

THE PROJECTIVE GEOMETRY OF A GROUP

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ABSTRACT. We show that the pair $(\mathcal{P}(\Omega), \text{Gras}(\Omega))$ given by the power set $\mathcal{P} = \mathcal{P}(\Omega)$ and by the “Grassmannian” $\text{Gras}(\Omega)$ of all subgroups of an arbitrary group Ω behaves very much like a projective space $\mathbb{P}(W)$ and its dual projective space $\mathbb{P}(W^*)$ of a vector space W . More precisely, we generalize several results from the case of the abelian group $\Omega = (W, +)$ (cf. [BeKi10a]) to the case of a general group Ω . Most notably, pairs of subgroups (a, b) of Ω parametrize *torsor* and *semitorsor* structures on \mathcal{P} . The rôle of associative algebras and $-$ -pairs from [BeKi10a] is now taken by analogs of *near-rings*.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

1.1. Projective geometry of an abelian group. Before explaining our general results, let us briefly recall the classical case of projective geometry of a vector space W : let $\mathcal{X} = \mathbb{P}(W)$ be the projective space of W and $\mathcal{X}' = \mathbb{P}(W^*)$ be its dual projective space (space of hyperplanes). The “duality” between \mathcal{X} and \mathcal{X}' is encoded on two levels

- (1) on the level of *incidence structures*: an element $x = [v] \in \mathbb{P}W$ is *incident* with an element $a = [\alpha] \in \mathbb{P}W^*$ if “ x lies on a ”, i.e., if $\alpha(v) = 0$; otherwise we say that they are *remote* or *transversal*, and we then write $x \top a$;
- (2) on the level of *(linear) algebra*: the set a^\top of elements $x \in \mathcal{X}$ that are transversal to a is, in a completely natural way, an *affine space*.

In [BeKi10a], the second point has been generalized: for any pair $(a, b) \in \mathcal{X}' \times \mathcal{X}'$, the intersection $U_{ab} := a^\top \cap b^\top$ of two “affine cells” carries a natural torsor structure. Recall that “torsors are for groups what affine spaces are for vector spaces”.¹

Definition 1.1. A *semitorsor* is a set G together with a map $G^3 \rightarrow G$, $(x, y, z) \mapsto (xyz)$ such that the following identity, called the *para-associative law*, holds:

$$(T1) \quad (xy(zuv)) = (x(uzv)) = ((xyz)uv).$$

A *torsor* is a *semitorsor* in which, moreover, the following idempotent law holds:

$$(T2) \quad (xxy) = y = (yxx).$$

Fixing the middle element y in a torsor G , we get a group law $xz := (xyz)$ with neutral element y , and every group is obtained in this way. Similarly, semitorsors give rise to semigroups, but the converse is more complicated. The torsors $U_a := U_{aa}$

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¹The concept used here goes back to J. Certainé [Cer43]; there are several equivalent versions, known under various other names such as *group*, *heap*, or *principal homogeneous space*.

are the underlying torsors of the affine space a^\top , hence are abelian, whereas for $a \neq b$, the torsors U_{ab} are in general non-commutative. Thus, in a sense, the torsors U_{ab} are *deformations* of the abelian torsor U_a . More generally, in [BeKi10a] all this is done for a pair $(\mathcal{X}, \mathcal{X}')$ of *dual Grassmannians*, not only for projective spaces.

1.2. Projective geometry of a general group. In the present work, the *commutative* group $(W, +)$ will be replaced by an arbitrary group Ω (however, in order to keep formulas easily readable, we will still use an additive notation for the group law of Ω). It turns out, then, that the rôle of \mathcal{X} is taken by the power set $\mathcal{P}(\Omega)$ of *all* subsets of Ω , and the one of \mathcal{X}' by the “Grassmannian” of all subgroups of Ω . We call Ω the “background group”, or just *the background*. Its subsets will be denoted by small latin letters a, b, x, y, \dots and, if possible, elements of such sets by corresponding greek letters: $\alpha \in a$, $\xi \in x$, and so on. As said above, “projective geometry on Ω ” in our sense has two ingredients which we are going to explain now

- (1) a (fairly weak) *incidence* (or rather: *non-incidence*) *structure*, and
- (2) a much more relevant *algebraic* structure consisting of a collection of torsors and semi-torsors.

Definition 1.2. *The projective geometry of a group $(\Omega, +)$ is its power set $\mathcal{P} := \mathcal{P}(\Omega)$. We say that a pair $(x, y) \in \mathcal{P}^2$ is left transversal if every $\omega \in \Omega$ admits a unique decomposition $\omega = \xi + \eta$ with $\xi \in x$ and $\eta \in y$. We then write $x \top y$. We say that the pair (x, y) is right transversal if $y \top x$, and we let*

$$x^\top := \{y \in \mathcal{P} \mid x \top y\}, \quad {}^\top x := \{y \in \mathcal{P} \mid y \top x\}.$$

The “(non-) incidence structure” thus defined is not very interesting in its own right; however, in combination with the algebraic torsor structures it becomes quite powerful. There are two, in a certain sense “pure”, special cases to consider; the general case is a sort of mixture of these two: let a, b be two subgroups of Ω ,

- (A) the *transversal case* $a \top b$: then $a^\top \cap {}^\top b$ is a torsor of “bijection type”,
- (B) the *singular case* $a = b$; it corresponds to “pointwise torsors” ${}^\top a$ and b^\top .

Prototypes for (A) are torsors of the type $G = \text{Bij}(X, Y)$ (set of bijections $f : X \rightarrow Y$ between two sets X and Y), with torsor structure $(fgh) := f \circ g^{-1} \circ h$, and prototypes for (B) are torsors of the type $G = \text{Map}(X, A)$ (set of maps from X to A), where A is a torsor and X a set, together with their natural “pointwise product”.

Case (A) arises, if, when $a \top b$, we identify Ω with the cartesian product $a \times b$; then elements $z \in {}^\top b$ can be identified with “left graphs” $\{(\alpha, Z\alpha) \mid \alpha \in a\}$ of maps $Z : a \rightarrow b$. The map Z is bijective iff this graph belongs to a^\top . Therefore $G := {}^\top b \cap a^\top$ carries a natural torsor structure of “bijection type”: as a torsor, it is isomorphic to $\text{Bij}(a, b)$. It may be empty; if it is non-empty, then it is isomorphic to (the underlying torsor of) the group $\text{Bij}(a, a)$. Note that the structure of this group does not involve the one of Ω , indeed, the group structure of Ω enters here only implicitly, via the identification of Ω with $a \times b$.

On the other hand, in the “singular case” (B), the set ${}^\top a$ is naturally identified with the set of sections of the canonical projection $\Omega \rightarrow \Omega/a$, and this set is a torsor of pointwise type, modelled on the “pointwise group” of all maps $f : \Omega/a \rightarrow a$. It

is abelian iff so is a . Indeed, such torsors correspond precisely to the “affine cells” from usual projective geometry.

1.3. The “balanced” torsors U_{ab} and the “unbalanced” torsors U_a . Following the ideas developed in [BeKi10a], we consider the general torsors U_{ab} as a sort of “deformation of the pure case (B) in direction of (A)”. However, for treating the case of a non-commutative group Ω we need several important modifications of the setting from [BeKi10a]: first of all, the projective geometry \mathcal{P} and its “dual” $Gras(\Omega)$ are no longer the same objects (the subset $Gras(\Omega) \subset \mathcal{P}$ is no longer stable under the various torsor laws); next, for $a = b$, we have to distinguish between several versions of torsor laws, those that one can deform easily, called *balanced*, and those which seem to be more rigid and which we call *unbalanced*. The most conceptual way to present this is via the following algebraic *structure maps*:

Definition 1.3. *The structure maps of $(\Omega, +)$ are the maps $\Gamma : \mathcal{P}^5 \rightarrow \mathcal{P}$ and $\Sigma : \mathcal{P}^4 \rightarrow \mathcal{P}$ defined, for $x, a, y, b, z \in \mathcal{P}$, by*

$$\Gamma(x, a, y, b, z) := \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \xi = \omega + \beta, \quad \eta = \alpha + \omega + \beta, \quad \zeta = \alpha + \omega \end{array} \right\},$$

$$\Sigma(a, x, y, z) := \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \xi \in x, \exists \eta \in y, \exists \zeta \in z, \exists \beta, \beta' \in b : \\ \xi = \omega + \beta, \quad \eta = \omega + \beta' + \beta, \quad \zeta = \omega + \beta' \end{array} \right\}.$$

For a fixed pair $(a, b) \in \mathcal{P}^2$, resp. a fixed element $b \in \mathcal{P}$, we let

$$(xyz)_{ab} := \Gamma(x, a, y, b, z), \quad (xyz)_b := \Sigma(b, x, y, z).$$

The following is a main result of the present work (Theorems 5.1 and 7.2):

Theorem 1.4. *Assume (a, b) is a pair of subgroups of Ω . Then the following holds:*

- (1) *The map $(x, y, z) \mapsto (xyz)_b$ defines a torsor structure on the set ${}^\top b$. We denote this torsor by U_b . It is isomorphic to the torsor of sections of the projection $\Omega \rightarrow \Omega/b$.*
- (2) *The map $(x, y, z) \mapsto (xyz)_{ab}$ defines a torsor structure on the set $a^\top \cap {}^\top b$. We denote this torsor by U_{ab} . If, moreover, $a \top b$, then it is isomorphic to the group of bijections of a .*

Just as the torsor structures considered in [BeKi10a], these torsor laws extend to semitorator laws onto the whole projective geometry, in the same way as the group law of $\text{Bij}(X)$ for a set X extends to a semigroup structure on $\text{Map}(X, X)$:

Theorem 1.5. *Assume (a, b) is a pair of subgroups of Ω . Then the following holds:*

- (1) *The map $(x, y, z) \mapsto (xyz)_b$ defines a semitorator structure on \mathcal{P} .*
- (2) *The map $(x, y, z) \mapsto (xyz)_{ab}$ defines a semitorator structure on \mathcal{P} .*

We call the torsors U_{ab} *balanced*, and the torsors U_b *unbalanced*. If Ω is non-abelian, the torsor U_{bb} is different from U_b – the latter are in general not members of a two-parameter family. This is due to the fact that the system of three equations defining Γ , called the *structure equations*,

$$(1.1) \quad \left\{ \begin{array}{l} \zeta = \alpha + \omega \\ \eta = \alpha + \omega + \beta \\ \xi = \omega + \beta \end{array} \right\}$$

is of a more symmetric nature than the one defining Σ . We come back to this item below (Subsection 1.6).

1.4. Affine picture. As usual in projective geometry, a “projective statement” may be translated into an “affine statement” by choosing some “affinization” of \mathcal{P} . Thus one can rewrite the torsor law of U_{ab} by an “affine formula” (Theorem 8.1). Here is a quite instructive special case: consider two arbitrary groups, $(V, +)$ and $(W, +)$, and fix a group homomorphism $A : W \rightarrow V$. Let $G := \text{Map}(V, W)$ be the set of all maps from V to W . Then it is an easy exercise to show that

$$(1.2) \quad X \cdot_A Y := Y + X \circ (\text{id}_V + A \circ Y)$$

defines an associative product on G (where $+$ is the pointwise “sum” of maps), with neutral element the “zero map” 0 , and which gives a group law on the set

$$(1.3) \quad G_A := \{X \in G \mid \text{id}_V + A \circ X \text{ is bijective} \}.$$

The parameter A is a sort of “deformation parameter”: if $A = 0$, we get pointwise addition; if A is an isomorphism, then G_A is in fact isomorphic to the usual group $\text{Bij}(V)$. If $V = W = \mathbb{R}^n$, then one may do the same construction using continuous, or smooth, maps, and thus gets a deformation of the abelian (additive) group G of vector fields to the highly non-commutative group G_A of diffeomorphisms of \mathbb{R}^n . If V, W are linear spaces and X, Z linear maps, then (1.2) gives us back the law $X + XAY + Y$ considered in [BeKi10a].

1.5. Distributive laws and near-rings. The reason why we are also interested in the unbalanced torsors is that they are natural (being spaces of sections of a principal bundle over a homogeneous space) and interact nicely with the balanced structures: there is a base-point free version of a *right distributive law* which makes the whole object a torsor-analog of a *near-ring* or a “generalized ring” (cf. [Pi77]):

Definition 1.6. *A (right) near-ring is a set N together with two binary operations, denoted by $+$ and \cdot , such that:*

- (1) $(N, +)$ is a group (not necessarily abelian),
- (2) (N, \cdot) is a semigroup,
- (3) we have the right distributive law $(x + y) \cdot z = x \cdot z + y \cdot z$.

A typical example is the set N of self-maps of a group $(G, +)$, where \cdot is composition and $+$ pointwise “addition”. In our context, Γ takes the role of the product \cdot , and Σ takes the one of the “addition” $+$ (cf. Theorem 8.3):

Theorem 1.7. *Let (a, b) be a pair of subgroups of Ω . Then we have the following left distributive law relating the unbalanced and the balanced torsor structures: for all $x, y \in U_{ab}$ and $u, v, w \in U_b$,*

$$(xy(uvw)_b)_{ab} = ((xyu)_{ab}(xyv)_{ab}(xyw)_{ab})_b.$$

Essentially, this means that $\top b$ looks like a ternary version of a near-ring, whose “multiplicative” structure now depends on an additional parameter y . As usual for near-rings, there is just one distributive law: the other distributive law does not hold! Compare with (1.2) which is “affine” in X , but not in Y .

1.6. Symmetry. The preceding theorem makes it obvious that the definition of Γ involves some arbitrary choices: there is no reason why left distributivity should be preferred to right distributivity! Indeed, if instead of Γ , we looked at the map $\check{\Gamma}$ obtained by using everywhere the opposite group law of $(\Omega, +)$, then we would get “right” instead of “left distributivity”. Thus Γ and $\check{\Gamma}$ are, in a certain sense, “equivalent”. In the same way, there is no reason to prefer the groups a or b to their opposite groups: in the structure equations we might replace α or β by their negatives, without changing the whole theory. Thus we are led to consider several versions of the fundamental equations as “essentially equivalent”. We investigate this item in Section 9: there are in fact 24 *signed* (i.e., essentially equivalent) versions of the structure equations on which a certain subgroup \mathbf{V} of the permutation group S_6 , permuting the six variables of (1.1), acts simply transitively (Theorem 9.6). We call \mathbf{V} the *Big Klein Group*² since it plays exactly the same rôle for the structure equations as the usual Klein Group V does for a single torsor structure (cf. Lemma 9.2). The group \mathbf{V} is isomorphic to S_4 , sitting inside S_6 as the subgroup preserving the partition of six letters in three subsets $\{\xi, \zeta\}$, $\{\alpha, \beta\}$ and $\{\eta, \omega\}$ (Lemma 9.4). Permutations from \mathbf{V} leave invariant the general shape of (1.1) and introduce just certain sign changes for some of the variables. If we are willing to neglect such sign changes – like, for instance, in the “projective” framework of [BeKi10a], where one can rescale by any invertible scalar – then the whole theory becomes invariant under these permutations.³ This explains partially why the associative geometries from [BeKi10a] (and their Jordan theoretic analogs) have such a high degree of symmetry (cf. the “symmetry” and “duality principles” for Jordan theory, [Lo75]). If we agree to neglect sign changes only with respect to α and β (which is reasonable since in Theorem 1.4 we assume that a and b are *subgroups*, hence $\alpha \in a$ iff $-\alpha \in a$, and same for b), then we obtain as invariance group again a usual Klein Group V , and the orbit under \mathbf{V} has $24/4 = 6$ elements. This, in turn, is completely analogous to the behavior of the classical *cross-ratio* under S_4 , which is invariant under V and takes generically 6 different values under permutations.

1.7. Further topics. Because of its generality, the approach presented in this work is likely to interact with many other mathematical theories. In the last section we mention some questions arising naturally in this context, and we refer to Section 4 of [BeKi10a] for some more remarks of a similar kind.

Notation. Throughout this paper, Ω is a (possibly non-commutative) group, called the *background*, whose group law will be written additively. Its neutral element will be denoted by o . We denote by $\mathcal{P} := \mathcal{P}(\Omega)$ its power set, by $\mathcal{P}^o := (\Omega)$ the set of subsets of Ω containing the neutral element o , and by $Gras(\Omega)$ the Grassmannian of Ω (the set of all subgroups of Ω). *Transversality*, as defined in Definition 1.2 above, is denoted by $x \top y$.

²translated from the German *Grosse Klein Gruppe*

³A side remark: the author cannot help feeling being reminded by this situation to CPT-invariance in physics, where a very similar phenomenon occurs.

2. STRUCTURE MAPS AND STRUCTURE SPACE

Definition 2.1. The structure maps of a group $(\Omega, +)$ are the maps $\Gamma : \mathcal{P}^5 \rightarrow \mathcal{P}$, $\check{\Gamma} : \mathcal{P}^5 \rightarrow \mathcal{P}$, $\Sigma : \mathcal{P}^4 \rightarrow \mathcal{P}$ and $\check{\Sigma} : \mathcal{P}^4 \rightarrow \mathcal{P}$ defined, for $x, a, y, b, z \in \mathcal{P}$, by

$$(2.1) \quad \Gamma(x, a, y, b, z) := \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \eta = \alpha + \omega + \beta, \zeta = \alpha + \omega, \xi = \omega + \beta \end{array} \right\},$$

$$(2.2) \quad \check{\Gamma}(x, a, y, b, z) := \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \eta = \beta + \omega + \alpha, \zeta = \omega + \alpha, \xi = \beta + \omega \end{array} \right\},$$

$$(2.3) \quad \Sigma(a, x, y, z) := \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \xi \in x, \exists \eta \in y, \exists \zeta \in z, \exists \beta, \beta' \in b : \\ \xi = \omega + \beta, \eta = \omega + \beta' + \beta, \zeta = \omega + \beta' \end{array} \right\},$$

$$(2.4) \quad \check{\Sigma}(a, x, y, z) := \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \xi \in x, \exists \eta \in y, \exists \zeta \in z, \exists \beta, \beta' \in b : \\ \xi = \beta + \omega, \eta = \beta + \beta' + \omega, \zeta = \beta' + \omega \end{array} \right\}.$$

It is obvious that the set $\mathcal{P}^o(\Omega)$ of subsets containing o is stable under each of these maps, and the corresponding restrictions of the four maps will also be called structure maps.

Note that $\check{\Gamma}$, resp. $\check{\Sigma}$, is obtained from Γ , resp. Σ simply by replacing the group law in Ω by the opposite group law. Hence, if Ω is abelian, we have $\Gamma = \check{\Gamma}$ and $\Sigma = \check{\Sigma}$. Moreover, if Ω is abelian, we obviously have

$$(2.5) \quad \Gamma(x, a, y, a, z) = \Sigma(a, x, y, z) = \Sigma(a, z, y, x).$$

For general Ω , the defining equations immediately imply the *symmetry relation*

$$(2.6) \quad \check{\Gamma}(z, b, y, a, x) = \Gamma(x, a, y, b, z).$$

Definition 2.2. The system (1.1) of three equations for six variables in Ω is called the structure equations. We say that another system of equations is equivalent to the structure equations if it has the same set of solutions, called the structure space of the group $(\Omega, +)$:

$$\Gamma := \left\{ (\xi, \zeta; \alpha, \beta; \eta, \omega) \in \Omega^6 \mid \eta = \alpha + \omega + \beta, \quad \zeta = \alpha + \omega, \quad \xi = \omega + \beta \right\}.$$

By definition, the opposite structure space is the structure space of Ω^{opp} :

$$\check{\Gamma} := \left\{ (\xi, \zeta; \alpha, \beta; \eta, \omega) \in \Omega^6 \mid \eta = \beta + \omega + \alpha, \quad \zeta = \omega + \alpha, \quad \xi = \beta + \omega \right\}.$$

The sets $\Sigma \subset \Omega^5$ and $\check{\Sigma} \subset \Omega^5$ can be defined similarly.

Lemma 2.3. The following systems are all equivalent to the structure equations:

$$(2.7) \quad \left\{ \begin{array}{l} \alpha = \eta - \xi \\ \omega = \xi - \eta + \zeta \\ \beta = -\zeta + \eta \end{array} \right\}$$

$$(2.8) \quad \left\{ \begin{array}{l} \eta = \alpha + \omega + \beta \\ \eta = \alpha + \xi \\ \eta = \zeta + \beta \end{array} \right\} \quad \left\{ \begin{array}{l} \eta = \zeta - \omega + \xi \\ \eta = \alpha + \xi \\ \eta = \zeta + \beta \end{array} \right\}$$

$$(2.9) \quad \left\{ \begin{array}{l} \omega = \xi - \eta + \zeta \\ \omega = \xi - \beta \\ \omega = -\alpha + \zeta \end{array} \right\} \quad \left\{ \begin{array}{l} \omega = -\alpha + \eta - \beta \\ \omega = \xi - \beta \\ \omega = -\alpha + \zeta \end{array} \right\}$$

$$(2.10) \quad \left\{ \begin{array}{l} \alpha = \zeta + \beta - \xi \\ \alpha = \eta - \xi \\ \alpha = \zeta - \omega \end{array} \right\} \quad \left\{ \begin{array}{l} \alpha = \eta - \beta - \omega \\ \alpha = \eta - \xi \\ \alpha = \zeta - \omega \end{array} \right\}$$

$$(2.11) \quad \left\{ \begin{array}{l} \beta = -\zeta + \alpha + \xi \\ \beta = -\omega + \xi \\ \beta = -\zeta + \eta \end{array} \right\} \quad \left\{ \begin{array}{l} \beta = -\omega - \alpha + \eta \\ \beta = -\omega + \xi \\ \beta = -\zeta + \eta \end{array} \right\}$$

$$(2.12) \quad \left\{ \begin{array}{l} \xi = -\alpha + \zeta + \beta \\ \xi = -\alpha + \eta \\ \xi = \omega + \beta \end{array} \right\} \quad \left\{ \begin{array}{l} \xi = \omega - \zeta + \eta \\ \xi = -\alpha + \eta \\ \xi = \omega + \beta \end{array} \right\}$$

$$(2.13) \quad \left\{ \begin{array}{l} \zeta = \eta - \xi + \omega \\ \zeta = \eta - \beta \\ \zeta = \alpha + \omega \end{array} \right\} \quad \left\{ \begin{array}{l} \zeta = \alpha + \xi - \beta \\ \zeta = \eta - \beta \\ \zeta = \alpha + \omega \end{array} \right\}$$

$$(2.14) \quad \left\{ \begin{array}{l} \eta = \alpha + \xi \\ \beta = -\omega + \xi \\ \zeta = \alpha + \omega \end{array} \right\} \quad \left\{ \begin{array}{l} \alpha = \eta - \xi \\ \zeta = \eta - \beta \\ \omega = \xi - \beta \end{array} \right\} \quad \left\{ \begin{array}{l} \alpha = \zeta - \omega \\ \xi = \omega + \beta \\ \eta = \zeta + \beta \end{array} \right\}$$

The proof is by completely elementary computations. Obviously, the structure space has certain symmetry properties with respect to permutations. This will be investigated in more detail in Section 9. Note also that, if Ω is abelian, the structure equations are \mathbb{Z} -linear, and hence can be written in matrix form

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \omega \\ \beta \end{pmatrix} = \begin{pmatrix} \zeta \\ \eta \\ \xi \end{pmatrix}.$$

Equations (2.7) then correspond to the inverse of this matrix.

3. THE SEMITORSOR LAWS

Theorem 3.1. *Assume that a and b are two subgroups of a group $(\Omega, +)$. Then the power set \mathcal{P} and its subset \mathcal{P}^o become semitorsors under the ternary compositions*

$$\begin{aligned} \mathcal{P}^3 &\rightarrow \mathcal{P}, & (x, y, z) &\mapsto (xyz)_{ab} := \Gamma(x, a, y, b, z), \\ \mathcal{P}^3 &\rightarrow \mathcal{P}, & (x, y, z) &\mapsto (xyz)_{ab}^{\check{}} := \check{\Gamma}(x, a, y, b, z), \\ \mathcal{P}^3 &\rightarrow \mathcal{P}, & (x, y, z) &\mapsto (xyz)_b := \Sigma(b, x, y, z), \\ \mathcal{P}^3 &\rightarrow \mathcal{P}, & (x, y, z) &\mapsto (xyz)_b^{\check{}} := \check{\Sigma}(b, x, y, z). \end{aligned}$$

We denote these semitorsors by \mathcal{P}_{ab} , $\check{\mathcal{P}}_{ab}$, \mathcal{P}_b , $\check{\mathcal{P}}_b$, respectively.

Proof. We prove, for $x, y, z \in \mathcal{P}(\Omega)$, the identity

$\Gamma(x, a, u, b, \Gamma(y, a, v, b, z)) = \Gamma(x, a, \Gamma(v, a, y, b, u), b, z) = \Gamma(\Gamma(x, a, u, b, y), a, v, b, z)$,
i.e., the semitorsor law for $(xyz)_{ab}$. For the proof, note that the definition of $\Gamma(x, a, y, b, z)$ can be written somewhat shorter, as follows:

$$(3.1) \quad \Gamma(x, a, y, b, z) = \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha \in a, \exists \beta \in b : \\ \alpha + \omega + \beta \in y, \quad \alpha + \omega \in z, \quad \omega + \beta \in x \end{array} \right\},$$

and similarly for $\check{\Gamma}$. We refer to this description as (a, b) -description. Using this, we have, on the one hand,

$$\begin{aligned} \Gamma(x, a, u, b, \Gamma(y, a, v, b, z)) &= \\ &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha \in a, \exists \beta \in b : \\ \alpha + \omega \in \Gamma(y, a, v, b, z), \alpha + \omega + \beta \in u, \omega + \beta \in x \end{array} \right\} \\ &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha \in a, \exists \beta \in b, \exists \alpha' \in a, \exists \beta' \in b : \\ \alpha + \omega + \beta \in u, \omega + \beta \in x, \alpha' + \alpha + \omega \in z, \\ \alpha' + \alpha + \omega + \beta' \in v, \alpha + \omega + \beta' \in y \end{array} \right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Gamma(x, a, \Gamma(v, a, y, b, u), b, z) &= \\ &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha'' \in a, \exists \beta'' \in b : \\ \alpha'' + \omega \in z, \alpha'' + \omega + \beta'' \in \Gamma(v, a, y, b, u), \omega + \beta'' \in x \end{array} \right\} \\ &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha'' \in a, \exists \beta'' \in b, \exists \alpha''' \in a, \exists \beta''' \in b : \\ \alpha'' + \omega \in z, \omega + \beta'' \in x, \alpha''' + \alpha'' + \omega + \beta'' \in u, \\ \alpha'' + \omega + \beta'' + \beta''' \in v, \alpha''' + \alpha'' + \omega + \beta'' + \beta''' \in y \end{array} \right\}. \end{aligned}$$

Via the change of variables $\alpha'' = \alpha' + \alpha$, $\alpha''' = \alpha'$, $\beta'' = \beta$, $\beta''' = -\beta + \beta'$, we see that these two subsets of Ω are the same. (Here we use that a and b are groups!) This proves the first defining equality of a semitorsor for Γ . Since Ω^{opp} is again a group, it holds also for $\check{\Gamma}$. The second equality now follows from the first one, using the symmetry relation (2.6).

Now consider the product $(xyz)_b$. Similarly as above, we have

$$(3.2) \quad \Sigma(b, x, y, z) = \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \beta, \beta' \in b : \\ \omega + \beta \in x, \omega + \beta' + \beta \in y, \omega + \beta' \in z \end{array} \right\}$$

Using (3.2), we have on the one hand,

$$\begin{aligned} (x, u, (y, v, z)_b)_b &= \\ &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha \in b, \exists \beta \in b : \\ \omega + \alpha \in (y, v, z)_b, \omega + \alpha + \beta \in u, \omega + \beta \in x \end{array} \right\} \\ &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha \in b, \exists \beta \in b, \exists \alpha' \in b, \exists \beta' \in b : \\ \omega + \alpha + \beta \in u, \omega + \beta \in x, \omega + \alpha + \alpha' \in z, \\ \omega + \alpha + \alpha' + \beta' \in v, \omega + \alpha + \beta' \in y \end{array} \right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (x, (v, y, u)_b z)_b &= \\
 &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha'' \in b, \exists \beta'' \in b : \\ \omega + \alpha'' \in z, \omega + \alpha'' + \beta'' \in (v, y, u)_b, \omega + \beta'' \in x \end{array} \right\} \\
 &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha'' \in b, \exists \beta'' \in b, \exists \alpha''' \in b, \exists \beta''' \in b : \\ \omega + \alpha'' \in z, \omega + \beta'' \in x, \omega + \alpha'' + \beta'' + \alpha''' \in u, \\ \omega + \alpha'' + \beta'' + \beta''' \in v, \omega + \alpha'' + \beta'' + \alpha''' + \beta''' \in y \end{array} \right\}.
 \end{aligned}$$

Via the change of variables $\beta = \beta''$, $\beta' = \beta'' + \beta'''$, $\alpha = \alpha'' + \beta'' + \alpha''' - \beta'''$, $\alpha' = \beta'' - \alpha''' - \beta''$, we see that these two subsets of Ω are the same. This proves the first defining equality of a semitorator for $(xyz)_b$. The proof of the other defining equality, as well as for the semitorator structure $(xyz)_b$, are similar. \square

Definition 3.2. We call the semitorators $\mathcal{P}_{ab}, \check{\mathcal{P}}_{ab}$ balanced and $\mathcal{P}_b, \check{\mathcal{P}}_b$ unbalanced.

By the symmetry relation (2.6), $\check{\mathcal{P}}_{ba}$ is the opposite semitorator of \mathcal{P}_{ab} (where, for any semitorator (xyz) , the *opposite* law is just (zyx)), whereas $\check{\mathcal{P}}_b$ is *not* the opposite semitorator of \mathcal{P}_b . Thus, given a subgroup $b \subset \Omega$, we have in general six different semitorator laws on \mathcal{P} : $\mathcal{P}_{bb}, \mathcal{P}_b, \check{\mathcal{P}}_b$, along with their opposite laws. If Ω is commutative, then of course these six semitorator laws coincide. More generally:

Theorem 3.3. Assume that a and b are central subgroups of Ω . Then:

- (1) $\mathcal{P}_{ab} = \check{\mathcal{P}}_{ab} = \mathcal{P}_{ba}^{opp}$ and $\mathcal{P}_b = \check{\mathcal{P}}_b = \check{\mathcal{P}}^{opp} = \mathcal{P}_b^{opp}$,
- (2) $Gras(\Omega)$ is stable under all ternary laws from Theorem 3.1.

Proof. The first statement follows immediately from the definitions, and the second by writing the structure equations for $\omega + \omega'$, resp. for $-\omega$, with $\omega, \omega' \in \Gamma(x, a, y, b, z)$, and using that variables from a and b commute with the others. \square

Note that our condition is sufficient, but not necessary with respect to item (2): for instance, if Ω is a direct product of a and b (as a group), then the result of the next section implies that $Gras(\Omega)$ is a subsemitorator of \mathcal{P}_{ab} . For general subgroups a, b , this is no longer true: the subsemitorator generated by $Gras(\Omega)$ will be strictly bigger.

4. THE TRANSVERSAL CASE: COMPOSITION OF RELATIONS IN GROUPS

Recall the definition of (*left*) transversality (Definition 1.2), denoted by $a \top b$. For a fixed transversal pair, we may identify Ω as a set with $a \times b$ via $(\alpha, \beta) \mapsto \alpha + \beta$. Then (by definition) the power set $\mathcal{P}(\Omega)$ is identified with the set $\text{Rel}(a, b)$ of *relations between a and b* .

Theorem 4.1. Let (a, b) be a pair of left transversal subgroups of Ω . Then the ternary composition $z \circ y^{-1} \circ x$ of relations $x, y, z \in \mathcal{P} = \text{Rel}(a, b)$ is given by

$$z \circ y^{-1} \circ x = \Gamma(x, a, y, b, z).$$

If a and b commute, then $Gras(\Omega)$ is stable under this ternary law.

Proof. The computation is the same as in [BeKi10a], Lemma 2.1, by respecting the possible non-commutativity of Ω . Recall first that, if A, B, C, \dots are any sets, we can *compose relations*: for $x \in \text{Rel}(A, B)$, $y \in \text{Rel}(B, C)$,

$$y \circ x := yx := \{(u, w) \in A \times C \mid \exists v \in B : (u, v) \in x, (v, w) \in y\}.$$

Composition is associative: both $(z \circ y) \circ x$ and $z \circ (y \circ x)$ are equal to

$$(4.1) \quad z \circ y \circ x = \{(u, w) \in A \times D \mid \exists (v_1, v_2) \in y : (u, v_1) \in x, (v_2, w) \in z\}.$$

The *reverse relation* of x is

$$x^{-1} := \{(w, v) \in B \times A \mid (v, w) \in x\}.$$

For $x, y, z \in \text{Rel}(A, B)$, we get another relation between A and B by $zy^{-1}x$. (This ternary composition satisfies the para-associative law, and hence relations between sets A and B form a semitorso; no structure on the sets A or B is needed here.) Coming back to $\Omega = a \times b$, and switching to an additive notation, we get

$$\begin{aligned} zy^{-1} \circ x &= \left\{ \omega = (\alpha', \beta') \in \Omega \mid \begin{array}{l} \exists \eta = (\alpha'', \beta'') \in y : \\ (\alpha', \beta'') \in x, (\alpha'', \beta') \in z \end{array} \right\} \\ &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha', \alpha'' \in a, \exists \beta', \beta'' \in b, \exists \eta \in y, \exists \xi \in x, \exists \zeta \in z : \\ \omega = (\alpha', \beta'), \eta = (\alpha'', \beta''), \xi = (\alpha', \beta''), \zeta = (\alpha'', \beta') \end{array} \right\} \\ &= \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha', \alpha'' \in a, \exists \beta', \beta'' \in b, \exists \eta \in y, \exists \xi \in x, \exists \zeta \in z : \\ \omega = \alpha' + \beta', \eta = \alpha'' + \beta'', \xi = \alpha' + \beta'', \zeta = \alpha'' + \beta' \end{array} \right\}. \end{aligned}$$

Now use that a and b are transversal *subgroups* of Ω : then the description of $zy^{-1}x$ can be rewritten, by introducing the new variables $\alpha := \alpha' - \alpha''$, $\beta := -\beta'' + \beta'$ (which belong again to a , resp. to b , since these are subgroups)

$$zy^{-1}x = \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha', \alpha \in a, \exists \beta', \beta \in b, \exists \eta \in y, \exists \xi \in x, \exists \zeta \in z : \\ \omega = \alpha' + \beta', \eta = -\alpha + \omega - \beta, \xi = \omega - \beta, \zeta = -\alpha + \omega \end{array} \right\}.$$

Since $a \top b$, the first condition ($\exists \alpha' \in a, \beta' \in b : \omega = \alpha' + \beta'$) in the preceding description is always satisfied and can hence be omitted in the description of $zy^{-1}x$. Thus

$$zy^{-1}x = \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \alpha', \alpha \in a, \exists \beta', \beta \in b, \exists \eta \in y, \exists \xi \in x, \exists \zeta \in z : \\ \eta = -\alpha + \omega - \beta, \xi = \omega - \beta, \zeta = -\alpha + \omega \end{array} \right\} = \Gamma(x, a, y, b, z).$$

Finally, if a and b commute, then the bijection $\Omega \cong a \times b$ is also a group homomorphism. Since subgroups in a direct product of groups form a monoid under composition of relations, it follows that $\text{Gras}(\Omega)$ is stable under the ternary composition map. \square

Recall that maps give rise to relations via their *graphs*. In our setting:

Definition 4.2. Assume (x, y) is a left-transversal pair of subsets of Ω : $x \top y$, and let $F : x \rightarrow y$ be a map. The (left) graph of F is the subset

$$G_F := \{\xi + F(\xi) \mid \xi \in x\} \subset \Omega,$$

and if $y \top x$, we define the right graph of $F : x \rightarrow y$ to be

$$\check{G}_F := \{F(\xi) + \xi \mid \xi \in x\} \subset \Omega.$$

Lemma 4.3. *Let b be a subgroup of Ω , and y a subset such that $y \top b$. Then there are natural bijections between the following sets:*

- (1) the set ${}^\top b$,
- (2) the set of sections $\sigma : \Omega/b \rightarrow \Omega$ of the canonical projection $\pi : \Omega \rightarrow \Omega/b$,
- (3) the set $\text{Map}(y, b)$ of maps from y to b .

More precisely, the bijection between (1) and (2) is given by the correspondence between σ and the image of σ , and the one between (1) and (3) by $\text{Map}(y, b) \rightarrow {}^\top b$, $F \mapsto G_F$. If, moreover, y is a subgroup, then we have:

(B) the map F is bijective iff $y \top G_F$.

Proof. Consider the equivalence relation given on Ω by $\omega \sim \omega'$ iff $-\omega + \omega' \in b$. Then $x \top b$ if, and only if, x is a set of representatives for this equivalence relation. As for any equivalence relation, it follows therefore that ${}^\top b$ is in bijection with the set of sections of the canonical projection $\Omega \rightarrow \Omega/\sim$. Now let $y = \sigma(\Omega/b)$ and $x = \sigma'(\Omega/b)$ for two sections σ, σ' . Then $f := -\sigma + \sigma'$ is a map $\Omega/b \rightarrow b$. Conversely, given $f : \Omega/b \rightarrow b$, $\sigma' := \sigma + f$ is another section, whose image is precisely the left graph of the map $F := f \circ \pi|_y : y \rightarrow b$.

For the last statement, assume that y is a subgroup and that F is bijective. If $\omega = \eta + \beta$ with $\beta = F(\eta')$, we have the decomposition $\omega = \eta - \eta' + \eta' + F(\eta')$ with $\eta - \eta' \in y$ and $\eta' + F(\eta') \in G_F$ which is unique since F is injective. Hence $y \top G_F$. The converse is proved similarly. \square

In a similar way, if $b \top y$, every element of b^\top is of the form \check{G}_F with a unique map $F : y \rightarrow b$.

Theorem 4.4. *Let (a, b) be a pair of left-transversal subgroups of a group Ω . Then $a^\top \cap {}^\top b$ is a subsemitorator of \mathcal{P}_{ab} , and it is actually a torsor, denoted by U_{ab} and naturally isomorphic to the torsor of bijections $F : a \rightarrow b$ with its usual torsor law*

$$(XYZ) = Z \circ Y^{-1} \circ X.$$

*In other words, if one fixes a bijection $Y : a \rightarrow b$ in order to identify a and b , then U_{ab} is the torsor corresponding to the group of all bijections of a .*⁴

Proof. By the lemma, a relation $r \in \mathcal{P}$ belongs to $a^\top \cap {}^\top b$ if, and only if, it is the left graph of a bijection $F : a \rightarrow b$. Since composition of maps corresponds precisely to the composition of their graphs, the claim now follows from Theorem 4.1. \square

Definition 4.5. *A transversal triple of subgroups is a triple of subgroups (a, b, c) of Ω such that a and b commute, and $a \top b$, $b \top c$ and $c \top a$.*

Theorem 4.6. *Let (a, b, c) be a triple of subgroups such that $a \top b$ and a, b commute. Then (a, b, c) is a transversal triple if and only if $\Omega \cong a \times a$, with a the first, b the second factor and c the diagonal. The subset $U'_{ab} := U_{ab} \cap \text{Gras}(\Omega)$ of U_{ab} is a torsor with base point c , isomorphic to the group $\text{Aut}(a)$ of group automorphisms of a .*

Proof. The assumption implies that $\Omega \cong a \times b$ as a group. Let $c \in U_{ab}$ be the graph of the bijective map $F : b \rightarrow a$. Then $c \subset \Omega$ is a subgroup if and only if F is a group morphism, and the claim follows from the preceding results. \square

⁴To be precise, we get the opposite of the “usual” composition. This convention, used already in [BeKi10a], is in keeping with certain formulas from Jordan theory.

5. THE SINGULAR CASE : POINTWISE TORSORS

Now we turn to the “singular” case $a = b$. In this case, Γ has to be replaced by Σ , and torsors of bijections by “pointwise torsors”:

Theorem 5.1. *Assume b is a subgroup of Ω , and let $y \in \mathcal{P}$ such that $y \top b$. Then there are natural isomorphisms between the following torsors:*

- (1) the set ${}^\top b$, which is a subsemitorator of \mathcal{P}_b , and which becomes a torsor, denoted by U_b , with the induced law,
- (2) the torsor of all (images) of sections $\sigma : \Omega/b \rightarrow \Omega$ of the canonical projection $\Omega \rightarrow \Omega/b$ with pointwise torsor structure $(\sigma\sigma'\sigma'')(u) = \sigma(u) - \sigma'(u) + \sigma''(u)$,
- (3) the torsor $\text{Map}(y, b)$ of maps from y to b , with its pointwise torsor structure.

Similar statements hold for $\check{U}_b = b^\top$, which can be identified with sections of the projection $\Omega \rightarrow b \backslash \Omega$, together with pointwise torsor structure.

Proof. On the level of sets, these bijections have been established in Lemma 4.3. It is immediately checked that the set of sections of the projection is stable under the pointwise torsor structure $(\sigma\sigma'\sigma'')(u) = \sigma(u) - \sigma'(u) + \sigma''(u)$ (as well as under its opposite torsor structure), and it is clear, then, that the bijection between (2) and (3) becomes an isomorphism of torsors with pointwise torsor structures.

In order to show that these torsor structures agree with the law described in Theorem 3.1, note that, by a change of variables, the unbalanced semitorator law can also be written

$$(5.1) \quad (xyz)_b = \left\{ \omega \in \Omega \mid \begin{array}{l} \exists \xi \in x, \exists \eta \in y, \exists \zeta \in z, \exists \beta, \beta' \in b : \\ \omega = \xi - \eta + \zeta, \quad \zeta = \eta + \beta, \quad \xi = \eta + \beta' \end{array} \right\}.$$

Given three sections $\sigma, \sigma', \sigma''$, we let $\xi = \sigma(u)$, $\eta = \sigma'(u)$, $\zeta = \sigma''(u)$ so that $\omega := (\sigma\sigma'\sigma'')(u) = \sigma(u) - \sigma'(u) + \sigma''(u) = \xi - \eta + \zeta$, which is the first condition in (5.1). The other two conditions just say that ξ , η and ζ belong to the same coset $\eta + b$. Thus the ternary structures from (1), (2) and (3) agree; since (2) and (3) are torsors, the semitorator law from (1) actually defines a torsor structure on ${}^\top b$.

The same arguments apply to sections of $\Omega \rightarrow b \backslash \Omega$, defining the two torsor structures on b^\top . \square

6. AN OPERATOR CALCULUS ON GROUPS

In this section we generalize the various “projection operators” used in [BeKi10a] to the case of general groups. If Ω is abelian, then all operators are \mathbb{Z} -linear, or \mathbb{Z} -affine, maps; in general, however, they will *not* be endomorphisms of Ω . In the following, all sums $F + G$ and differences $F - G$ of maps $F, G : \Omega \rightarrow \Omega$ are pointwise sums, resp. differences, and hence one has to respect orders in such expressions.

Definition 6.1. *Assume a and x are subsets of a group Ω such that $a \top x$. The left, resp. right projection operators are defined by*

$$\begin{aligned} P_x^a : \Omega &\rightarrow \Omega, & \omega = \alpha + \xi &\mapsto \xi \\ \check{P}_a^x : \Omega &\rightarrow \Omega, & \omega = \alpha + \xi &\mapsto \alpha \end{aligned}$$

where $\alpha \in a$, $\xi \in x$.

Remark. By dropping the transversality conditions, some results of this section still hold if one replaces projectors by *generalized projectors*, i.e., by relations $P_a^x \subset \Omega^2$ of the form $\{(\eta, \omega) \in \Omega^2 \mid \exists \alpha \in a, \xi \in x : \eta = \alpha + \xi, \omega = \xi\}$, for arbitrary $x, a \in \mathcal{P}$ (see [BeKi10b] for the case of abelian Ω .) This will be taken up elsewhere.

Lemma 6.2. *Let $a, b, x, y \in \mathcal{P}$ such that $a \top x, y$ and $a, b \top x$. Then*

- i) $\check{P}_a^x + P_x^a = \text{id}_\Omega$, that is, $\check{P}_a^x = \text{id}_\Omega - P_x^a$ and $P_x^a = -\check{P}_a^x + \text{id}_\Omega$,
- ii) $P_x^a \circ P_x^b = P_x^b$; in particular, P_x^a is idempotent: $(P_x^a)^2 = P_x^a$,
- iii) if a is a subgroup, then $P_x^a \circ P_y^a = P_x^a$.

Proof. i) If $\omega = \alpha + \xi$ with $\alpha \in a, \xi \in x$, then $\xi = P_x^a(\omega)$ and $\alpha = \check{P}_a^x(\omega)$, whence the claim. (Note: in general, $P_x^a + \check{P}_a^x$ will be different from the identity map!)

ii) is obvious, and iii) is proved by decomposing $\omega = \alpha + \xi = \alpha' + \eta$; then $P_x^a \circ P_y^a(\omega) = P_x^a(\eta) = \xi = P_x^a(\omega)$ since $\eta = -\alpha' + \alpha + \xi$. \square

6.1. The “unbalanced operators”.

Definition 6.3. *Let $a, b, x, y, z \in \mathcal{P}$ such that $x, y \top b$ and $a \top x, y$. We define two types of transvection operators (from y to x , along b , reps. a) by*

$$\begin{aligned} T_{x,y}^b &:= \text{id} - P_b^x + P_b^y = \check{P}_{x,y}^b + P_b^y = \check{P}_x^b - \check{P}_y^b + \text{id} \\ \check{T}_{x,y}^a &:= \check{P}_a^y - \check{P}_a^x + \text{id} = \check{P}_a^y + P_x^a = \text{id} - P_y^a + P_x^a. \end{aligned}$$

Notation is chosen such that, for any operator $A : \Omega \rightarrow \Omega$ which can be written as a “word” (iterated sum) in projection operators, \check{A} denotes the corresponding operator obtained by replacing Ω by Ω^{opp} .

Lemma 6.4. *Assume a, b are subgroups, $x, y \top b$ and $a \top x, y$. Then, for all $z \in \mathcal{P}$,*

$$\begin{aligned} \Sigma(b, x, y, z) &= T_{xy}^b(z), \\ \check{\Sigma}(a, x, y, z) &= \check{T}_{xy}^a(z). \end{aligned}$$

In particular, it follows that, if also $u \top b$,

$$T_{x,x}^b = \text{id}_\Omega, \quad T_{x,u}^b \circ T_{u,y}^b = T_{x,y}^b.$$

Thus $\{T_{x,y}^b \mid x, y \in \top b\}$ is a group, isomorphic to $\top b$ as a torsor. Similar remarks apply to $\check{T}_{x,y}^b$ with respect to b^\top .

Proof. $(\check{P}_x^b - \check{P}_y^b + \text{id})(z)$ is the set of $\omega \in \Omega$ that can be written $\omega = \xi - \eta + \zeta$ with $\zeta \in z, \xi = P_x^b(\zeta), \eta = P_y^b(\zeta)$. The last two conditions mean that there are $\beta, \beta' \in b$ with $\zeta = \xi + \beta$ and $\zeta = \eta + \beta'$. Since b is a subgroup, we see that ω satisfies precisely the three conditions from (5.1), whence the first claim. From the torsor property, we get now $T_{x,u}^b(T_{u,y}^b(z)) = T_{x,y}^b(z)$. Since this holds in particular for singletons $z = \{\zeta\}$, the identity $T_{x,u}^b \circ T_{u,y}^b = T_{x,y}^b$ for operators on Ω follows. The remaining claims are clear. \square

6.2. The “balanced operators”.

Definition 6.5. Let $a, b, x, y, z \in \mathcal{P}$. If $a \top x$ and $z \top b$, we define the middle multiplication operators by

$$\begin{aligned} M_{xabz} &:= P_x^a - \text{id} + \check{P}_z^b \\ &= P_x^a - P_b^z \\ &= -\check{P}_a^x + \text{id} - P_b^z \\ &= -\check{P}_a^x + \check{P}_z^b. \end{aligned}$$

If $a \top x$ and $y \top b$, we define the left multiplication operators by

$$L_{xayb} := -\check{P}_a^x \circ \check{P}_y^b + \text{id},$$

and if $(-a) \top y$ and $z \top b$, we define the right multiplication operators by

$$R_{aybz} := \text{id} - P_b^z \circ P_y^{-a}.$$

Lemma 6.6. Let $a, b, x, y, z \in \mathcal{P}$.

- i) If $a \top x$ and $z \top b$, then $\Gamma(x, a, y, b, z) = M_{xabz}(y)$.
- ii) If $a \top x$ and $y \top b$, then $\Gamma(x, a, y, b, z) = L_{xayb}(z)$.
- iii) If $z \top b$ and $(-a) \top y$, then $\Gamma(x, a, y, b, z) = R_{aybz}(x)$.

Proof. i) $(P_x^a - \text{id} + \check{P}_z^b)(y)$ is the set of all $\omega \in \Omega$ that can be written in the form $\omega = \xi - \eta + \zeta$ with $\xi = P_x^a(\eta)$ and $\zeta = \check{P}_z^b(\eta)$ for some $\eta \in y$. This means, in turn, that there is $\alpha \in a$ and $\zeta \in z$ such that $\eta = \alpha + \xi$ and $\eta = \zeta + \beta$. Summing up, ω satisfies exactly the three conditions from (2.7), hence they describe $\Gamma(x, a, y, b, z)$, whence the equality of both sets.

ii) $(-\check{P}_a^x \circ \check{P}_y^b + \text{id})(z)$ is the set of all $\omega \in \Omega$ that can be written as $\omega = -\alpha + \zeta$ with $\alpha = P_a^x \circ \check{P}_y^b(\zeta)$ and $\zeta \in z$. This means that there is $\xi \in x$ such that $\eta = \alpha + \xi$ for $\eta = \check{P}_y^b(\zeta)$. And this means that there is $\eta \in y$ and $\beta \in b$ such that $\zeta = \eta - \beta$. Thus we end up with the three conditions

$$\omega = -\alpha + \zeta, \quad \eta = \alpha + \xi, \quad \zeta = \eta - \beta$$

which are equivalent to the structure equations.

iii) $(\text{id} - P_b^z \circ P_y^{-a})(x)$ is the set of all $\omega \in \Omega$ that can be written as $\omega = \xi - \beta$ with $\xi \in x$ and $\beta := P_b^z(P_y^{-a}(\xi))$. This means that there is $\beta \in b$ and $\zeta \in z$ such that $\eta := P_y^{-a}(\xi) = \zeta + \beta$. And this means that there is $\alpha \in a$ and $\eta \in y$ such that $\xi = -\alpha + \eta$. Again, the three conditions thus obtained,

$$\omega = \xi - \beta, \quad \eta = \zeta + \beta, \quad \xi = -\alpha + \eta$$

are equivalent to the structure equations. (Note: it is not assumed in this lemma that a or b are groups.) \square

Lemma 6.7. Let $a, b, x, y, z \in \mathcal{P}$. Then:

- i) Assume b contains the origin of Ω . If $(-a) \top y$ and $y \top b$, then $R_{ayby} = \text{id}$, hence $\Gamma(x, a, y, b, y) = x$ for all $x \in \mathcal{P}$.
- ii) If $x \top b$ and $a \top x$, then $L_{xaxb} = \text{id}$, hence $\Gamma(x, a, x, b, z) = z$ for all $z \in \mathcal{P}$.

Proof. i) We have $P_b^y(y) = \{o\}$ since $o \in y$, hence $R_{ayby} = \text{id} - P_b^y P_y^{-a} = \text{id}$.

- ii) $L_{xaxb} = -\check{P}_a^x \check{P}_x^b + \text{id} = -(\text{id} - P_x^a) \circ \check{P}_x^b + \text{id} = -(\check{P}_x^b - \check{P}_x^b) + \text{id} = \text{id}$ \square

6.3. The “canonical kernel”.

Definition 6.8. *The canonical kernel is the family of maps defined by*

$$\begin{aligned} K_{x,y}^a &:= P_x^a|_y : y \rightarrow x, \quad \eta \mapsto P_x^a(\eta), \\ \check{K}_{x,y}^b &:= \check{P}_x^b|_y : y \rightarrow x, \quad \eta \mapsto \check{P}_x^b(\eta), \end{aligned}$$

where $x, y \in \mathcal{P}$, $a, b \in \text{Gras}(\Omega)$ with $a \top x$ and $x \top b$. We let

$$B_y^{a,x,b} := K_{y,x}^a \circ \check{K}_{x,y}^b = P_y^a \circ \check{P}_x^b|_y : y \rightarrow y, \quad \eta \mapsto P_y^a \circ \check{P}_x^b(\eta).$$

Lemma 6.9. *Let $x, y \in \mathcal{P}$, $a, b \in \text{Gras}(\Omega)$ with $a \top x$ and $y \top b$. Then $K_{x,y}^a : y \rightarrow x$ is bijective iff $a \top y$, and if this holds, then, for all $\eta \in y$,*

$$K_{x,y}^a(\eta) = \check{T}_{x,y}^a(\eta) = L_{xayb}(\eta).$$

Similarly, $\check{K}_{y,x}^b : x \rightarrow y$ is bijective iff $x \top b$, and if this holds then, for $\xi \in x$,

$$\check{K}_{y,x}^b(\xi) = T_{y,x}^b(\xi) = R_{axyb}(\xi).$$

It follows that $a \top x$ if and only if $B_y^{b,x,a} : y \rightarrow y$ is bijective, and if this is the case,

$$B_y^{b,x,a} = \check{T}_{y,x}^a \circ T_{x,y}^b|_y : y \rightarrow y, \quad (B_y^{b,x,a})^{-1} = T_{y,x}^b \circ \check{T}_{x,y}^a : y \rightarrow y.$$

Proof. It is clear that y is another set of representatives for $a \setminus \Omega$ iff the projection from y to x is a bijection. If this holds, then

$$\check{T}_{x,y}^a(\eta) = (\check{P}_a^y + P_x^a)(\eta) = P_x^a(\eta),$$

$$L_{xayb}(\eta) = -\check{P}_a^x \check{P}_y^b(\eta) + \eta = (-\check{P}_a^x + \text{id})\eta = P_x^a(\eta).$$

The second claim is proved in the same way, and the last statement follows. \square

If x, y, a, b are vector lines in \mathbb{R}^2 , then $B_y^{a,x,b}$ is the linear map obtained by first projecting y along b onto x and then back onto y along a . Here is a quite general description that applies to the case of vector spaces, where one sees the close relation with the so-called *Bergman operators* known in Jordan theory:

Lemma 6.10. *Assume Ω is a direct product of its subgroups y and b , and let a be a subgroup such that $y \top a$ and $x \in \mathcal{P}$ such that $x \top b$. Realize $x = G_X$ as a graph of a map $X : y \rightarrow b$ and a as a graph of a group homomorphism $A : b \rightarrow y$. Then*

$$B_y^{a,x,b} = \text{id}_y - A \circ X : y \rightarrow y.$$

In particular, $x \top a$ if, and only if, $\text{id}_y - A \circ X$ is bijective.

Proof. We have to show that, for all $\eta \in y$, $P_y^a(\check{P}_x^b \eta) = \eta - AX\eta$. Indeed: since $\eta = \eta + X\eta - X\eta$ is a decomposition according to $x \top b$, we have $\check{P}_x^b \eta = \eta + X\eta$. Next, we decompose $\eta + X\eta = \eta - AX\eta + AX\eta + X\eta$ with $\eta - AX\eta \in y$ and $AX\eta + X\eta \in G_A = a$, whence $P_y^a(\check{P}_x^b \eta) = P_y^a(\eta + X\eta) = \eta - AX\eta$. \square

7. THE BALANCED TORSORS U_{ab}

Theorem 7.1. *Fix a pair (a, b) of subgroups of Ω . Then the maps*

$$\Pi_{ab}^+ : a^\top \times {}^\top b \times a^\top \rightarrow a^\top, \quad (x, y, z) \mapsto \Gamma(x, a, y, b, z)$$

$$\Pi_{ab}^- : {}^\top b \times a^\top \times {}^\top b \rightarrow {}^\top b, \quad (x, y, z) \mapsto \Gamma(x, a, y, b, z)$$

are well-defined.

Proof. Assume $a^\top x$ and use the bijection $\Omega \cong a \times x$ in order to write other subsets as graphs: let $a^\top z$ and write $z = G_Z$ with a map $Z : x \rightarrow a$, and let $y^\top b$. We have to show that $L_{xayb}(z)$ belongs again to a^\top , that is, that it can be written as is a graph. In order to prove this, let $\zeta = Z\xi + \xi \in z$, where $\xi \in x$. Then

$$L_{xayb}(Z\xi + \xi) = (-\check{P}_a^x \circ \check{P}_y^b + \text{id})(Z\xi + \xi) = -\check{P}_a^x \circ \check{P}_y^b(Z\xi + \xi) + Z\xi + \xi.$$

Define the map

$$F : x \rightarrow a, \quad \xi \mapsto -\check{P}_a^x \circ \check{P}_y^b(Z\xi + \xi)$$

so that $L_{xayb}(\zeta) = F(\xi) + Z(\xi) + \xi$, hence $L_{xayb}z$ is the graph of $F + Z : x \rightarrow a$, and it follows that Π_{ab}^+ is well-defined.

Next assume that $x^\top b$, $z^\top b$ and $a^\top y$ and identify $\Omega \cong z \times b$. Write $x = G_X$ as graph of a map $X : z \rightarrow b$. We have to show that $R_{aybz}x$ belongs again to ${}^\top b$. Let $\xi = \zeta + X\zeta \in x$, so

$$R_{aybz}(\zeta + X\zeta) = (\text{id} - P_b^z \circ P_y^a)(\zeta + X\zeta) = \zeta + X\zeta - P_b^z(P_y^a(\zeta + X\zeta))$$

and as above we see that $R_{aybz}x$ is the graph of a map $X + F : z \rightarrow b$, hence belongs to ${}^\top b$. Thus Π_{ab}^- is well-defined. \square

Remark. See [BeKi10a], Theorem 1.8 for the case of abelian Ω and linear maps: in this case, one can give a more explicit form for F in terms of block matrices.

Theorem 7.2. *Let Ω be a group and (a, b) a pair of subgroups of Ω . Then*

- (1) *the set $a^\top \cap {}^\top b = \{x \in \mathcal{P} \mid a^\top x, x^\top b\}$ is a subsemitorator of \mathcal{P}_{ab} , and with respect to the induced law it is a torsor, which we will denote by U_{ab} .*
- (2) *$b^\top \cap {}^\top a$ is a subsemitorator of $\check{\mathcal{P}}_{ab}$, and with the induced law it becomes a torsor, denoted by \check{U}_{ab} ,*
- (3) *the torsor \check{U}_{ba} is the opposite torsor of U_{ab} : $\check{U}_{ba} = U_{ab}^{opp}$.*

Proof. (1) The fact that $a^\top \cap {}^\top b \subset \mathcal{P}_{ab}$ is a subsemitorator follows directly from the preceding theorem. The idempotent laws are satisfied by Lemma 6.7. Thus U_{ab} is a torsor. Now (2) and (3) follow by the symmetry relation (2.6). Note that the underlying sets of \check{U}_{ba} and U_{ab} obviously agree. \square

Definition 7.3. *The tautological bundle of $\mathcal{P}(\Omega)$ is the set*

$$\hat{\mathcal{P}} = \hat{\mathcal{P}}(\Omega) := \{(y, \eta) \mid y \in \mathcal{P}(\Omega), \eta \in y\},$$

and the map $\pi : \hat{\mathcal{P}} \rightarrow \mathcal{P}$, $(y, \eta) \mapsto y$ is called the canonical projection.

Theorem 7.4. *Let Ω be a group, (a, b) a pair of subgroups of Ω , and fix $y \in U_{ab}$, considered as neutral element of the group (U_{ab}, y) defined by the preceding theorem. Then there are natural left actions*

$$\begin{aligned} U_{ab} \times \mathcal{P} &\rightarrow \mathcal{P}, & (x, z) &\mapsto x.z := \Gamma(x, a, y, b, z), \\ U_{ab} \times \hat{\mathcal{P}} &\rightarrow \hat{\mathcal{P}}, & (x, (z, \zeta)) &\mapsto x.(z, \zeta) := (x.z, L_{xayb}(\zeta)), \end{aligned}$$

by “bundle maps”, i.e., we have $\pi(x.(z, \zeta)) = x.(\pi(z, \zeta))$. Over a^\top , this action can be trivialized: it is given in terms of the canonical kernel by (whith $\eta \in y$)

$$x.(y, \eta) = (x.y, K_{x,y}^a(\eta)).$$

Similarly, we have natural right actions of U_{ab} on \mathcal{P} and on $\hat{\mathcal{P}}$, which commute with the left actions. In particular, U_{ab} acts by conjugation on the fiber over the neutral element, and this action is given by the explicit formula

$$U_{ab} \times y \rightarrow y, \quad \eta \mapsto x\eta x^{-1} = L_{xayb} \circ R_{axby}(\eta) = B_y^{a,x,b}(\eta).$$

Proof. Concerning the left action, everything amounts to proving the following identity for operators on Ω , with $x, x' \in U_{ab}$:

$$(7.1) \quad L_{xayb} \circ L_{x'ayb} = L_{\Gamma(x,a,y,b,x'), ayb}.$$

Note that the operator on the right hand side is well-defined since $\Gamma(x, a, y, b, x') \in U_{ab}$, by the preceding theorem. Now, para-associativity (Theorem 3.1, combined with Lemma 6.6) shows that, applied to any subset $z \subset \omega$, both operators give the same result. Taking for z singletons, it follows that the operators coincide. The proof for the right action is similar, and the fact that both actions commute again amounts to an operator identity

$$(7.2) \quad L_{xayb} \circ R_{ax'by} = R_{ax'by} \circ L_{xayb},$$

which is proved by the same arguments as (7.1). For $x = x'$, we use the definition of the canonical kernel (Definition 6.8) and get the action by conjugation. \square

Theorem 7.5. *Assume (a, b) is a pair of central subgroups. Then the set $Gras_{ab} := Gras(\Omega) \cap U_{ab}$ is a subtorsor of U_{ab} which acts from the left and from the right on the Grassmannian $Gras(\Omega)$ and on the Grassmann tautological bundle*

$$\widehat{Gras}(\Omega) := \{(x, \xi) \mid x \in Gras(\Omega), \xi \in x\}.$$

Proof. This follows by combining the preceding result with Theorem 3.3. \square

8. DISTRIBUTIVE LAW AND “AFFINE PICTURE”

The following fairly explicit description of the group law of U_{ab} is the analog of the “affine picture” from the abelian and linear case given in Section 1 of [BeKi10a]:

Theorem 8.1. *Let (a, b) be a pair of subgroups of Ω and $x, y, z \in \mathcal{P}$ such that $x, y, z \top b$ and $a \top x, y$. Write x and z as left graphs with respect to the decomposition $\Omega \cong y \times b$, i.e., $x = G_X$, $z = G_Z$ with maps $X, Z : y \rightarrow b$. Then*

$$\Gamma(G_X, a, y, b, G_Z) = G_{X+Z \circ B_y^{a,x,b}},$$

i.e., $\Gamma(x, a, y, b, z)$ is the graph of the map $X + Z \circ B_y^{a,x,b} : y \rightarrow b$, where $B_y^{a,x,b} : y \rightarrow y$ is the canonical kernel (Definition 6.8).

This “affine formula” may be written, by identifying x with X and z with Z ,

$$(8.1) \quad X \cdot_{a,y,b} Z = X + Z \circ B_y^{a,X,b}.$$

Here $y = o^+$, $b = o^-$ are fixed “basepoints”, and a is the “deformation parameter”.

Proof. Recall from Lemma 6.6 that

$$\Gamma(x, a, y, b, z) = (P_x^a - \text{id} + \check{P}_z^b)(y) = \{P_x^a(\eta) - \eta + \check{P}_z^b(\eta) \mid \eta \in y\}.$$

Let $\eta \in y$. Since $P_x^a(\eta) \in x = G_X$ and $\check{P}_z^b(\eta) \in z = G_Z$, there exist unique $\eta' \in y$ and $\eta'' \in y$ such that

$$P_x^a(\eta) = \eta' + X\eta', \quad \check{P}_z^b(\eta) = \eta'' + Z\eta''.$$

We determine η' and η'' as functions of η : since $\eta' \in y$ and $X\eta' \in b$, we have, by definition of the projection,

$$\eta' = \check{P}_y^b(P_x^a(\eta)) = B^{-1}(\eta),$$

where $B := B_y^{a,x,b} : y \rightarrow y$ is the canonical kernel (Definition 6.8), and in the same way, using Lemma 6.2, we get

$$\eta'' = \check{P}_y^b \check{P}_z^b(\eta) = \check{P}_y^b \eta = \eta,$$

whence $\check{P}_z^b(\eta) = \eta + Z\eta$. Since the operator $B : y \rightarrow y$ is bijective (Lemma 6.9), we can make a change of variables $\eta' = B^{-1}\eta$, $\eta = B\eta'$, and we get

$$\begin{aligned} (P_x^a - \text{id} + \check{P}_z^b)(\eta) &= \eta' + X\eta' - \eta + \eta + Z\eta \\ &= \eta' + X\eta' + ZB\eta' \\ &= \eta' + (X + Z \circ B)\eta', \end{aligned}$$

and hence, invoking Lemma 6.6, $\Gamma(x, a, y, b, z)$ is equal to

$$(P_x^a - \text{id} + \check{P}_z^b)(y) = \{\eta' + (X + Z \circ B_y^{a,x,b})\eta' \mid \eta' \in y\},$$

that is, to the (left) graph of the map $X + Z \circ B_y^{a,x,b} : y \rightarrow b$. \square

Remark. One may turn everything also the other way round: assume $(b, +)$ is a group, y a set, and let $G := \text{Map}(y, b)$. Assume given a map $B : G \rightarrow \text{Map}(y, y)$, $X \mapsto B^X$ such that $B^0 = \text{id}_y$, and define a binary law on G by

$$X \cdot Z := X \cdot_B Z := X + Z \circ B^X,$$

where $+$ denotes “pointwise addition” in G . It is straightforward to show that this law is associative iff B becomes a homomorphism in the sense that

$$B^{X+Z \circ B^X} = B^X \circ B^Z$$

(cf. [Pi77], p. 243, for a similar construction in the context of near-fields). The neutral element is the zero map 0 , and an element X in G is invertible iff $B^X : y \rightarrow y$ is bijective, and then its inverse is the “quasi-inverse”

$$X^{-1} := -X \circ (B^X)^{-1}.$$

As a special case, all this works if y and b are groups, $A : b \rightarrow y$ a group homomorphism and $B^X = \text{id}_y + A \circ X$, namely, this is the affine picture of the following

Corollary 8.2. *Assume that y is a subgroup commuting with b , and write $a = G_A$ with a group homomorphism $A : b \rightarrow y$. Then Formula (8.1) reads*

$$\Gamma(G_X, G_A, y, b, G_Z) = G_{X+Z \circ (\text{id}_y - A \circ X)} ,$$

and $U_{ab} \rightarrow \text{Bij}(y)^{op}$, $G_X \mapsto \text{id}_y - AX$ is a group homomorphism.

Proof. Write (8.1), using that by Lemma 6.10, $B_y^{a,X,b} = \text{id}_y - A \circ X$. The homomorphism property follows from Theorem 7.4.⁵ \square

Theorem 8.3. *Let (a, b) be a pair of subgroups of Ω . Then we have the following “left distributive law”:* for all $x, y \in U_{ab}$ and $u, v, w \in U_b$,

$$(xy(uvw))_b = ((xyu)_{ab}(xyv)_{ab}(xyw)_{ab})_b.$$

In other words, left multiplications L_{xayb} from U_{ab} are automorphisms of the torsor U_b . Similarly, right multiplications from \check{U}_{ba} are automorphisms of the torsor U_a .

Proof. Let $u, v, w \in b$, and denote by uppercase letters the corresponding maps $y \rightarrow b$. Then the law of the “pointwise torsor” U_b is simply described by the pointwise torsor structure $U - V + W$ of maps from y to b (Theorem 5.1). The claim now follows from Theorem 8.1:

$$\begin{aligned} X + (U - V + W) \circ B^x &= X + U \circ B^x - V \circ B^x + W \circ B^x \\ &= (X + U \circ B^x) - (X + V \circ B^x) + (X + W \circ B^x). \end{aligned}$$

The “dual” statement follows by replacing Ω by Ω^{opp} . \square

In general, the law of U_{ab} is *not* right distributive: the laws $xz = \Gamma(x, a, y, b, z)$ and $x + z = \Sigma(b, x, y, z)$ define a *near-ring*, and not a ring (cf. Definition 1.6).

9. PERMUTATION SYMMETRIES

We have already mentioned (remark after Lemma 2.3) that the structure map Γ and the structure space $\mathbf{\Gamma}$ have certain invariance, or “covariance”, properties with respect to permutations. We start by a simple remark on torsors.

Definition 9.1. *The torsor graph of a torsor $(G, (\))$ is*

$$\mathbf{T} := \mathbf{T}(G) := \{(\xi, \eta, \zeta, \omega) \in G^4 \mid \omega = (\xi\eta\zeta)\}.$$

Using an additive notation, the torsor graph of a group $(\Omega, +)$ is thus given by

$$\mathbf{T} = \mathbf{T}(\Omega) = \{(\xi, \eta, \zeta, \omega) \in \Omega^4 \mid \omega = \xi - \eta + \zeta\}.$$

Lemma 9.2. *The torsor graph of a torsor is invariant under the Klein four-group generated by the two double-transpositions (12)(34) and (13)(24).*

Proof. In additive notation, this follows immediately from the fact that the torsor equation $\omega = \xi - \eta + \zeta$ is equivalent to $\eta = \zeta - \omega + \xi$ and to $\zeta = \eta - \xi + \omega$.

For an intrinsic proof, without fixing a base point, note that symmetry under (13)(24) is equivalent so saying that the middle multiplication operators $M_{xz}(y) = (xyz)$ are invertible with inverse M_{zx} , and symmetry under (12)(34) is equivalent

⁵ We have to use the opposite group structure on $\text{Bij}(y)$ in order to be in keeping with our convention on U_{ab} , cf. footnote 4

so saying that the left multiplication operators $L_{xy}(z) = (xyz)$ are invertible with inverse L_{yx} (cf. Appendix A of [BeKi10a]). \square

Recall the definition of the structure space $\mathbf{\Gamma}(\Omega)$ (Definition 2.2) and the equivalent versions of the structure equations (Lemma 2.3). Note that in System (2.9) the “torsor equation” appears, hence the “ $(\xi, \eta, \zeta, \omega)$ -projection”

$$\mathbf{\Gamma}(\Omega) \rightarrow \mathbf{T}(\Omega), \quad (\xi, \zeta; \alpha, \beta; \eta, \omega) \mapsto (\xi, \eta, \zeta, \omega)$$

is well-defined. Concerning other variables, the “torsor equation” also appears, modulo certain sign changes. The relevant symmetry group here is a subgroup \mathbf{V} of the permutation group S_6 playing a similar role as the Klein four-group $V \subset S_4$ in the preceding lemma:

Definition 9.3. *The Big Klein group is the subgroup \mathbf{V} of permutations $\sigma \in A_6$, acting on six letters $\{\alpha, \beta, \xi, \zeta, \eta, \omega\}$, and preserving the partition*

$$\{\alpha, \beta, \xi, \zeta, \eta, \omega\} = A_1 \cup A_2 \cup A_3, \quad A_1 := \{\alpha, \beta\}, \quad A_2 := \{\xi, \zeta\}, \quad A_3 := \{\eta, \omega\},$$

i.e., for $i = 1, 2, 3$, there is $i' \in \{1, 2, 3\}$ with $\sigma(A_i) = A_{i'}$.

Lemma 9.4. *The Big Klein group \mathbf{V} is isomorphic to S_4 , and its action on six letters is equivalent to the natural action of S_4 on the set K of all two-element subsets of $\{1, 2, 3, 4\}$.*

Proof. We fix the following correspondence between our six letters and K :

$$\alpha = \{1, 2\}, \quad \beta = \{3, 4\}, \quad \xi = \{1, 3\}, \quad \zeta = \{2, 4\}, \quad \eta = \{1, 4\}, \quad \omega = \{2, 3\}.$$

The natural action of S_4 induces a homomorphism $S_4 \rightarrow S_6$, letting act S_4 on the six letters α, \dots, ω . This homomorphism is obviously injective, and its image belongs to \mathbf{V} (note that each transposition from S_4 acts by a double-transposition of these six letters, hence the image belongs to A_6). Let us prove that the homomorphism is surjective: from the very definition of \mathbf{V} we get a homomorphism $\mathbf{V} \rightarrow S_3$, sending σ to the permutation $i \mapsto i'$. The kernel of this homomorphism is a Klein four-group, and one easily constructs a section $S_3 \rightarrow \mathbf{V}$, so that $|\mathbf{V}| = 24 = |S_4|$, whence the claim. \square

Definition 9.5. *A vector $s = (s_1, \dots, s_6)$ with $s_i \in \{\pm 1\}$ will be called a sign vector. Given a sign vector s , the subspace*

$$\mathbf{\Gamma}^s := \left\{ (\xi, \zeta; \alpha, \beta; \eta, \omega) \in \Omega^6 \mid (s_1\xi, s_2\zeta; s_3\alpha, s_4\beta; s_5\eta, s_6\omega) \in \mathbf{\Gamma} \right\}$$

is called a signed version of the structure space.

Since $\Omega \rightarrow \Omega$, $v \mapsto -v$ is an antiautomorphism of Ω , we see that the opposite structure space is a signed version of $\mathbf{\Gamma}$, namely

$$\check{\mathbf{\Gamma}} = \mathbf{\Gamma}^{(-1, -1; -1, -1; -1, -1)}.$$

Theorem 9.6. *The Big Klein group transforms $\mathbf{\Gamma}$ into signed versions of $\mathbf{\Gamma}$: for each $\sigma \in \mathbf{V}$ there exists a sign vector $s = s(\sigma)$ such that $\sigma.\mathbf{\Gamma} = \mathbf{\Gamma}^{s(\sigma)}$. More precisely, we have the following table of elements $\sigma \in \mathbf{V}$ (given together with their corresponding element in S_4 , under the isomorphism from Lemma 9.4) and corresponding sign vectors $s(\sigma)$:*

| <i>element of $V \subset S_4$</i> | <i>corresponding element $\sigma \in \mathbf{V}$</i> | <i>corresponding sign vector $s(\sigma)$</i> |
|--|---|---|
| id | id | (1, 1; 1, 1; 1, 1) |
| (12)(34) | ($\xi\zeta$)($\eta\omega$) | (1, 1; -1, -1; 1, 1) |
| (13)(24) | ($\alpha\beta$)($\eta\omega$) | (-1, -1; 1, 1; -1, -1) |
| (14)(23) | ($\alpha\beta$)($\xi\zeta$) | (-1, -1; -1, -1; -1, -1) |
| <i>element of $A_4 \setminus V$</i> | <i>corresponding element $\sigma \in \mathbf{V}$</i> | <i>corresponding sign vector $s(\sigma)$</i> |
| (123) | ($\alpha\omega\xi$)($\beta\eta\zeta$) | (1, -1; -1, 1; -1, -1) |
| (132) | ($\alpha\xi\omega$)($\beta\zeta\eta$) | (-1, 1; -1, -1; -1, 1) |
| (124) | ($\alpha\zeta\eta$)($\beta\xi\omega$) | (-1, 1; 1, 1; 1, -1) |
| (142) | ($\alpha\eta\zeta$)($\beta\omega\xi$) | (-1, 1; 1, -1; 1, 1) |
| (134) | ($\alpha\omega\zeta$)($\beta\eta\xi$) | (1, -1; -1, 1; 1, 1) |
| (143) | ($\alpha\zeta\omega$)($\beta\xi\eta$) | (1, -1; 1, 1; 1, -1) |
| (234) | ($\alpha\xi\eta$)($\beta\zeta\omega$) | (1, -1; -1, -1; -1, 1) |
| (243) | ($\alpha\eta\xi$)($\beta\omega\zeta$) | (-1, 1; 1, -1; -1, -1) |
| <i>transposition in S_4</i> | <i>corresponding element $\sigma \in \mathbf{V}$</i> | <i>corresponding sign vector $s(\sigma)$</i> |
| (12) | ($\xi\omega$)($\zeta\eta$) | (1, 1; 1, -1; 1, 1) |
| (13) | ($\alpha\omega$)($\beta\eta$) | (1, -1; 1, -1; 1, -1) |
| (14) | ($\alpha\zeta$)($\beta\xi$) | (1, 1; 1, 1; 1, -1) |
| (23) | ($\alpha\xi$)($\beta\zeta$) | (-1, -1; -1, -1; -1, 1) |
| (24) | ($\alpha\eta$)($\beta\omega$) | (-1, 1; 1, -1; 1, -1) |
| (34) | ($\xi\eta$)($\zeta\omega$) | (1, 1; -1, 1; 1, 1) |
| <i>elt. of order 4 in S_4</i> | <i>corresponding element $\sigma \in \mathbf{V}$</i> | <i>corresponding sign vector $s(\sigma)$</i> |
| (1234) | ($\alpha\omega\beta\eta$)($\xi\zeta$) | (1, -1; -1, 1; -1, 1) |
| (1243) | ($\alpha\zeta\beta\xi$)($\eta\omega$) | (-1, -1; 1, 1; 1, -1) |
| (1324) | ($\alpha\beta$)($\xi\omega\zeta\eta$) | (-1, -1; 1, -1; -1, -1) |
| (1342) | ($\alpha\xi\beta\zeta$)($\eta\omega$) | (1, 1; -1, -1; -1, 1) |
| (1423) | ($\alpha\beta$)($\xi\eta\zeta\omega$) | (-1, -1; -1, 1; -1, -1) |
| (1432) | ($\alpha\eta\beta\omega$)($\xi\zeta$) | (-1, 1; 1, -1; -1, 1) |

Proof. The proof is by direct computation: take any of the systems from (2.8) – (2.13) in Lemma 2.3, replace variables according to σ ; the system thus obtained agrees, up to sign changes, with some other among the systems from (2.8) – (2.13), and, by comparing, one can read off the sign vector. \square

A glance at the tables shows that elements from A_4 induce an even number of sign changes, and elements of $S_4 \setminus A_4$ an odd number of sign changes. The sign vectors given in the tables form an S_4 -torsor under the induced action of \mathbf{V} . If Ω is non-commutative, then different sign vectors s give rise to different spaces $\mathbf{\Gamma}^s$; if Ω is commutative, then $\mathbf{\Gamma} = \check{\mathbf{\Gamma}}$, but (except for some degenerate examples) this is the only case in which two signed versions of the structure space coincide. If, in the abelian case, we work only with *linear* subspaces (subgroups), then we may ignore all sign changes, and hence all 24 permutations can be considered as equivalent. In the general case, the following statements also follow by direct inspection from the tables.

Definition 9.7. *A subset x in a group $(\Omega, +)$ is a symmetric subset if $-x = x$.*

Theorem 9.8. *For any group $(\Omega, +)$ and $(x, a, y, b, z) \in \mathcal{P}^5$, we have:*

- (1) *(behavior of Γ under the Klein group $\{\text{id}, (14), (14)(23), (23)\} \subset S_4$):*

$$\Gamma(b, z, y, x, a) = -\Gamma(x, a, y, b, z)$$

$$\Gamma(z, b, y, a, x) = \check{\Gamma}(x, a, y, b, z)$$

$$\Gamma(a, x, y, z, b) = -\check{\Gamma}(x, a, y, b, z),$$
and if x, a, y, b, z are symmetric subsets, then $\Gamma(x, a, y, b, z) = \Gamma(a, x, y, z, b)$.
- (2) *If we consider sign changes in α or β as negligible (and the corresponding signed versions as “equivalent”), then the equivalence class of Γ is invariant under the Klein group $\{\text{id}, (12)(34), (12)(34)\} \subset S_4$.*

Remark. Following [BeKi10a], Theorem 2.11 and Remark 2.12, the second item may be reformulated in another way: invariance under (12), that is, under $(\xi\omega)(\zeta\eta)$, amounts to the fact that the inverse of the relation $\mathbf{l}_{xayb} \subset \Omega^2$ (which, as in [BeKi10a], generalizes the operator L_{xayb} for arbitrary $(x, a, y, b) \in \mathcal{P}^4$) is given by the relation \mathbf{l}_{yaxb} . Similarly, invariance under (34) amounts to the analog for right translations, and invariance under (12)(34) to the analog for middle multiplication operators:

$$(\mathbf{l}_{xayb})^{-1} = \mathbf{l}_{yaxb}, \quad (\mathbf{r}_{aybz})^{-1} = \mathbf{r}_{azby}, \quad (\mathbf{m}_{xabz})^{-1} = \mathbf{m}_{zabx}.$$

Note that this is the exact analog of Lemma 9.2; however, since a, b need not be transversal subgroups, these relations now apply to *semitorsors* as well. In a certain sense, this means that our semitorsors have the same “symmetry type as a torsor” – a property which certainly distinguishes them from “arbitrary” semitorsors.

10. GENERALIZED LATTICE STRUCTURES

For abelian groups, Theorem 2.4 of [BeKi10a] establishes a close link between the structure map Γ and the *lattice of subgroups*. For non-abelian groups Ω , the subgroups form no longer a lattice. Nevertheless, the two set-theoretic operations

$$x \wedge y := x \cap y, \quad x + y := \{\omega \in \Omega \mid \exists \xi \in x, \exists \eta \in y : \omega = \xi + \eta\}$$

behave very much like “meet” (\wedge) and “join” (\vee), as shows the following analog of Theorem 2.4 of [BeKi10a]:

Theorem 10.1. *Let x, a, y, b, z be subgroups of Ω . Then we have:*

- (1) *values of Γ on the “diagonal $x = y$ ”:*

$$\begin{aligned} \Gamma(x, a, x, b, z) &= \left((x \wedge a) + z \right) \wedge (x + b) \\ &= (x \wedge a) + \left(z \wedge (x + b) \right), \end{aligned}$$

- (2) *values of Γ on the “diagonal $a = z$ ”:*

$$\begin{aligned} \Gamma(x, a, y, b, a) &= \left((x \wedge (a + y)) + b \right) \wedge a \\ &= \left(x + ((y + a) \wedge b) \right) \wedge a, \end{aligned}$$

- (3) *values of Γ on the “diagonal $b = z$ ”:*

$$\Gamma(x, a, y, b, b) = \left((a + (y \wedge b)) \wedge x \right) + b$$

$$= \left(a \wedge (x + (y \wedge b)) \right) + b .$$

This implies, in particular, that for all $x, a, y, b \in \text{Gras}(\Omega)$,

$$\begin{aligned} \Gamma(x, a, x, b, x) &= x \\ \Gamma(a, a, y, b, b) &= a + b \\ \Gamma(b, a, y, b, a) &= a \wedge b . \end{aligned}$$

Proof. The technical details of the the proofs are exactly as in [BeKi10a], loc. cit., by respecting the possible non-commutativity of Ω (and hence of the operation $+$), so let us here only prove the first equality from item (1). As in loc. cit. it is always understood that $\alpha \in a$, $\xi \in x$, $\beta \in b$, $\eta \in y$, $\zeta \in z$. We use System (2.9).

Let $\omega \in \Gamma(x, a, x, b, z)$, then $\omega = \xi - \beta$, hence $\omega \in (x + b)$, and $\omega = \xi - \eta + \zeta$ with $v := \omega - \zeta = \xi - \eta \in x$ (since $x = y$). On the other hand, $v = \omega - \zeta = -\alpha \in a$, whence $\omega = v + \zeta$ with $v \in (x \wedge a)$, proving one inclusion. (Note that we have used that x and a are subgroups and that b is symmetric.)

Conversely, let $\omega \in ((x \wedge a) + z) \wedge (x + b)$. Then $\omega = \xi + \beta = \alpha + \zeta$ with $\alpha \in (x \wedge a)$. Let $\eta := -\alpha + \xi$. Then $\eta \in x$, and $\omega = \xi + \beta = \alpha + \eta + \beta$, hence $\omega \in \Gamma(x, a, x, b, z)$.

The remaining proofs are similar. Note that, since the systems in (2.8)–(2.14) come in pairs, there are in fact two different expressions for one diagonal value. For the final conclusion, one uses the “absorption laws” in the following form:

Lemma 10.2. Recall that $\mathcal{P}^o = \{x \in \mathcal{P}(\Omega) \mid o \in x\}$.

- (1) If $y \in \mathcal{P}^o$, then $x \wedge (x + y) = x$.
- (2) If $y \in \mathcal{P}^o$ and $x \in \text{Gras}(\Omega)$, then $x + (x \wedge y) = x = (x \wedge y) + x$.

□

Remark. It is remarkable that for subgroups x, a, z a “non-commutative modular law” is still satisfied (cf. Remark 2.5 of [BeKi10a]): by letting $b = x$ in (1), we get

$$((x \wedge a) + z) \wedge x = (x \wedge a) + (z \wedge x) ,$$

and letting $a = x$ we get the “dual modular law”

$$x + (z \wedge (x + b)) = (x + z) \wedge (x + b) .$$

More generally, one has the impression that the formulas express some sort of “duality” between the operations \wedge and $+$. This would be rather mysterious since, at a first glance, the definitions of \wedge and $+$ look quite “non-symmetric”.

11. FINAL REMARKS

Since *groups* and *projective spaces* are foundational concepts in mathematics, the present approach is likely to interact with many areas of mathematics. The author’s original motivation came from *non-associative algebra*, and in particular *Jordan theory* – this domain does not really belong to today’s mathematical mainstream, and thus the shape of the approach presented here may seem quite unusual for a “normal user of group theory”. The reader may find more motivation and general remarks in the introductory and concluding sections of preceding work, e.g., [BeKi10a, BeKi10b, BeNe05, BeL08, Be02].

In the following, let us add some short comments on aspects for which the generalization of the framework to general groups and general torsors of mappings may be relevant. Most importantly, one needs to study *categorical aspects* more systematically; we hope to come back to such items in subsequent work.

11.1. Morphisms. Projective spaces or general Grassmannians can be turned into categories in two essentially different ways (cf. [Be02, BeKi10a]). The same is true in the present context. One may conjecture that an analog of the “fundamental theorem of projective geometry” holds: *every automorphism of $(\mathcal{P}(\Omega), \Gamma)$ is induced by an automorphism of Ω* (if Ω is not too small). However, the main difference is that now, in the non-abelian case, “interior maps” (left-, right- and middle multiplication operators) are no longer morphisms in either of the two categories. One may wonder whether they are morphisms in yet another sense.

11.2. Antiautomorphisms, involutions. For the case of commutative Ω , see [BeKi10b]. If Ω is non-commutative, the situation seems to change drastically: first of all, in principle, each permutation of the 5 arguments of Γ gives rise to its own notion of “anti-homomorphism”. The simplest case corresponds to anti-homomorphisms on the level of Ω : they correspond to “usual” morphisms between Γ and $\check{\Gamma}$. For instance, the inversion map of Ω induces an “involution” of this kind. It corresponds to the permutation $(\xi\zeta)(\alpha\beta)$, which belongs to \mathbf{V} . On the other hand, looking at permutations not belonging to \mathbf{V} , one may ask whether anti-homomorphisms corresponding to the permutation $(\xi\zeta)$ exist: these would generalize the involutions considered in [BeKi10b]. In the commutative case, they are given by *orthocomplementation maps*. In particular, they induce *lattice antiautomorphisms*. As mentioned above, our formulas suggest that some kind of duality in this sense exists; on the other hand, there seems to be no hope to generalize orthocomplementation maps in some obvious way.

11.3. Subobjects. A *subspace of $\mathcal{P}(\Omega)$* is a subset $\mathcal{Y} \subset \mathcal{P}(\Omega)$ stable under Γ . Such sets may be defined by algebraic conditions (cf. Theorem 3.3), or by topological or differential conditions: e.g., if Ω is a topological or Lie group (and in particular, for $\Omega = \mathbb{R}^{2n}$), we may consider spaces \mathcal{Y} of *closed* subsets or of *smooth* or *algebraic submanifolds*. One expects that Γ will have “singularities”, so one possibly has to exclude some “singular sets”: outside such sets we expect Γ to be fairly regular (cf. related results in [BeNe05]). The analogy with near-rings suggests also to look at subspaces corresponding to *near-fields*.

11.4. Ideals, inner ideals, intrinsic subspaces. As in rings or near-rings, or in Jordan algebraic structures, one may define notions of certain subobjects playing the rôle of various kinds of “ideals”, and which will be of importance for a systematic “structure theory”. Such ideals are defined as subobjects \mathcal{Y} defining conditions like the “inner ideal condition” $\Gamma(\mathcal{Y}, \mathcal{P}, \mathcal{Y}, \mathcal{P}, \mathcal{Y}) \subset \mathcal{Y}$. In a Jordan theoretic context, such sets have been characterized in [BeL08] as “intrinsic subspaces”.

11.5. Products. Our construction is compatible with *direct products*. For instance, $\mathcal{P}(\Omega_1) \times \mathcal{P}(\Omega_2)$ is a subspace of $\mathcal{P}(\Omega_1 \times \Omega_2)$. Here, the case $\Omega_1 = \Omega_2 = \Omega$ is particularly important since elements of $\mathcal{P}(\Omega \times \Omega)$ are nothing but *relations* on Ω ,

and hence may play the rôle of *endomorphisms*, as discussed above. This situation is characterized by the existence of *transversal triples* (Theorem 4.6) and of projections that are endomorphisms.

11.6. Axiomatic approach, base points, and equivalence of categories. Following [BeKi10a], one may consider as *base point* a fixed pair (o^+, o^-) of transversal subgroups, and then look at the “pair of ternary near-rings” (π^+, π^-) defined by Theorem 7.1 as a sort of “tangent object”. In which sense can one say, then, that the theory of such objects is equivalent to the theory of $\mathcal{P}(\Omega)$ – is there an equivalence of categories between “tangent objects” and “geometries with base point”?

11.7. Symmetric spaces, and Jordan theory. Unlike the three “diagonals” mentioned in Theorem 10.1, the “diagonal $x = z$ ” is not related to lattice theory, but rather to *symmetric spaces*: in any torsor, the ternary composition gives rise to a binary map $\mu(x, y) := (xyx)$, which (in the case of a Lie group) is precisely the underlying “symmetric space structure” (in the sense of [Lo69]). Thus, automatically, our torsors U_{ab} give rise to families of symmetric spaces. Other symmetric spaces can be constructed from these in presence of an *involution*. Therefore, symmetric spaces are a main geometric ingredient for the theory corresponding to the “diagonal $x = y$ ”, which, in turn, would be a kind of “non-commutative Jordan theory”. Many of the surprising features of Jordan theory are due to the fact that, because of the symmetry $(\alpha\zeta)(\xi\beta)$ (Theorem 9.8), this theory essentially is the same as the theory of the “diagonal $a = b$ ”, i.e., of the family of torsors U_{aa} : thus there should be some kind of duality between certain families of symmetric spaces and certain families of torsors.

11.8. Flag geometries. For Jordan theory, the *projective geometry of a Lie algebra* defined in [BeNe04] gives a useful “universal geometric model”. The construction is similar in spirit to the present work; however, it is not clear at all how to carry it out on the level of groups. Essentially, one needs a definition of an analog of our maps Γ and Σ for (*short*) *flags*, that is, for pairs of subsets (x, x') with $o \in x \subset x' \subset \Omega$, instead of single subsets. Some results by J. Chenal ([Ch09]) point in the direction that such constructions should be possible for general spaces of finite flags.

11.9. Reductive groups, finite groups. For a reductive Lie group Ω , how is the projective geometry of Ω related to well-known structure theory? Is there some link with the notion of *building*? Is the projective geometry of the *Weyl group* related in some definite way to the projective geometry of Ω ?

11.10. Supersymmetry. We have the impression that Section 9 on “symmetry” is closely related to the topic of *supersymmetry*: indeed, the behavior of the “signed versions” under permutations reminds the “sign rule” of supersymmetry. On a very fundamental level, our map Γ takes account of the principle that there is no reason to prefer a group to its opposite group: in principle, both should play symmetric rôles. However, in presence of additional structure (two or more operations) this symmetry may be broken: e.g., there may be “left-distributivity”, but not “right-distributivity”. As the tables in Theorem 9.6 show, a complete book-keeping of such situations is not entirely trivial. The next level in such a book-keeping should

be reached when we are dealing with $\mathcal{P}(\Omega \times \Omega)$: namely, subsets of $\Omega \times \Omega$ are just relations on Ω , and here again, there is no reason to prefer “usual” composition to its opposite; in other words, we have to fix choices and conventions for the structure maps $\Gamma_{\Omega \times \Omega}$ of the group $\Omega \times \Omega$, and similarly for iterated products Ω^k . *Supersymmetry* might turn out to be one of the sign-rules that are used for such book-keeping of iterated products.

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