

# A vector bundle proof of Poncelet theorem

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## Abstract

In the town of Saratov where he was prisoner, Poncelet, continuing the work of Euler and Steiner on polygons simultaneously inscribed in a circle and circumscribed around another circle, proved the following generalization.

**Theorem** *Let  $C$  and  $D$  be two smooth conics in  $\mathbb{P}^2(\mathbb{C})$ . If  $D$  passes through the  $\binom{n}{2}$  vertices of a complete polygon with  $n$  sides tangent to  $C$  then  $D$  passes through the vertices of infinitely many such polygons.*

According to Berger [Be] this theorem is the nicest result about the geometry of conics. Even if it is, there are few proofs of it. To my knowledge there are only three. The first proof, published in 1822 and based on infinitesimal deformations, is due to Poncelet ([Po]). Later, Jacobi proposed a new proof based on finite order points on elliptic curves; his proof, certainly the most famous, is explained in a modern way and in detail by Griffiths and Harris ([GH]). In 1870 Weyr proved a Poncelet theorem in space (more precisely for two quadrics) that implies the one above when one quadric is a cone; this proof is explained by Barth and Bauer ([BB]).

Our aim in this short note is to involve vector bundles techniques to propose a new proof of this celebrated result. Poncelet did not appreciate Jacobi's for the reason that it was too far from the geometric intuition. I guess that he would not appreciate our proof either for the same reason.

## 1 Preliminaries

In all this text the ground field is  $\mathbb{C}$ . A set of  $n$  vertices connected by  $n$  distinct lines form a  $n$ -gon, when a set consisting of  $n$  distinct lines with their  $\binom{n}{2}$  vertices form a complete  $n$ -gon.

We say that a  $n$ -gon (respectively a complete  $n$ -gon) is inscribed in a given curve if this curve passes through the  $n$ -vertices (respectively the  $\binom{n}{2}$  vertices). We say that a  $n$ -gon, or a complete  $n$ -gon, is circumscribed around a smooth conic  $C$  if the sides of the polygon, i.e. the  $n$  lines, are tangent to the  $C$ .

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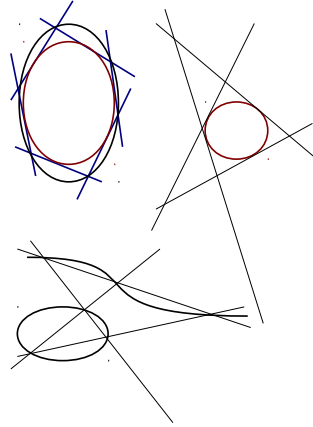


Figure 1: A complete 5-gon inscribed in a cubic, a complete 4-gon circumscribed around a conic and a pentagon simultaneously inscribed in a conic and circumscribed around another

## 2 Schwarzenberger bundles

First of all let us introduce a vector bundle  $E_{n,C}$  naturally associated to any set of  $n$  lines tangent to a fixed smooth conic  $C \subset \mathbb{P}^2$  (they were defined by Schwarzenberger in [S]). A set of  $n$  lines tangent to  $C$  corresponds by projective duality to a degree  $n$  divisor on the dual conic  $C^\vee \subset \mathbb{P}^{2\vee}$ . According to the isomorphism  $C^\vee \simeq \mathbb{P}^1$  we can consider the subvariety  $X \subset \mathbb{P}^2 \times \mathbb{P}^1$  defined by the equation  $x_0u^2 + x_1uv + x_2v^2 = 0$  ( $(x_0, x_1, x_2)$  are the homogeneous coordinates on  $\mathbb{P}^2$  and  $(u, v)$  the homogeneous coordinates on  $\mathbb{P}^1$ ) and the projections  $p$  and  $q$  respectively on  $\mathbb{P}^2$  and  $\mathbb{P}^1$ .

$$\begin{array}{ccc} X & \xrightarrow{q} & \mathbb{P}^1 \\ p \downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

The variety  $X$  is a double cover of  $\mathbb{P}^2$  ramified along  $C$ . If  $x \in \mathbb{P}^1$  then  $p(q^{-1}(x))$  is a line in  $\mathbb{P}^2$  tangent to  $C$ . If, instead of considering a point, we are considering a line bundle on  $\mathbb{P}^1$  we will find a vector bundle of rank two on  $\mathbb{P}^2$  by taking the direct image of its inverse image. Moreover, following Schwarzenberger, we know a very explicit resolution of this bundle. Indeed, tensorizing the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(-1, -2) \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$



is surjective, there exists a non-zero section  $t \in H^0(E_{n,C})$  (i.e. another  $n$ -gon) such that the determinant

$$\mathcal{O}_{\mathbb{P}^2}^2 \xrightarrow{(s,t)} E_{n,C}$$

is the equation of  $S$ . This proves the theorem since any linear combination of  $s$  and  $t$  vanish along the vertices of a complete  $n$ -gon.  $\square$

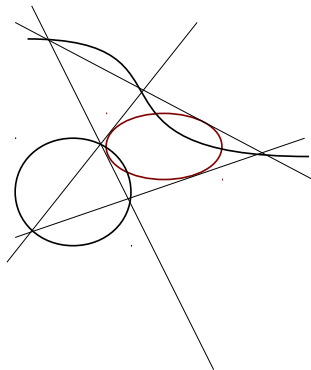


Figure 2: A complete circumscribed 4 gon and a cubic Poncelet curve

These curves described by Darboux are called *Poncelet curves*. When  $n = 5$  they are the so-called Luröth quartics (see [OS]).

## 4 Poncelet theorem

Let us now consider  $n$ -gons that are simultaneously inscribed in a smooth conic and circumscribed around a other one. For these configurations Poncelet proved ([Po], page 362) :

**Theorem 4.1.** *Let  $C \subset \mathbb{P}^2$  and  $D \subset \mathbb{P}^2$  be two smooth conics such that there exist one  $n$ -gon inscribed in  $D$  and circumscribed around  $C$ . Then there are infinitely many of such  $n$ -gons.*

*Proof.* Let us consider one such  $n$ -gon. It is the union of  $n$  lines  $l_1, \dots, l_n$  with the  $\binom{n}{2}$  vertices  $l_i \cap l_j$  for  $1 \leq i, j \leq n$  and  $i \neq j$ . There is a non-zero section  $s \in H^0(E_{n,C})$  vanishing along the vertices of these lines. We denote by  $Z(s)$  the set of these vertices.

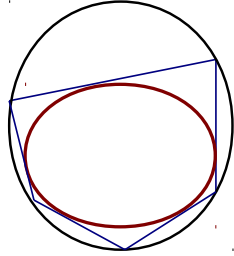


Figure 3: pentagon inscribed and circumscribed

Let us tensorize the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow E_{n,C} \longrightarrow \mathcal{I}_{Z(s)}(n-1) \longrightarrow 0$$

by  $\mathcal{O}_D$ . Since  $D \cap Z(s)$  consists in  $n$ -points, it induces the following decomposition of  $E_{n,C}$  along  $D$  :

$$E_{n,C} \otimes \mathcal{O}_D = \mathcal{O}_D\left(\frac{n-2}{2}\right) \oplus \mathcal{O}_D\left(\frac{n}{2}\right).$$

According to this decomposition, we consider the following exact sequence :

$$0 \longrightarrow F \longrightarrow E_{n,C} \longrightarrow \mathcal{O}_D\left(\frac{n-2}{2}\right) \longrightarrow 0$$

where  $F$  is a rank two vector bundle over  $\mathbb{P}^2$ . Taking the cohomology long exact sequence we verify immediately that  $h^0(F) \geq 2$ . Then, let us consider a pencil of sections of  $F$  and also the pencil of sections of  $E_{n,C}$  induced by it. We obtain a commutative diagram:

$$\begin{array}{ccccccc} \mathcal{O}_{\mathbb{P}^2}^2 & \xlongequal{\quad} & \mathcal{O}_{\mathbb{P}^2}^2 & & & & \\ \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & F & \longrightarrow & E_{n,C} & \longrightarrow & \mathcal{O}_D\left(\frac{n-2}{2}\right) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & \mathcal{L}_2 & \longrightarrow & \mathcal{O}_D\left(\frac{n-2}{2}\right) \longrightarrow 0. \end{array}$$

The sheaf  $\mathcal{L}_2$  is supported by a curve  $\Gamma_2$  of degree  $(n-1)$  that is the determinant of a pencil of sections of  $E_{n,C}$ . This curve  $\Gamma_2$  is a Poncelet curve. Then a general

point on  $\Gamma_2$  is a vertex of a complete  $n$ -gon inscribed in  $\Gamma_2$  and circumscribed around  $C$ . Moreover any intersection point of the  $n$  lines forming the  $n$ -gon with  $\Gamma_2$  is a vertex of this  $n$ -gon (it is clear by Bézout theorem since  $n(n-1) = 2 \times \binom{n}{2}$ ). Let  $\Gamma_1$  be the curve supporting the sheaf  $\mathcal{L}_1$ . We have of course  $\Gamma_2 = D \cup \Gamma_1$ . Then  $D$  is an irreducible component of a Poncelet curve and by the way any general point on  $D$  is the vertex of complete  $n$ -gon inscribed in  $\Gamma_2$ . Then this configuration meets the conic  $D$  in at least (because there are  $n$  lines) and at most (because they are vertices and the decomposition of the bundle along  $D$  is fixed)  $n$ -points, so exactly  $n$ -points, each counting doubly.  $\square$

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