$ tends to the confluent hypergeometric limit function ${}_0\mathrm{F}_1\big(c;Z\big)$ as $a,b\to\infty$ and $z\to 0$ in such a way that $abz\to Z$. Consequently, as $\varepsilon\to 0$,

$$\varphi_{\lambda/\varepsilon}^{(\alpha,\alpha)}(\varepsilon x)={}_2\mathrm{F}_1\Big(\alpha+\frac{1}{2}+i\frac{\lambda}{2\varepsilon},\alpha+\frac{1}{2}-i\frac{\lambda}{2\varepsilon};\alpha+1;-(\sinh\varepsilon x)^2\Big)$$

tends to the normalized Bessel function

$$\mathrm{j}_\alpha(\lambda x)={}_0\mathrm{F}_1\Big(\alpha+1;-\big(\frac{\lambda x}{2}\big)^2\Big)=\Gamma(\alpha+1)\sum_{m=0}^{+\infty}\frac{(-1)^m}{m!\,\Gamma(\alpha+1+m)}\,\big(\frac{\lambda x}{2}\big)^{2m},$$

hence

$$\mathrm{G}_{\lambda/\varepsilon}^{(\alpha,\alpha)}(\varepsilon x)=\varphi_{\lambda/\varepsilon}^{(\alpha,\alpha)}(\varepsilon x)+\frac{2\alpha+1+i\lambda/\varepsilon}{4\,(\alpha+1)}\sinh(2\varepsilon x)\,\varphi_{\lambda/\varepsilon}^{(\alpha,\alpha)}(\varepsilon x)$$

tends to

$$\mathrm{E}_\alpha(i\lambda,x)=\mathrm{j}_\alpha(\lambda x)+\frac{i\lambda x}{2\,(\alpha+1)}\,\mathrm{j}_{\alpha+1}(\lambda x).$$

The latter expression is the so–called Dunkl kernel in dimension 1, whose product formula was obtained in [56]:

$$\mathrm{E}_\alpha(i\lambda,x)\,\mathrm{E}_\alpha(i\lambda,y)=\int_{\mathbb{R}}\mathrm{E}_\alpha(i\lambda,z)\,k_\alpha(x,y,z)\,|z|^{2\alpha+1}\,dz\,, \tag{3.3.11}$$

where

$$k_\alpha(x,y,z)=2^{-2\alpha}\,\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\,\big[\,1-\varsigma_{x,y,z}+\varsigma_{z,y,x}+\varsigma_{x,z,y}\,\big]$$
$$\times\,\frac{[\,(x+y+z)\,(-x+y+z)\,(x-y+z)\,(x+y-z)\,]^{\,\alpha-\frac{1}{2}}}{|\,x\,y\,z\,|^{\,2\alpha}}$$

with

$$\varsigma_{x,y,z}=\begin{cases}\frac{x^2+y^2-z^2}{2\,x\,y} & \text{if }\ x\,y\neq 0\,,\\ 0 & \text{if }\ x\,y=0\,,\end{cases}$$

hence

$$k_\alpha(x,y,z)=2^{-2\alpha-1}\,\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\,\frac{[(x+y+z)(-x+y+z)(x-y+z)(x+y-z)]^{\alpha-\frac{1}{2}}}{|xyz|^{2\alpha}}$$
$$\times\,\frac{(x+y+z)(-x+y+z)(x-y+z)}{xyz}\,. \tag{3.3.12}$$

Here is an immediate consequence of (3.3.8) and (3.3.12).

Lemma 3.3.6. *For every* $\alpha > -\frac{1}{2}$ *and* $x, y, z \in \mathbb{R}^*$, *we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{2\alpha+2}\, \mathcal{K}_{\alpha,\alpha}(\varepsilon x, \varepsilon y, \varepsilon z) = k_\alpha(x, y, z).$$

We deduce the following result, which was announced in the abstract and in the introduction.

Corollary 3.3.7. *The product formula* (3.3.11) *is the rational limit of the product formula* (3.3.9). *More precisely,* (3.3.11) *is obtained by replacing* λ *by* λ/ε *and* (x, y) *by* $(\varepsilon x, \varepsilon y)$ *in* (3.3.9), *and by letting* $\varepsilon \to 0$.

Theorem 3.3.8. *Let* $x, y \in \mathbb{R}$.

(i) For $\alpha \geq \beta \geq -\frac{1}{2}$ *with* $\alpha > -\frac{1}{2}$, *we have* $\operatorname{supp} \mu_{x,y}^{(\alpha,\beta)} \subset I_{x,y}$.

(ii) For $\alpha \geq \beta \geq -\frac{1}{2}$ *with* $\alpha > -\frac{1}{2}$, *we have* $\mu_{x,y}^{(\alpha,\beta)}(\mathbb{R}) = 1$.

(iii) For $\alpha > \beta > -\frac{1}{2}$, *we have* $\|\mu_{x,y}^{(\alpha,\beta)}\| \leq 4 + \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\dfrac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)}$.

(iv) For $\alpha = \beta > -\frac{1}{2}$, *we have* $\|\mu_{x,y}^{(\alpha,\alpha)}\| \leq \frac{5}{2}$.

Proof. (i) is obvious.
(ii) This claim follows from Theorems 3.3.2 and 3.3.5 and the fact that $\mathrm{G}_{i\rho}^{(\alpha,\beta)} \equiv 1$.
(iii) From the proof of Theorem 3.3.2, we may rewrite the product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$ as follows:

$$\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(y) = \int_{I_{x,y}} \mathrm{G}_\lambda^{(\alpha,\beta)}(z)\, \widetilde{\mathcal{K}}_{\alpha,\beta}(x, y, z)\, A_{\alpha,\beta}(|z|)\, dz \,+\, \mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x, y),$$

where $\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}$ is given by (3.3.5) and

$$\widetilde{\mathcal{K}}_{\alpha,\beta}(x, y, z) := M_{\alpha,\beta}\, \big|\sinh x\, \sinh y\, \sinh z\,\big|^{-2\alpha} \times \int_0^\pi (1 - \sigma_{x,y,z}^\chi + \sigma_{x,z,y}^\chi + \sigma_{z,y,x}^\chi)\, g(x, y, z, \chi)_+^{\alpha-\beta-1}\, (\sin\chi)^{2\beta}\, d\chi.$$

By [14, Proposition 2.7], we have

$$\int_{I_{x,y}} |\, \widetilde{\mathcal{K}}_{\alpha,\beta}(x, y, z)|\, A_{\alpha,\beta}(|z|)\, dz \leq 4.$$

On the other hand, using the product formula (3.2.6) for the Jacobi functions, we may rewrite $\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}$ as follows :

$$\begin{aligned}
\mathcal{I}_{\lambda,2}^{(\alpha,\beta)}(x, y) &= \frac{\rho(\rho+i\lambda)}{8(\alpha+1)^2}\, \sinh 2x\, \sinh 2y\, \varphi_\lambda^{(\alpha+1,\beta+1)}(x)\, \varphi_\lambda^{(\alpha+1,\beta+1)}(y)\\
&= \frac{\rho(\rho+i\lambda)}{8(\alpha+1)^2} \sinh 2x \sinh 2y \int_0^1\!\!\int_0^\pi \varphi_\lambda^{(\alpha+1,\beta+1)}(\operatorname{arg\,cosh} |\gamma(x, y, r, \psi)|)\, dm_{\alpha+1,\beta+1}(r, \psi)\\
&= \frac{\rho}{4(\alpha+1)} \sinh 2x \sinh 2y \int_0^1\!\!\int_0^\pi \frac{\mathrm{G}_{\lambda,\mathrm{o}}^{(\alpha,\beta)}(\operatorname{arg\,cosh} |\gamma(x, y, r, \psi)|)}{|\gamma(x, y, r, \psi)| \sqrt{|\gamma(x, y, r, \psi)|^2 - 1}}\, dm_{\alpha+1,\beta+1}(r, \psi),
\end{aligned}$$

where $\gamma(x,y,r,\psi)=\cosh x\cosh y+\sinh x\sinh y\, re^{i\psi}$. In order to conclude, it remains for us to prove the following inequality

$$\frac{\rho}{4(\alpha+1)}\sinh 2x\sinh 2y\int_0^1\!\!\int_0^\pi\frac{dm_{\alpha+1,\beta+1}(r,\psi)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}}\leq\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac12)}\frac{\Gamma(\beta+\frac12)}{\Gamma(\beta+1)}.$$

By expressing $|\gamma(x,y,r,\psi)|$ and $dm_{\alpha+1,\beta+1}$, the left hand side becomes

$$\begin{aligned}
&\frac{\rho}{4(\alpha+1)}\sinh 2x\ \sinh 2y\int_0^1\!\!\int_0^\pi\frac{dm_{\alpha+1,\beta+1}(r,\psi)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}}\\
&=\frac{\rho}{4(\alpha+1)}\frac{2\,\Gamma(\alpha+2)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac32)}\ \sinh 2x\ \sinh 2y\int_0^1\!\!\int_0^\pi(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta+2}\\
&\times\ \frac{1}{\sqrt{(\cosh x\cosh y+r\cos\psi\sinh x\sinh y)^2+(r\sin\psi\sinh x\sinh y)^2}}\\
&\times\ \frac{1}{\sqrt{(\cosh x\cosh y+r\cos\psi\sinh x\sinh y)^2+(r\sin\psi\sinh x\sinh y)^2-1}}\,r\,dr\,d\psi\\
&=\frac{\rho\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac32)}\int_0^1\!\!\int_0^\pi(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta+2}\frac{dr\,d\psi}{\sqrt{U+\cos\psi}\sqrt{V+\cos\psi}},
\end{aligned}$$

where

$$U=\frac{\cosh^2\!x\cosh^2\!y+r^2\sinh^2\!x\sinh^2\!y}{2r\cosh x\cosh y\sinh x\sinh y},$$

and

$$V=\frac{\cosh^2\!x\cosh^2\!y+r^2\sinh^2\!x\sinh^2\!y-1}{2r\cosh x\cosh y\sinh x\sinh y}.$$

Since

$$U-1>V-1=\frac{(\cosh x\cosh y-r\sinh x\sinh y)^2-1}{2r\cosh x\cosh y\sinh x\sinh y}\geq 0,$$

we can estimate

$$\begin{aligned}
&\frac{\rho}{\alpha+1}\sinh x\cosh x\sinh y\cosh y\int_0^1\!\!\int_0^\pi\frac{dm_{\alpha+1,\beta+1}(r,\psi)}{|\gamma(x,y,r,\psi)|\sqrt{|\gamma(x,y,r,\psi)|^2-1}}\\
&\leq\frac{\rho\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac32)}\int_0^1\!\!\int_0^\pi(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta+2}(1+\cos\psi)^{-1}drd\psi\\
&=\frac{\rho}{2}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac32)}\,\frac{\Gamma(\beta+\frac12)}{\Gamma(\beta+1)}\leq\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac12)}\,\frac{\Gamma(\beta+\frac12)}{\Gamma(\beta+1)}\,,
\end{aligned}$$

using classical formulas for the Beta and Gamma functions.
(iv) is proved in a similar way, using the product formula (3.2.8) for $\varphi_\lambda^{(\alpha,\alpha)}$ instead of (3.2.6). □

Remark 3.3.9. *The measure* $\mu_{x,y}^{(\alpha,\beta)}$ *is not positive, for any* $\alpha \geq \beta > -\frac{1}{2}$ *and* $x, y > 0$. *More precisely, let us show that* $\mathcal{K}_{\alpha,\beta}(x,y,z) < 0$ *if* $-x-y < z < -|x-y|$, *while* $\mathcal{K}_{\alpha,\beta}(x,y,z) > 0$ *if* $|x-y| < z < x+y$. *In the limit case* $\alpha = \beta$, *our claim follows immediately from the expression* (3.3.8). *Thus we may restrict to the case* $\alpha > \beta$. *Assume first that* $-x-y < z < -|x-y|$ *and let us split up*

$$\mathcal{K}_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta}\,(\sinh x\,\sinh y\,\sinh(-z))^{-2\alpha}\Big[\mathcal{K}^{(1)}_{\alpha,\beta}(x,y,z) + \mathcal{K}^{(2)}_{\alpha,\beta}(x,y,z)\Big],$$

where

$$\mathcal{K}^{(1)}_{\alpha,\beta}(x,y,z) = \int_0^\pi g(x,y,-z,\chi)_+^{\alpha-\beta-1}\big(1-\sigma^\chi_{x,y,z} + \sigma^\chi_{x,z,y} + \sigma^\chi_{z,y,x}\big)(\sin\chi)^{2\beta}\,d\chi$$

and

$$\mathcal{K}^{(2)}_{\alpha,\beta}(x,y,z) = \frac{\rho}{\beta+1/2}\,\coth x\coth y\coth z\int_0^\pi g(x,y,-z,\chi)_+^{\alpha-\beta-1}(\sin\chi)^{2\beta+2}\,d\chi\,.$$

On one hand, $\coth x\,\coth y\,\coth z < -1$ *and*

$$\int_0^\pi g(x,y,-z,\chi)_+^{\alpha-\beta-1}\,(\sin\chi)^{2\beta+2}\,d\chi > 0\,, \tag{3.3.13}$$

as the change of variables (3.2.9) *holds for* χ *in an interval starting at* 0, *where*

$$g(x,y,-z,\chi) = \sinh^2 x\,\sinh^2 y\,(1-r^2) > 0\,.$$

Hence $\mathcal{K}^{(2)}_{\alpha,\beta}(x,y,z) < 0$. *On the other hand, as*

$$\begin{aligned}\varrho^\chi(x,y,z) &= 1-\sigma^\chi_{x,y,z} + \sigma^\chi_{x,z,y} + \sigma^\chi_{z,y,x}\\ &= \tfrac{1}{\sinh x\sinh y\sinh z}\big[\sinh x\sinh y\sinh z + \sinh x\cosh y\cosh z\\ &\qquad + \cosh x\sinh y\cosh z - \cosh x\cosh y\sinh z\\ &\qquad + \tfrac{\cos\chi}{2}(-\sinh 2x - \sinh 2y + \sinh 2z)\big]\end{aligned}$$

is a decreasing function of χ, *we have*

$$\begin{aligned}\varrho^\chi(x,y,z) &\leq \varrho^0(x,y,z)\\ &= \tfrac{4}{\sinh x\sinh y\sinh z}\,\sinh\tfrac{x+y+z}{2}\,\sinh\tfrac{-x+y+z}{2}\,\sinh\tfrac{x-y+z}{2}\,\cosh\tfrac{x+y-z}{2}\\ &< 0\,.\end{aligned}$$

Hence $\mathcal{K}^{(2)}_{\alpha,\beta}(x,y,z) < 0$. *When* $|x-y| < z < x+y$, *the positivity of* $\mathcal{K}_{\alpha,\beta}(x,y,z)$ *is proved along the same lines. If* $\sinh 2z \leq \sinh 2x + \sinh 2y$, *we have now*

$$\varrho^\chi(x,y,z) \geq \varrho^0(x,y,z) > 0$$

while, if $\sinh 2z \geq \sinh 2x + \sinh 2y$,

$$\begin{aligned}\varrho^{\chi}(x,y,z) &\geq \varrho^{\pi}(x,y,z)\\ &= \tfrac{4}{\sinh x \sinh y \sinh z}\, \cosh \tfrac{x+y+z}{2}\, \cosh \tfrac{-x+y+z}{2}\, \cosh \tfrac{x-y+z}{2}\, \sinh \tfrac{x+y-z}{2}\\ &> 0\,.\end{aligned}$$

3.4 Generalized translations and convolution product

Let us denote by $\mathcal{C}_c(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ with compact support.

Let $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha > -\frac{1}{2}$. The Opdam–Cherednik transform is the Fourier transform in the trigonometric Dunkl setting. It is defined for $f \in \mathcal{C}_c(\mathbb{R})$ by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)\, \mathrm{G}_{\lambda}^{(\alpha,\beta)}(-x)\, A_{\alpha,\beta}(|x|)\, dx \qquad \forall\, \lambda \in \mathbb{C} \tag{3.4.1}$$

and the inverse transform writes

$$\mathcal{J}g(x) = \int_{\mathbb{R}} g(\lambda)\, \mathrm{G}_{\lambda}^{(\alpha,\beta)}(x) \left(1 - \frac{\rho}{i\lambda}\right) \frac{d\lambda}{8\pi\, |\mathrm{c}_{\alpha,\beta}(\lambda)|^2}\,.$$

Here $A_{\alpha,\beta}$ and $\mathrm{c}_{\alpha,\beta}$ are given by (3.2.11) and (3.2.3). See [51] for more details.

The Fourier transform $\mathcal{F}$ can be expressed in terms of the Jacobi transform

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_0^{+\infty} f(x)\, \varphi_{\lambda}^{(\alpha,\beta)}(x)\, A_{\alpha,\beta}(x)\, dx\,. \tag{3.4.2}$$

More precisely :

Lemma 3.4.1. *For $\lambda \in \mathbb{C}$ and $f \in \mathcal{C}_c(\mathbb{R})$, we have*

$$\mathcal{F}(f)(\lambda) = 2\, \mathcal{F}_{\alpha,\beta}(f_{\mathrm{e}})(\lambda) + 2\,(\rho + i\lambda)\, \mathcal{F}_{\alpha,\beta}(Jf_{\mathrm{o}})(\lambda)\,,$$

where f_{e} (resp. f_{o}) denotes the even (resp. odd) part of f, and

$$Jf_{\mathrm{o}}(x) := \int_{-\infty}^{x} f_{\mathrm{o}}(t)\, dt\,.$$

Proof. Write $f = f_{\mathrm{e}} + f_{\mathrm{o}}$. Firstly, if $\lambda = -i\rho$, then

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)\, A_{\alpha,\beta}(|x|)\, dx = 2\, \mathcal{F}_{\alpha,\beta}(f_{\mathrm{e}})(i\rho)\,.$$

Secondly, if $\lambda \neq -i\rho$, we have

$$\mathcal{F}(f)(\lambda) = 2\,\mathcal{F}_{\alpha,\beta}(f_{\mathrm{e}})(\lambda) + \frac{2}{\rho - i\lambda}\int_0^{+\infty} f_{\mathrm{o}}(x)\,\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x)\,A_{\alpha,\beta}(x)\,dx\,.$$

Recall the Jacobi operator

$$\Delta_{\alpha,\beta} = \frac{1}{A_{\alpha,\beta}(x)}\frac{\partial}{\partial x}\Big[A_{\alpha,\beta}(x)\frac{\partial}{\partial x}\Big] = \frac{\partial^2}{\partial x^2} + \Big[(2\alpha+1)\coth x + (2\beta+1)\tanh x\Big]\frac{\partial}{\partial x}\,.$$

By integration by parts, we obtain

$$\begin{aligned}&\int_0^{+\infty} f_{\mathrm{o}}(x)\,\frac{\partial}{\partial x}\varphi_\lambda^{(\alpha,\beta)}(x)\,A_{\alpha,\beta}(x)\,dx\\ &= -\int_0^{+\infty}\varphi_\lambda^{(\alpha,\beta)}(x)\,\frac{1}{A_{\alpha,\beta}(x)}\,\frac{\partial}{\partial x}\Big[A_{\alpha,\beta}(x)\,\frac{\partial}{\partial x}\,Jf_{\mathrm{o}}(x)\Big]A_{\alpha,\beta}(x)\,dx\\ &= -\,\mathcal{F}_{\alpha,\beta}(\Delta_{\alpha,\beta}Jf_{\mathrm{o}})(\lambda) = (\rho^2+\lambda^2)\,\mathcal{F}_{\alpha,\beta}(Jf_{\mathrm{o}})(\lambda)\,.\end{aligned}$$

□

The following Plancherel formula was proved by Opdam [51, Theorem 9.13(3)]:

$$\begin{aligned}\int_{\mathbb{R}}|f(x)|^2A_{\alpha,\beta}(|x|)\,dx &= \int_0^{+\infty}\big(\,|\mathcal{F}(f)(\lambda)|^2 + |\mathcal{F}(\check{f})(\lambda)|^2\,\big)\,\frac{d\lambda}{16\pi\,|\mathrm{c}_{\alpha,\beta}(\lambda)|^2}\\ &= \int_{\mathbb{R}}\mathcal{F}(f)(\lambda)\,\overline{\mathcal{F}(\check{f})(-\lambda)}\,(1-\frac{\rho}{i\lambda})\,\frac{d\lambda}{8\pi\,|\mathrm{c}_{\alpha,\beta}(\lambda)|^2}\,,\end{aligned}$$

where $\check{f}(x) := f(-x)$. The following result is obtained by specializing [57, Theorem 4.1].

Theorem 3.4.2. *The Opdam–Cherednik transform $\mathcal{F}$ and its inverse $\mathcal{J}$ are topological isomorphisms between the Schwartz space $\mathcal{S}_{\alpha,\beta}(\mathbb{R}) = (\cosh x)^{-\rho}\,\mathcal{S}(\mathbb{R})$ and the Schwartz space $\mathcal{S}(\mathbb{R})$. Recall that $\rho = \alpha+\beta+1$.*

Let us denote by $\mathcal{C}_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$.

Definition 3.4.3. *Let $x \in \mathbb{R}$ and let $f \in \mathcal{C}_b(\mathbb{R})$. For $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$, we define the generalized translation operator $\tau_x^{(\alpha,\beta)}$ by*

$$\tau_x^{(\alpha,\beta)}f(y) \,=\, \int_{\mathbb{R}} f(z)\,d\mu_{x,y}^{(\alpha,\beta)}(z)\,,$$

where $d\mu_{x,y}^{(\alpha,\beta)}$ is given by (3.3.2) for $\alpha > \beta$, and by (3.3.10) for $\alpha = \beta$.

The following properties are clear. However for completeness we will sketch their proof.

Proposition 3.4.4. *Let* $\alpha \geq \beta \geq -\frac{1}{2}$ *with* $\alpha \neq -\frac{1}{2}$, $x, y \in \mathbb{R}$ *and* $f \in \mathcal{C}_b(\mathbb{R})$. *Then*
(i) $\tau_x^{(\alpha,\beta)} f(y) = \tau_y^{(\alpha,\beta)} f(x)$.
(ii) $\tau_0^{(\alpha,\beta)} f = f$.
(iii) $\tau_x^{(\alpha,\beta)} \tau_y^{(\alpha,\beta)} = \tau_y^{(\alpha,\beta)} \tau_x^{(\alpha,\beta)}$.
(iv) $\tau_x^{(\alpha,\beta)} \mathrm{G}_\lambda^{(\alpha,\beta)}(y) = \mathrm{G}_\lambda^{(\alpha,\beta)}(x)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(y)$.
If we suppose also that f *belongs to* $\mathcal{C}_c(\mathbb{R})$, *then*
(v) $\mathcal{F}(\tau_x^{(\alpha,\beta)} f)(\lambda) = \mathrm{G}_\lambda^{(\alpha,\beta)}(x)\, \mathcal{F}(f)(\lambda)$.
(vi) $\mathrm{T}^{(\alpha,\beta)}\, \tau_x^{(\alpha,\beta)} = \tau_x^{(\alpha,\beta)}\, \mathrm{T}^{(\alpha,\beta)}$.

Proof. (i) follows from the property $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(y,x,z)$.
(ii) follows from the fact that $\mathcal{K}_{\alpha,\beta}(0,y,z) = \delta_y(z)$.
(iii) follows from the fact that the function

$$H(x_1,y_1,x_2,y_2) := \int_{\mathbb{R}} \mathcal{K}_{\alpha,\beta}(x_1,y_1,z)\, \mathcal{K}_{\alpha,\beta}(x_2,y_2,z)\, A_{\alpha,\beta}(|z|)\, dz$$

is symmetric in the four variables.
(iv) follows from the product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$.
(v) For $f \in \mathcal{C}_c(\mathbb{R})$, we have

$$\begin{aligned}
\mathcal{F}(\tau_x^{(\alpha,\beta)} f)(\lambda) &= \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(y)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(-y)\, A_{\alpha,\beta}(|y|)\, dy \\
&= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} f(z)\, \mathcal{K}_{\alpha,\beta}(x,y,z)\, A_{\alpha,\beta}(|z|)\, dz \Big] \mathrm{G}_\lambda^{(\alpha,\beta)}(-y)\, A_{\alpha,\beta}(|y|)\, dy \\
&= \int_{\mathbb{R}} f(z) \Big[\int_{\mathbb{R}} \mathrm{G}_\lambda^{(\alpha,\beta)}(-y)\, \mathcal{K}_{\alpha,\beta}(x,y,z)\, A_{\alpha,\beta}(|y|)\, dy \Big] A_{\alpha,\beta}(|z|)\, dz\,.
\end{aligned}$$

Since $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(x,-z,-y)$, it follows from the product formula that

$$\begin{aligned}
\mathcal{F}(\tau_x^{(\alpha,\beta)} f)(\lambda) &= \mathrm{G}_\lambda^{(\alpha,\beta)}(x) \int_{\mathbb{R}} f(z)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(-z)\, A_{\alpha,\beta}(|z|)\, dz \\
&= \mathrm{G}_\lambda^{(\alpha,\beta)}(x)\, \mathcal{F}(f)(\lambda).
\end{aligned}$$

(vi) This property follows from the injectivity of $\mathcal{F}$ and the fact that $\tau_x^{(\alpha,\beta)}(\mathrm{T}^{(\alpha,\beta)} f)$ and $\mathrm{T}^{(\alpha,\beta)}(\tau_x^{(\alpha,\beta)} f)$ have the same Fourier transform, namely

$$\lambda \longmapsto i\, \lambda\, \mathrm{G}_\lambda^{(\alpha,\beta)}(x)\, \mathcal{F}(f)(\lambda)\,.$$

□

Remark 3.4.5. *Generalized translations in the Dunkl setting were first introduced by Trimèche, using transmutation operators. This approach is resumed in [48], which deals with a generalization of Dunkl analysis in dimension* 1.

Lemma 3.4.6. *Let* $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}, A_{\alpha,\beta}(|z|)dz)$ *and* $x \in \mathbb{R}$. *Then*

$$\|\tau_x^{(\alpha,\beta)} f\|_p \leq C_{\alpha,\beta} \, \|f\|_p \,, \tag{3.4.3}$$

where

$$C_{\alpha,\beta} = \begin{cases} 4 + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+1)} & \text{if } \alpha > \beta > -\frac{1}{2}\,, \\ \frac{5}{2} & \text{if } \alpha = \beta > -\frac{1}{2}\,. \end{cases} \tag{3.4.4}$$

Proof. The inequality (3.4.3) follows from Theorem 3.3.8. More precisely, the cases $p = 1$ and $p = \infty$ are elementary, while the intermediate case $1 < p < \infty$ is obtained by interpolation or by using Hölder's inequality, as follows:

$$\begin{aligned} \|\tau_x^{(\alpha,\beta)} f\|_p^p \leq & \Big(\int_{\mathbb{R}} |\mathcal{K}_{\alpha,\beta}(x,y,z)| \, A_{\alpha,\beta}(|z|) \, dz \Big)^{p-1} \\ & \times \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{K}_{\alpha,\beta}(x,y,z)| \, |f(z)|^p \, A_{\alpha,\beta}(|z|) \, A_{\alpha,\beta}(|y|) \, dz \, dy \\ & \leq C_{\alpha,\beta}^p \, \|f\|_p^p \,. \end{aligned}$$

□

Definition 3.4.7. *The convolution product of suitable functions* f *and* g *is defined by*

$$(f *_{\alpha,\beta} g)(x) = \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y) \, g(y) \, A_{\alpha,\beta}(|y|) \, dy \,.$$

Remark 3.4.8. *It is clear that this convolution product is both commutative and associative:*

(i) $f *_{\alpha,\beta} g = g *_{\alpha,\beta} f$.
(ii) $(f *_{\alpha,\beta} g) *_{\alpha,\beta} h = f *_{\alpha,\beta} (g *_{\alpha,\beta} h)$.

For every $a > 0$, let us denote by $\mathcal{D}_a(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ which are supported in $[-a, a]$.

Proposition 3.4.9. *Let* $f \in \mathcal{D}_a(\mathbb{R})$ *and* $g \in \mathcal{D}_b(\mathbb{R})$. *Then* $f *_{\alpha,\beta} g \in \mathcal{D}_{a+b}(\mathbb{R})$ *and*

$$\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) = \mathcal{F}(f)(\lambda) \, \mathcal{F}(g)(\lambda) \,.$$

Proof. By definition we have

$$\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y)\, g(y)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(-x)\, A_{\alpha,\beta}(|x|)\, A_{\alpha,\beta}(|y|)\, dx\, dy\,.$$

Using the product formula for $\mathrm{G}_\lambda^{(\alpha,\beta)}$ and Remark 3.4.8, we deduce that

$$\begin{aligned}\mathcal{F}(f *_{\alpha,\beta} g)(\lambda) &= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} \mathrm{G}_\lambda^{(\alpha,\beta)}(-x)\, \mathcal{K}_{\alpha,\beta}(-z,-y,-x) \\ &\quad \times\ A_{\alpha,\beta}(|x|)\, A_{\alpha,\beta}(|y|)\, A_{\alpha,\beta}(|z|)\, dx\, dy\, dz \\ &= \int_{\mathbb{R}} f(z)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(-z)\, A_{\alpha,\beta}(|z|)\, dz \int_{\mathbb{R}} g(y)\, \mathrm{G}_\lambda^{(\alpha,\beta)}(-y)\, A_{\alpha,\beta}(|y|)\, dy \\ &= \mathcal{F}(f)(\lambda)\, \mathcal{F}(g)(\lambda)\,.\end{aligned}$$

□

By standard arguments, the following statement follows from Lemma 3.4.6.

Proposition 3.4.10. *Assume that* $1 \le p, q, r \le \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. *Then, for every* $f \in L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$ *and* $g \in L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, *we have* $f *_{\alpha,\beta} g \in L^r(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$, *and*

$$\| f *_{\alpha,\beta} g \|_r \le C_{\alpha,\beta}\, \|f\|_p\, \|g\|_q\,,$$

where $C_{\alpha,\beta}$ *is as in* (3.4.4).

3.5 The Kunze–Stein phenomenon

This remarkable phenomenon was first observed by Kunze and Stein [45] for the group $G = SL(2,\mathbb{R})$ equipped with its Haar measure. They proved that

$$L^p(G) * L^2(G) \subset L^2(G) \quad \forall\, 1 \le p < 2\,.$$

By such an inclusion, we mean the existence of a constant $C_p > 0$ such that the following inequality holds:

$$\| f * g \|_2 \le C_p\, \|f\|_p\, \|g\|_2 \quad \forall\, f \in L^p(G),\ \forall\, g \in L^2(G).$$

This result was generalized by Cowling [27] to all connected noncompact semisimple Lie groups with finite center. We prove the following analog in our setting (we understand that Trimèche has recently extended this result to higher dimensions).

Theorem 3.5.1. *Let* $1 \leq p < 2 < q \leq \infty$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \tag{3.5.1}$$

and

$$L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)\,. \tag{3.5.2}$$

Proof. (i) Let $f, g \in \mathcal{C}_c(\mathbb{R})$. Then, by the Plancherel formula, we have

$$\begin{aligned}
&\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)|^2(x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= \int_{\mathbb{R}^+} |\mathcal{F}(f *_{\alpha,\beta} g)(\lambda)|^2 \frac{d\lambda}{16\pi\, |\mathrm{c}(\lambda)|^2} + \int_{\mathbb{R}^+} |\mathcal{F}(f *_{\alpha,\beta} g)\check{}\,(\lambda)|^2 \frac{d\lambda}{16\pi\, |\mathrm{c}(\lambda)|^2} \\
&\leq \sup_{\lambda\in\mathbb{R},\, w\in\{\pm 1\}} |\mathcal{F}(w\cdot g)(\lambda)|^2 \Big[\int_{\mathbb{R}^+} |\mathcal{F}(f)(\lambda)|^2 \frac{d\lambda}{16\pi\, |\mathrm{c}(\lambda)|^2} + \int_{\mathbb{R}^+} |\mathcal{F}(\check{f})(\lambda)|^2 \frac{d\lambda}{16\pi\, |\mathrm{c}(\lambda)|^2} \Big] \\
&= \sup_{\lambda\in\mathbb{R},\, w\in\{\pm 1\}} |\mathcal{F}(w\cdot g)(\lambda)|^2\, \|f\|_2^2\,.
\end{aligned}$$

Here we have used the fact that $\mathcal{F}(f *_{\alpha,\beta} g)\check{} = \mathcal{F}(\check{f})\,\mathcal{F}(\check{g})$. Next, if $1 \leq p < 2$ and $2 < q \leq \infty$ are dual indices, we estimate

$$\begin{aligned}
|\mathcal{F}(w\cdot g)(\lambda)| &\leq \int_{\mathbb{R}} |g(wx)|\, |\mathrm{G}_\lambda^{(\alpha,\beta)}(-x)|\, A_{\alpha,\beta}(|x|)dx \\
&\leq \|g\|_p\, \|\mathrm{G}_\lambda^{(\alpha,\beta)}\|_q
\end{aligned}$$

using Hölder's inequality. We conclude by using Lemma 3.5.2 below, which implies that $\|\mathrm{G}_\lambda^{(\alpha,\beta)}\|_q$ is bounded uniformly in $\lambda \in \mathbb{R}$.
(ii) Let $f, g, k \in \mathcal{C}_c(\mathbb{R})$. Using the Cauchy-Schwartz inequality and (3.5.1), we get

$$\begin{aligned}
\Big| \int_{\mathbb{R}} (f *_{\alpha,\beta} g)(x)\, k(x)\, A_{\alpha,\beta}(|x|)\, dx \Big| &\leq C\, \|g\|_2\, \|\, f * \check{k}\, \|_2 \\
&\leq C_p\, \|f\|_2\, \|g\|_2\, \|k\|_p\,.
\end{aligned}$$

Hence $\|\, f *_{\alpha,\beta} g\, \|_q \leq C_q\, \|f\|_2\, \|g\|_2\,.$ ∎

Lemma 3.5.2. (i) *The function* $\mathrm{G}_0^{(\alpha,\beta)}$ *is stricly positive and is bounded above by*

$$\begin{cases} C\,(1+x)\, e^{-\rho x} & \text{if } x \geq 0\,, \\ C\, e^{\rho x} & \text{if } x \leq 0\,. \end{cases}$$

(ii) *For every* $\lambda \in \mathbb{R}$ *and* $x \in \mathbb{R}$, *we have*

$$\big|\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\big| \leq \mathrm{G}_0^{(\alpha,\beta)}(x)\,.$$

Proof. These estimates are proved in full generality in [57] (see Lemma 3.1, Proposition 3.1.a and Theorem 3.2). For the reader's convenience, we include a proof in dimension 1.
(i) Firstly, by specializing (3.2.1) and (3.1.2) for $\lambda\!=\!0$, we obtain

$$\varphi_0^{(\alpha,\beta)}(x) = {}_2\mathrm{F}_1\big(\tfrac{\rho}{2},\tfrac{\alpha-\beta+1}{2};\alpha\!+\!1;\tanh^2\!x\big)\,(\cosh x)^{-\rho} \tag{3.5.3}$$

and

$$\mathrm{G}_0^{(\alpha,\beta)}(x) = \varphi_0^{(\alpha,\beta)}(x) + \frac{\frac{\rho}{2}}{\alpha\!+\!1}\sinh x\cosh x\,\varphi_0^{(\alpha+1,\beta+1)}(x)\,. \tag{3.5.4}$$

It is clear that (3.5.3) is stricly positive, hence (3.5.4) when $x\!\ge\!0$. By looking more carefully at their expansions, we observe that the expression

$$\Psi_1(x) = {}_2\mathrm{F}_1\big(\tfrac{\rho}{2},\tfrac{\alpha-\beta+1}{2};\alpha\!+\!1;\tanh^2\!x\big) = \sum\nolimits_{n=0}^{+\infty}\frac{(\frac{\rho}{2})_n\,(\frac{\alpha-\beta+1}{2})_n}{(\alpha\!+\!1)_n\,n!}\,(\tanh x)^{2n}$$

is stricly larger than the expression

$$\begin{aligned}\Psi_2(x) &= \frac{\frac{\rho}{2}}{\alpha\!+\!1}\,{}_2\mathrm{F}_1\big(\tfrac{\rho}{2}\!+\!1,\tfrac{\alpha-\beta+1}{2};\alpha+2;\tanh^2\!x\big)\\ &= \sum\nolimits_{n=0}^{+\infty}\frac{(\frac{\rho}{2})_{n+1}(\frac{\alpha-\beta+1}{2})_n}{(\alpha\!+\!1)_{n+1}\,n!}\,(\tanh x)^{2n}\,.\end{aligned}$$

Hence

$$\mathrm{G}_0^{(\alpha,\beta)}(x) = (\cosh x)^{-\rho}\big\{\Psi_1(x)+\tanh x\,\Psi_2(x)\big\} > (\cosh x)^{-\rho}\big\{\Psi_1(x)-\Psi_2(x)\big\}$$

is strictly positive on $\mathbb{R}$. Secondly, by combining (3.5.4) with (3.2.5), we obtain

$$\mathrm{G}_0^{(\alpha,\beta)}(x) = \frac{2^{\rho+2}\,\Gamma(\alpha\!+\!1)}{\Gamma(\frac{\rho}{2})\,\Gamma(\frac{\alpha-\beta+1}{2})}\,x\,e^{-\rho\,x} + \mathcal{O}\big(e^{-\rho\,x}\big) \qquad \text{as } x\to+\infty$$

and

$$\mathrm{G}_0^{(\alpha,\beta)}(x) = \mathcal{O}\big(e^{\rho\,x}\big) \qquad \text{as } x\to-\infty\,,$$

which yields the announced upper bounds.
(ii) Consider the quotient

$$\mathrm{Q}_\lambda^{(\alpha,\beta)}(x) = \tfrac{\mathrm{G}_\lambda^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\,.$$

By using the equation (3.1.1) for $\mathrm{G}_\lambda^{(\alpha,\beta)}$ and $\mathrm{G}_0^{(\alpha,\beta)}$, we obtain

$$\begin{aligned}\tfrac{\partial}{\partial x}\,\mathrm{Q}_\lambda^{(\alpha,\beta)}(x) &= \tfrac{\frac{\partial}{\partial x}\,\mathrm{G}_\lambda^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)} - \mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\,\tfrac{\frac{\partial}{\partial x}\,\mathrm{G}_0^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\\ &= \big\{(\alpha\!-\!\beta)\coth x + (2\beta\!+\!1)\coth 2x + \rho\big\}\,\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(-x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\,\big\{\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x) - \mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\big\}\\ &\quad + i\lambda\,\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\,.\end{aligned}$$

Hence

$$\begin{aligned}\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|^2 &= 2\ \mathrm{Re}\left[\tfrac{\partial}{\partial x}\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\,\overline{\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)}\right]\\ &= -\,2\left\{(\alpha-\beta)\coth x+(2\beta+1)\coth 2x+\rho\right\}\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(-x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\\ &\quad\times\left\{\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|^2-\mathrm{Re}\left[\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\,\overline{\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)}\right]\right\}\end{aligned}$$

and

$$\begin{aligned}\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|^2 &= -\,2\left\{(\alpha-\beta)\coth x+(2\beta+1)\coth 2x-\rho\right\}\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(-x)}\\ &\quad\times\left\{\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|^2-\mathrm{Re}\left[\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\,\overline{\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)}\right]\right\}.\end{aligned}$$

Thus, for every $x>0$, we have

$$\begin{aligned}\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|^2 &\le -\,2\left\{(\alpha-\beta)\coth x+(2\beta+1)\coth 2x+\rho\right\}\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(-x)}{\mathrm{G}_0^{(\alpha,\beta)}(x)}\\ &\quad\times\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|\left\{\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|-\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|\right\}\\ &\le 0\end{aligned}\tag{3.5.5}$$

if $\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|\ge\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|$ and

$$\begin{aligned}\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|^2 &\le -\,2\left\{(\alpha-\beta)\coth x+(2\beta+1)\coth 2x-\rho\right\}\tfrac{\mathrm{G}_0^{(\alpha,\beta)}(x)}{\mathrm{G}_0^{(\alpha,\beta)}(-x)}\\ &\quad\times\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|\left\{\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|-\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|\right\}\\ &\le 0\end{aligned}\tag{3.5.6}$$

if $|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)|\ \ge\ |\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)|$. As real analytic functions of x, $|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)|^2$ and $|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)|^2$ coincide either everywhere or on a discrete subset of $\mathbb{R}$ with no accumulation point. In the first case, $|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)|^2=|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)|^2$ is a decreasing function of x on $[0,+\infty)$, according to (3.5.5) or (3.5.6). In the second case, consider the continuous and piecewise differentiable function

$$M(x)=\max\left\{\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|^2,\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|^2\right\}$$

on $[0,+\infty)$. Firstly, if $\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|>\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|$, then

$$\tfrac{\partial}{\partial x}M(x)=\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|^2<0,$$

according to (3.5.5). Secondly, if $\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right|<\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|$, then

$$\tfrac{\partial}{\partial x}M(x)=\tfrac{\partial}{\partial x}\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|^2<0,$$

according to (3.5.6). Thirdly, if $\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right| = \left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(-x)\right|$ for some $x>0$, then M has left and right derivatives at x, which are nonpositive, according to (3.5.5) and (3.5.6). Thus M is a decreasing function on $[0,+\infty)$. In all cases, we conclude in particular that, for every $x\in\mathbb{R}$,

$$\left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(x)\right| \leq \left|\mathrm{Q}_\lambda^{(\alpha,\beta)}(0)\right| = 1 \quad \text{i.e.} \quad \left|\mathrm{G}_\lambda^{(\alpha,\beta)}(x)\right| \leq \mathrm{G}_0^{(\alpha,\beta)}(x).$$

□

The following results are deduced by interpolation and duality from Theorem 3.5.1 and Proposition 3.4.10.

Corollary 3.5.3. *(i) Let* $1\leq p<q\leq 2$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx).$$

(ii) Let $1<p<2$ *and* $p<q\leq \frac{p}{2-p}$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx).$$

(iii) Let $2<p,q<\infty$ *such that* $\frac{q}{2}\leq p<q$. *Then*

$$L^p(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) *_{\alpha,\beta} L^{q'}(\mathbb{R}, A_{\alpha,\beta}(|x|)dx) \subset L^q(\mathbb{R}, A_{\alpha,\beta}(|x|)dx).$$

3.6 A special orthogonal system

In this section we construct an orthogonal basis of $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$ and we compute its Opdam–Cherednik transform. As limits, we recover the Hermite functions constructed by Rosenblum [55].

Proposition 3.6.1. *Let* $\alpha\geq\beta\geq-\frac{1}{2}$ *with* $\alpha>-\frac{1}{2}$. *For any fixed* $\delta>0$, *consider the sequence of functions*

$$\begin{cases} \mathrm{H}_{2n}^\delta(x) = (\cosh x)^{-\alpha-\beta-\delta-2}\, \mathrm{P}_n^{(\alpha,\delta)}(1-2\tanh^2 x), \\ \mathrm{H}_{2n+1}^\delta(x) = (\cosh x)^{-\alpha-\beta-\delta-2}\, \mathrm{P}_n^{(\alpha+1,\delta-1)}(1-2\tanh^2 x)\, \tanh x, \end{cases} \tag{3.6.1}$$

whose definition involves the Jacobi polynomials (3.2.17). *Then* $\{\mathrm{H}_n^\delta\}_{n\in\mathbb{N}}$ *is an orthogonal basis of* $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$.

Proof. To begin with, let us prove the orthogonality of $\{\mathrm{H}_n^\delta\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}, A_{\alpha,\beta}(|x|)dx)$. Firstly, by oddness

$$\int_{\mathbb{R}} \mathrm{H}_{2m}^\delta(x)\, \mathrm{H}_{2n+1}^\delta(x)\, A_{\alpha,\beta}(|x|)\, dx = 0.$$

Secondly, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{H}^{\delta}_{2m}(x)\, \mathrm{H}^{\delta}_{2n}(x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= 2\int_0^1 \mathrm{P}_m^{(\alpha,\delta)}(1-2y^2)\, \mathrm{P}_n^{(\alpha,\delta)}(1-2y^2)\, y^{2\alpha+1}\,(1-y^2)^{\delta}\, dy \\
&= 2^{-\alpha-\delta-1}\int_{-1}^1 \mathrm{P}_m^{(\alpha,\delta)}(z)\, \mathrm{P}_n^{(\alpha,\delta)}(z)\,(1-z)^{\alpha}\,(1+z)^{\delta}\, dz \\
&= \frac{\Gamma(\alpha+n+1)\,\Gamma(\delta+n+1)}{(\alpha+\delta+2n+1)\; n!\; \Gamma(\alpha+\delta+n+1)}\;\delta_{m,n}\,,
\end{aligned}$$

by performing the changes of variables $y=\tanh x$, $z=1-y^2$ and by using the orthogonality of Jacobi polynomials (see for instance [1]) :

$$\begin{aligned}
&\int_{-1}^{+1} \mathrm{P}_m^{(\alpha,\delta)}(z)\, \mathrm{P}_n^{(\alpha,\delta)}(z)\,(1-z)^{\alpha}\,(1+z)^{\delta}\, dz \\
&= 2^{\alpha+\delta+1}\,\frac{\Gamma(\alpha+n+1)\,\Gamma(\delta+n+1)}{(\alpha+\delta+2n+1)\; n!\; \Gamma(\alpha+\delta+n+1)}\;\delta_{m,n}.
\end{aligned}$$

Thirdly, by the same arguments

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{H}^{\delta}_{2m+1}(x)\, \mathrm{H}^{\delta}_{2n+1}(x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= 2\int_0^1 \mathrm{P}_m^{(\alpha+1,\delta-1)}(1-2y^2)\, \mathrm{P}_n^{(\alpha+1,\delta-1)}(1-2y^2)\, y^{2\alpha+3}\,(1-y^2)^{\delta-1}\, dy \\
&= 2^{-\alpha-\delta-1}\int_{-1}^1 \mathrm{P}_m^{(\alpha+1,\delta-1)}(z)\, \mathrm{P}_n^{(\alpha+1,\delta-1)}(z)\,(1-z)^{\alpha+1}\,(1+z)^{\delta-1}\, dz \\
&= \frac{\Gamma(\alpha+n+2)\,\Gamma(\delta+n)}{(\alpha+\delta+2n+1)\, n!\, \Gamma(\alpha+\delta+n+1)}\;\delta_{m,n}.
\end{aligned}$$

Let us turn to the completeness of $\{\mathrm{H}^{\delta}_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R},A_{\alpha,\beta}(|x|)dx)$. Recall (see for instance [1]) that the Jacobi polynomials $\{\mathrm{P}_n^{(\tilde\alpha,\tilde\delta)}\}_{n\in\mathbb{N}}$ span a dense subspace of $L^2(\,]-1,1[\,,(1-z)^{\tilde\alpha}(1+z)^{\tilde\delta}dz)$. By the above changes of variables, we deduce that $\{\mathrm{H}^{\delta}_{2n}\}_{n\in\mathbb{N}}$ and $\{\mathrm{H}^{\delta}_{2n+1}\}_{n\in\mathbb{N}}$ span dense subspaces of $L^2(\mathbb{R},A_{\alpha,\beta}(|x|)dx)_{\mathrm{e}}$ and $L^2(\mathbb{R},A_{\alpha,\beta}(|x|)dx)_{\mathrm{o}}$ respectively. □

Remark 3.6.2. *In* (3.6.1), *let us replace δ by ε^{-2}, x by εx and let $\varepsilon\searrow 0$. As*

$$(\cosh\varepsilon x)^{-\alpha-\beta-\varepsilon^{-2}-2} \longrightarrow e^{-\frac{x^2}{2}}$$

and

$$\mathrm{P}_n^{(a,b+\varepsilon^{-2})}(1-2\tanh^2\varepsilon x) = \tfrac{(a+1)_n}{n!}\,{}_2\mathrm{F}_1(-n,a+b+\varepsilon^{-2}+n+1;a+1;\tanh^2\varepsilon x)$$

tends to the Laguerre polynomial

$$\mathrm{L}_n^a(x^2) = \tfrac{(a+1)_n}{n!}\,{}_1\mathrm{F}_1(-n\,;a+1;x^2),$$

we recover in the limit the even and odd Hermite functions constructed by Rosenblum [55, Definition 3.4] in the rational Dunkl setting:

$$\begin{cases} \mathrm{H}_{2n}^{\varepsilon^{-2}}(\varepsilon x) \longrightarrow e^{-\frac{x^2}{2}}\,\mathrm{L}_n^{\alpha}(x^2)\,, \\ \varepsilon^{-1}\,\mathrm{H}_{2n+1}^{\varepsilon^{-2}}(\varepsilon x) \longrightarrow e^{-\frac{x^2}{2}}\,\mathrm{L}_n^{\alpha+1}(x^2)\,x\,. \end{cases}$$

Theorem 3.6.3. *The Opdam–Cherednik transform of $\{H_n^\delta\}_{n\in\mathbb{N}}$ is given by*

$$\begin{aligned} \mathcal{F}(\mathrm{H}_{2n}^\delta)(\lambda) = \frac{(-1)^n}{n!}\,&\frac{\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{\Gamma(\frac{\alpha+\beta+\delta}{2}+n+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+n+1)} \\ &\times\,\mathrm{P}_n\Big(-\frac{\lambda^2}{4};\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\alpha+\beta+1}{2},\frac{\alpha-\beta+1}{2}\Big)\,, \end{aligned} \tag{3.6.2}$$

and

$$\begin{aligned} \mathcal{F}(\mathrm{H}_{2n+1}^\delta)(\lambda) = \frac{(-1)^n}{n!}\,&\frac{(\rho+i\lambda)\,\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{2\,\Gamma(\frac{\alpha+\beta+\delta}{2}+n+2)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+n+1)} \\ &\times\mathrm{P}_n\Big(-\frac{\lambda^2}{4};\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\alpha+\beta+3}{2},\frac{\alpha-\beta+1}{2}\Big) \end{aligned} \tag{3.6.3}$$

where

$$\begin{aligned} \mathrm{P}_n(t^2;a,b,c,d) = (a+b)_n\,(a+c)_n\,(a+d)_n& \\ \times\,{}_4\mathrm{F}_3\Big(&{-n,a+b+c+d+n-1,a+t,a-t \atop a+b,a+c,a+d};1\Big) \end{aligned}$$

denotes the Wilson polynomials.

Proof. By evenness, the Opdam–Cherednik transform $\mathcal{F}(\mathrm{H}_{2n}^\delta)$ coincides with the Jacobi transform $\mathcal{F}_{\alpha,\beta}(\mathrm{H}_{2n}^\delta)$. Thus (3.6.2) amounts to Formula (9.4) in [47]. Let us recall its proof, which was sketched in [47, Section 9] and which will be used for (3.6.3). On one hand, we expand

$$\begin{aligned} \mathrm{H}_{2n}^\delta(x) &= (-1)^n\,(\cosh x)^{-\alpha-\beta-\delta-2}\,\mathrm{P}_n^{(\delta,\alpha)}(2\tanh^2 x-1) \\ &= (-1)^n\,\frac{(\delta+1)_n}{n!}\,(\cosh x)^{-\alpha-\beta-\delta-2}\,{}_2\mathrm{F}_1(-n,\alpha+\delta+n+1;\delta+1;\cosh^{-2}x) \\ &= \frac{(-1)^n}{n!}\,(\delta+1)_n\sum_{m=0}^{n}\frac{(-n)_m\,(\alpha+\delta+n+1)_m}{(\delta+1)_m\,m!}\,(\cosh x)^{-\alpha-\beta-\delta-2m-2} \end{aligned} \tag{3.6.4}$$

in negative powers of $\cosh x$, using the symmetry

$$\mathrm{P}_n^{(\alpha,\delta)}(x) = (-1)^n\, \mathrm{P}_n^{(\delta,\alpha)}(-x)$$

and the definition (3.2.17) of Jacobi polynomials. On the other hand, recall the following Jacobi transform [47, Formula (9.1)] :

$$\int_0^{+\infty} (\cosh x)^{-\alpha-\beta-\mu-1}\, \varphi_\lambda^{(\alpha,\beta)}(x)\, A_{\alpha,\beta}(x)\, dx = \frac{\Gamma(\alpha+1)\,\Gamma(\frac{\mu+i\lambda}{2})\,\Gamma(\frac{\mu-i\lambda}{2})}{2\,\Gamma(\frac{\alpha+\beta+\mu+1}{2})\,\Gamma(\frac{\alpha-\beta+\mu+1}{2})}\,. \tag{3.6.5}$$

We conclude by combining (3.6.4) and (3.6.5) :

$$\begin{aligned}
\mathcal{F}(\mathrm{H}_{2n}^\delta)(\lambda) &= \int_{\mathbb{R}} \mathrm{H}_{2n}^\delta(x)\, \varphi_\lambda^{(\alpha,\beta)}(x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= \frac{(-1)^n}{n!}\,(\delta+1)_n \sum_{m=0}^n \frac{(-n)_m\,(\alpha+\delta+n+1)_m}{(\delta+1)_m\, m!} \\
&\quad \times \underbrace{2\int_0^{+\infty} (\cosh x)^{-\alpha-\beta-\delta-2m-2}\, \varphi_\lambda^{(\alpha,\beta)}(x)\, A_{\alpha,\beta}(x)\, dx}_{\frac{\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2}+m)\,\Gamma(\frac{\delta+1-i\lambda}{2}+m)}{\Gamma(\frac{\alpha+\beta+\delta}{2}+m+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+m+1)}} \\
&= \frac{(-1)^n}{n!}\, \frac{\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{\Gamma(\frac{\alpha+\beta+\delta}{2}+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+1)}\,(\delta+1)_n \\
&\quad \times \underbrace{\sum_{m=0}^n \frac{(-n)_m\,(\alpha+\delta+n+1)_m\,(\frac{\delta+1+i\lambda}{2})_m\,(\frac{\delta+1-i\lambda}{2})_m}{(\delta+1)_m\,(\frac{\alpha+\beta+\delta}{2}+1)_m\,(\frac{\alpha-\beta+\delta}{2}+1)_m\, m!}}_{{}_4\mathrm{F}_3\left(\begin{smallmatrix}-n,\,\alpha+\delta+n+1,\,\frac{\delta+1+i\lambda}{2},\,\frac{\delta+1-i\lambda}{2}\\ \delta+1,\,\frac{\alpha+\beta+\delta}{2}+1,\,\frac{\alpha-\beta+\delta}{2}+1\end{smallmatrix};1\right)} \\
&= \frac{(-1)^n}{n!}\, \frac{\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{\Gamma(\frac{\alpha+\beta+\delta}{2}+n+1)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+n+1)} \\
&\quad \times \mathrm{P}_n\Big(-\frac{\lambda^2}{4};\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\alpha+\beta+1}{2},\frac{\alpha-\beta+1}{2}\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{H}_{2n+1}^\delta(x) &= (-1)^n\,(\cosh x)^{-\alpha-\beta-\delta-2}\,\mathrm{P}_n^{(\delta-1,\alpha+1)}(2\tanh^2 x-1)\,\tanh x \\
&= \frac{(-1)^n}{n!}\,(\delta)_n\,(\sinh x)\,(\cosh x)^{-\alpha-\beta-\delta-3}\,{}_2\mathrm{F}_1(-n,\alpha+\delta+n+1;\delta;\cosh^{-2}x) \\
&= \frac{(-1)^n}{n!}\,(\delta)_n\,(\sinh x)\sum_{m=0}^n \frac{(-n)_m\,(\alpha+\delta+n+1)_m}{(\delta)_m\, m!}\,(\cosh x)^{-\alpha-\beta-\delta-2m-3},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}(\mathrm{H}^\delta_{2n+1})(\lambda) &= \int_{\mathbb{R}} \mathrm{H}^\delta_{2n+1}(x)\, \mathrm{G}^{(\alpha,\beta)}_{\lambda,\mathrm{o}}(x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= \frac{\rho+i\lambda}{4(\alpha+1)} \int_{\mathbb{R}} \mathrm{H}^\delta_{2n+1}(x)\, \varphi^{(\alpha+1,\beta+1)}_\lambda(x)\, (\sinh 2x)\, A_{\alpha,\beta}(|x|)\, dx \\
&= \frac{\rho+i\lambda}{\alpha+1} \int_0^{+\infty} (\sinh x \cosh x)^{-1}\, \mathrm{H}^\delta_{2n+1}(x)\, \varphi^{(\alpha+1,\beta+1)}_\lambda(x)\, A_{\alpha+1,\beta+1}(x)\, dx \\
&= \frac{(-1)^n}{n!} \frac{\rho+i\lambda}{\alpha+1} (\delta)_n \sum_{m=0}^n \frac{(-n)_m\, (\alpha+\delta+n+1)_m}{(\delta)_m\, m!} \\
&\times \underbrace{\int_0^{+\infty} (\cosh x)^{-\alpha-\beta-\delta-2m-4}\, \varphi^{(\alpha+1,\beta+1)}_\lambda(x)\, A_{\alpha+1,\beta+1}(x)\, dx}_{\frac{\Gamma(\alpha+2)\,\Gamma(\frac{\delta+1+i\lambda}{2}+m)\,\Gamma(\frac{\delta+1-i\lambda}{2}+m)}{2\,\Gamma(\frac{\alpha+\beta+\delta}{2}+m+2)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+m+1)}} \\
&= \frac{(-1)^n}{n!} \frac{(\rho+i\lambda)\,\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{2\,\Gamma(\frac{\alpha+\beta+\delta}{2}+2)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+1)} (\delta)_n \\
&\times \underbrace{\sum_{m=0}^n \frac{(-n)_m\,(\alpha+\delta+n+1)_m\,(\frac{\delta+1+i\lambda}{2})_m\,(\frac{\delta+1-i\lambda}{2})_m}{(\delta)_m\,(\frac{\alpha+\beta+\delta}{2}+2)_m\,(\frac{\alpha-\beta+\delta}{2}+1)_m\, m!}}_{{}_4\mathrm{F}_3\left(\begin{smallmatrix}-n,\,\alpha+\delta+n+1,\,\frac{\delta+1+i\lambda}{2},\,\frac{\delta+1-i\lambda}{2}\\ \delta,\,\frac{\alpha+\beta+\delta}{2}+2,\,\frac{\alpha-\beta+\delta}{2}+1\end{smallmatrix};1\right)} \\
&= \frac{(-1)^n}{n!} \frac{(\rho+i\lambda)\,\Gamma(\alpha+1)\,\Gamma(\frac{\delta+1+i\lambda}{2})\,\Gamma(\frac{\delta+1-i\lambda}{2})}{2\,\Gamma(\frac{\alpha+\beta+\delta}{2}+n+2)\,\Gamma(\frac{\alpha-\beta+\delta}{2}+n+1)} \\
&\times \mathrm{P}_n\Big(-\frac{\lambda^2}{4}; \frac{\delta+1}{2}, \frac{\delta-1}{2}, \frac{\alpha+\beta+3}{2}, \frac{\alpha-\beta+1}{2}\Big).
\end{aligned}$$

□

By comparing the Opdam–Cherednik transform of H^δ_n with the particular case $\mathrm{H}^\delta_0(x) = (\cosh x)^{-\alpha-\beta-\delta-2}$, we obtain the following Rodrigues type formula.

Corollary 3.6.4. *Consider the polynomials*

$$\begin{cases}
\tilde{\mathrm{P}}^\delta_{2n}(t) = \frac{(-1)^n}{n!\,(\frac{\alpha+\beta+\delta}{2}+1)_n\,(\frac{\alpha-\beta+\delta}{2}+1)_n}\, \mathrm{P}_n\big(\frac{t^2}{4}; \frac{\delta+1}{2}, \frac{\delta+1}{2}, \frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+1}{2}\big), \\
\tilde{\mathrm{P}}^\delta_{2n+1}(t) = \frac{(-1)^n\,(\rho+t)}{2\,n!\,(\frac{\alpha+\beta+\delta}{2}+1)_{n+1}\,(\frac{\alpha-\beta+\delta}{2}+1)_n}\, \mathrm{P}_n\big(\frac{t^2}{4}; \frac{\delta+1}{2}, \frac{\delta-1}{2}, \frac{\alpha+\beta+3}{2}, \frac{\alpha-\beta+1}{2}\big).
\end{cases}$$

Then

$$\mathrm{H}^\delta_n(x) = \tilde{\mathrm{P}}^\delta_n(\mathrm{T}^{(\alpha,\beta)}_x)(\cosh x)^{-\alpha-\beta-\delta-2} \qquad \forall\, n\in\mathbb{N}. \tag{3.6.6}$$

In other words, by replacing in the expansion of the polynomial $\tilde{\mathrm{P}}_n^\delta(t)$ *the variable* t *by the Dunkl–Cherednik operator* $T^{(\alpha,\beta)}$*, one obtains a differential–difference operator, whose action on the function* $(\cosh x)^{-\alpha-\beta-\delta-2}$ *yields the function* $\mathrm{H}_n^\delta(x)$*.*

Chapitre 4

Contributions à l'analyse de Dunkl trigonométrique unidimensionnelle : transformation d'Abel, noyau de la chaleur, et équation de Schrödinger

Ce chapitre reprend la prépublication [7].

Résumé: En analyse de Dunkl trigonométrique unidimensionnelle, on étudit la transformation d'Abel afin de l'utiliser pour estimer le noyau de la chaleur en temps complexe. Ce qui permet en temps imaginaire pure de déduire des inégalités dispersives et de Strichartz pour l'équation de Schrödinger linéaire. Ainsi les appliquer dans l'étude du problème bien-posé dans le cas non-linéaire.

Abstract: In one–dimensional trigonometric Dunkl analysis, we first study the inverse Abel transform and next apply it to estimate the heat kernel in complex time. For imaginary time, we deduce in particular dispersive and Strichartz inequalities for the linear Schrödinger equation and apply them to well–posedness in the nonlinear case.

4.1 Introduction

One–dimensional trigonometric Dunkl theory is a two–parameter deformation of classical Fourier analysis on $\mathbb{R}$, which includes Jacobi function theory and in particular radial analysis on hyperbolic spaces or on Damek–Ricci spaces. In this paper we consider the heat equation

$$\begin{cases} \partial_t u(x,t) - \Delta_x u(x,t) = F(x,t) & \forall\,(x,t)\in\mathbb{R}\times\mathbb{R}^+ \\ u(x,0) = f(x) & \forall\, x\in\mathbb{R} \end{cases} \tag{4.1.1}$$

and the Schrödinger equation

$$\begin{cases} i\,\partial_t u(x,t) + \Delta_x u(x,t) = F(x,t) & \forall\,(x,t)\in\mathbb{R}\times\mathbb{R} \\ u(x,0) = f(x) & \forall\, x\in\mathbb{R} \end{cases} \tag{4.1.2}$$

associated to the trigonometric Dunkl Laplacian

$$\begin{aligned} \Delta f(x) &= \Delta_{\alpha,\beta} f(x) \\ &= f''(x) + \big\{(2\alpha{+}1)\coth x + (2\beta{+}1)\tanh x\big\}\, f'(x) + \rho^2 f(x) \\ &+ \left\{-\frac{2\alpha+1}{\sinh^2 x} + \frac{2\beta+1}{\cosh^2 x}\right\} \frac{f(x)-f(-x)}{2} \end{aligned} \tag{4.1.3}$$

on $\mathbb{R}$. Here $\alpha\geq\beta\geq-\frac{1}{2}$ with $\alpha\neq-\frac{1}{2}$ and $\rho=\alpha+\beta+1>0$. Solutions to these equations are provided by the heat kernel $h_t(x)$ $(t>0)$ and by the Schrödinger kernel $s_t(x)=h_{it}(x)$ $(t\in\mathbb{R}^*)$. More precisely, let

$$e^{t\,\Delta} f(x) = h_t * f(x) \quad\text{and}\quad e^{i\,t\,\Delta} f(x) = s_t * f(x),$$

where $*=*_{\alpha,\beta}$ denotes the trigonometric Dunkl convolution product (see for instance [6]). Then (4.1.1) is solved by

$$u(x,t) = e^{t\,\Delta} f(x) + \int_0^t e^{(t-s)\,\Delta_x} F(x,s)\,ds$$

and (4.1.2) by

$$u(x,t) = e^{i\,t\,\Delta} f(x) - i\int_0^t e^{i\,(t-s)\,\Delta_x} F(x,s)\,ds\,. \tag{4.1.4}$$

Therefore a fundamental problem consists in understanding the behavior of these two kernels. A main result in this paper is the following sharp global upper bound for the complex time heat kernel: There exists a constant $C>0$ such that

$$|h_t(x)| \,\leq\, \begin{cases} C\,|t|^{-\frac{3}{2}}\,(1{+}|x|)\, e^{-\rho|x|}\, e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})} & \text{if }\ |t|\geq 1{+}|x| \\ C\,|t|^{-\alpha-1}\,(1{+}|x|)^{\alpha+\frac{1}{2}}\, e^{-\rho|x|}\, e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})} & \text{if }\ |t|\leq 1{+}|x| \end{cases} \tag{4.1.5}$$

for every $x\in\mathbb{R}$ and $t\in\mathbb{C}^*$ with $\operatorname{Re} t\geq 0$. This estimate is obtained as in [28, 2, 35, 3, 4] by applying the so–called inverse Abel transform $\mathcal{A}^{-1}$ to $x\mapsto e^{-x^2/4t}$. Notice that $\mathcal{A}^{-1}$ is again a composition of differential and integral transforms, but that it involves now generically two integral transforms instead of one at most, which makes it harder to work with.

As an important application, we study the Schrödinger equation (4.1.2), following closely [3] (see also [41]). We obtain first the dispersive inequality

$$\|e^{it\Delta}\|_{L^{\tilde{q}'}\to L^q} \lesssim \begin{cases} |t|^{-2(\alpha+1)\max\left\{\frac{1}{2}-\frac{1}{q},\frac{1}{2}-\frac{1}{\tilde{q}}\right\}} & \text{if } \ 0<|t|<1 \\ |t|^{-\frac{3}{2}} & \text{if } \ |t|\geq 1 \end{cases} \tag{4.1.6}$$

for every $2<q,\tilde{q}\leq\infty$, using for large t a version of the Kunze–Stein phenomenon established in [6]. Here $L^q=L^q_{\alpha,\beta}(\mathbb{R})$ denotes the space of L^q functions on $\mathbb{R}$ with respect to the measure $|\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1}dx$. We deduce next Strichartz estimates for a wider range of indices than in the Euclidean case and we prove eventually the well–posedness of the semilinear Schrödinger equation

$$\begin{cases} i\,\partial_t u(x,t)+\Delta_x u(x,t)=F(u(x,t)) \qquad \forall\,(x,t)\in\mathbb{R}\times\mathbb{R}\,, \\ u(x,0)=f(x) \qquad \forall\, x\in\mathbb{R}\,, \end{cases} \tag{4.1.7}$$

with powerlike nonlinearities $F(u)\sim|u|^\gamma$ $(1<\gamma\leq 1+\frac{2}{\alpha+1})$.

Let us describe the content of our paper. In Section 4.2 we briefly recall trigonometric Dunkl theory in dimension 1 and its connection with Jacobi functions. The next three sections are devoted to the Abel transform $\mathcal{A}$ in our setting, its inverse $\mathcal{A}^{-1}$ and its dual $\mathcal{A}^*$. More precisely, in Section 4.3 we express $\mathcal{A}$ and $\mathcal{A}^{-1}$ in terms of Weyl fractional transforms, in Section 4.4 we compute explicitely the kernel $K(x,y)=K^{(\alpha,\beta)}(x,y)$ of the integral transform $\mathcal{A}$, and in Section 4.5 we express $\mathcal{A}^*$ in terms of Riemann–Liouville fractional transforms. This part overlaps with the recent work [32] where the kernel of $\mathcal{A}^*$, called the *intertwining operator*, is computed from the formula $G_\lambda=\mathcal{A}^*\big(e^{i\lambda\cdot}\big)$. For the sake of completeness, we include full details of our dual approach and in the appendix we check that our results agree. In Section 4.6 we prove the upper bound (4.1.5) and in Section 4.7 we show that it is optimal when $t>0$ and $x\in\mathbb{R}$ or when $t\in i\,\mathbb{R}^*$ and $|x|\to+\infty$. In Section 4.8 we deduce the dispersive inequality (4.1.6) in Theorem 4.8.4 and the Strichartz inequality in Theorem 4.8.5. We conclude in Section 4.9 with the local and global well–posedness of NLS (4.1.7).

4.2 Setting and notation

Let $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$. The Dunkl–Cherednik operator $T = T^{(\alpha,\beta)}$ on $\mathbb{R}$ is given by

$$Tf(x) = f'(x) + \Big\{(2\alpha+1)\coth x + (2\beta+1)\tanh x\Big\}\,\frac{f(x)-f(-x)}{2} - \rho f(-x),$$

where $\rho = \alpha+\beta+1$. Notice that, in Cherednik's notation,

$$Tf(x) = f'(x) + \Big\{\frac{2k_1}{1-e^{-2x}} + \frac{4k_2}{1-e^{-4x}}\Big\}\{f(x)-f(-x)\} - (k_1+2k_2)f(x),$$

with $\alpha = k_1+k_2-\frac{1}{2}$ and $\beta = k_2-\frac{1}{2}$. As general references, we use [52] for the trigonometric Dunkl theory and [47] for Jacobi functions.

For all $\lambda \in \mathbb{C}$, the Opdam hypergeometric functions $G_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}$ are defined as the following normalized eigenfunctions

$$\begin{cases} TG_\lambda{}^{(\alpha,\beta)}(x) = i\,\lambda\, G_\lambda^{(\alpha,\beta)}(x)\,, \\ G_\lambda^{(\alpha,\beta)}(0) = 1\,. \end{cases}$$

These functions are closely related to Jacobi functions or to the Gauss hypergeometric function. More precisely,

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{(\alpha,\beta)}(x) + \frac{\rho+i\lambda}{2(\alpha+1)}\sinh x\cosh x\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x), \tag{4.2.1}$$

where $\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\Big(\frac{\rho+i\lambda}{2},\frac{\rho-i\lambda}{2};\alpha+1;-\sinh^2 x\Big)$.

For $f \in \mathcal{C}_c^\infty(\mathbb{R})$, the Opdam–Cherednik transform is defined by

$$\mathcal{F}(f)(\lambda) = \int_{-\infty}^{+\infty} f(x)\, G_\lambda^{(\alpha,\beta)}(-x)\, A_{\alpha,\beta}(|x|)\, dx, \tag{4.2.2}$$

where

$$A_{\alpha,\beta}(x) = (\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}. \tag{4.2.3}$$

By substituting (4.2.1) into (4.2.2), we obtain

$$\begin{aligned}
\mathcal{F}f(\lambda) &= \int_{-\infty}^{+\infty} f(x)\Big[\varphi_\lambda^{(\alpha,\beta)}(x) - \frac{\rho+i\lambda}{2(\alpha+1)}\sinh x\cosh x\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\Big]\,A_{\alpha,\beta}(|x|)\,dx \\
&= \int_{-\infty}^{+\infty} f_{\text{even}}(x)\,\varphi_\lambda^{(\alpha,\beta)}(x)\,(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}\,dx \\
&\quad - \frac{\rho+i\lambda}{2(\alpha+1)}\int_{-\infty}^{+\infty} f_{\text{odd}}(x)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\sinh x\,(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+2}\,dx \\
&= 2\int_0^{+\infty} f_{\text{even}}(x)\,\varphi_\lambda^{(\alpha,\beta)}(x)\,(\sinh x)^{2\alpha+1}\,(\cosh x)^{2\beta+1}\,dx \\
&\quad - \frac{\rho+i\lambda}{\alpha+1}\int_0^{+\infty}\frac{f_{\text{odd}}(x)}{\sinh x\cosh x}\,\varphi_\lambda^{(\alpha+1,\beta+1)}(x)\,(\sinh x)^{2\alpha+3}\,(\cosh x)^{2\beta+3}\,dx\,.
\end{aligned}$$

Here f_{even} (resp. f_{odd}) denotes the even (resp. odd) part of the function f. Using the Jacobi transform, which is defined for even functions by

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = 2^{2\rho}\int_0^{+\infty} f(x)\,\varphi_\lambda^{(\alpha,\beta)}(x)\,A_{\alpha,\beta}(x)\,dx, \tag{4.2.4}$$

we deduce that

$$\mathcal{F}f(\lambda) = 2^{-2\rho+1}\,\mathcal{F}_{\alpha,\beta}f_{\text{even}}(\lambda) - \frac{\rho+i\lambda}{\alpha+1}\,2^{-2\rho-4}\,\mathcal{F}_{\alpha+1,\beta+1}\Big(\frac{f_{\text{odd}}}{\sinh\cosh}\Big)(\lambda). \tag{4.2.5}$$

4.3 The Abel transform and its inverse

Recall [47] that the Jacobi transform (4.2.4) is the composition

$$\mathcal{F}_{\alpha,\beta} = \mathcal{F}_0\circ\mathcal{A}_{\alpha,\beta} \tag{4.3.1}$$

of the Euclidean Fourier transform $\mathcal{F}_0 g(\lambda) = \int_{\mathbb{R}} g(x)\,e^{-i\lambda x}dx$ on $\mathbb{R}$ preceded by the Abel transform

$$\mathcal{A}_{\alpha,\beta} = \tfrac{2^{3\alpha+\frac{1}{2}}\,\Gamma(\alpha+1)}{\sqrt{\pi}}\,\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac{1}{2}}, \tag{4.3.2}$$

in the Jacobi setting. Here $\mathcal{W}^\tau_\mu$ denotes the Weyl fractional transform, which is defined for $\tau>0$, $\text{Re}\mu>0$ and even functions by

$$\mathcal{W}^\tau_\mu h(x) = \frac{\tau}{\Gamma(\mu)}\int_{|x|}^{+\infty} h(t)(\cosh\tau t - \cosh\tau x)^{\mu-1}\sinh\tau t\,dt. \tag{4.3.3}$$

Note that, for fixed $\tau>0$, (4.3.3) extends holomorphically to $\mu\in\mathbb{C}$ and gives rise to a one–parameter group of transforms $\{\mathcal{W}^\tau_\mu\}_{\mu\in\mathbb{C}}$. In particular, $\mathcal{W}^\tau_\mu\circ\mathcal{W}^\tau_\nu = \mathcal{W}^\tau_{\mu+\nu}$,

$\mathcal{W}_0^\tau = \mathrm{Id}$, and

$$\mathcal{W}_{-1}^\tau h(x) = -\frac{\partial}{\partial(\cosh \tau x)} h(x) = -\frac{1}{\tau}\frac{1}{\sinh \tau x}\frac{\partial}{\partial x} h(x).$$

Define the Abel transform in the trigonometric Dunkl setting by

$$\mathcal{A} = \mathcal{F}_0^{-1} \circ \mathcal{F}, \tag{4.3.4}$$

where $\mathcal{F}$ is the Opdam–Cherednik transform (4.2.2). Let us first express $\mathcal{A}$ in terms of $\mathcal{A}_{\alpha,\beta}$. Consider the first term on the right hand side of (4.2.5). By (4.3.1), we have

$$\mathcal{F}_{\alpha,\beta} f_{\text{even}}(\lambda) = \mathcal{F}_0\left(\mathcal{A}_{\alpha,\beta} f_{\text{even}}\right)(\lambda) = \int_{-\infty}^{+\infty} (\mathcal{A}_{\alpha,\beta} f_{\text{even}})(x)\, e^{-i\lambda x}\, dx.$$

Consider the second term on the right hand side of (4.2.5). Setting $g = \mathcal{A}_{\alpha+1,\beta+1}(\frac{f_{\text{odd}}}{\sinh\cosh})$, we have

$$\begin{aligned}(\rho + i\lambda)\mathcal{F}_{\alpha+1,\beta+1}\Big(\frac{f_{\text{odd}}}{\sinh\cosh}\Big)(\lambda) &= (\rho+i\lambda)\int_{-\infty}^{+\infty} g(x)\, e^{-i\lambda x} dx \\ &= \int_{-\infty}^{+\infty} g(x)\,(\rho - \frac{\partial}{\partial x})\, e^{-i\lambda x} dx \\ &= \rho\int_{-\infty}^{+\infty} g(x)\, e^{-i\lambda x}\, dx + \int_{-\infty}^{+\infty} g'(x)\, e^{-i\lambda x}\, dx\,.\end{aligned}$$

Hence, it follows from the definition (4.3.4) of $\mathcal{A}$, together with (4.2.5), that

$$\begin{aligned}\mathcal{A}f = &\overbrace{2^{-2\rho+1}\mathcal{A}_{\alpha,\beta} f_{\text{even}} - \tfrac{2^{-2\rho-4}\rho}{\alpha+1}\,\mathcal{A}_{\alpha+1,\beta+1}(\tfrac{f_{\text{odd}}}{\sinh\cosh})}^{(\mathcal{A}f)_{\text{even}}} \\ &\underbrace{-\tfrac{2^{-2\rho-4}}{\alpha+1}\Big(\mathcal{A}_{\alpha+1,\beta+1}(\tfrac{f_{\text{odd}}}{\sinh\cosh})\Big)'}_{(\mathcal{A}f)_{\text{odd}}}.\end{aligned} \tag{4.3.5}$$

Let us next proceed towards the inverse Abel transform. Using the expression (4.3.2) of $\mathcal{A}_{\alpha,\beta}$, we may rewrite (4.3.5) as

$$\mathcal{A}f(x) = c_{\alpha,\beta}\Big[\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac{1}{2}}\big) f_{\text{even}}(x) - \tfrac{1}{4}\big(\tfrac{\partial}{\partial x}+\rho\big)\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac{3}{2}}\big)\big(\tfrac{f_{\text{odd}}}{\sinh\cosh}\big)(x)\Big],$$

where

$$c_{\alpha,\beta} = \frac{2^{\alpha-2\beta-\frac{1}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}}. \tag{4.3.6}$$

Further, by (4.3.3), we have

$$\mathcal{W}^2_1 h(x) = 4\int_{|x|}^{+\infty} h(t)\, \sinh t\, \cosh t\, dt\,.$$

Thus $\mathcal{W}_1^2(\frac{h}{\sinh\cosh})(x) = 4\,\mathcal{I}h(x)$, where

$$\mathcal{I}h(x) = \int_{|x|}^{+\infty} h(t)\,dt\,. \tag{4.3.7}$$

Notice that

$$\mathcal{I}(f_{\rm odd})(x) = \int_x^{+\infty} f_{\rm odd}(t)\,dt = -\int_{-\infty}^{x} f_{\rm odd}(t)\,dt$$

is an even function of $x\in\mathbb{R}$. Eventually we may rewrite the odd part $(\mathcal{A}f)_{\rm odd}$ of the Abel transform as

$$\begin{aligned}(\mathcal{A}f)_{\rm odd}(x) &= -\tfrac{c_{\alpha,\beta}}{4}\,\tfrac{\partial}{\partial x}\,\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac32}\big)\big(\tfrac{f_{\rm odd}}{\sinh\cosh}\big)(x)\\ &= -\tfrac{c_{\alpha,\beta}}{4}\,\tfrac{\partial}{\partial x}\,\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\circ\mathcal{W}^2_1\big)\big(\tfrac{f_{\rm odd}}{\sinh\cosh}\big)(x)\\ &= -\,c_{\alpha,\beta}\,\tfrac{\partial}{\partial x}\,\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\circ\mathcal{I}\big)f_{\rm odd}(x)\,.\end{aligned}$$

By inversion, we get

$$f_{\rm odd}(x) = -\,c_{\alpha,\beta}^{-1}\,\tfrac{\partial}{\partial x}\,\big(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\circ\mathcal{I}\big)(\mathcal{A}f)_{\rm odd}(x)\,. \tag{4.3.8}$$

Similarly, the even part $(\mathcal{A}f)_{\rm even}$ of the Abel transform writes

$$\begin{aligned}(\mathcal{A}f)_{\rm even}(x) &= 2^{-2\rho+1}\mathcal{A}_{\alpha,\beta}(f_{\rm even})(x) - \tfrac{2^{-2\rho-4}\rho}{\alpha+1}\,\mathcal{A}_{\alpha+1,\beta+1}\big(\tfrac{f_{\rm odd}}{\sinh\cosh}\big)(x)\\ &= c_{\alpha,\beta}\big[\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\big)f_{\rm even}(x) - \rho\,\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\circ\mathcal{I}\big)f_{\rm odd}(x)\big]\end{aligned}$$

and we obtain by inversion

$$f_{\rm even}(x) - \rho\,\mathcal{I}f_{\rm odd}(x) = c_{\alpha,\beta}^{-1}\,\big(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\big)(\mathcal{A}f)_{\rm even}(x)\,. \tag{4.3.9}$$

By applying $\mathcal{I}$ to (4.3.8), (4.3.9) becomes eventually

$$f_{\rm even}(x) = c_{\alpha,\beta}^{-1}\,\big[\big(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\big)(\mathcal{A}f)_{\rm even}(x) - \rho\,\big(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\circ\mathcal{I}\big)(\mathcal{A}f)_{\rm odd}(x)\big].$$

In conclusion we have obtained the following formulas.

Theorem 4.3.1. *Let* $\alpha\geq\beta\geq\frac12$ *with* $\alpha\neq-\frac12$, *and let* $\rho=\alpha+\beta+1$.

(1) The Abel transform is given by

$$\begin{aligned}\mathcal{A}f(x) = c_{\alpha,\beta}\Big[&\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\big)f_{\rm even}(x)\\ &- \big(\tfrac{\partial}{\partial x}-\rho\big)\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\circ\mathcal{I}\big)f_{\rm odd}(x)\Big],\end{aligned} \tag{4.3.10}$$

where $\mathcal{W}^\tau_\mu$ *denotes the Weyl transform* (4.3.3), *the integral transform* $\mathcal{I}$ *is given by* (4.3.7), *and the constant* $c_{\alpha,\beta}$ *by* (4.3.6).

(2) The inverse Abel transform is given by

$$\mathcal{A}^{-1}f(x) = c_{\alpha,\beta}^{-1}\Big[\big(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\big)f_{\text{even}}(x) + \big(\tfrac{\partial}{\partial x}-\rho\big)\big(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\circ\mathcal{I}\big)f_{\text{odd}}(x)\Big]. \tag{4.3.11}$$

Remark 4.3.2. *In the limit cases* $\alpha=\beta>-\frac12$ *and* $\alpha>\beta=-\frac12$*, formulas* (4.3.10) *and* (4.3.11) *reduce to the following expressions, involving a single Weyl transform.*

(i) Assume that $\alpha=\beta>-\frac12$. *Then*

$$\mathcal{A}f(x) = c_{\alpha,\alpha}\Big[\mathcal{W}^2_{\alpha+\frac12}f_{\text{even}}(x) - \big(\tfrac{\partial}{\partial x}+\rho\big)\big(\mathcal{W}^2_{\alpha+\frac12}\circ\mathcal{I}\big)f_{\text{odd}}(x)\Big]$$

and

$$\mathcal{A}^{-1}f(x) = c_{\alpha,\alpha}^{-1}\Big[\mathcal{W}^2_{-\alpha-\frac12}f_{\text{even}}(x) - \big(\tfrac{\partial}{\partial x}+\rho\big)\big(\mathcal{W}^2_{-\alpha-\frac12}\circ\mathcal{I}\big)f_{\text{odd}}(x)\Big].$$

(ii) Assume that $\alpha>\beta=-\frac12$. *Then*

$$\mathcal{A}f(x) = c_{\alpha,-\frac12}\Big[\mathcal{W}^1_{\alpha+\frac12}f_{\text{even}}(x) - \big(\tfrac{\partial}{\partial x}+\rho\big)\big(\mathcal{W}^1_{\alpha+\frac12}\circ\mathcal{I}\big)f_{\text{odd}}(x)\Big]$$

and

$$\mathcal{A}^{-1}f(x) = c_{\alpha,-\frac12}^{-1}\Big[\mathcal{W}^1_{-\alpha-\frac12}f_{\text{even}}(x) - \big(\tfrac{\partial}{\partial x}+\rho\big)\big(\mathcal{W}^1_{-\alpha-\frac12}\circ\mathcal{I}\big)f_{\text{odd}}(x)\Big].$$

4.4 Integral representation of the Abel transform

Assume that $\alpha>\beta>-\frac12$. According to the definition (4.3.3) of the Weyl transform, we have, for all $x\geq 0$,

$$\begin{aligned}
&\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\big)\,f_{\text{even}}(x)\\
&= \tfrac{1}{\Gamma(\alpha-\beta)}\int_x^{+\infty}\mathcal{W}^2_{\beta+\frac12}f_{\text{even}}(z)\,(\cosh z-\cosh x)^{\alpha-\beta-1}\,\sinh z\,dz\\
&= \tfrac{2}{\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac12)}\int_x^{+\infty}(\cosh z-\cosh x)^{\alpha-\beta-1}\,\sinh z\\
&\times\Big[\int_z^{+\infty}f_{\text{even}}(t)\,(\cosh 2t-\cosh 2z)^{\beta-\frac12}\,\sinh 2t\,dt\Big]dz\\
&= \tfrac{2}{\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac12)}\int_x^{+\infty}f_{\text{even}}(t)\,\sinh 2t\\
&\times\Big[\int_x^t(\cosh z-\cosh x)^{\alpha-\beta-1}\big(\cosh 2t-\cosh 2z\big)^{\beta-\frac12}\,\sinh z\,dz\Big]dt\,.
\end{aligned}$$

Since $\mathcal{I}(f_{\rm odd})$ is an even function, we deduce that

$$\begin{aligned}
&\Big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\circ\mathcal{I}\Big)\, f_{\rm odd}(x)\\
&= \tfrac{2}{\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac12)} \int_x^{+\infty}\!\int_t^{+\infty} f_{\rm odd}(y)\,dy\\
&\times \int_x^t (\cosh z-\cosh x)^{\alpha-\beta-1}\,(\cosh 2t-\cosh 2z)^{\beta-\frac12}\,\sinh 2t\,\sinh z\,dz\,dt\\
&= \tfrac{2}{\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac12)} \int_x^{+\infty} f_{\rm odd}(y)\\
&\times \int_x^y (\cosh z-\cosh x)^{\alpha-\beta-1}\,\sinh z\,\underbrace{\int_z^y (\cosh 2t-\cosh 2z)^{\beta-\frac12}\,\sinh 2t\,dt}_{\frac{1}{2\beta+1}\,(\cosh 2y-\cosh 2z)^{\beta+\frac12}}\,dz\,dy\\
&= \tfrac{1}{\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac32)} \int_x^{+\infty} f_{\rm odd}(y)\\
&\times \Big[\int_x^y (\cosh z-\cosh x)^{\alpha-\beta-1}\,(\cosh 2y-\cosh 2z)^{\beta+\frac12}\,\sinh z\,dz\Big]dy\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\tfrac{\partial}{\partial x}\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac32}\circ\mathcal{I}\big)\, f_{\rm odd}(x)\\
&= -\tfrac{1}{\Gamma(\alpha-\beta-1)\,\Gamma(\beta+\frac12)}\,\sinh x\int_x^{+\infty} f_{\rm odd}(y)\,\times\\
&\times \Big[\int_x^y (\cosh z-\cosh x)^{\alpha-\beta-2}\,(\cosh 2y-\cosh 2z)^{\beta+\frac12}\,\sinh z\,dz\Big]dy\,.
\end{aligned}$$

By substituting $f_{\rm even}(x)=\frac{f(x)+f(-x)}{2}$ and $f_{\rm odd}(x)=\frac{f(x)-f(-x)}{2}$ in the expressions of $\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\big)f_{\rm even}(x)$, $\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\circ\mathcal{I}\big)f_{\rm odd}(x)$, and $\frac{\partial}{\partial x}\big(\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac12}\circ\mathcal{I}\big)f_{\rm odd}(x)$, we can rewrite as follows the expression (4.3.10) of the Abel transform, for every $x\geq 0$,

$$\begin{aligned}
\mathcal{A}f(x) &= \int_x^{+\infty} f(y)\,\big\{\,K_1^{(\alpha,\beta)}(x,y)+K_2^{(\alpha,\beta)}(x,y)+K_3^{(\alpha,\beta)}(x,y)\,\big\}\,dy\\
&\quad+\int_x^{+\infty} f(-y)\,\big\{\,K_1^{(\alpha,\beta)}(x,y)-K_2^{(\alpha,\beta)}(x,y)-K_3^{(\alpha,\beta)}(x,y)\,\big\}\,dy\\
&\quad=\int_{|y|>x} f(y)\,K^{(\alpha,\beta)}(x,y)\,dy\,,
\end{aligned}$$

where

$$K^{(\alpha,\beta)}(x,y)=K_1^{(\alpha,\beta)}(x,y)+K_2^{(\alpha,\beta)}(x,y)+K_3^{(\alpha,\beta)}(x,y) \tag{4.4.1}$$

and

$$K_1^{(\alpha,\beta)}(x,y)=\tfrac{2^{\,\alpha-2\beta-\frac12}\,\Gamma(\alpha+1)}{\sqrt{\pi}\;\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac12)}\;\sinh 2|y|$$

$$\times\int_{|x|}^{|y|}(\cosh 2y-\cosh 2z)^{\beta-\frac12}\,(\cosh z-\cosh x)^{\alpha-\beta-1}\;\sinh z\,dz\,,$$

$$K_2^{(\alpha,\beta)}(x,y)=-\;\tfrac{2^{\,\alpha-2\beta-\frac32}\,\rho\,\Gamma(\alpha+1)}{\sqrt{\pi}\;\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac32)}\,\operatorname{sign}(y)$$

$$\times\int_{|x|}^{|y|}(\cosh 2y-\cosh 2z)^{\beta+\frac12}\,(\cosh z-\cosh x)^{\alpha-\beta-1}\;\sinh z\,dz\,,$$

$$K_3^{(\alpha,\beta)}(x,y)=\tfrac{2^{\,\alpha-2\beta-\frac32}\,\Gamma(\alpha+1)}{\sqrt{\pi}\;\Gamma(\alpha-\beta-1)\,\Gamma(\beta+\frac32)}\,\operatorname{sign}(y)\;\sinh x$$

$$\times\int_{|x|}^{|y|}(\cosh 2y-\cosh 2z)^{\beta+\frac12}\,(\cosh z-\cosh x)^{\alpha-\beta-2}\;\sinh z\,dz\,.$$

Let us eventually express the kernels $K_1^{(\alpha,\beta)}$, $K_2^{(\alpha,\beta)}$ and $K_3^{(\alpha,\beta)}$ in terms of the classical hypergeometric function. By using the formula

$$\int_{|x|}^{|y|}(\cosh 2y-\cosh 2z)^{\beta-\frac12}\,(\cosh z-\cosh x)^{\alpha-\beta-1}\;\sinh z\,dz$$

$$=2^{\,2\beta-1}\,\tfrac{\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac12)}{\Gamma(\alpha+\frac12)}\,(\cosh y)^{\beta-\frac12}\,(\cosh y-\cosh x)^{\alpha-\frac12}$$

$$\times\,{}_2F_1\big(\tfrac12+\beta,\tfrac12-\beta;\alpha+\tfrac12;\tfrac{\cosh y\,-\,\cosh x}{2\,\cosh y}\big)$$

(see [47, (5.50) & (5.60)]) and the well known symmetry

$${}_2F_1(a,b\,;c\,;u)=(1-u)^{c-a-b}\,{}_2F_1(c-a,c-b\,;c\,;u)\,,$$

we get

$$K_1^{(\alpha,\beta)}(x,y)=2^{\,\alpha-\frac32}\,\tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\;\Gamma(\alpha+\frac12)}\;\sinh(2|y|)\,(\cosh y)^{\beta-\frac12}\,(\cosh y-\cosh x)^{\alpha-\frac12}$$

$$\times\,{}_2F_1\big(\tfrac12+\beta,\tfrac12-\beta\,;\alpha+\tfrac12\,;\tfrac{\cosh y\,-\,\cosh x}{2\,\cosh y}\big)$$

$$=\tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\;\Gamma(\alpha+\frac12)}\,\sinh|y|\,(\cosh y)^{\beta-\alpha+1}\,(\cosh^2 y-\cosh^2 x)^{\alpha-\frac12}$$

$$\times\,{}_2F_1\big(\alpha-\beta,\rho-1\,;\alpha+\tfrac12\,;\tfrac{\cosh y\,-\,\cosh x}{2\,\cosh y}\big)\,.$$

Similarly,

$$K_2^{(\alpha,\beta)}(x,y)=-\,\tfrac{\rho\,\Gamma(\alpha+1)}{2\,\sqrt{\pi}\;\Gamma(\alpha+\frac32)}\,\operatorname{sign}(y)\,(\cosh y)^{\beta-\alpha}\,(\cosh^2 y-\cosh^2 x)^{\alpha+\frac12}$$

$$\times\,{}_2F_1\big(\alpha-\beta,\rho+1\,;\alpha+\tfrac32\,;\tfrac{\cosh y\,-\,\cosh x}{2\,\cosh y}\big)$$

and

$$K_3^{(\alpha,\beta)}(x,y) = \tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac12)}\,\sinh x\,\mathrm{sign}(y)\,(\cosh y)^{\beta-\alpha+1}\,(\cosh^2 y-\cosh^2 x)^{\alpha-\frac12}$$
$$\times\,{}_2F_1\big(\alpha-\beta-1,\rho\,;\alpha+\tfrac12\,;\tfrac{\cosh y-\cosh x}{2\cosh y}\big)\,.$$

The above formulas are eventually extended to $x<0$ by evenness and oddness:

$$\begin{aligned}
\mathcal{A}f(x) &= (\mathcal{A}f)_{\mathrm{even}}(x) + (\mathcal{A}f)_{\mathrm{odd}}(x)\\
&= (\mathcal{A}f)_{\mathrm{even}}(|x|) - (\mathcal{A}f)_{\mathrm{odd}}(|x|)\\
&= \int_{|x|}^{+\infty} f(y)\,\big\{\,K_1^{(\alpha,\beta)}(|x|,y) + K_2^{(\alpha,\beta)}(|x|,y) - K_3^{(\alpha,\beta)}(|x|,y)\big\}\,dy\\
&+ \int_{-\infty}^{-|x|} f(y)\,\big\{\,K_1^{(\alpha,\beta)}(|x|,y) + K_2^{(\alpha,\beta)}(|x|,y) - K_3^{(\alpha,\beta)}(|x|,y)\big\}\,dy\\
&= \int_{|x|}^{+\infty} f(y)\,\big\{\,K_1^{(\alpha,\beta)}(x,y) + K_2^{(\alpha,\beta)}(x,y) + K_3^{(\alpha,\beta)}(x,y)\big\}\,dy\\
&+ \int_{-\infty}^{-|x|} f(y)\,\big\{\,K_1^{(\alpha,\beta)}(x,y) + K_2^{(\alpha,\beta)}(x,y) + K_3^{(\alpha,\beta)}(x,y)\big\}\,dy\\
&= \int_{|y|>|x|} f(y)\,K^{(\alpha,\beta)}(x,y)\,dy\,.
\end{aligned}$$

In conclusion, for all $\alpha>\beta>-\frac12$ and $x\in\mathbb{R}$, we have

$$\mathcal{A}f(x) \;=\; \int_{|y|>|x|} K^{(\alpha,\beta)}(x,y)\,f(y)\,dy\,.$$

In the limit case $\alpha=\beta>-\frac12$, the kernel expression (4.4.1) reduces to

$$K^{(\alpha,\alpha)}(x,y) = \tfrac{2^{-\alpha-\frac12}\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac12)}\,\mathrm{sign}(y)\,(\cosh 2y-\cosh 2x)^{\alpha-\frac12}\,(e^{2x}-e^{-2y}) \tag{4.4.2}$$

and, in the limit case $\alpha>\beta=-\frac12$, to

$$K^{(\alpha,-\frac12)}(x,y) = \tfrac{2^{\alpha-\frac12}\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac12)}\mathrm{sign}(y)\,(\cosh y-\cosh x)^{\alpha-\frac12}\,(e^{x}-e^{-y})\,. \tag{4.4.3}$$

In conclusion we have proved the following result.

Theorem 4.4.1. *The Abel transform* $\mathcal{A}$ *writes*

$$\mathcal{A}f(x) \;=\; \int_{|y|>|x|} K(x,y)\,f(y)\,dy \qquad \forall\; x\in\mathbb{R}\,,$$

where $K(x,y)=K^{(\alpha,\beta)}(x,y)$ *is given by* (4.4.1), (4.4.2) *and* (4.4.3).

Remark 4.4.2. *Using a dual approach, Gallardo and Trimèche [32] computed recently the kernel of the dual Abel transform $\mathcal{A}^*$. We compare our results in the Appendix.*

4.5 The dual Abel transform

Recall that the Riemann–Liouville fractional transform $\mathcal{R}^{\tau}_{\mu}$ is defined for $\tau > 0$, $\mathrm{Re}\mu > 0$ and even functions by

$$\mathcal{R}^{\tau}_{\mu} f(x) = \tfrac{1}{\Gamma(\mu)} \int_0^{|x|} f(t)\,(\cosh \tau x - \cosh \tau t)^{\mu-1} dt\,.$$

Then $\mathcal{R}^{\tau}_{\mu}$ is the dual transform of $\mathcal{W}^{\tau}_{\mu}$ in the following sense:

$$\int_0^{+\infty} \mathcal{W}^{\tau}_{\mu} f(x)\, g(x)\, dx = \int_0^{+\infty} f(t)\, \mathcal{R}^{\tau}_{\mu}\, g(t)\, d(\cosh \tau t)\,.$$

Define the transform $\mathcal{A}^*$ by

$$\mathcal{A}^* g(y) = A_{\alpha,\beta}(|y|)^{-1} \int_{|x|<|y|} K^{(\alpha,\beta)}(x,y)\, g(x)\, dx\,.$$

Then $\mathcal{A}^*$ is the dual transform of $\mathcal{A}$ in the following sense:

$$\int_{\mathbb{R}} \mathcal{A} f(x)\, g(x)\, dx = \int_{\mathbb{R}} f(y)\, \mathcal{A}^* g(y)\, A_{\alpha,\beta}(|y|)\, dy\,.$$

Let us express $\mathcal{A}^*$ in term of Riemann-Liouville transforms. Firsty, it is clear that

$$\int_{\mathbb{R}} \mathcal{A}(f_{\text{even}})(x)\, g_{\text{odd}}(x)\, dx = 0\,.$$

Secondly,

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{A}(f_{\text{even}})(x)\, g_{\text{even}}(x)\, dx &= 2\, c_{\alpha,\beta} \int_0^{+\infty} \big(\mathcal{W}^1_{\alpha-\beta} \circ \mathcal{W}^2_{\beta+\frac12}\big) f_{\text{even}}(x)\, g_{\text{even}}(x)\, dx \\
&= 2\, c_{\alpha,\beta} \int_0^{+\infty} \big(\mathcal{W}^2_{\beta+\frac12} f_{\text{even}}\big)(x)\, \big(\mathcal{R}^1_{\alpha-\beta}\, g_{\text{even}}\big)(x)\, \sinh x\, dx \\
&= 4\, c_{\alpha,\beta} \int_0^{+\infty} f_{\text{even}}(x)\, \mathcal{R}^2_{\beta+\frac12}\big(\sinh\, \mathcal{R}^1_{\alpha-\beta}\, g_{\text{even}}\big)(x)\, \sinh 2x\, dx \\
&= 2\, c_{\alpha,\beta} \int_{\mathbb{R}} f_{\text{even}}(x)\, \mathcal{R}^2_{\beta+\frac12}\big(|\sinh|\, \mathcal{R}^1_{\alpha-\beta}\, g_{\text{even}}\big)(x)\, \sinh 2|x|\, dx\,.
\end{aligned}$$

Thirdly,

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{A}(f_{\text{odd}})(x)\, g_{\text{even}}(x)\, dx &= -\tfrac{1}{2}\,\rho\, c_{\alpha,\beta} \int_0^{+\infty} \big(\mathcal{W}^1_{\alpha-\beta} \circ \mathcal{W}^2_{\beta+\frac{3}{2}}\big)\big(\tfrac{f_{\text{odd}}}{\sinh\cosh}\big)(x)\, g_{\text{even}}(x)\, dx \\ &= -\,2\,\rho\, c_{\alpha,\beta} \int_0^{+\infty} f_{\text{odd}}(x)\, \mathcal{R}^2_{\beta+\frac{3}{2}}\big(\sinh\, \mathcal{R}^1_{\alpha-\beta}\, g_{\text{even}}\big)(x)\, dx \\ &= -\,\rho\, c_{\alpha,\beta} \int_{\mathbb{R}} f_{\text{odd}}(x)\, \operatorname{sign}(x)\, \mathcal{R}^2_{\beta+\frac{3}{2}}\big(|\sinh|\, \mathcal{R}^1_{\alpha-\beta}\, g_{\text{even}}\big)(x)\, dx\,.\end{aligned}$$

Fourthly,

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{A}(f_{\text{odd}})(x)\, g_{\text{odd}}(x)\, dx &= -\tfrac{1}{2}\, c_{\alpha,\beta} \int_0^{+\infty} \tfrac{\partial}{\partial x}\, \big(\mathcal{W}^1_{\alpha-\beta} \circ \mathcal{W}^2_{\beta+\frac{3}{2}}\big)\big(\tfrac{f_{\text{odd}}}{\sinh\cosh}\big)(x)\, g_{\text{odd}}(x)\, dx \\ &= 2\, c_{\alpha,\beta} \int_0^{+\infty} f_{\text{odd}}(x)\, \mathcal{R}^2_{\beta+\frac{3}{2}}\big(\sinh\, \mathcal{R}^1_{\alpha-\beta}\, \tfrac{\partial}{\partial x}\, g_{\text{odd}}\big)(x)\, dx \\ &= c_{\alpha,\beta} \int_{\mathbb{R}} f_{\text{odd}}(x)\, \operatorname{sign}(x)\, \mathcal{R}^2_{\beta+\frac{3}{2}}\big(|\sinh|\, \mathcal{R}^1_{\alpha-\beta}\, \tfrac{\partial}{\partial x}\, g_{\text{odd}}\big)(x)\, dx\,.\end{aligned}$$

In summary, we have obtained the following expression for the dual Abel transform :

$$\begin{aligned}\mathcal{A}^* g(x) = c_{\alpha,\beta}\, A_{\alpha,\beta}(|x|)^{-1} \Big\{ & 4\cosh x\, \sinh|x|\, \mathcal{R}^2_{\beta+\frac{1}{2}}\big(|\sinh|\, \mathcal{R}^1_{\alpha-\beta}\, g_{\text{even}}\big)(x) \\ & -\rho\, \operatorname{sign}(x)\, \mathcal{R}^2_{\beta+\frac{3}{2}}\big(|\sinh|\, \mathcal{R}^1_{\alpha-\beta}\, g_{\text{even}}\big)(x) + \operatorname{sign}(x)\, \mathcal{R}^2_{\beta+\frac{3}{2}}\big(|\sinh|\, \mathcal{R}^1_{\alpha-\beta}\, \tfrac{\partial}{\partial x}\, g_{\text{odd}}\big)(x) \Big\}.\end{aligned}$$

In order to simplify the above expression of $\mathcal{A}^*$, let us introduce the fractional transforms

$$\mathcal{V}^\tau_\mu h(x) = \tfrac{1}{\Gamma(\mu)} \int_{-x}^{x} (\cosh\tau x - \cosh\tau z)^{\mu-1}\, h(z)\, dz\,. \tag{4.5.1}$$

Notice that, when h is odd, then $\mathcal{V}^\tau_\mu h = 0$. Moreover, when h is even, then $\mathcal{V}^\tau_\mu h$ is odd and $\mathcal{V}^\tau_\mu h(x) = 2\,\mathcal{R}^\tau_\mu h(x)$ for all $x \geq 0$. In conclusion, we have proved the following result.

Theorem 4.5.1. *The dual Abel transform $\mathcal{A}^*$ writes*

$$\begin{aligned}\mathcal{A}^* g(x) = c_{\alpha,\beta}\, A_{\alpha,\beta}(|x|)^{-1} \Big\{ & \cosh\, \sinh\, \mathcal{V}^2_{\beta+\frac{1}{2}}\big(\sinh\, \mathcal{V}^1_{\alpha-\beta}\, g\big)(x) \\ & -\tfrac{\rho}{4}\, \mathcal{V}^2_{\beta+\frac{3}{2}}\big(\sinh \mathcal{V}^1_{\alpha-\beta}\, g\big)(x) + \tfrac{1}{4}\, \mathcal{V}^2_{\beta+\frac{3}{2}}\big(\sinh \mathcal{V}^1_{\alpha-\beta}\, g'\big)(x) \Big\},\end{aligned}$$

where $A_{\alpha,\beta}$ is given by (4.2.3), *$\mathcal{V}^\tau_\mu$ by* (4.5.1), *and $c_{\alpha,\beta}$ by* (4.3.6).

4.6 Upper bound for the heat kernel in complex time

Consider the homogeneous heat equation

$$\begin{cases} \partial_t u(x,t) = \Delta_x u(x,t) & \forall\,(x,t)\in\mathbb{R}\times\mathbb{R}^+ \\ u(x,0) = f(x) & \forall\, x\in\mathbb{R} \end{cases} \tag{4.6.1}$$

associated to the trigonometric Dunkl Laplacian (4.1.3) on $\mathbb{R}$. Under sensible growth conditions, its solution is given by $u(t,x) = e^{t\Delta}f(x) = h_t * f(x)$, where $h_t(x) = \mathcal{F}^{-1}\big(e^{-t\lambda^2}\big)(x)$ is the heat kernel and $* = *_{\alpha,\beta}$ denotes the trigonometric Dunkl convolution product (see for instance [6]). For imaginary time, (4.6.1) becomes the homogeneous Schrödinger equation

$$\begin{cases} -i\,\partial_t u(x,t) = \Delta_x u(x,t) & \forall\,(x,t)\in\mathbb{R}\times\mathbb{R}\,, \\ u(x,0) = f(x) & \forall\, x\in\mathbb{R}\,, \end{cases}$$

whose solution is given by $u(x,t) = e^{it\Delta}f(x) = s_t{*}f(x)$ with $s_t(x){=}\mathcal{F}^{-1}\big(e^{-it\lambda^2}\big)(x)$.

Remark 4.6.1. *Chouchene, Gallardo and Mili have studied in [25] the heat equation* (4.6.1)*, the heat kernel and the Markov process associated to the so–called Jacobi–Dunkl Laplacian*

$$\begin{aligned} \widetilde{\Delta} f(x) &= \big(\Delta_{\alpha,\beta} - \rho^2\big) f(x) \\ &= f''(x) + \big\{(2\alpha{+}1)\coth x + (2\beta{+}1)\tanh x\big\}\, f'(x) \\ &\quad + \Big\{-\frac{2\,\alpha\,+\,1}{\sinh^2 x} + \frac{2\,\beta\,+\,1}{\cosh^2 x}\Big\}\,\frac{f(x)\,-\,f(-x)}{2} \end{aligned}$$

on $\mathbb{R}$. In particular, it is shown that the closure of $\widetilde{\Delta}$ on $C_0(\mathbb{R})$ generates a strongly continuous Markovian semigroup $\{H_t^{(\alpha,\beta)}, t\geq 0\}$ with a strictly positive kernel $h_t^{(\alpha,\beta)}$. However, a closed expression of the heat kernel is given only in the case $\alpha=\frac{1}{2}$ and $\beta=-\frac{1}{2}$.

Theorem 4.6.2. (Upper bounds for the complex time heat kernel) *The following global estimate holds for $x\in\mathbb{R}$ and $t\in\mathbb{C}^*$ with $\operatorname{Re} t\geq 0$:*

$$|h_t(x)| \lesssim \begin{cases} |t|^{-\frac{3}{2}}\,(1{+}|x|)\,e^{-\rho|x|}\,e^{-\frac{x^2}{4}\operatorname{Re}(\frac{1}{t})} & \text{if } |t|\geq 1{+}|x|\,, \\ |t|^{-\alpha-1}\,(1{+}|x|)^{\alpha+\frac{1}{2}}\,e^{-\rho|x|}\,e^{-\frac{x^2}{4}\operatorname{Re}(\frac{1}{t})} & \text{if } |t|\leq 1{+}|x|\,. \end{cases} \tag{4.6.2}$$

The rest of this section is devoted to the proof of the above theorem in the case $\alpha>\beta>-\frac{1}{2}$. Notice first that the heat kernel $h_t(x)=\mathcal{F}^{-1}(e^{-t\lambda^2})(x)$ is an even function of x and that we may therefore restrict to $x\geq 0$. According to Theorem 4.3.1,

$$h_t(x)=\tfrac{2^{2\beta-\alpha-\frac{1}{2}}}{\Gamma(\alpha+1)}\,t^{-\frac{1}{2}}\left(\mathcal{W}^2_{-\beta-\frac{1}{2}}\circ\mathcal{W}^1_{-\alpha+\beta}\right)\left(e^{-\frac{z^2}{4t}}\right)(x)\,.$$

Recall that the Weyl transform $\mathcal{W}^\tau_\mu$ is given by

$$\mathcal{W}^\tau_\mu f(x)=\tfrac{1}{\Gamma(\mu+k)}\int_{|x|}^{+\infty}\left(-\tfrac{\partial}{\partial(\cosh\tau s)}\right)^k f(s)\,(\cosh\tau s-\cosh\tau x)^{\mu+k-1}\,d(\cosh\tau s)$$

for every $\mu\in\mathbb{C}$ and $k\in\mathbb{N}$ such that $\mathrm{Re}\mu+k>0$. Thus

$$\begin{aligned}&\left(\mathcal{W}^2_{-\beta-\frac{1}{2}}\circ\mathcal{W}^1_{-\alpha+\beta}\right)f(x)=\tfrac{1}{\Gamma(n-\beta-\frac{1}{2})}\\&\times\int_x^{+\infty}\left(-\tfrac{\partial}{\partial(\cosh 2s)}\right)^n\mathcal{W}^1_{-\alpha+\beta}f(s)\,(\cosh 2s-\cosh 2x)^{n-\beta-\frac{3}{2}}\,d(\cosh 2s)\,,\end{aligned}\tag{4.6.3}$$

where $n=[\beta+\frac{1}{2}]+1\in\mathbb{N}^*$, so that $0<n-\beta-\frac{1}{2}\leq 1$. Since

$$\left(-\tfrac{\partial}{\partial(\cosh 2s)}\right)^n=\sum_{j=1}^n c_{n,j}\,(\cosh s)^{j-2n}\left(-\tfrac{\partial}{\partial(\cosh s)}\right)^j,$$

we have

$$\left(-\tfrac{\partial}{\partial(\cosh 2s)}\right)^n\mathcal{W}^1_{-\alpha+\beta}f(s)=\sum_{j=1}^n c_{n,j}\,(\cosh s)^{j-2n}\,\mathcal{W}^1_{-(\alpha-\beta+j)}f(s)\,,\tag{4.6.4}$$

with $c_{n,n}=2^{-n}$. Next

$$\begin{aligned}&\mathcal{W}^1_{-(\alpha-\beta+j)}f(x)=\tfrac{1}{\Gamma(m-\alpha+\beta)}\\&\times\int_s^{+\infty}\left(-\tfrac{1}{\sinh z}\tfrac{\partial}{\partial z}\right)^{m+j}f(z)\,(\cosh z-\cosh s)^{m-\alpha+\beta-1}\,\sinh z\,dz\,,\end{aligned}\tag{4.6.5}$$

where $m=[\alpha-\beta]+1\in\mathbb{N}^*$, so that $0<m-\alpha+\beta\leq 1$. Let us apply these formulad to $f(z)=e^{-z^2/4t}$ and, for this purpose, let us expand

$$\left(-\tfrac{1}{\sinh z}\tfrac{\partial}{\partial z}\right)^{m+j}e^{-\frac{z^2}{4t}}=e^{-\frac{z^2}{4t}}\sum_{k=1}^{m+j}t^{-k}\,f_k(z)\,.\tag{4.6.6}$$

The coefficients $f_k(z)$ are linear combinations of products $\varphi_{\ell_1}(z)\cdots\varphi_{\ell_k}(z)$, where

$$\varphi_\ell(z)=\left(-\tfrac{1}{\sinh z}\tfrac{\partial}{\partial z}\right)^\ell z^2,$$

and $\ell_1,\dots,\ell_k\in\mathbb{N}^*$ satisfy $\ell_1+\dots+\ell_k=m+j$. It is clear that

$$\varphi_\ell(z)=\mathrm{O}\left((1+z)\,e^{-\ell z}\right).$$

Hence

$$\left|\left(\tfrac{1}{\sinh z}\tfrac{\partial}{\partial z}\right)^{m+j} e^{-\frac{z^2}{4t}}\right| \lesssim \sum\nolimits_{k=1}^{m+j} t^{-k}\,(1+z)^k\, e^{-(m+j)z}\, e^{-\frac{z^2}{4}\operatorname{Re}\left(\frac{1}{t}\right)}$$

and therefore

$$\left|\left(\tfrac{1}{\sinh z}\tfrac{\partial}{\partial z}\right)^{m+j} e^{-\frac{z^2}{4t}}\right| \lesssim \left\{\tfrac{1+z}{t} + \left(\tfrac{1+z}{t}\right)^{m+j}\right\} e^{-(m+j)z}\, e^{-\frac{z^2}{4}\operatorname{Re}\left(\frac{1}{t}\right)}\,. \tag{4.6.7}$$

We shall use repeatedly the following elementary estimates:

$$\sinh z \asymp \tfrac{z}{z+1}\, e^{z}\,, \tag{4.6.8}$$

$$\begin{aligned}
\cosh z - \cosh s &= 2\,\sinh\tfrac{z-s}{2}\,\sinh\tfrac{z+s}{2}\\
&\asymp \tfrac{z-s}{1+z-s}\, e^{\frac{z-s}{2}}\,\tfrac{z+s}{1+z+s}\, e^{\frac{z+s}{2}}\\
&\asymp \tfrac{z-s}{1+z-s}\,\tfrac{z}{z+1}\, e^{z} \qquad (4.6.9)\\
&\asymp \begin{cases} \frac{z^2-s^2}{1+s}\, e^{s} & \text{if } s<z<s+1\,,\\ e^{z} & \text{if } z\ge s+1\,, \end{cases} \qquad (4.6.10)
\end{aligned}$$

$$\tfrac{z\coth z-1}{\sinh z} \asymp z\, e^{-z}\,. \tag{4.6.11}$$

By combining (4.6.5), (4.6.7), (4.6.8), (4.6.9), we obtain

$$\begin{aligned}
&\left|\mathcal{W}^1_{-(\alpha-\beta+j)}\left(e^{-\frac{z^2}{4t}}\right)(s)\right|\\
&\lesssim \int_s^{+\infty}\left\{\tfrac{1+z}{|t|}+\left(\tfrac{1+z}{|t|}\right)^{m+j}\right\}\left(\tfrac{z-s}{1+z-s}\right)^{m-\alpha+\beta-1}\left(\tfrac{z}{z+1}\right)^{m-\alpha+\beta} e^{-(\alpha-\beta+j)\,z}\, e^{-\frac{z^2}{4}\operatorname{Re}\left(\frac{1}{t}\right)}\, dz\,,
\end{aligned}$$

hence

$$\left|\mathcal{W}^1_{-(\alpha-\beta+j)}\left(e^{-\frac{z^2}{4t}}\right)(s)\right| \lesssim \left\{\tfrac{1+s}{|t|}+\left(\tfrac{1+s}{|t|}\right)^{m+j}\right\} e^{-(\alpha-\beta+j)\,s}\, e^{-\frac{s^2}{4}\operatorname{Re}\left(\frac{1}{t}\right)}\,, \tag{4.6.12}$$

after performing the change of variables $z=s+u$ and using the elementary inequalities $s+u\le 1+s+u\le(1+s)(1+u)$. Thus, according to (4.6.4),

$$\left|\left(\tfrac{\partial}{\partial(\cosh 2s)}\right)^n \mathcal{W}^1_{-\alpha+\beta}\left(e^{-\frac{z^2}{4t}}\right)(s)\right| \lesssim \left\{\tfrac{1+s}{|t|}+\left(\tfrac{1+s}{|t|}\right)^{m+n}\right\} e^{-(\alpha-\beta+2n)\,s}\, e^{-\frac{s^2}{4}\operatorname{Re}\left(\frac{1}{t}\right)}\,. \tag{4.6.13}$$

By combining this time (4.6.3), (4.6.8), (4.6.9), (4.6.13), we obtain similarly

$$\begin{aligned}
&\left|\left(\mathcal{W}^2_{-\beta-\frac{1}{2}}\circ\mathcal{W}^1_{-\alpha+\beta}\right)\left(e^{-\frac{z^2}{4t}}\right)(x)\right|\\
&\lesssim \int_x^{+\infty}\left\{\tfrac{1+s}{|t|}+\left(\tfrac{1+s}{|t|}\right)^{m+n}\right\} e^{-\rho s}\left(\tfrac{s}{s+1}\right)^{n-\beta-\frac{1}{2}}\left(\tfrac{s-x}{1+s-x}\right)^{n-\beta-\frac{3}{2}} e^{-\frac{s^2}{4}\operatorname{Re}\left(\frac{1}{t}\right)}\, ds\,,
\end{aligned}$$

hence

$$\left|\left(\mathcal{W}^2_{-\beta-\frac12}\circ\mathcal{W}^1_{-\alpha+\beta}\right)\left(e^{-\frac{z^2}{4t}}\right)(x)\right| \lesssim \left\{\tfrac{1+x}{|t|}+\left(\tfrac{1+x}{|t|}\right)^{m+n}\right\} e^{-\rho x}\, e^{-\frac{x^2}{4}\operatorname{Re}(\frac1t)}\,,$$

after performing the change of variables $s=x+u$ and using the elementary inequalities $x+u\le 1+x+u\le(1+x)(1+u)$. In summary, we have obtained so far the following upper bounds for the heat kernel:

$$|h_t(x)| \lesssim \begin{cases} |t|^{-\frac32}\,(1+x)\,e^{-\rho x}\,e^{-\frac{x^2}{4}\operatorname{Re}(\frac1t)} & \text{if } |t|\ge 1+x\,,\\ |t|^{-m-n-\frac12}\,(1+|x|)^{m+n}\,e^{-\rho x}\,e^{-\frac{x^2}{4}\operatorname{Re}(\frac1t)} & \text{if } |t|\le 1+x\,. \end{cases}$$

In order to improve the second estimate, let us rewrite (4.6.6) as follows:

$$\begin{aligned}\left(-\tfrac{1}{\sinh z}\tfrac{\partial}{\partial z}\right)^{m+j} e^{-\frac{z^2}{4t}} &= e^{-\frac{z^2}{4t}} \sum\nolimits_{k=1}^{m+j-1} t^{-k}\, f_k(z)\\ &\quad + t^{-(m+j-1)}\left(\tfrac12\tfrac{z}{\sinh z}\right)^{m+j-1}\left(-\tfrac{1}{\sinh z}\tfrac{\partial}{\partial z}\right) e^{-\frac{z^2}{4t}}\,.\end{aligned} \tag{4.6.14}$$

The contribution to $\mathcal{W}^1_{-(\alpha-\beta+j)}\left(e^{-\frac{z^2}{4t}}\right)$ of the sum in (4.6.14) is estimated as in (4.6.12):

$$\begin{aligned}&\left|\int_s^{+\infty} e^{-\frac{z^2}{4t}} \sum_{k=1}^{m+j-1} t^{-k} f_k(z)\,(\cosh z-\cosh s)^{m-\alpha+\beta-1}\sinh z\,dz\right|\\ &\lesssim \left(\tfrac{1+s}{|t|}\right)^{m+j-1} e^{-(\alpha-\beta+j)\,s}\, e^{-\frac{s^2}{4}\operatorname{Re}(\frac1t)}\,.\end{aligned} \tag{4.6.15}$$

In order to estimate the contribution of the last term in (4.6.14), let us split up the integral

$$I = \int_s^{+\infty}\left(\tfrac{z}{\sinh z}\right)^{m+j-1}\left(\tfrac{\partial}{\partial z}\,e^{-\frac{z^2}{4t}}\right)(\cosh z-\cosh s)^{m-\alpha+\beta-1}\,dz = I_1+I_2+I_3 \tag{4.6.16}$$

according to

$$\int_s^{+\infty} = \int_s^{\sqrt{s^2+|t|}} + \int_{\sqrt{s^2+|t|}}^{s+1} + \int_{s+1}^{+\infty}$$

and let us use again the estimates (4.6.8), (4.6.9), (4.6.10) and (4.6.11). On one hand,

$$\begin{aligned}|I_1| &\lesssim |t|^{-1}\,(1+s)^{\alpha-\beta+j}\,e^{-(\alpha-\beta+j)\,s}\int_s^{\sqrt{s^2+|t|}}(z^2-s^2)^{m-\alpha+\beta-1}\,e^{-\frac{z^2}{4}\operatorname{Re}(\frac1t)}\,z\,dz\\ &\lesssim |t|^{m-\alpha+\beta-1}\,(1+s)^{\alpha-\beta+j}\,e^{-(\alpha-\beta+j)\,s}\,e^{-\frac{s^2}{4}\operatorname{Re}(\frac1t)}\,.\end{aligned} \tag{4.6.17}$$

On the other hand, after an integration by parts, the expression

$$I_2 + I_3 = \Big\{ \int_{\sqrt{s^2+|t|}}^{s+1} + \int_{s+1}^{+\infty} \Big\} \left(\tfrac{z}{\sinh z}\right)^{m+j-1} \left(\tfrac{\partial}{\partial z}\, e^{-\frac{z^2}{4t}}\right) (\cosh z - \cosh s)^{m-\alpha+\beta-1}\, dz$$

becomes

$$\begin{aligned} I_2 + I_3 = &\Big[e^{-\frac{z^2}{4t}} \left(\tfrac{z}{\sinh z}\right)^{m+j-1} (\cosh z - \cosh s)^{m-\alpha+\beta-1} \Big]_{\sqrt{s^2+|t|}}^{+\infty} \\ &+ \Big\{ \int_{\sqrt{s^2+|t|}}^{s+1} + \int_{s+1}^{+\infty} \Big\} e^{-\frac{z^2}{4t}} (\cosh z - \cosh s)^{m-\alpha+\beta-2} \left(\tfrac{z}{\sinh z}\right)^{m+j-2} \\ &\times \Big[(m+j-1) \left(\tfrac{z \coth z - 1}{\sinh z}\right) (\cosh z - \cosh s) - (m-\alpha+\beta-1)\, z \Big]\, dz. \end{aligned} \tag{4.6.18}$$

It is clear that

$$\begin{aligned} &\Big| \Big[e^{-\frac{z^2}{4t}} \left(\tfrac{z}{\sinh z}\right)^{m+j-1} (\cosh z - \cosh s)^{m-\alpha+\beta-1} \Big]_{\sqrt{s^2+|t|}}^{+\infty} \Big| \\ &\qquad \lesssim |t|^{m-\alpha+\beta-1} (1+s)^{\alpha-\beta+j}\, e^{-(\alpha-\beta+j)\, s}\, e^{-\frac{s^2}{4} \mathrm{Re}(\frac{1}{t})}\,. \end{aligned} \tag{4.6.19}$$

The first integral on the right hand side of (4.6.18) is estimated as follows:

$$\begin{aligned} &\Big| \int_{\sqrt{s^2+|t|}}^{s+1} e^{-\frac{z^2}{4t}} (\cosh z - \cosh s)^{m-\alpha+\beta-2} \left(\tfrac{z}{\sinh z}\right)^{m+j-2} \\ &\times \Big[(m+j-1)\, \tfrac{z \coth z - 1}{\sinh z} (\cosh z - \cosh s) + (m-\alpha+\beta-1)\, z \Big] dz \Big| \\ &\lesssim (1+s)^{m+j-2}\, e^{-(\alpha-\beta+j)\, s}\, e^{-\frac{s^2}{4} \mathrm{Re}(\frac{1}{t})} \int_{\sqrt{s^2+|t|}}^{s+1} \left(\tfrac{z^2 - s^2}{1+s}\right)^{m-\alpha+\beta-2} \left(\tfrac{z^2 - s^2}{1+s} + 1\right) z\, dz \\ &\lesssim (1+s)^{\alpha-\beta+j}\, e^{-(\alpha-\beta+j)\, s}\, e^{-\frac{s^2}{4} \mathrm{Re}(\frac{1}{t})} \int_{\sqrt{s^2+|t|}}^{s+1} (z^2 - s^2)^{m-\alpha+\beta-2}\, z\, dz \\ &\lesssim |t|^{m-\alpha+\beta-1} (1+s)^{\alpha-\beta+j}\, e^{-(\alpha-\beta+j)\, s}\, e^{-\frac{s^2}{4} \mathrm{Re}(\frac{1}{t})}\,. \end{aligned} \tag{4.6.20}$$

Let us turn to the second integral on the right hand side of (4.6.18):

$$\begin{aligned} &\Big| \int_{s+1}^{+\infty} e^{-\frac{z^2}{4t}} (\cosh z - \cosh s)^{m-\alpha+\beta-2} \left(\tfrac{z}{\sinh z}\right)^{m+j-2} \\ &\times \Big[(m+j-1)\, \tfrac{z \coth z - 1}{\sinh z} (\cosh z - \cosh s) - (m-\alpha+\beta-1)\, z \Big] dz \Big| \\ &\lesssim e^{-\frac{s^2}{4} \mathrm{Re}(\frac{1}{t})} \int_{s+1}^{+\infty} (1+z)^{m+j-1}\, e^{-(\alpha-\beta+j)\, z}\, dz\,. \end{aligned} \tag{4.6.21}$$

The last integral is comparable to $(1+s)^{m+j-1}\, e^{-(\alpha-\beta+j)\, s}$, which is

$$\mathrm{O}\left(|t|^{m-\alpha+\beta-1} (1+s)^{\alpha-\beta+j}\, e^{-(\alpha-\beta+j)\, s} \right), \tag{4.6.22}$$

as $|t|\le 1+x\le 1+s$. In summary, (4.6.17), (4.6.19), (4.6.20), (4.6.21), (4.6.22) yield the following estimate of the integral (4.6.16):

$$|I| \lesssim |t|^{m-\alpha+\beta-1}\,(1+s)^{\alpha-\beta+j}\,e^{-(\alpha-\beta+j)\,s}e^{-\frac{s^2}{4}\mathrm{Re}(\frac{1}{t})}\,.$$

By combining this result with (4.6.15) and (4.6.4), we obtain first

$$\big|\,\mathcal{W}^1_{-(\alpha-\beta)-j}\big(e^{-\frac{z^2}{4t}}\big)(s)\,\big| \lesssim \big(\tfrac{1+s}{|t|}\big)^{\alpha-\beta+j}\,e^{-(\alpha-\beta+j)\,s}\,e^{-\frac{s^2}{4}\mathrm{Re}(\frac{1}{t})} \tag{4.6.23}$$

and next

$$\big|\,\big(\tfrac{\partial}{\partial(\cosh 2s)}\big)^{\ell}\,\mathcal{W}^1_{-\alpha+\beta}\big(e^{-\frac{z^2}{4t}}\big)(s)\,\big| \lesssim \big(\tfrac{1+s}{|t|}\big)^{\alpha-\beta+\ell}\,e^{-(\alpha-\beta+2\ell)s}\,e^{-\frac{s^2}{4}\mathrm{Re}(\frac{1}{t})}$$

for every $\ell\in\mathbb{N}$, in particular for $\ell=n$ and $\ell=n-1$. Let us turn eventually to the expression

$$\big(\mathcal{W}^2_{-\beta-\frac{1}{2}}\circ\mathcal{W}^1_{-\alpha+\beta}\big)\big(e^{-\frac{z^2}{4t}}\big)$$

which coincides, according to (4.6.3) and up to a constant, with

$$J \;=\int_x^{+\infty}\tfrac{\partial}{\partial(\cosh 2s)}\,h(s,t)\,(\cosh 2s-\cosh 2x)^{n-\beta-\frac{3}{2}}\,d(\cosh 2s)\,, \tag{4.6.24}$$

where

$$h(s,t)=\big(\tfrac{\partial}{\partial(\cosh 2s)}\big)^{n-1}\,\mathcal{W}^1_{-\alpha+\beta}\,\big(e^{-\frac{z^2}{4t}}\big)(s)\,.$$

This integral is handled as (4.6.16). Let us split up $J=J_1+J_2+J_3$ according to

$$\int_x^{+\infty}=\int_x^{\sqrt{x^2+|t|}}+\int_{\sqrt{x^2+|t|}}^{x+1}+\int_{x+1}^{+\infty}\,.$$

The first integral is estimated as follows:

$$\begin{aligned}
|J_1| \;&=\Big|\int_x^{\sqrt{x^2+|t|}}\big(\tfrac{\partial}{\partial(\cosh 2s)}\big)^{n}\,\mathcal{W}^1_{-\alpha+\beta}\,\big(e^{-\frac{z^2}{4t}}\big)(s)\,(\cosh 2s-\cosh 2x)^{n-\beta-\frac{3}{2}}\,d(\cosh 2s)\Big|\\
&\lesssim |t|^{-\alpha+\beta-n}\,(1+x)^{\alpha+\frac{1}{2}}\,e^{-\rho\,x}\,e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})}\int_x^{\sqrt{x^2+|t|}}(s^2-x^2)^{n-\beta-\frac{3}{2}}\,s\,ds\\
&\lesssim |t|^{-\alpha-\frac{1}{2}}\,(1+x)^{\alpha+\frac{1}{2}}\,e^{-\rho\,x}\,e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})}\,.
\end{aligned}$$

As far as the two other integrals are concerned, we begin with an integration by parts:

$$\begin{aligned}
J_2+J_3=&\Big[\,h(s,t)\,(\cosh 2s-\cosh 2x)^{n-\beta-\frac{3}{2}}\,\Big]_{\sqrt{x^2+|t|}}^{+\infty}\\
&-2\,(n-\beta-\tfrac{3}{2})\,\Big\{\int_{\sqrt{x^2+|t|}}^{x+1}+\int_{x+1}^{+\infty}\Big\}\,h(s,t)\,(\cosh 2s-\cosh 2x)^{n-\beta-\frac{5}{2}}\,\sinh 2s\,ds\,.
\end{aligned}$$

Notice that $-1<n-\beta-\frac{3}{2}\leq 0$. Firstly,

$$\Big|\Big[\,h(s,t)\,(\cosh 2s-\cosh 2x)^{n-\beta-\frac{3}{2}}\,\Big]_{\sqrt{x^2+|t|}}^{+\infty}\Big| \lesssim |t|^{-\alpha-\frac{1}{2}}\,(1+x)^{\alpha+\frac{1}{2}}\,e^{-\rho\,x}\,e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})}\,.$$

Secondly,

$$\begin{aligned}&\Big|\,(n-\beta-\tfrac{3}{2})\int_{\sqrt{x^2+|t|}}^{x+1} h(s,t)\,(\cosh 2s-\cosh 2x)^{n-\beta-\frac{5}{2}}\,\sinh 2s\,ds\,\Big|\\ &\lesssim |t|^{-\alpha+\beta-n+1}\,(1+x)^{\alpha+\frac{1}{2}}\,e^{-\rho\,x}\,e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})}\,(-n+\beta+\tfrac{3}{2})\int_{\sqrt{x^2+|t|}}^{x+1}(s^2-x^2)^{n-\beta-\frac{5}{2}}\,s\,ds\\ &\lesssim |t|^{-\alpha-\frac{1}{2}}\,(1+x)^{\alpha+\frac{1}{2}}\,e^{-\rho\,x}\,e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})}\,.\end{aligned}$$

Thirdly,

$$\begin{aligned}&\Big|\int_{x+1}^{+\infty} h(s,t)\,(\cosh 2s-\cosh 2x)^{n-\beta-\frac{5}{2}}\,\sinh 2s\,ds\,\Big|\\ &\lesssim |t|^{-\alpha+\beta-n+1}\int_{x+1}^{+\infty}(1+s)^{\alpha-\beta+n-1}\,e^{-\rho\,s}\,e^{-\frac{s^2}{4}\mathrm{Re}(\frac{1}{t})}\,ds\\ &\lesssim |t|^{-\alpha+\beta-n+1}\,(1+x)^{\alpha-\beta+n-1}\,e^{-\rho\,x}\,e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})}\\ &\lesssim |t|^{-\alpha-\frac{1}{2}}\,(1+x)^{\alpha+\frac{1}{2}}\,e^{-\rho\,x}\,e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})}\,.\end{aligned}$$

The last inequality is due to the fact that $\big(\frac{1+x}{|t|}\big)^{n-\beta-\frac{3}{2}}\leq 1$. In summary, we obtain the following estimate for the integral (4.6.24) when $|t|\leq 1+x$:

$$|J|\,\lesssim\,|t|^{-\alpha-\frac{1}{2}}\,(1+x)^{\alpha+\frac{1}{2}}\,e^{-\rho\,x}\,e^{-\frac{x^2}{4}\mathrm{Re}(\frac{1}{t})}\,.$$

This concludes the proof of Theorem 4.6.2.

Remark 4.6.3. *Recall from Remark 4.3.2 that, in the limit cases* $\alpha=\beta>-\frac{1}{2}$ *and* $\alpha>\beta=-\frac{1}{2}$*, the inverse Abel transform involves at most one integral transform. Therefore the estimates of the heat kernel* $h_t(x)$ *and of the Schrödinger kernel* $s_t(x)=h_{it}(x)$ *are obtained as in [28, 2, 35, 3, 4].*

4.7 Lower bound for the heat and for the Schrödinger kernel

Theorem 4.7.1. (Lower bound for the heat kernel) *For every* $t>0$ *and* $x\in\mathbb{R}$*, we have*

$$|h_t(x)|\gtrsim\begin{cases}t^{-\frac{3}{2}}\,(1+|x|)\,e^{-\rho|x|} & \text{if }\ t\geq 1+|x|\\ t^{-\alpha-1}\,(1+|x|)^{\alpha+\frac{1}{2}}\,e^{-\rho|x|} & \text{if }\ 0<t\leq 1+|x|,\end{cases}$$

where $\rho = \alpha + \beta + 1$.

Proof. Adapted from [2, Section 5]. ☐

The following is an immediate consequence of Theorem 4.6.2 and Theorem 4.7.1

Corollary 4.7.2. *For every* $t > 0$ *and* $x \in \mathbb{R}$, *we have*

$$|h_t(x)| \asymp \begin{cases} t^{-\frac{3}{2}}\,(1+|x|)\,e^{-\rho|x|} & \text{if } t \geq 1+|x| \\ t^{-\alpha-1}\,(1+|x|)^{\alpha+\frac{1}{2}}\,e^{-\rho|x|} & \text{if } 0 < t \leq 1+|x|, \end{cases}$$

where $\rho = \alpha + \beta + 1$.

Theorem 4.7.3. (Lower bound for the Schrödinger kernel) *For every* $t \in \mathbb{R}^*$, *there exist constants* $c > 1$ *and* $C > 0$ *such that*

$$|s_t(x)| \geq C\,|t|^{-\alpha-1}\,|x|^{\alpha+\frac{1}{2}}\,e^{-\rho\,|x|} \quad \forall\ |x| \geq 1+c\,|t|,$$

where $\rho = \alpha + \beta + 1$.

Proof. Adapted from [4, Lemma 5.1]. ☐

Corollary 4.7.4. *For every* $t \in \mathbb{R}^*$, *there exists a constant* $c > 1$ *such that*

$$|s_t(x)| \asymp |t|^{-\alpha-1}\,|x|^{\alpha+\frac{1}{2}}\,e^{-\rho\,|x|} \quad \forall\ |x| \geq 1+c\,|t|,$$

where $\rho = \alpha + \beta + 1$.

Proof. The upper bound follows from Theorem 4.6.2 and the lower bound follows from Theorem 4.7.3. ☐

4.8 Dispersive and Strichartz inequalities

In this section, we adapt to our setting the analysis carried out in [3, 4, 5] for dispersive equations on hyperbolic spaces and more generally on Damek–Ricci spaces. Let us denote by $L^p_{\alpha,\beta}(\mathbb{R})$ the Lebesgue space on $\mathbb{R}$ corresponding to the norm

$$\|f\|_{L^p_{\alpha,\beta}} = \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p A_{\alpha,\beta}(|x|)\,dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| & \text{if } p = \infty. \end{cases}$$

The following statement is an immediate consequence of Theorem 4.6.2.

Corollary 4.8.1. *Let* $2<r\leq\infty$. *Then*

$$||s_t||_{L^r_{\alpha,\beta}} \lesssim \begin{cases} |t|^{-\alpha-1} & \text{if } 0<|t|<1\,, \\ |t|^{-\frac{3}{2}} & \text{if } |t|\geq 1\,. \end{cases}$$

Let us turn our attention to $L^q = L^q_{\alpha,\beta}$ mapping properties of the Schrödinger propagator $e^{it\Delta}$, where $\Delta=\Delta_{\alpha,\beta}$ denotes the trigonometric Dunkl Laplacian (4.1.3). Notice first that $e^{it\Delta}$ is a unitary operator on L^2. Recall next the following L^r mapping properties of the trigonometric Dunkl convolution product

$$f*g = f*_{\alpha,\beta}\, g = \mathcal{F}^{-1}\big(\mathcal{F}f\,\mathcal{F}g\big).$$

Proposition 4.8.2. (*Young's inequality* [6, Proposition 4.10]) *Let* $1\leq p,q,r\leq\infty$ *such that* $\frac{1}{p}+\frac{1}{q}-\frac{1}{r}=1$. *Then*

$$L^p_{\alpha,\beta}(\mathbb{R}) *_{\alpha,\beta} L^q_{\alpha,\beta}(\mathbb{R}) \subset L^r_{\alpha,\beta}(\mathbb{R})\,.$$

By this inclusion, we mean that there exists a constant $C>0$ *such that*

$$\|\, f*g\,\|_{L^r} \leq C\,\|f\|_{L^p}\,\|g\|_{L^q} \quad \forall\, f\in L^p,\, \forall\, g\in L^q.$$

Proposition 4.8.3. (*Kunze–Stein phenomenon* [6, Corollary 5.3.iii]) *Let* $2<p,q<\infty$ *such that* $\frac{q}{2}\leq p<q$. *Then*

$$L^p_{\alpha,\beta}(\mathbb{R}) *_{\alpha,\beta} L^{q'}_{\alpha,\beta}(\mathbb{R}) \subset L^q_{\alpha,\beta}(\mathbb{R})\,.$$

Theorem 4.8.4. *Let* $2<q,\tilde{q}\leq\infty$. *Then*

$$\|\, e^{it\Delta}\,\|_{L^{\tilde{q}'}\to L^q} \lesssim \begin{cases} |t|^{-2(\alpha+1)\max\left\{\frac{1}{2}-\frac{1}{q},\frac{1}{2}-\frac{1}{\tilde{q}}\right\}} & \text{if } 0<|t|<1\,, \\ |t|^{-\frac{3}{2}} & \text{if } |t|\geq 1\,. \end{cases} \tag{4.8.1}$$

Proof. Assume first that $|t|>0$ is small. On one hand, by L^2 conservation,

$$\|\, e^{it\Delta}\,\|_{L^2\to L^2} = 1\,. \tag{4.8.2}$$

On the other hand, it follows from Proposition 4.8.2 and from Corollary 4.8.1 that, for every $2<r\leq\infty$,

$$\|\, e^{it\Delta}\,\|_{L^1\to L^r} \lesssim \|s_t\|_{L^r} \lesssim |t|^{-(\alpha+1)}\,, \tag{4.8.3}$$

$$\|\, e^{it\Delta}\,\|_{L^{r'}\to L^\infty} \lesssim \|s_t\|_{L^r} \lesssim |t|^{-(\alpha+1)}\,. \tag{4.8.4}$$

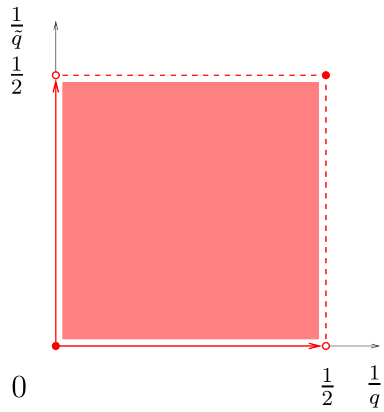

Figure 4.1: Interpolation for $|t|$ small

By complex interpolation between (4.8.2), (4.8.3) and (4.8.4), we deduce the announced estimate, for every $2<q,\tilde{q}\leq\infty$:

$$\| e^{it\Delta} \|_{L^{\tilde{q}'}\to L^q} \lesssim |t|^{-2(\alpha+1)\max\left\{\frac{1}{2}-\frac{1}{q},\frac{1}{2}-\frac{1}{\tilde{q}}\right\}} . \tag{4.8.5}$$

Assume next that $|t|$ is large. On one hand, it follows from Proposition 4.8.3 and from Corollary 4.8.1 that, for every $2<q<\infty$,

$$\| e^{it\Delta} \|_{L^{q'}\to L^q} \lesssim |t|^{-\frac{3}{2}} . \tag{4.8.6}$$

On the other hand, it follows again from Proposition 4.8.2 and from Corollary 4.8.1 that, for every $2<r\leq\infty$,

$$\| e^{it\Delta} \|_{L^1\to L^r} \lesssim \|s_t\|_{L^r} \lesssim |t|^{-\frac{3}{2}} , \tag{4.8.7}$$

$$\| e^{it\Delta} \|_{L^{r'}\to L^\infty} \lesssim \|s_t\|_{L^r} \lesssim |t|^{-\frac{3}{2}} . \tag{4.8.8}$$

By complex interpolation between (4.8.6), (4.8.7) and (4.8.8), we deduce again the announced estimate, for every $2<q,\tilde{q}\leq\infty$:

$$\| e^{it\Delta} \|_{L^{\tilde{q}'}_{\alpha,\beta}\to L^q_{\alpha,\beta}} \lesssim |t|^{-\frac{3}{2}} . \tag{4.8.9}$$

This concludes the proof of Theorem 4.8.4. ☐

Consider the inhomogeneous linear Schrödinger equation

$$\begin{cases} i\,\partial_t u(x,t) + \Delta_x u(x,t) = F(x,t) \qquad \forall\,(x,t)\in\mathbb{R}\times\mathbb{R}\,, \\ u(x,0) = f(x) \qquad \forall\,x\in\mathbb{R}\,, \end{cases} \tag{4.8.10}$$

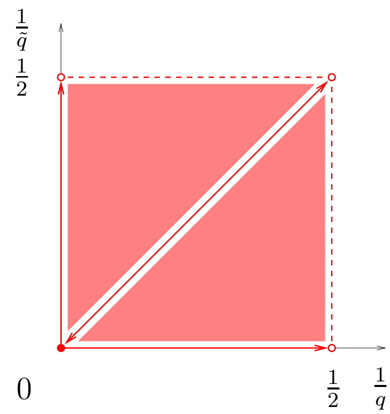

Figure 4.2: Interpolation for $|t|$ large

whose solution is given by Duhamel's formula

$$u(x,t) = e^{i\,t\,\Delta} f(x) + \int_0^t e^{i\,(t-s)\,\Delta_x} F(x,s)\,ds\,. \tag{4.8.11}$$

Let

$$T_\alpha = \left\{ \left(\tfrac1p, \tfrac1q\right) \in \left(0, \tfrac12\right] \times \left(0, \tfrac12\right) \,\middle|\, \tfrac1p \geq (\alpha+1)\left(\tfrac12 - \tfrac1q\right) \right\} \cup \left\{ \left(0, \tfrac12\right) \right\}.$$

A pair (p,q) is called *admissible* if $\left(\frac1p, \frac1q\right) \in T_\alpha$.

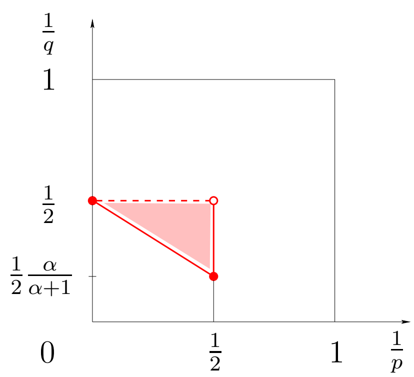

Figure 4.3: Admissible set for $\alpha > 0$

Theorem 4.8.5. *Let* (p,q) *and* $(\tilde p, \tilde q)$ *be two admissible pairs. Then, for every solution* u *to* (4.8.10)*, we have*

$$\|u(x,t)\|_{L^p_t L^q_x} \lesssim \|f(x)\|_{L^2_x} + \|F(x,t)\|_{L^{\tilde p'}_t L^{\tilde q'}_x}\,.$$

Remark 4.8.6. *In this statement, we may restrict to any time interval containing* 0*.*

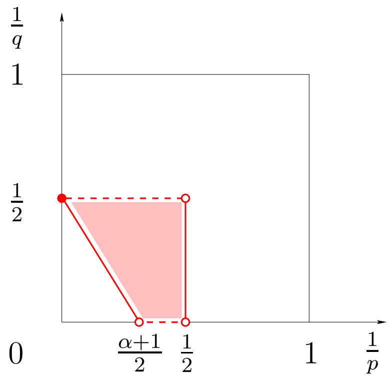

Figure 4.4: Admissible set for $-\frac{1}{2} \le \alpha \le 0$

Proof. We follow the TT^* strategy developped by Ginibre and Velo [34] and perfected by Keel and Tao [44]. Consider the operator

$$Tf(x,t) = e^{it\Delta_x} f(x) \tag{4.8.12}$$

and its formal adjoint

$$T^*F(x) = \int_{-\infty}^{+\infty} e^{-is\Delta_x} F(x,s)\, ds\,. \tag{4.8.13}$$

We shall first prove the $L_s^{p'} L_x^{q'} \to L_t^p L_x^q$ boundedness of

$$TT^*F(x,t) = \int_{-\infty}^{+\infty} e^{i(t-s)\Delta_x} F(x,s)\, ds\,, \tag{4.8.14}$$

for any admissible pair (p,q), which is equivalent to the $L_x^2 \to L_t^p L_x^q$ boundedness of T or to the $L_s^{p'} L_x^{q'} \to L_x^2$ boundedness of T^*. We may disregard the trivial case $(p,q) = (2,\infty)$. Using Theorem 4.8.4, we estimate

$$\begin{aligned} \Big\| \int_{\mathbb{R}} e^{i(t-s)\Delta_x} F(x,s)\, ds \Big\|_{L_t^p L_x^q} &\le \Big\| \int_{\mathbb{R}} \| e^{i(t-s)\Delta_x} F(x,s) \|_{L_x^q} ds \Big\|_{L_t^p} \\ &\lesssim \Big\| \int_{|t-s|\ge 1} |t-s|^{-\frac{3}{2}} \big\| F(x,s) \big\|_{L_x^{q'}} ds \Big\|_{L_t^p} \\ &\quad + \Big\| \int_{0<|t-s|<1} |t-s|^{-2(\alpha+1)(\frac{1}{2}-\frac{1}{q})} \big\| F(x,s) \big\|_{L_x^{q'}} ds \Big\|_{L_t^p}\,. \end{aligned}$$

On one hand, the convolution kernel $|t-s|^{-\frac{3}{2}} \mathbf{1}_{\{|t-s|\ge 1\}}$ on $\mathbb{R}$ defines a bounded operator from $L_s^{p_1}$ to $L_t^{p_2}$, for all $1 \le p_1 \le p_2 \le \infty$, in particular from $L_s^{p'}$ to L_t^p, for all $2 \le p \le \infty$. On the other hand, as long as $\frac{1}{q} > \frac{1}{2}\frac{\alpha}{\alpha+1}$, the convolution kernel

$|t-s|^{-2(\alpha+1)(\frac{1}{2}-\frac{1}{q})}\,\mathbf{1}_{\{0<|t-s|<1\}}$ defines a bounded operator from $L^{p_1}_s$ to $L^{p_2}_t$, for all $1<p_1,p_2<\infty$ such that $0\le\frac{1}{p_1}-\frac{1}{p_2}\le 1-2(\alpha+1)(\frac{1}{2}-\frac{1}{q})$, in particular from $L^{p'}_s$ to L^{p}_t, for all $2\le p<\infty$ such that $\frac{1}{p}\ge(\alpha+1)(\frac{1}{2}-\frac{1}{q})$. This argument breaks down at the endpoint $(\frac{1}{p},\frac{1}{q})=(\frac{1}{2},\frac{1}{2}\frac{\alpha}{\alpha+1})$, when $\alpha>0$, and we use instead the refined analysis carried out in [44], that we recall now. Thus, assume that $\alpha>0$ and that $2\le q,\tilde q<\infty$. First of all, observe that the $L^2_tL^{\tilde q'}_x\to L^2_tL^q_x$ boundedness of the linear operator

$$\mathcal{T}F(x,t)=\int_{|s-t|<1}e^{\,i(t-s)\Delta_x}F(x,s)\,ds$$

amounts to the $L^2_tL^{\tilde q'}_x\times L^2_tL^{q'}_x\to\mathbb{C}$ boundedness of the Hermitian form

$$\mathcal{B}^1(F,G)=\iint_{|s-t|<1}\int_{\mathbb{R}}e^{\,i(t-s)\Delta_x}F(x,s)\,\overline{G(x,t)}\,A_{\alpha,\beta}(|x|)\,dx\,ds\,dt \tag{4.8.15}$$

$$=\iint_{|s-t|<1}\int_{\mathbb{R}}e^{-is\Delta_x}F(x,s)\,\overline{e^{-it\Delta_x}G(x,t)}\,A_{\alpha,\beta}(|x|)\,dx\,ds\,dt\,. \tag{4.8.16}$$

Next, split up dyadically

$$\iint_{|s-t|<1}=\sum_{j\in\mathbb{N}}\iint_{2^{-j-1}\le|s-t|<2^{-j}}$$

and $\mathcal{B}^1=\sum_{j\in\mathbb{N}}\mathcal{B}^1_j$ accordingly. For every $j\in\mathbb{N}$, split up further

$$F(x,s)=\sum_{k\in\mathbb{Z}}\underbrace{\mathbf{1}_{[k2^{-j},(k+1)2^{-j})}(s)\,F(x,s)}_{F^{(j)}_k(x,s)}\ \text{ and }\ G(x,t)=\sum_{\ell\in\mathbb{Z}}\underbrace{\mathbf{1}_{[\ell 2^{-j},(\ell+1)2^{-j})}(t)\,G(x,t)}_{G^{(j)}_\ell(x,t)}.$$

Then, notice the orthogonality

$$\|F\|_{L^2L^{\tilde q'}}=\Big\{\sum\nolimits_{k=-\infty}^{+\infty}\|\,F^{(j)}_k\|^2_{L^2L^{\tilde q'}}\Big\}^{1/2},\quad \|G\|_{L^2L^{q'}}=\Big\{\sum\nolimits_{\ell=-\infty}^{+\infty}\|\,G^{(j)}_\ell\|^2_{L^2L^{q'}}\Big\}^{1/2}$$

and the almost orthogonality

$$\mathcal{B}^1_j(F,G)=\sum\nolimits_{\substack{k,\ell\in\mathbb{Z}\\|k-\ell|\le 1}}\mathcal{B}^1_j(F^{(j)}_k,G^{(j)}_\ell)\,.$$

By applying Lemma 4.8.7 below, we deduce that

$$\sup\nolimits_{j\in\mathbb{N}}2^{\,\kappa(q,\tilde q)j}\,|\mathcal{B}^1_j(F,G)|\lesssim\|F\|_{L^2L^{\tilde q'}}\|G\|_{L^2L^{q'}}$$

for all $2\le q,\tilde q<\infty$ satisfying the condition (4.8.20) below. By real interpolation, we obtain the improved estimate

$$\sum\nolimits_{j\in\mathbb{N}}2^{\,\kappa(q,\tilde q)j}\,|\mathcal{B}^1_j(F,G)|\lesssim\|F\|_{L^2L^{\tilde q'}}\|G\|_{L^2L^{q'}} \tag{4.8.17}$$

for all $2<q,\tilde q<\infty$ satisfying the condition (4.8.20) below. In particular,

$$|\mathcal{B}^1(F,G)| \le \sum\nolimits_{j\in\mathbb{N}} |\mathcal{B}^1_j(F,G)| \lesssim \|F\|_{L^2L^{q'}}\|G\|_{L^2L^{q'}} \tag{4.8.18}$$

when $q=\tilde q=2\frac{\alpha+1}{\alpha}$. This concludes the proof of the $L^{p'}_tL^{q'}_x\to L^p_tL^q_x$ boundedness of the operator TT^* given by (4.8.14) for every admissible pair (p,q).

In summary, the operators T and T^*, given by (4.8.12) and (4.8.13), are bounded from L^2_x into $L^p_tL^q_x$ and from $L^{\tilde p'}_sL^{\tilde q'}_x$ into L^2_x, for all admissible pairs (p,q) and $(\tilde p,\tilde q)$. Hence TT^* is bounded from $L^{\tilde p'}_sL^{\tilde q'}_x$ into $L^p_tL^q_x$ and it remains for us to prove the same result for the truncated operator

$$\widetilde{TT^*}F(x,t) = \int_{-\infty}^{t} e^{\,i(t-s)\Delta_x}F(x,s)\,ds\,. \tag{4.8.19}$$

According to the Christ–Kiselev Lemma [26], this holds true as long as $\tilde p'<p$. Thus we are left with the case where $p=\tilde p=2$ and where $2<q,\tilde q<\infty$ satisfy $\frac1q,\frac1{\tilde q}\ge\frac12\frac{\alpha}{\alpha+1}$. Let us resume the analysis carried out in [44] and recalled above. Split up again

$$\widetilde{TT^*}F(x,t) = \underbrace{\int_{-\infty}^{t-1} e^{\,i(t-s)\Delta_x}F(x,s)\,ds}_{\mathcal{T}_\infty F(x,t)} + \sum_{j\in\mathbb{N}}\underbrace{\int_{t-2^{-j}}^{t-2^{-j-1}} e^{\,i(t-s)\Delta_x}F(x,s)\,ds}_{\mathcal{T}_jF(x,t)},$$

and further

$$F(x,s) = \sum_{k\in\mathbb{Z}}\underbrace{\mathbf{1}_{[k2^{-j},(k+1)2^{-j})}(s)\,F(x,s)}_{F^{(j)}_k(x,s)}.$$

On one hand,

$$\begin{aligned}\big\|\mathcal{T}_\infty F\big\|_{L^2_tL^q_x} &\le \Big\|\int_{-\infty}^{t-1}\big\|\,e^{\,i(t-s)\Delta_x}F(x,s)\,\big\|_{L^q_x}\,ds\,\Big\|_{L^2_t}\\ &\lesssim \Big\|\int_{-\infty}^{t-1}(t-s)^{-\frac32}\,\big\|F(x,s)\big\|_{L^{\tilde q'}_x}\,ds\,\Big\|_{L^2_t} \lesssim \big\|F(x,s)\big\|_{L^2_sL^{\tilde q'}_x}.\end{aligned}$$

On the other hand, if $\alpha>0$ and $q=\tilde q=2\frac{\alpha+1}{\alpha}$, the proof of (4.8.18) yields the boundedness of the Hermitian form $\mathcal{B}^1=\sum_{j\in\mathbb{N}}\mathcal{B}^1_j$ associated to the operator $\mathcal{T}=\sum_{j\in\mathbb{N}}\mathcal{T}_j$. If $q\ne\tilde q$, we don't need the real interpolation step (4.8.17) and we deduce directly from Lemma 4.8.7 that

$$\begin{aligned}|\mathcal{B}^1(F,G)| &\le \sum\nolimits_{j\in\mathbb{N}}\sum\nolimits_{\substack{k,\ell\in\mathbb{Z}\\|k-\ell|\le1}}|\mathcal{B}^1_j(F^{(j)}_k,G^{(j)}_\ell)|\\ &\lesssim \Big\{\sum\nolimits_{j\in\mathbb{N}}2^{-\kappa(q,\tilde q)j}\Big\}\Big\{\sum\nolimits_{k\in\mathbb{Z}}\|F^{(j)}_k\|^2\Big\}^{1/2}\Big\{\sum\nolimits_{\ell\in\mathbb{Z}}\|F^{(j)}_\ell\|^2\Big\}^{1/2}\\ &\lesssim \|F\|_{L^2L^{\tilde q'}}\|G\|_{L^2L^{q'}}\,.\end{aligned}$$

Notice indeed that $\kappa(q,\tilde{q})>0$ under the assumptions $2<q\neq\tilde{q}<\infty$ and $\frac{1}{q},\frac{1}{\tilde{q}}\geq\frac{1}{2}\frac{\alpha}{\alpha+1}$. □

Lemma 4.8.7. *Let $2\!\leq\! q,\tilde{q}\!<\!\infty$ such that*

$$|\log q-\log\tilde{q}\,|<\log(\alpha{+}1)-\log\alpha \tag{4.8.20}$$

if $\alpha>0$ and with no further assumption if $-\frac{1}{2}\leq\alpha\leq 0$. Then, with the above notation,

$$\big|\mathcal{B}_j^1(F_k^{(j)},G_\ell^{(j)})\big|\lesssim 2^{-\kappa(q,\tilde{q})j}\|F_k^{(j)}\|_{L^2L^{\tilde{q}'}}\|G_\ell^{(j)}\|_{L^2L^{q'}} \tag{4.8.21}$$

for every $j\!\in\!\mathbb{N}$ and for every $k,\ell\!\in\!\mathbb{Z}$, where $\kappa(q,\tilde{q})\!=\!(\alpha{+}1)(\frac{1}{q}{+}\frac{1}{\tilde{q}})\!-\!\alpha$.

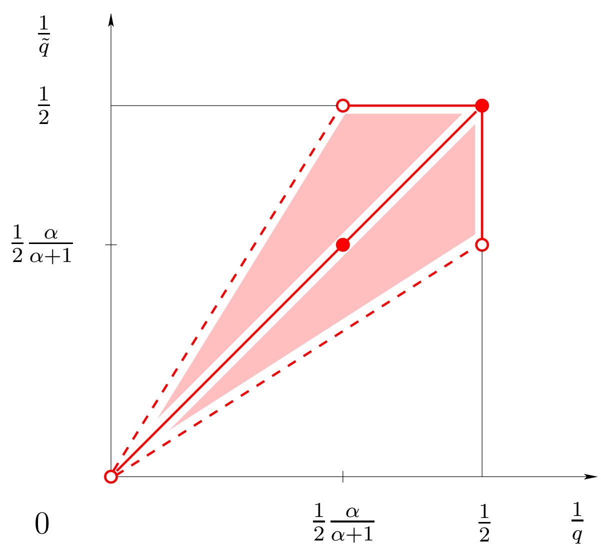

Figure 4.5: Lemma 4.8.7 in the case $\alpha\!>\!0$

Proof. This estimate is obtained by complex interpolation between the following cases :

(a) $q\!=\!2$ and $2\!\leq\!\tilde{q}\!<\!\infty$ with $\frac{1}{\tilde{q}}\!>\!\frac{1}{2}\frac{\alpha}{\alpha+1}$,
(b) $2\!\leq\! q\!<\!\infty$ with $\frac{1}{q}\!>\!\frac{1}{2}\frac{\alpha}{\alpha+1}$ and $\tilde{q}\!=\!2$,
(c) $2\!<\!q\!=\!\tilde{q}\!<\!\infty$.

• *Case* (a) : We estimate the expression (4.8.16) of $\mathcal{B}_j^1(F_k^{(j)},G_\ell^{(j)})$ using the Cauchy–Schwarz inequality, L^2 conservation, the $L_s^{\tilde{p}'}L_x^{\tilde{q}'}\to L_x^2$ boundedness of (4.8.13) with

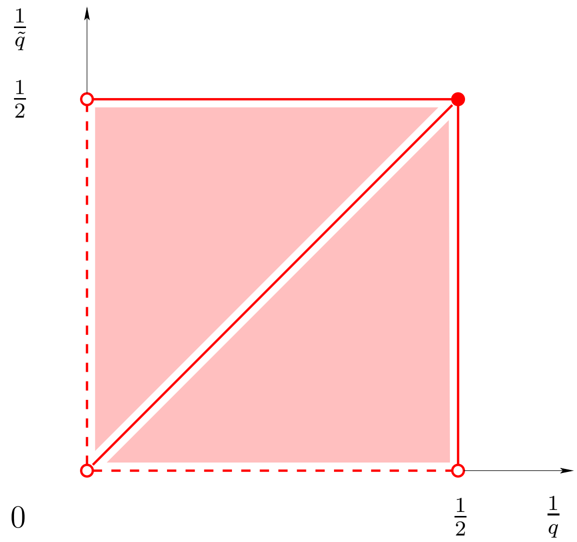

Figure 4.6: Lemma 4.8.7 in the case $-\frac{1}{2}\leq\alpha\leq 0$

$\frac{1}{\tilde{p}}=\frac{\alpha+1}{2}(\frac{1}{2}-\frac{1}{\tilde{q}})$ and Hölder's inequality. Specifically,

$$\begin{aligned}
\big|\mathcal{B}_j^1(F_k^{(j)},G_\ell^{(j)})\big| &\leq \sup_{t\in\mathbb{R}} \Big\| \int_{2^{-j-1}\leq|s-t|<2^{-j}} e^{-is\Delta_x} F_k^{(j)}(x,s)\,ds \Big\|_{L^2_x} \big\| e^{-it\Delta_x} G_\ell^{(j)}(x,t) \big\|_{L^1_t L^2_x} \\
&\lesssim \sup_{t\in\mathbb{R}} \big\| \mathbf{1}_{(t-2^{-j},\,t-2^{-j-1}]\cup[t+2^{-j-1},\,t+2^{-j})}(s)\, F_k^{(j)}(x,s) \big\|_{L^{\tilde{p}'}_s L^{\tilde{q}'}_x} \big\| G_\ell^{(j)}(x,t) \big\|_{L^1_t L^2_x} \\
&\lesssim 2^{-\kappa(2,\tilde{q})j} \,\| F_k^{(j)}\|_{L^2 L^{\tilde{q}'}} \| G_\ell^{(j)}\|_{L^2L^2}\,,
\end{aligned}$$

with $\kappa(2,\tilde{q})=\frac{1}{\tilde{p}'}$.

- *Case* (b) is handled similarly.
- *Case* (c): We estimate this time the expression (4.8.15) of $\mathcal{B}_j^1(F_k^{(j)},G_\ell^{(j)})$ using Theorem 4.8.4 and Hölder's inequality. Specifically,

$$\begin{aligned}
\big|\mathcal{B}_j^1(F_k^{(j)},G_\ell^{(j)})\big| &\leq \iint_{2^{-j-1}\leq|s-t|<2^{-j}} \big\| e^{i(t-s)\Delta_x} F_k^{(j)}(x,s) \big\|_{L^q_x} \big\| G_\ell^{(j)}(x,t) \big\|_{L^{q'}_x} \\
&\lesssim 2^{2(\alpha+1)(\frac{1}{2}-\frac{1}{q})j} \big\| F_k^{(j)}(x,s) \big\|_{L^1_s L^{q'}_x} \big\| G_\ell^{(j)}(x,t) \big\|_{L^1_t L^{q'}_x} \\
&\lesssim 2^{-\kappa(q,q)j} \big\| F_k^{(j)}(x,s) \big\|_{L^2_s L^{q'}_x} \big\| G_\ell^{(j)}(x,t) \big\|_{L^2_t L^{q'}_x}\,,
\end{aligned}$$

with $\kappa(q,q)=(\alpha+1)\frac{2}{q}-\alpha$.

This concludes the proof of Lemma 4.8.7. □

4.9 Well–posedness results for NLS

Strichartz estimates are used to prove local and global well–posedness results for the corresponding nonlinear equation. Following [3] we prove in this section some first results in this direction for the Cauchy problem

$$\begin{cases} i\,\partial_t u(x,t)+\Delta_x u(x,t)=F(u(x,t)) & \forall\,(x,t)\in\mathbb{R}\times\mathbb{R} \\ u(x,0)=f(x) & \forall\,x\in\mathbb{R} \end{cases} \tag{4.9.1}$$

with powerlike nonlinearities $F(u)\sim|u|^\gamma$. Specifically we assume that there exist constants $C\geq 0$ and $\gamma>1$ such that

$$|F(u)|\leq C\,|u|^\gamma \quad\text{and}\quad |F(u)-F(v)|\leq C\,(|u|^{\gamma-1}+|v|^{\gamma-1})\,|u-v|\,. \tag{4.9.2}$$

We recall the definition of well–posedness:

Definition 4.9.1. *The NLS equation* (4.9.1) *is locally well–posed in* $L^2_{\alpha,\beta}(\mathbb{R})$ *if, for any bounded subset* B *of* $L^2_{\alpha,\beta}(\mathbb{R})$*, there exist* $T>0$ *and a Banach space* X *(depending possibly on* T *and) continuously embedded into* $\mathcal{C}([-T,T\,];L^2_{\alpha,\beta}(\mathbb{R}))$ *such that:*

(i) for any Cauchy data $f\in B$, (4.9.1) *has a unique solution* $u\in X$;

(ii) the map $f\mapsto u$ *is continuous from* B *in* X.

We say that the equation is globally well–posed if these properties hold with $T=\infty$.

Theorem 4.9.2. *Assume* (4.9.2)*. Then, for every* $1<\gamma\leq 1+\frac{2}{\alpha+1}$*, the NLS* (4.9.1) *is globally well–posed for small* $L^2=L^2_{\alpha,\beta}$ *data. Moreover, in the subcritical case* $1<\gamma<1+\frac{2}{\alpha+1}$*, the NLS* (4.9.1) *is locally well–posed for arbitrary* L^2 *data.*

Proof. Let $u=\Phi(v)$ be the solution to the Cauchy problem

$$\begin{cases} i\,\partial_t u(x,t)+\Delta_x u(x,t)=F(v(x,t))\,, \\ u(x,0)=f(x)\,, \end{cases}$$

which is given by

$$u(x,t)=e^{i\,t\,\Delta_x}f(x)+\int_0^t e^{i\,(s-t)\,\Delta_x}F(v(x,s))\,ds\,.$$

By Theorem 4.8.5, the following Strichartz estimate holds:

$$\|\,u(x,t)\,\|_{L^\infty_t L^2_x}+\|\,u(x,t)\,\|_{L^p_t L^q_x}\leq C\,\|\,f(x)\,\|_{L^2_x}+C\,\|\,F(v(x,s))\,\|_{L^{\tilde{p}'}_s L^{\tilde{q}'}_x} \tag{4.9.3}$$

(p,q) and $(\tilde p,\tilde q)$ are admissible pairs, which amounts to the conditions

$$\begin{cases} 2\le p,q\le\infty \quad \text{satisfy } \frac{\delta}{p}+\frac{\alpha+1}{q}=\frac{\alpha+1}{2}, \ \text{ for some } \delta\in(0,1]\,,\\ 2\le \tilde p,\tilde q\le\infty \quad \text{satisfy } \frac{\tilde\delta}{\tilde p}+\frac{\alpha+1}{\tilde q}=\frac{\alpha+1}{2}\,, \ \text{ for some } \tilde\delta\in(0,1]\,. \end{cases} \tag{4.9.4}$$

By hypothesis on the nonlinearity we have

$$\|F(v)\|_{L_t^{\tilde p'}L_x^{\tilde q'}} \le C\,\|\,|v|^\gamma\,\|_{L_t^{\tilde p'}L_x^{\tilde q'}} \le C\,\|v\|^\gamma_{L_t^{\gamma\tilde p'}L_x^{\gamma\tilde q'}}.$$

Thus, we have

$$\|u\|_{L_t^\infty L_x^2}+\|u\|_{L_t^pL_x^q} \le C\,\|f\|_{L^2}+C\,\|v\|^\gamma_{L_t^{\gamma\tilde p'}L_x^{\gamma\tilde q'}} \tag{4.9.5}$$

In order to remain within the same functions space on the left and the right hand sides of (4.9.5), we assume that

$$\gamma\,\tilde p' = p \quad\text{and}\quad \gamma\,\tilde q' = q\,.$$

It is easy to see that these conditions hold true if we choose for instance

$$\begin{cases} 0<\delta=\tilde\delta\le 1,\\ \gamma=1+\frac{2\,\delta}{\alpha+1},\\ p=\tilde p=q=\tilde q=\gamma+1. \end{cases} \tag{4.9.6}$$

For such a choice of parameters, Φ maps $L^\infty(\mathbb{R},L^2_{\alpha,\beta}(\mathbb{R}))\cap L^p(\mathbb{R},L^q_{\alpha,\beta}(\mathbb{R}))$ into itself. More precisely, Φ maps $X=\mathcal{C}(\mathbb{R},L^2_{\alpha,\beta}(\mathbb{R}))\cap L^p(\mathbb{R},L^q_{\alpha,\beta}(\mathbb{R}))$ into itself. Since X is a Banach space for the norm

$$\|v\|_X = \|u\|_{L_t^\infty L_x^2}+\|v\|_{L_t^pL_x^q},$$

it is enough for us to show that Φ is a contraction in the closed ball

$$X_\varepsilon=\{\,u\in X\mid \|u\|_X\le\varepsilon\,\}$$

when $\varepsilon>0$ and $\|f\|_{L^2}$ are sufficiently small. Let $v_1,v_2\in X$ and $u_1=\Phi(v_1)$, $u_2=\Phi(v_2)$. By applying Hölder's inequality, we obtain

$$\begin{aligned}\|\,u_1-u_2\,\|_X &\le C\,\|\,F(v_1)-F(v_2)\,\|_{L_t^{\tilde p'}L_x^{\tilde q'}}\\ &\le C\,\|\,|\,v_1-v_2\,|\,(|v_1|^{\gamma-1}+|v_2|^{\gamma-1})\,\|_{L_t^{\tilde p'}L_x^{\tilde q'}}\\ &\le C\,\|\,v_1-v_2\,\|_{L_t^pL_x^q}\,\{\,\|v_1\|^{\gamma-1}_{L_t^pL_x^q}+\|v_2\|^{\gamma-1}_{L_t^pL_x^q}\,\}\,.\end{aligned}$$

Hence

$$\|\,u_1-u_2\,\|_X \le C\,\|\,v_1-v_2\,\|_X\,\{\,\|v_1\|_X^{\gamma-1}+\|v_2\|_X^{\gamma-1}\,\}. \tag{4.9.7}$$

If we assume that $\|v_1\|_X \le \varepsilon$, $\|v_2\|_X \le \varepsilon$ and $\|f\|_{L^2} \le \eta$, where $\varepsilon > 0$ and $\eta > 0$ will be specified below, then (4.9.5) and (4.9.7) yield

$$\|u_1\|_X \le C\,\eta + C\,\varepsilon^\gamma\,, \quad \|u_2\|_X \le C\,\eta + C\,\varepsilon^\gamma \quad \text{and} \quad \|\,u_1 - u_2\,\|_X \le 2\,C\,\varepsilon^{\gamma-1}\,\|\,v_1 - v_2\,\|_X\,.$$

Hence

$$\|u_1\|_X \le \varepsilon\,, \quad \|u_2\|_X \le \varepsilon \quad \text{and} \quad \|\,u_1 - u_2\,\|_X \le \tfrac{1}{2}\,\|\,v_1 - v_2\,\|_X$$

provided that $C\,\varepsilon^{\gamma-1} \le \frac{1}{4}$ and $C\,\eta \le \frac{\varepsilon}{4}$, which can be achieved by choosing ε and η small enough. We conclude by applying the fixed point theorem in the complete metric space X_ε.

Moreover, in the subcritical case $\gamma < 1 + \frac{2}{\alpha+1}$, we prove in the same way the local well–posedness of (4.9.1) in L^2 for arbitrarily large initial data $f \in L^2 = L^2_{\alpha,\beta}$. Specifically, we restrict to a small time interval $I = [-T, T\,]$ and proceed as above, except that we increase $\tilde{\delta} \in (\delta, 1]$ and $\tilde{p} = \frac{\tilde{\delta}}{\delta}\,p$ accordingly, and that we apply in addition Hölder's inequality in t. This way we obtain on one hand

$$\|u\|_X \le C\,\|f\|_{L^2} + C\,T^\lambda\,\|v\|_X^\gamma\,,$$

where $X = \mathcal{C}(I, L^2_{\alpha,\beta}(\mathbb{R})) \cap L^p(I, L^q_{\alpha,\beta}(\mathbb{R}))$ and $\lambda = 1 - \frac{\gamma}{p} - \frac{1}{\tilde{p}} > 0$, and on the other hand

$$\|\,u_1 - u_2\,\|_X \le C\,T^\lambda\,\|\,v_1 - v_2\,\|_X\,\big\{\|v_1\|_X^{\gamma-1} + \|v_2\|_X^{\gamma-1}\big\}\,.$$

Hence the map Φ is a contraction in the closed ball

$$X_M = \{\,u \in X \mid \|u\|_X \le M\,\}$$

provided that $C\,\|f\|_{L^2} \le \frac{3}{4}\,M$ and $C\,T^\lambda\,M^{\gamma-1} \le \frac{1}{4}$, which can be achieved by choosing first $M > 0$ large enough and next $T > 0$ small enough. We conclude as above. □

Remark 4.9.3. *Notice that T depends only on the $L^2_{\alpha,\beta}(\mathbb{R})$ norm of the initial data f:*

$$T = 3^{\frac{\gamma-1}{\lambda}}\,4^{-\frac{\gamma}{\lambda}}\,C^{-\frac{\gamma}{\lambda}}\,\|f\|_{L^2}^{-\frac{\gamma-1}{\lambda}}\,.$$

4.10 Appendix

In this appendix, we compare the kernel expression $K^{(\alpha,\beta)}(x,y)$ in Theorem 4.4.1 for the Abel transform $\mathcal{A}$ with the kernel expression $\widetilde{K}^{(\alpha,\beta)}(x,y)$ obtained by Gallardo and Trimeche in [32, Theorem 2.5] for the dual Abel transform $\mathcal{A}^*$. More precisely, as

$$G_\lambda^{(\alpha,\beta)}(x) = \int_{-|x|}^{|x|} \widetilde{K}^{(\alpha,\beta)}(x,y)\,e^{\,i\,\lambda\,y}\,dy\,,$$

our aim is to show that

$$\widetilde{K}^{(\alpha,\beta)}(x,y) = A_{\alpha,\beta}(|x|)^{-1} K^{(\alpha,\beta)}(-y,-x)\,, \tag{4.10.1}$$

where $A_{\alpha,\beta}$ denotes the weight (4.2.3). Assume first that $\alpha > \beta > -\frac{1}{2}$ and recall in this case that

$$\widetilde{K}^{(\alpha,\beta)} = \widetilde{K}_1^{(\alpha,\beta)}(x,y) + \widetilde{K}_2^{(\alpha,\beta)}(x,y) + \widetilde{K}_3^{(\alpha,\beta)}(x,y)\,,$$

where

$$\begin{aligned}\widetilde{K}_1^{(\alpha,\beta)}(y,x) = {} & \tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,(\sinh|x|)^{-2\alpha}\,(\cosh x)^{-\alpha-\beta}\,(\cosh^2 x-\cosh^2 y)^{\alpha-\frac{1}{2}}\\ & \times {}_2F_1\big(\alpha-\beta,\alpha+\beta\,;\alpha+\tfrac{1}{2}\,;\tfrac{\cosh x-\cosh y}{2\,\cosh x}\big)\,,\end{aligned}$$

$$\begin{aligned}\widetilde{K}_2^{(\alpha,\beta)}(x,y) = {} & \tfrac{\rho}{2}\,\tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{3}{2})}\,\mathrm{sign}(x)\,(\sinh|x|)^{-2\alpha-1}\,(\cosh x)^{-\alpha-\beta-1}\,(\cosh^2 x-\cosh^2 y)^{\alpha+\frac{1}{2}}\\ & \times {}_2F_1\big(\alpha-\beta,\alpha+\beta+2\,;\alpha+\tfrac{3}{2}\,;\tfrac{\cosh x-\cosh y}{2\,\cosh x}\big)\,,\end{aligned}$$

and

$$\begin{aligned}\widetilde{K}_3^{(\alpha,\beta)}(x,y) &= -\tfrac{1}{\rho}\,\tfrac{\partial}{\partial y}\,\widetilde{K}_2^{(\alpha,\beta)}(x,y)\\ &= \tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,\mathrm{sign}(x)\sinh y\,(\sinh|x|)^{-2\alpha-1}(\cosh x)^{-\alpha-\beta}(\cosh^2 x-\cosh^2 y)^{\alpha-\frac{1}{2}}\\ &\times \Big\{\tfrac{\cosh y}{\cosh x}\,{}_2F_1\big(\alpha-\beta,\alpha+\beta+2\,;\alpha+\tfrac{3}{2}\,;\tfrac{\cosh x-\cosh y}{2\,\cosh x}\big)\\ &\quad + \tfrac{(\alpha-\beta)\,(\alpha+\beta+2)}{(\alpha+\frac{1}{2})\,(\alpha+\frac{3}{2})}\,\tfrac{\cosh^2 x-\cosh^2 y}{4\,\cosh^2 x}\\ &\quad \times {}_2F_1\big(\alpha-\beta+1,\alpha+\beta+3\,;\alpha+\tfrac{5}{2}\,;\tfrac{\cosh x-\cosh y}{2\,\cosh x}\big)\Big\}\,.\end{aligned}$$

By using the relation (see for instance [1, (2.5.2)])

$$\begin{aligned}{}_2F_1(a,b\,;c\,;u) = {} & \big(1-\tfrac{a+b+1}{c}\,u\big)\,{}_2F_1(a+1,b+1\,;c+1;u)\\ & + \tfrac{(a+1)\,(b+1)}{c\,(c+1)}\,u(1-u)\,{}_2F_1(a+2,b+2\,;c+2\,;u)\end{aligned}$$

with $a=\alpha-\beta$, $b=\alpha+\beta+1=\rho$, $c=\alpha+\frac{1}{2}$ and $u=\frac{\cosh x-\cosh y}{2\,\cosh x}=\frac{1}{2}\big(1-\frac{\cosh y}{\cosh x}\big)$, we obtain

$$\begin{aligned}\widetilde{K}_3^{(\alpha,\beta)}(x,y) = {} & \tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,\mathrm{sign}(x)\sinh y\\ & \times (\sinh|x|)^{-2\alpha-1}(\cosh x)^{-\alpha-\beta}(\cosh^2 x-\cosh^2 y)^{\alpha-\frac{1}{2}}\\ & \times {}_2F_1\big(\alpha-\beta,\rho\,;\alpha+\tfrac{1}{2}\,;\tfrac{\cosh x-\cosh y}{2\,\cosh x}\big)\,.\end{aligned}$$

We conclude that, for $\alpha>\beta>-\frac{1}{2}$,

$$\begin{aligned} A_{\alpha,\beta}(|x|)^{-1}\,K_1^{(\alpha,\beta)}(-y,-x) &= \widetilde{K}_1^{(\alpha,\beta)}(x,y)\,,\\ A_{\alpha,\beta}(|x|)^{-1}\,K_2^{(\alpha,\beta)}(-y,-x) &= \widetilde{K}_2^{(\alpha,\beta)}(x,y)\,,\\ A_{\alpha,\beta}(|x|)^{-1}\,K_3^{(\alpha,\beta)}(-y,-x) &= \widetilde{K}_3^{(\alpha,\beta)}(x,y)\,. \end{aligned}$$

In the limit case $\alpha=\beta>-\frac{1}{2}$, we have on one hand

$$\begin{aligned} A_{\alpha,\alpha}(|x|)^{-1}\,K^{(\alpha,\alpha)}(-y,-x) = 2^{\,\alpha+\frac{1}{2}}\,\tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,&\operatorname{sign}(x)\,(\sinh 2|x|)^{-2\alpha-1}\\ &\times(\cosh 2x-\cosh 2y)^{\alpha-\frac{1}{2}}\,(e^{\,2x}-e^{-2y}) \end{aligned}$$

and on the other hand

$$\begin{aligned} \widetilde{K}^{(\alpha,\alpha)}(x,y) = 2^{\,\alpha+\frac{1}{2}}\,\tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,&\operatorname{sign}(x)\,(\sinh 2|x|)^{-2\alpha-1}\\ &\times(\cosh 2x-\cosh 2y)^{\alpha-\frac{1}{2}}\,(e^{\,2x}-e^{-2y})\,. \end{aligned}$$

Similarly, in the limit case $\alpha>\beta=-\frac{1}{2}$,

$$\begin{aligned} A_{\alpha,-\frac{1}{2}}(|x|)^{-1}\,K^{(\alpha,-\frac{1}{2})}(-y,-x) = 2^{\,\alpha-\frac{1}{2}}\,\tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,&\operatorname{sign}(x)\,(\sinh|x|)^{-2\alpha-1}\\ &\times(\cosh x-\cosh y)^{\alpha-\frac{1}{2}}\,(e^{\,x}-e^{-y}) \end{aligned}$$

and

$$\begin{aligned} \widetilde{K}^{(\alpha,-\frac{1}{2})}(x,y) = 2^{\,\alpha-\frac{1}{2}}\,\tfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\,&\operatorname{sign}(x)\,(\sinh|x|)^{-2\alpha-1}\\ &\times(\cosh x-\cosh y)^{\alpha-\frac{1}{2}}\,(e^{\,x}-e^{-y})\,. \end{aligned}$$

This proves our claim (4.10.1).

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Fatma Ayadi épouse Ben Said

Analyse harmonique et équation de Schrödinger associées au laplacien de Dunkl trigonométrique

Cette thèse est constituée de trois chapitres. Le premièr chapitre porte sur l'examen des conditions de validité du principe d'équipartition de l'énergie totale de la solution de l'équation des ondes associée au laplacien de Dunkl trigonométrique. Enfin, nous établissons le comportement asymptotique de l'équipartition dans le cas général. Les résultats de cette partie ont fait l'objet de la publication [8]. Le deuxième chapitre, publié avec J.Ph. Anker et M. Sifi [6], montre que les fonctions d'Opdam dans le cas de rang 1 satisfont à une formule produit. Cela nous a permis de définir une structure de convolution du genre hypergroupe. En particulier, on montre que cette convolution satisfait l'analogue du phénomène de Kunze-Stein. Le dernier chapitre est consacrée à l'étude des propriétés dispersives et estimations de Strichartz pour la solution de l'équation de Schrödinger associée au laplacien de Dunkl trigonométrique unidimensionnel [7]. Cette étude commence par des estimations optimales du noyau de la chaleur et de Schrödinger. À l'aide de ces résultats, ainsi que les outils d'analyse harmonique dévellopée dans le chapitre 2, on montre des éstimées de type Strichartz qui permettent de trouver des conditions d'admissibilité pour des équations de Schrödinger semi-linéaires.

Mots clés : Laplacien de Dunkl trigonométrique, formule produit, équation des ondes, équation de la chaleur, équation de Schrödinger, estimations de Strichartz.

Harmonic analysis and Schrödinger equation associated with the trigonometric Dunkl Laplacian

This thesis consists of three chapters. The first one is concerned with energy properties of the wave equation associated with the trigonometric Dunkl Laplacian. We establish the conservation of the total energy, the strict equipartition of energy under suitable assumptions and the asymptotic equipartition in the general case. These results were published in [8]. The second chapter, in collaboration with J.Ph. Anker and M. Sifi [6], shows that Opdam's functions in the rank one case satisfy a product formula. We then define and study a convolution structure related to Opdam's functions. In particular, we prove that this convolution fulfills a Kunze-Stein type phenomena. The last chapter deals with dispersive and Strichartz estimates for the linear Schrödinger equation associated with the one dimensional trigonometric Dunkl Laplacian [7]. We establish sharp estimates for the heat kernel in complex time, and therefore for the Schrödinger kernel. We then use these estimates together with tools from chapter 2 to deduce dispersive and Strichartz inequalities for the linear Schrödinger equation and apply them to well–posedness in the nonlinear case.

Keywords: Trigonometric Dunkl Laplacian, product formula, wave equation, heat equation, Schrödinger equation, Strichartz estiamtes.

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