

Strong solutions for a 1D viscous bilayer Shallow Water model

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Abstract

In this paper, we consider a viscous bilayer shallow water model in one space dimension that represents two superposed immiscible fluids. For this model, we prove the existence of strong solutions in a periodic domain. The initial heights are required to be bounded above and below away from zero and we get the same bounds for every time. Our analysis is based on the construction of approximate system which satisfy the BD entropy and on the method developed by A. MELLET and A. VASSEUR to obtain the existence of global strong solutions for the one dimensional Navier-Stokes equations.

Keywords. Strong solutions, shallow water, viscous models, bilayer, stability.

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1 Introduction.

In this paper, we study the global existence in time of strong solutions to the following viscous bilayer shallow water model:

$$\partial_t h_1 + \partial_x(h_1 v_1) = 0, \quad (1)$$

$$\partial_t(h_1 v_1) + \partial_x(h_1 v_1^2) + gh_1 \partial_x h_1 + rgh_1 \partial_x h_2 - \nu_1 \partial_x(h_1 \partial_x v_1) = 0, \quad (2)$$

$$\partial_t h_2 + \partial_x(h_2 v_2) = 0, \quad (3)$$

$$\partial_t(h_2 v_2) + \partial_x(h_2 v_2^2) + gh_2 \partial_x h_2 + gh_2 \partial_x h_1 - \nu_2 \partial_x(h_2 \partial_x v_2) = 0. \quad (4)$$

where $(t, x) \in (0, T) \times \Omega$, and Ω is a periodic domain in one dimension. These equations represent a system composed of two layers of immiscible fluids. Index 1 refers to the deeper layer and index 2 to the upper layer, see Figure 1.

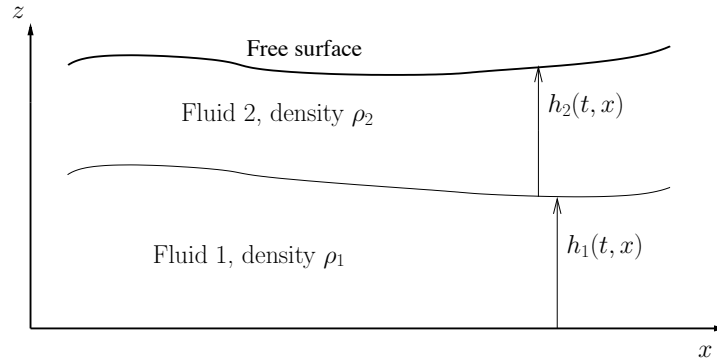


Figure 1: Notations for the bilayer model

We denote by ρ_1 and ρ_2 the densities of each layer of fluid, and r is their ratio $r = \rho_2/\rho_1 < 1$. The quantities ν_1 and ν_2 are the respective kinematic viscosity, that is $\nu_i = \mu_i/\rho_i$ where μ_i is the dynamic viscosity. This model is formally derived in [21] in the two dimensional case. Such a model appear naturally in geophysical flows, see for example [1, 6].

Let us give a non-exhaustive list of the results on the existence of solutions to such kinds of equations. The existence of solutions to the one dimensional Navier-Stokes equations (which includes the shallow water equations) has been studied by many authors. In the case of a constant viscosity, the existence of weak solution was first studied in [13] with smooth initial data close

to equilibrium. The case of discontinuous data was solved in [9, 10, 11, 22]. In particular, in [11], D. HOFF proved the existence of weak solutions with large discontinuous initial data that can have different limits at $x = \pm\infty$. But it requires the initial data to be bounded away from zero. The first global existence result for initial density which can vanish was shown in [16]. This result was extended later in [7] to the full compressible Navier-Stokes equations.

More recently, in [2, 3, 4], D. BRESCH and B. DESJARDINS obtained the existence of a global weak solution for a 2D shallow water and a Korteweg systems with a diffusion term of the type $\operatorname{div}(hD(u))$. The key point that gave them this result is an entropy inequality namely BD entropy. This BD entropy has been extended in [5] to more general Navier-Stokes equations with an algebraic relation between the shear and the bulk viscosity coefficients. But the authors used quadratic friction terms and capillary effects to get the stability of the system. In [18], using this inequality again, A. MELLET and A. VASSEUR obtained the stability of global weak solutions for the compressible isentropic Navier-Stokes equations without any regularizing term. Another consequence of this BD entropy is that it gives control on some negative power of the density in one dimension. This nice control was shown in [17] and led the authors to prove that vacuum cannot arise if there is no vacuum at the initial time. With this result, they obtained the existence of strong solutions for the one dimensional viscous compressible Navier-Stokes equations. However, the admissible viscosity must be some power of the density, which does not take the shallow water system into account.

Let us go back to the shallow water equations, and more precisely to bilayer systems. An existence result on a one dimensional bilayer shallow water model was studied in [19]. The authors obtained the existence, uniqueness and some smoothness of weak solutions under the assumption that the data are sufficiently small. But the bilayer shallow water they considered is not exactly the one we study here. In this paper, we are interested in another system of bilayer immiscible fluid obtained by derivation in [21]. These equations have been studied in [20, 23]; the authors proved the existence of a global weak solution for viscous bilayer Shallow Water models with the BD inequality (but in two dimensions).

This paper deals with strong solutions, in one dimension. The main existence result is stated in Section 2. To prove this statement, we construct approximate solutions in Section 3 following the work performed in [15, 16], based on [12]. Indeed in [15], the authors obtained the existence of strong so-

lutions for the one dimensional compressible Navier-Stokes equations, which include the case of shallow water. They proved that there exists some time after which the density is bounded by below by a positive constant and the weak solution becomes a strong one. Here, we prove that our approximate system verifies the BD entropy, which gives the lower bound for the water heights. Next, in Section 4, we obtain the existence of strong solutions for the approximate system by using the regularity theorem for smooth data given in [14], and we pass to the limit. The last part (A) is dedicated to the proofs of the classical energy and BD entropy inequalities.

2 Main result

Theorem 2.1. *Let the initial data $h_{i0} = h_i|_{t=0}$ and $v_{i0} = v_i|_{t=0}$ satisfy the conditions*

$$\begin{aligned} 0 < \underline{c}_0 \leq h_{i0} \leq \bar{c}_0, \quad h_{i0}v_{i0} = q_{i0} \\ h_{i0} \in H^1(\Omega), \quad v_{i0} \in H^1(\Omega), \end{aligned} \quad (5)$$

where \underline{c}_0 and \bar{c}_0 are some positive constants. Assume that the viscosities ν_1 and ν_2 verify the relation

$$\nu_1 > \frac{r(\nu_1 + \nu_2)}{2}(2 - \alpha), \quad \nu_2 > \frac{\nu_1 + \nu_2}{2}(2 - \alpha)$$

with $1 > \alpha > 2r/(1 + r)$. Then the problem (1)–(4) admits a strong solution (h_1, h_2, v_1, v_2) such that $h_i \in L^\infty(0, T; H^1(\Omega))$, $v_i \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ and $\partial_t v_i \in L^2(0, T; L^2(\Omega))$.

To prove this theorem, we construct approximate smooth solutions in the next section, following the work performed in [15].

3 Approximate smooth solutions

In this section, we consider the approximate system defined as follows:

$$\partial_t h_{1,\varepsilon} + \partial_x(h_{1,\varepsilon}v_{1,\varepsilon}) = 0, \quad (6)$$

$$\begin{aligned} \partial_t(h_{1,\varepsilon}v_{1,\varepsilon}) + \partial_x(h_{1,\varepsilon}v_{1,\varepsilon}^2) + gh_{1,\varepsilon}\partial_x h_{1,\varepsilon} \\ + rgh_{1,\varepsilon}\partial_x h_{2,\varepsilon} - \nu_1\partial_x(\psi_\varepsilon(h_{1,\varepsilon})\partial_x v_{1,\varepsilon}) = 0, \end{aligned} \quad (7)$$

$$\partial_t h_{2,\varepsilon} + \partial_x(h_{2,\varepsilon}v_{2,\varepsilon}) = 0, \quad (8)$$

$$\begin{aligned} \partial_t(h_{2,\varepsilon}v_{2,\varepsilon}) + \partial_x(h_{2,\varepsilon}v_{2,\varepsilon}^2) + gh_{2,\varepsilon}\partial_x h_{2,\varepsilon} \\ + gh_{2,\varepsilon}\partial_x h_{1,\varepsilon} - \nu_2\partial_x(\psi_\varepsilon(h_{2,\varepsilon})\partial_x v_{2,\varepsilon}) = 0. \end{aligned} \quad (9)$$

where $\psi_\varepsilon(h_{i,\varepsilon}) = h_{i,\varepsilon} + \varepsilon h_{i,\varepsilon}^\beta$ (for $i = 1, 2$), ε is a small parameter satisfying

$$\varepsilon < \frac{1-r}{1+r}, \quad (10)$$

β belongs to $(0, 1/2)$. We also assume the following relation on the viscosities:

$$\nu_1 > \frac{r(\nu_1 + \nu_2)}{2}(1 + \varepsilon), \quad \nu_2 > \frac{\nu_1 + \nu_2}{2}(1 + \varepsilon) \quad (11)$$

This type of approximate systems can be constructed by a mollifying process as in [8, 11]. We suppose that the initial data $h_{i,\varepsilon}|_{t=0} = h_{i_0,\varepsilon}$, $h_{i,\varepsilon}v_{i,\varepsilon}|_{t=0} = q_{i_0,\varepsilon} \in \mathcal{C}(\overline{\Omega})$ satisfy (5) and that:

$$h_{i_0,\varepsilon} \rightarrow h_{i_0} \text{ in } L^1(\Omega), h_{i_0,\varepsilon}^{1/2} \rightarrow h_{i_0}^{1/2} \text{ in } H^1(\Omega), \quad \frac{q_{i_0,\varepsilon}^2}{h_{i_0,\varepsilon}} \rightarrow \frac{q_{i_0}^2}{h_{i_0}} \text{ in } L^1(\Omega) \quad (12)$$

Remark 3.1. Let us define $\varphi_\varepsilon(h_{i,\varepsilon}) = h_{i,\varepsilon} + \frac{\varepsilon}{\beta} h_{i,\varepsilon}^\beta$. Then we have

$$h_{i,\varepsilon} \varphi'_\varepsilon(h_{i,\varepsilon}) = \psi_\varepsilon(h_{i,\varepsilon}). \quad (13)$$

This relation will be used to get the BD entropy, see A.

We assume that the initial energy associated to the approximate system (6)–(9) is taken such that

$$\begin{aligned} \mathcal{E}_0 &= \frac{1}{2} \int_{\Omega} h_{1_0,\varepsilon} |v_{1_0,\varepsilon}|^2 + \frac{r}{2} \int_{\Omega} h_{2_0,\varepsilon} |v_{2_0,\varepsilon}|^2 + \frac{g(1-r)}{2} \int_{\Omega} |h_{1_0,\varepsilon}|^2 \\ &\quad + \frac{rg}{2} \int_{\Omega} |h_{1_0,\varepsilon} + h_{2_0,\varepsilon}|^2 \leq C\varepsilon^2 \leq C \end{aligned}$$

and

$$\mathcal{F}_0 = \frac{1}{2} \int_{\Omega} \left| \frac{\partial_x \varphi_\varepsilon(h_{1_0,\varepsilon})}{\sqrt{h_{1_0,\varepsilon}}} \right|^2 + \frac{1}{2} \int_{\Omega} \left| \frac{\partial_x \varphi_\varepsilon(h_{2_0,\varepsilon})}{\sqrt{h_{2_0,\varepsilon}}} \right|^2 \leq C\varepsilon^2 \leq C$$

for $\varepsilon > 0$ small enough.

3.1 Energy inequalities

In this section, we give the classical energy inequality and the BD entropy for the solution of System (6)–(9).

Proposition 3.2. *If $(h_{1,\varepsilon}, h_{2,\varepsilon}, v_{1,\varepsilon}, v_{2,\varepsilon})$ is a smooth solution of System (6)–(9), then the following classical inequality holds:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} h_{1,\varepsilon} |v_{1,\varepsilon}|^2 + \frac{r}{2} \frac{d}{dt} \int_{\Omega} h_{2,\varepsilon} |v_{2,\varepsilon}|^2 \\ & + \nu_1 \int_{\Omega} \psi_{\varepsilon}(h_{1,\varepsilon}) (\partial_x v_{1,\varepsilon})^2 dx + r \nu_2 \int_{\Omega} \psi_{\varepsilon}(h_{2,\varepsilon}) (\partial_x v_{2,\varepsilon})^2 dx \\ & + \frac{g(1-r)}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon}|^2 + \frac{rg}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon} + h_{2,\varepsilon}|^2 \leq 0. \end{aligned} \quad (14)$$

Corollary 3.3. *The classical energy estimate gives the following uniform bounds:*

$$\begin{aligned} & \left\| \sqrt{h_{i,\varepsilon}} v_{i,\varepsilon} \right\|_{L^{\infty}(0,T;L^2(\Omega))} \leq C(T); & \|h_{i,\varepsilon}\|_{L^{\infty}(0,T;L^2(\Omega))} \leq C(T); \\ & \left\| \sqrt{\psi_{\varepsilon}(h_{i,\varepsilon})} \partial_x v_{i,\varepsilon} \right\|_{L^2(0,T;(L^2(\Omega))^2)} \leq C(T). \end{aligned} \quad (15)$$

As these bounds are not informative enough to get the existence of solutions, we write the BD entropy to have more information on the solution.

Proposition 3.4. *Let $(h_{1,\varepsilon}, h_{2,\varepsilon}, v_{1,\varepsilon}, v_{2,\varepsilon})$ be a smooth solution of (6)–(9), then the following mathematical BD entropy inequality holds:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} h_{1,\varepsilon} \left| v_{1,\varepsilon} + \nu_1 \frac{\partial_x \varphi_{\varepsilon}(h_{1,\varepsilon})}{h_{1,\varepsilon}} \right|^2 + \frac{r}{2} \frac{d}{dt} \int_{\Omega} h_{2,\varepsilon} \left| v_{2,\varepsilon} + \nu_2 \frac{\partial_x \varphi_{\varepsilon}(h_{2,\varepsilon})}{h_{2,\varepsilon}} \right|^2 \\ & + \frac{g(1-r)}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon}|^2 + \frac{rg}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon} + h_{2,\varepsilon}|^2 \\ & + \nu_1 g \int_{\Omega} \varphi'_{\varepsilon}(h_{1,\varepsilon}) (\partial_x h_{1,\varepsilon})^2 + \nu_2 r g \int_{\Omega} \varphi'_{\varepsilon}(h_{2,\varepsilon}) (\partial_x h_{2,\varepsilon})^2 \\ & + \nu_1 r g \int_{\Omega} \partial_x h_{2,\varepsilon} \partial_x \varphi_{\varepsilon}(h_{1,\varepsilon}) + \nu_2 r g \int_{\Omega} \partial_x h_{1,\varepsilon} \partial_x \varphi_{\varepsilon}(h_{2,\varepsilon}) \leq 0. \end{aligned} \quad (16)$$

Corollary 3.5. *Equation (16) can be integrated over $[0, t]$, for $t \in [0, T]$.*

Then we get:

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} h_{1,\varepsilon} \left| v_{1,\varepsilon} + \nu_1 \frac{\partial_x \varphi_\varepsilon(h_{1,\varepsilon})}{h_{1,\varepsilon}} \right|^2 + \frac{r}{2} \int_{\Omega} h_{2,\varepsilon} \left| v_{2,\varepsilon} + \nu_2 \frac{\partial_x \varphi_\varepsilon(h_{2,\varepsilon})}{h_{2,\varepsilon}} \right|^2 \\
& + \frac{g(1-r)}{2} \int_{\Omega} |h_{1,\varepsilon}|^2 + \frac{rg}{2} \int_{\Omega} |h_{1,\varepsilon} + h_{2,\varepsilon}|^2 \\
& + \nu_1 g \int_0^t \int_{\Omega} \varphi'_\varepsilon(h_{1,\varepsilon}) (\partial_x h_{1,\varepsilon})^2 + \nu_2 r g \int_0^t \int_{\Omega} \varphi'_\varepsilon(h_{2,\varepsilon}) (\partial_x h_{2,\varepsilon})^2 \quad (17) \\
& + \nu_1 r g \int_0^t \int_{\Omega} \partial_x h_{2,\varepsilon} \partial_x \varphi_\varepsilon(h_{1,\varepsilon}) + \nu_2 r g \int_0^t \int_{\Omega} \partial_x h_{1,\varepsilon} \partial_x \varphi_\varepsilon(h_{2,\varepsilon}) \\
& \leq \int_{\Omega} h_{10,\varepsilon} \left| v_{10,\varepsilon} + \nu_1 \frac{\partial_x \varphi_\varepsilon(h_{10,\varepsilon})}{h_{10,\varepsilon}} \right|^2 + \frac{r}{2} \int_{\Omega} h_{20,\varepsilon} \left| v_{20,\varepsilon} + \nu_2 \frac{\partial_x \varphi_\varepsilon(h_{20,\varepsilon})}{h_{20,\varepsilon}} \right|^2 \\
& + \frac{g(1-r)}{2} \int_{\Omega} |h_{10,\varepsilon}|^2 + \frac{rg}{2} \int_{\Omega} |h_{10,\varepsilon} + h_{20,\varepsilon}|^2 \\
& \leq c(\mathcal{E}_0 + \mathcal{F}_0) \leq C\varepsilon^2.
\end{aligned}$$

Propositions 3.2 and 3.4 are proved in A.

To obtain new bounds thanks to the BD entropy (in order to complete Corollary 3.3), it remains to control the last two terms of the left hand side of Equation (16). Since $\varphi'_\varepsilon(h_{i,\varepsilon}) > 1$ (for $i = 1, 2$), one can write:

$$\begin{aligned}
Q & = \nu_1 r g \int_{\Omega} |\partial_x h_{2,\varepsilon} \partial_x \varphi_\varepsilon(h_{1,\varepsilon})| + \nu_2 r g \int_{\Omega} |\partial_x h_{1,\varepsilon} \partial_x \varphi_\varepsilon(h_{2,\varepsilon})| \\
& \leq \nu_1 r g \int_{\Omega} |\partial_x \varphi_\varepsilon(h_{2,\varepsilon}) \partial_x \varphi_\varepsilon(h_{1,\varepsilon})| + \nu_2 r g \int_{\Omega} |\partial_x \varphi_\varepsilon(h_{1,\varepsilon}) \partial_x \varphi_\varepsilon(h_{2,\varepsilon})|,
\end{aligned}$$

and consequently

$$Q \leq \frac{\nu_1 r g + \nu_2 r g}{2} \int_{\Omega} |\partial_x \varphi_\varepsilon(h_{1,\varepsilon})|^2 + \frac{\nu_1 r g + \nu_2 r g}{2} \int_{\Omega} |\partial_x \varphi_\varepsilon(h_{2,\varepsilon})|^2.$$

Next, we split the two integrals by using the sets $\{h_{i,\varepsilon} \leq 1\}$ and $\{h_{i,\varepsilon} > 1\}$, with $i = 1$ for the first integral and $i = 2$ for the second one. Since $\beta < \frac{1}{2}$,

$\varphi'_\varepsilon(h_{i,\varepsilon}) \leq 1 + \varepsilon$ in the set $\{h_{i,\varepsilon} > 1\}$. Then, we have

$$\begin{aligned} Q \leq & \frac{\nu_1 r g + \nu_2 r g}{2} \int_{\{h_{1,\varepsilon} \leq 1\}} \left| \frac{\partial_x \varphi_\varepsilon(h_{1,\varepsilon})}{\sqrt{h_{1,\varepsilon}}} \right|^2 \\ & + \frac{\nu_1 r g + \nu_2 r g}{2} \int_{\{h_{1,\varepsilon} > 1\}} (1 + \varepsilon) \varphi'_\varepsilon(h_{1,\varepsilon}) (\partial_x(h_{1,\varepsilon}))^2 \\ & + \frac{\nu_1 r g + \nu_2 r g}{2} \int_{\{h_{2,\varepsilon} \leq 1\}} \left| \frac{\partial_x \varphi_\varepsilon(h_{2,\varepsilon})}{\sqrt{h_{2,\varepsilon}}} \right|^2 \\ & + \frac{\nu_1 r g + \nu_2 r g}{2} \int_{\{h_{2,\varepsilon} > 1\}} (1 + \varepsilon) \varphi'_\varepsilon(h_{2,\varepsilon}) (\partial_x(h_{2,\varepsilon}))^2. \end{aligned}$$

The two quantities $\frac{\nu_1 r g + \nu_2 r g}{2} \int_{\{h_{1,\varepsilon} > 1\}} (1 + \varepsilon) \varphi'_\varepsilon(h_{1,\varepsilon}) (\partial_x(h_{1,\varepsilon}))^2$ and $\frac{\nu_1 r g + \nu_2 r g}{2} \int_{\{h_{2,\varepsilon} > 1\}} (1 + \varepsilon) \varphi'_\varepsilon(h_{2,\varepsilon}) (\partial_x(h_{2,\varepsilon}))^2$ can be absorbed in the left hand side. Due to conditions (11) on the viscosities and (10) on ε , the quantities $(\nu_1 - \frac{r\nu_1 + r\nu_2}{2})g \int_\Omega (1 + \varepsilon) \varphi'_\varepsilon(h_{1,\varepsilon}) (\partial_x(h_{1,\varepsilon}))^2$ and $(r\nu_2 - \frac{r\nu_1 + r\nu_2}{2})g \int_\Omega (1 + \varepsilon) \varphi'_\varepsilon(h_{2,\varepsilon}) (\partial_x(h_{2,\varepsilon}))^2$ are indeed positive.

The last step is to develop the first two terms on the left hand side of Equation (17) and to use the fact that $2ab \leq \theta a^2 + \frac{1}{\theta} b^2$ for all $\theta > 0$. So, thanks to

Gronwall's lemma, the two quantities $\frac{r\nu_1 g + r\nu_2 g}{2} \int_{\{h_{i,\varepsilon} \leq 1\}} \left| \frac{\partial_x \varphi_\varepsilon(h_{i,\varepsilon})}{\sqrt{h_{i,\varepsilon}}} \right|^2$ can

be controlled, for $i = 1, 2$.

More precisely, we proved the following result:

Corollary 3.6. *The BD mathematical entropy inequality (Equation (17)) implies the following bounds:*

$$\begin{aligned} \left\| \frac{\partial_x \varphi_\varepsilon(h_{i,\varepsilon})}{\sqrt{h_{i,\varepsilon}}} \right\|_{L^\infty(0,T;L^2(\Omega))} &= 2 \left\| \psi_\varepsilon(h_{i,\varepsilon}) \partial_x(h_{i,\varepsilon}^{-1/2}) \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T)\varepsilon, \\ \left\| \varphi'_\varepsilon(h_{i,\varepsilon}) \partial_x h_{i,\varepsilon} \right\|_{L^2(0,T;L^2(\Omega))} &\leq C(T). \end{aligned} \quad (18)$$

3.2 Uniform bounds for the heights and velocities

Proposition 3.7. *Let $(h_{1,\varepsilon}, h_{2,\varepsilon}, v_{1,\varepsilon}, v_{2,\varepsilon})$ be a smooth solution of (6)–(9); then for all $T > 0$, there exist some constants $\bar{c}(T) > 0$ and $\underline{c}(T) > 0$ such that*

$$\underline{c}(T) \leq h_{i,\varepsilon}(x, t) \leq \bar{c}(T), \quad \forall x \in \Omega, t \in [0, T], \quad i = 1, 2. \quad (19)$$

The proof of the lower bound relies on two main results. The first one is the following remark:

Remark 3.8. *For $\tilde{h}_{i,\varepsilon}$ between fixed reference heights ($0 < h_i^- < \tilde{h}_{i,\varepsilon} < h_i^+$) for $i = 1, 2$, there exists a constant K such that*

$$h_{i,\varepsilon} + h_{i,\varepsilon}^2 \leq K(1 + (h_{i,\varepsilon} - \tilde{h}_{i,\varepsilon})^2), \quad (20)$$

$$\liminf_{h_{i,\varepsilon} \rightarrow 0} (h_{i,\varepsilon} - \tilde{h}_{i,\varepsilon})^2 \geq K^{-1}, \quad i = 1, 2. \quad (21)$$

The proof is also related to the following lemma derived in [11] and [18].

Lemma 3.9. *For every $T > 0$, there exist $\delta > 0$ and $R(T)$ such that for every $x_0 \in \mathbb{R}$ and $t_0 > 0$, there exists $x_1 \in [x_0 - R(T), x_0 + R(T)]$ with*

$$h_{i,\varepsilon}(x_1, t_0) > \delta.$$

Let us now detail the proof of Proposition 3.7.

Proof. Let us define $\ell_{i,\varepsilon} = \inf(h_{i,\varepsilon}, 1)$. We have $\partial_x \ell_{i,\varepsilon} = \partial_x h_{i,\varepsilon} \mathbf{1}_{\{h_{i,\varepsilon} \leq 1\}}$. Due to inequality (18), the following bound holds:

$$\left\| \psi_\varepsilon(\ell_{i,\varepsilon}) \partial_x (\ell_{i,\varepsilon}^{-1/2}) \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T)\varepsilon.$$

Since $\psi_\varepsilon(\ell_{i,\varepsilon}) = h_{i,\varepsilon} + \frac{\varepsilon}{\beta} h_{i,\varepsilon}^\beta \geq \frac{\varepsilon}{\beta} h_{i,\varepsilon}^\beta = \frac{\varepsilon}{\beta} \ell_{i,\varepsilon}^\beta$ in the set where $\{h_{i,\varepsilon} \leq 1\}$, we deduce that

$$\frac{\varepsilon}{\beta} \left\| \partial_x \ell_{i,\varepsilon}^{\beta-1/2} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq \left\| \psi_\varepsilon(\ell_{i,\varepsilon}) \partial_x (\ell_{i,\varepsilon}^{-1/2}) \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T)\varepsilon. \quad (22)$$

Hence by using Sobolev-Poincaré inequality, Lemma 3.9 and Equation (22) imply that $\ell_{i,\varepsilon}^{\beta-1/2}$ is bounded in $L^\infty(0, T; L^\infty(\Omega))$. More precisely, we obtain a lower bound for $\ell_{i,\varepsilon}$ (and for $h_{i,\varepsilon}$): $\ell_{i,\varepsilon} \geq (\beta C(T))^{-\frac{2}{1-2\beta}}$.

Next, we want to get a bound for $\sqrt{h_{i,\varepsilon}}$ in $L^\infty(0, T; W^{1,1}(\Omega))$. First, we remark that:

$$\begin{aligned}
\int_{\Omega} \left| \partial_x h_{i,\varepsilon}^{1/2} \right| dx &= \frac{1}{2} \int_{\Omega} \left| h_{i,\varepsilon}^{-1/2} \partial_x h_{i,\varepsilon} \right| dx \\
&\leq \left(\int_{\Omega} |h_{i,\varepsilon}| \right)^{1/2} \left(\int_{\Omega} \left| \frac{\partial_x h_{i,\varepsilon}}{h_{i,\varepsilon}} \right|^2 \right)^{1/2} \\
&\leq \frac{1}{2} \left(\int_{\Omega} |\sqrt{h_{i,\varepsilon}}|^2 dx \right)^{1/2} \left(\int_{\Omega} h_{i,\varepsilon} \left| \frac{\partial_x h_{i,\varepsilon}}{\sqrt{h_{i,\varepsilon}}} \right|^2 \frac{|\varphi'_\varepsilon(h_{i,\varepsilon})|^2}{|\psi_\varepsilon(h_{i,\varepsilon})|^2} dx \right)^{1/2} \\
&\leq \frac{1}{2} \frac{1}{\sqrt{\underline{c}(T)}} \left(\int_{\Omega} |\sqrt{h_{i,\varepsilon}}|^2 dx \right)^{1/2} \left(\int_{\Omega} \left| \frac{\partial_x \varphi_\varepsilon(h_{i,\varepsilon})}{\sqrt{h_{i,\varepsilon}}} \right|^2 \right)^{1/2},
\end{aligned}$$

as $\psi_{i,\varepsilon} \geq h_{i,\varepsilon}$ and $h_{i,\varepsilon}$ has a lower bound denoted $\underline{c}(T)$. Thanks to Equation (18), we obtain:

$$\int_{\Omega} \left| \partial_x h_{i,\varepsilon}^{1/2} \right| dx \leq \bar{c}(T).$$

Finally, we have $\partial_x \sqrt{h_{i,\varepsilon}} \in L^\infty(0, T; L^1(\Omega))$ and $\sqrt{h_{i,\varepsilon}} \in L^\infty(0, T; L^2(\Omega))$; this means that $\sqrt{h_{i,\varepsilon}} \in L^\infty(0, T; W^{1,1}(\Omega))$. To conclude, we use Sobolev embeddings to get the upper bound for $h_{i,\varepsilon}$. \square

Proposition 3.10. *For all $T > 0$, there exist some constants $\bar{k}_1(T), \bar{k}_2(T) > 0$ such that*

$$\|v_{i,\varepsilon}\|_{L^2((0,T),H^2(\Omega))} \leq \bar{k}_1(T), \quad \|\partial_t v_{i,\varepsilon}\|_{L^2((0,T),L^2(\Omega))} \leq \bar{k}_2(T). \quad (23)$$

Proof. Relations (15) and (19) imply that

$$\|v_{i,\varepsilon}\|_{L^\infty((0,T),L^2(\Omega))} \leq C_1(T) \text{ and } \|\partial_x v_{i,\varepsilon}\|_{L^2((0,T),L^2(\Omega))} \leq C_2(T)$$

for some positive constants $C_1(T), C_2(T)$. Moreover, since $v_{i,\varepsilon}$ belongs to $L^\infty((0, T), H^1(\Omega))$, thanks to Sobolev embeddings, we get:

$$\partial_t h_{i,\varepsilon} = -\partial_x(h_{i,\varepsilon} v_{i,\varepsilon}) \in L^2((0, T), L^2(\Omega)).$$

Now let us write the two momentum equations in the following form:

$$\begin{aligned}
&\partial_t v_{i,\varepsilon} - \partial_x \left(\frac{\psi_\varepsilon(h_{i,\varepsilon})}{h_{i,\varepsilon}} \partial_x v_{i,\varepsilon} \right) \\
&= -g \partial_x (h_{i,\varepsilon} + r_i h_{j,\varepsilon}) - v_{i,\varepsilon} \partial_x v_{i,\varepsilon} + \nu_i \frac{\partial_x h_{i,\varepsilon}}{h_{i,\varepsilon}^2} \psi_\varepsilon(h_{i,\varepsilon}) \partial_x v_{i,\varepsilon} \quad (24)
\end{aligned}$$

where $i, j = 1, 2, i \neq j, r_1 = r$ and $r_2 = 1$.

It remains to control the right hand side of this equation to deduce a better bound for the velocities $v_{i,\varepsilon}$.

The first term $g\partial_x(h_{i,\varepsilon} + r_i h_{j,\varepsilon})$ is bounded in $L^2((0, T), L^2(\Omega))$. Next, we have

$$\int_{\Omega} (\partial_x h_{i,\varepsilon})^2 = \int_{\Omega} \frac{(\varphi'_{\varepsilon}(h_{i,\varepsilon}))^2 (\partial_x h_{i,\varepsilon})^2}{(\varphi'_{\varepsilon}(h_{i,\varepsilon}))^2} \leq \int_{\Omega} \frac{h_{i,\varepsilon} (\partial_x (\varphi_{\varepsilon}(h_{i,\varepsilon})))^2}{(\varepsilon h_{i,\varepsilon}^{\beta-1/2})^2}.$$

Due to the BD entropy (18) and relation (19), the last term is bounded. So, $\partial_x h_{i,\varepsilon}$ is bounded in $L^{\infty}((0, T), L^2(\Omega))$.

Finally, by using Holder and interpolation inequalities, we get:

$$\begin{aligned} & \left\| \left(v_{i,\varepsilon} - \nu_i \frac{\partial_x h_{i,\varepsilon}}{h_{i,\varepsilon}^2} \psi_{\varepsilon}(h_{i,\varepsilon}) \right) \partial_x v_{i,\varepsilon} \right\|_{L^2((0,T), L^{4/3}(\Omega))} \\ & \leq \left\| v_{i,\varepsilon} - \nu_i \frac{\partial_x h_{i,\varepsilon}}{h_{i,\varepsilon}^2} \psi_{\varepsilon}(h_{i,\varepsilon}) \right\|_{L^{\infty}((0,T), L^2(\Omega))} \|\partial_x v_{i,\varepsilon}\|_{L^2((0,T), L^4(\Omega))}, \\ & \leq \left\| v_{i,\varepsilon} - \nu_i \frac{\partial_x h_{i,\varepsilon}}{h_{i,\varepsilon}^2} \psi_{\varepsilon}(h_{i,\varepsilon}) \right\|_{L^{\infty}((0,T), L^2(\Omega))} \\ & \quad \|\partial_x v_{i,\varepsilon}\|_{L^2((0,T), L^2(\Omega))}^{1/3} \|\partial_x v_{i,\varepsilon}\|_{L^2((0,T), W^{1,4/3}(\Omega))}^{2/3}, \\ & \leq C(T) \|\partial_x v_{i,\varepsilon}\|_{L^2((0,T), W^{1,4/3}(\Omega))}^{2/3}. \end{aligned}$$

Hence, thanks to standard regularity results for linear parabolic equations (see [14] for example), we obtain that

$$\|\partial_x v_{i,\varepsilon}\|_{L^2((0,T), W^{1,4/3}(\Omega))}^{1/3} \leq C(T).$$

As a consequence of this bound, we get that the second member of (24) is bounded in $L^2((0, T), L^2(\Omega))$. So $v_{i,\varepsilon}$ is bounded in $L^2((0, T), H^2(\Omega))$ and $\partial_t v_{i,\varepsilon}$ is bounded in $L^2((0, T), L^2(\Omega))$. \square

4 Proof of the existence result

Proposition 4.1. (Solonikov). *Let the functions $\psi_{\varepsilon}(h_{i,\varepsilon})$ be lower bounded:*

$$\psi_{\varepsilon}(h_{i,\varepsilon}) \geq \psi, \tag{25}$$

where ψ is some positive constant, and ν_1, ν_2 satisfying relation (11). Assume that the initial conditions $h_{i_0,\varepsilon}$ and $v_{i_0,\varepsilon}$ satisfy (5), with ε verifying condition (10). Then there exists $T_0 > 0$ depending on $\underline{c}_0, \bar{c}_0, \|h_{i_0,\varepsilon}\|_{H^1(\Omega)}$ and $\|v_{i_0,\varepsilon}\|_{H^1(\Omega)}$ such that system (6)–(9) has a unique solution $(h_{1,\varepsilon}, h_{2,\varepsilon}, v_{1,\varepsilon}, v_{2,\varepsilon})$ on $(0, T_0)$ satisfying

$$\begin{aligned} h_{i,\varepsilon} &\in L^\infty(0, T_1; H^1(\Omega)), & v_{i,\varepsilon} &\in L^2(0, T_1; H^2(\Omega)), \\ \partial_t v_{i,\varepsilon} &\in L^2(0, T_1; L^2(\Omega)), & \partial_t h_{i,\varepsilon} &\in L^2(0, T_1; L^2(\Omega)), \end{aligned}$$

for all $T_1 < T_0$.

Moreover, there exist some positive constants $\underline{c}(T)$ and $\bar{c}(T)$ such that

$$\underline{c}(T) \leq h_{i,\varepsilon}(t, x) \leq \bar{c}(T), \quad \forall t \in (0, T_0).$$

Proposition 4.2. *Let $h_{i_0,\varepsilon}$ and $v_{i_0,\varepsilon}$ satisfy (5); then the approximate system (6)–(9) admits a strong solution satisfying the energy and entropy inequalities.*

Proof. Let us define $\psi_{\varepsilon,n}(h_{i,\varepsilon}) = \max(\psi_\varepsilon(h_{i,\varepsilon}), 1/n)$ for $n \geq 1$; we have $\psi \leq \psi_{\varepsilon,n} \leq \psi + 1$. Moreover, for all $n \in \mathbb{N}^*$, $\psi_{\varepsilon,n} \geq \frac{1}{n} > 0$. Using Proposition 4.1, we can consider $(h_{1,\varepsilon}^n, h_{2,\varepsilon}^n, v_{1,\varepsilon}^n, v_{2,\varepsilon}^n)$ the strong solution of (6)–(9) where the functions $\psi_\varepsilon(h_{i,\varepsilon})$ are replaced by $\psi_{\varepsilon,n}(h_{i,\varepsilon})$; this solution exists at least for small time T_0 . Due to Proposition 3.7, $h_{i,\varepsilon}^n$ is uniformly bounded in n from below, and then we can take $T_0 = \infty$. Assume that $n > 1/\underline{c}(T)$; then $(h_{1,\varepsilon}^n, h_{2,\varepsilon}^n, v_{1,\varepsilon}^n, v_{2,\varepsilon}^n)$ is the solution of (6)–(9) with $\psi_{\varepsilon,n}(h_{i,\varepsilon}) = \psi_\varepsilon(h_{i,\varepsilon})$. \square

Proposition 4.3. *Let us define the velocities:*

$$v_{i,\varepsilon} = \frac{\sqrt{h_{i,\varepsilon} v_{i,\varepsilon}}}{\sqrt{h_{i,\varepsilon}}},$$

which is possible thanks to the lower bound on the heights. The strong solution of the approximate problem (6)–(9) converges to some expected strong solution (h_1, h_2, v_1, v_2) of the problem (1)–(4) as $\varepsilon \rightarrow 0^+$ in the sense:

$$\begin{aligned} h_{i,\varepsilon} &\rightarrow h_i \text{ in } C([0, T] \times \bar{\Omega}) \\ v_{i,\varepsilon} &\rightarrow v_i \text{ in } C([0, T] \times \bar{\Omega}) \\ \psi_\varepsilon(h_{i,\varepsilon}) &\rightarrow h_i \text{ in } C([0, T] \times \bar{\Omega}) \\ \sqrt{h_{i,\varepsilon} v_{i,\varepsilon}} &\rightarrow \sqrt{h_i v_i} \text{ in } L^{2+\theta}((0, T) \times \bar{\Omega}), \end{aligned}$$

θ being some positive constant.

Proof. Since we have $\underline{c}(T) \leq h_{i,\varepsilon} \leq \bar{c}(T)$ uniformly with respect to ε , the limit h_i is bounded and bounded away from zero. The limit system can then be divided by h_i and becomes parabolic with respect to the velocities v_i . Finally standard regularity results for parabolic systems can be used again to obtain that

$$\begin{aligned} h_i &\in L^\infty(0, T_1; H^1(\Omega)), & v_i &\in L^2(0, T_1; H^2(\Omega)), \\ \partial_t v_i &\in L^2(0, T_1; L^2(\Omega)), & \partial_t h_i &\in L^2(0, T_1; L^2(\Omega)). \end{aligned}$$

□

A Proofs of Propositions 3.2 and 3.4

This section is devoted to the proofs of the energy and entropy inequalities used in Section 3. For the classical energy, as usual, the idea is to multiply the momentum equation by the velocity. More precisely, we multiply the momentum equations (7) and (9) by $v_{1,\varepsilon}$ and $v_{2,\varepsilon}$ respectively, and we integrate by parts. For $i = 1, 2$, we obtain:

$$\begin{aligned} \int_{\Omega} (\partial_t(h_{1,\varepsilon}v_{1,\varepsilon}) + \partial_x(h_{1,\varepsilon}v_{1,\varepsilon}^2)) v_{1,\varepsilon} dx + g \int_{\Omega} (h_{1,\varepsilon}\partial_x h_{1,\varepsilon} + r h_{1,\varepsilon}\partial_x h_{2,\varepsilon}) v_{1,\varepsilon} dx \\ - \nu_1 \int_{\Omega} \partial_x(\psi_\varepsilon(h_{1,\varepsilon})\partial_x(v_{1,\varepsilon})) v_{1,\varepsilon} dx = 0, \end{aligned} \quad (26)$$

and

$$\begin{aligned} \int_{\Omega} (\partial_t(h_{2,\varepsilon}v_{2,\varepsilon}) + \partial_x(h_{2,\varepsilon}v_{2,\varepsilon}^2)) v_{2,\varepsilon} dx + g \int_{\Omega} (h_{2,\varepsilon}\partial_x h_{2,\varepsilon} + h_{2,\varepsilon}\partial_x h_{1,\varepsilon}) v_{2,\varepsilon} dx \\ - \nu_2 \int_{\Omega} \partial_x(\psi_\varepsilon(h_{1,\varepsilon})\partial_x(v_{2,\varepsilon})) v_{2,\varepsilon} dx = 0. \end{aligned} \quad (27)$$

We can reformulate some terms, namely (for $i = 1, 2$)

$$\int_{\Omega} (\partial_t(h_{i,\varepsilon}v_{i,\varepsilon}) + \partial_x(h_{i,\varepsilon}v_{i,\varepsilon}^2)) v_{i,\varepsilon} dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} h_{i,\varepsilon} |v_{i,\varepsilon}|^2 dx, \quad (28)$$

$$- \nu_i \int_{\Omega} (\partial_x(\psi_\varepsilon(h_{i,\varepsilon})\partial_x(v_{i,\varepsilon})) v_{i,\varepsilon} dx = \nu_i \int_{\Omega} \psi_\varepsilon(h_{i,\varepsilon}) (\partial_x(v_{i,\varepsilon}))^2 dx. \quad (29)$$

To obtain the energy inequality, we add (26) to (27) multiplied by r . We remark that:

$$g \int_{\Omega} h_{1,\varepsilon} \partial_x h_{1,\varepsilon} v_{1,\varepsilon} dx + rg \int_{\Omega} h_{1,\varepsilon} \partial_x h_{2,\varepsilon} v_{1,\varepsilon} dx + rg \int_{\Omega} h_{2,\varepsilon} \partial_x h_{2,\varepsilon} v_{2,\varepsilon} dx \\ + rg \int_{\Omega} h_{2,\varepsilon} \partial_x h_{1,\varepsilon} v_{2,\varepsilon} dx = \frac{g(1-r)}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon}|^2 + \frac{rg}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon} + h_{2,\varepsilon}|^2.$$

Thanks to this simplification, we obtain the classical energy estimate (14):

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} h_{1,\varepsilon} |v_{1,\varepsilon}|^2 + \frac{r}{2} \frac{d}{dt} \int_{\Omega} h_{2,\varepsilon} |v_{2,\varepsilon}|^2 + \nu_1 \int_{\Omega} \psi_{\varepsilon}(h_{1,\varepsilon}) (\partial_x v_{1,\varepsilon})^2 dx \\ + r\nu_2 \int_{\Omega} \psi_{\varepsilon}(h_{2,\varepsilon}) (\partial_x v_{2,\varepsilon})^2 dx + \frac{g(1-r)}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon}|^2 + \frac{rg}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon} + h_{2,\varepsilon}|^2 \leq 0.$$

The second proof of this section is the one of the BD entropy estimate (16). Multiplying the mass equations by $\varphi'_{\varepsilon}(h_{i,\varepsilon})$ (for $i = 1, 2$), we get

$$\partial_t \varphi_{\varepsilon}(h_{i,\varepsilon}) + \partial_x (\varphi_{\varepsilon}(h_{i,\varepsilon})) v_{i,\varepsilon} + \varphi'_{\varepsilon}(h_{i,\varepsilon}) h_{i,\varepsilon} \partial_x v_{i,\varepsilon} = 0.$$

Thanks to Equation (13), this relation reads

$$\partial_t \varphi_{\varepsilon}(h_{i,\varepsilon}) + \partial_x (\varphi_{\varepsilon}(h_{i,\varepsilon})) v_{i,\varepsilon} + \psi_{\varepsilon}(h_{i,\varepsilon}) \partial_x v_{i,\varepsilon} = 0.$$

We differentiate this equation with respect to x and introduce the corresponding viscosity coefficient to get

$$\partial_t \left(h_{i,\varepsilon} \nu_i \frac{\partial_x (\varphi_{\varepsilon}(h_{i,\varepsilon}))}{h_{i,\varepsilon}} \right) + \partial_x \left(h_{i,\varepsilon} \nu_i \frac{\partial_x (\varphi_{\varepsilon}(h_{i,\varepsilon}))}{h_{i,\varepsilon}} v_{i,\varepsilon} \right) \\ + \nu_i \partial_x (\psi_{\varepsilon}(h_{i,\varepsilon}) \partial_x (v_{i,\varepsilon})) = 0.$$

We add the momentum equation (Equation (7) for $i = 1$ and Equation (9) for $i = 2$) to the previous equality and we get

$$\partial_t \left(h_{1,\varepsilon} v_{1,\varepsilon} + h_{1,\varepsilon} \nu_1 \frac{\partial_x \varphi_{\varepsilon}(h_{1,\varepsilon})}{h_{1,\varepsilon}} \right) + \partial_x \left(h_{1,\varepsilon} v_{1,\varepsilon}^2 + h_{1,\varepsilon} \nu_1 \frac{\partial_x \varphi_{\varepsilon}(h_{1,\varepsilon})}{h_{1,\varepsilon}} v_{1,\varepsilon} \right) \\ + gh_{1,\varepsilon} \partial_x h_{1,\varepsilon} + rgh_{1,\varepsilon} \partial_x h_{2,\varepsilon} = 0, \quad (30)$$

$$\partial_t \left(h_{2,\varepsilon} v_{2,\varepsilon} + h_{2,\varepsilon} \nu_2 \frac{\partial_x \varphi_{\varepsilon}(h_{2,\varepsilon})}{h_{2,\varepsilon}} \right) + \partial_x \left(h_{2,\varepsilon} v_{2,\varepsilon}^2 + h_{2,\varepsilon} \nu_2 \frac{\partial_x \varphi_{\varepsilon}(h_{2,\varepsilon})}{h_{2,\varepsilon}} v_{2,\varepsilon} \right) \\ + gh_{2,\varepsilon} \partial_x h_{2,\varepsilon} + gh_{2,\varepsilon} \partial_x h_{1,\varepsilon} = 0. \quad (31)$$

We add Equation (30) multiplied by $\left(v_{1,\varepsilon} + \nu_1 \frac{\partial_x(\varphi_\varepsilon(h_{1,\varepsilon}))}{h_{1,\varepsilon}}\right)$ to Equation (31) multiplied by $r \left(v_{2,\varepsilon} + \nu_2 \frac{\partial_x(\varphi_\varepsilon(h_{2,\varepsilon}))}{h_{2,\varepsilon}}\right)$. Integrating over Ω , we obtain the BD entropy (16)

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} h_{1,\varepsilon} \left| v_{1,\varepsilon} + \nu_1 \frac{\partial_x \varphi_\varepsilon(h_{1,\varepsilon})}{h_{1,\varepsilon}} \right|^2 &+ \frac{r}{2} \frac{d}{dt} \int_{\Omega} h_{2,\varepsilon} \left| v_{2,\varepsilon} + \nu_2 \frac{\partial_x \varphi_\varepsilon(h_{2,\varepsilon})}{h_{2,\varepsilon}} \right|^2 \\ &+ \frac{g(1-r)}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon}|^2 + \frac{rg}{2} \frac{d}{dt} \int_{\Omega} |h_{1,\varepsilon} + h_{2,\varepsilon}|^2 \\ &+ \nu_1 g \int_{\Omega} \varphi'_\varepsilon(h_{1,\varepsilon}) (\partial_x h_{1,\varepsilon})^2 + \nu_2 r g \int_{\Omega} \varphi'_\varepsilon(h_{2,\varepsilon}) (\partial_x h_{2,\varepsilon})^2 \\ &+ \nu_1 r g \int_{\Omega} \partial_x h_{2,\varepsilon} \partial_x \varphi_\varepsilon(h_{1,\varepsilon}) + \nu_2 r g \int_{\Omega} \partial_x h_{1,\varepsilon} \partial_x \varphi_\varepsilon(h_{2,\varepsilon}) \leq 0. \end{aligned}$$

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