Spherical Hecke algebras for Kac-Moody groups over local fields

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Abstract

We define the spherical Hecke algebra $\mathcal{H}$ for an almost split Kac-Moody group G over a local non-archimedean field. We use the hovel $\mathscr{I}$ associated to this situation, which is the analogue of the Bruhat-Tits building for a reductive group. The stabilizer K of a special point on the standard apartment plays the role of a maximal open compact subgroup. We can define $\mathcal{H}$ as the algebra of $K-$bi-invariant functions on G with almost finite support. As two points in the hovel are not always in a same apartment, this support has to be in some large subsemigroup G^+ of G. We prove that the structure constants of $\mathcal{H}$ are polynomials in the cardinality of the residue field, with integer coefficients depending on the geometry of the standard apartment. Actually, our results apply to abstract "locally finite" hovels. When G is split, we prove that $\mathcal{H}$ is commutative.

Introduction

Let G be a connected reductive group over a local non-archimedean field $\mathcal{K}$ and K an open compact subgroup. The space $\mathcal{H}$ of complex functions on G, bi-invariant by K and with compact support is an algebra for the natural convolution product. Ichiro Satake [Sa63] studied this algebra $\mathcal{H}$ to define the spherical functions and proved, in particular, that $\mathcal{H}$ is commutative for good choices of K. We know now that one of the good choices for K is the fixator of some special vertex for the action of G on its Bruhat-Tits building $\mathscr{I}$, whose structure is explained in [BrT72]. Moreover $\mathcal{H}$, now called spherical Hecke algebra, may be entirely defined with $\mathscr{I}$, see *e.g.* [P06].

Kac-Moody groups are interesting generalizations of reductive groups and it is natural to try to generalize the spherical Hecke algebra to the case of a Kac-Moody group. But there is now no good topology on G and no good compact subgroup, so the "convolution product" has to be defined only with algebraic means. Alexander Braverman and David Kazhdan [BrK10] succeeded in defining such a spherical Hecke algebra, when G is split and untwisted affine. For a well chosen subgroup K, they define $\mathcal{H}$ as an algebra of $K-$bi-invariant complex functions with "almost finite" support. There are two new features: the support has to be in a subsemigroup G^+ of G and it is an infinite union of double classes. Hence, $\mathcal{H}$ is naturally a module over the ring of complex Laurent polynomials.

Our idea is to build this spherical Hecke algebra using the hovel associated to the almost split Kac-Moody group G that we built in [GR08], [Ro12] and [Ro13]. This hovel $\mathscr{I}$ is a set with an action of G and a covering by subsets called apartments, in one-to-one correspondence with the maximal split subtori, hence permuted transitively by G. Each apartment A is a finite dimensional real affine space and its stabilizer N in G acts on it via a generalized affine Weyl group $W = W^v \ltimes Y$ (where $Y \subset \overrightarrow{A}$ is a discrete subgroup of translations) which stabilizes a set $\mathcal{M}$ of affine hyperplanes called walls. So $\mathscr{I}$ looks much like the Bruhat-Tits building of a reductive group, but $\mathcal{M}$ is not a locally finite system of hyperplanes (as the root system Φ is infinite) and two points in $\mathscr{I}$ are not always in a same apartment (this is why $\mathscr{I}$ is called a hovel). There is on $\mathscr{I}$ a $G-$invariant preorder $\leq$ which induces on each apartment A the preorder given by the Tits cone $\mathcal{T} \subset \overrightarrow{A}$.

Now, we consider the fixator K in G of a special point 0 in a chosen standard apartment A. The spherical Hecke algebra $\mathcal{H}_R$ is a space of $K-$bi-invariant functions on G with values in a ring R, *i.e.* the space $\mathcal{H}_R^{\mathscr{I}}$ of $G-$invariant functions on $\mathscr{I}_0 \times \mathscr{I}_0$ where $\mathscr{I}_0 = G/K$ is the orbit of 0 in $\mathscr{I}$. The convolution product is easy to guess from this point of view: $(\varphi * \psi)(x,y) = \sum_{z\in\mathscr{I}_0} \varphi(x,z)\psi(z,y)$ (if this sum means something). As two points x, y in $\mathscr{I}$ are not always in a same apartment (*i.e.* the Cartan decomposition fails: $G \neq KNK$), we have to consider pairs $(x,y) \in \mathscr{I}_0 \times \mathscr{I}_0$, with $x \leq y$ (this implies that x, y are in a same apartment). For $\mathcal{H}_R$ this means that the support of $\varphi \in \mathcal{H}_R$ has to be in $K\backslash G^+/K$ where $G^+ = \{g \in G \mid 0 \leq g.0\}$ is a semigroup. Then $K\backslash G^+/K$ is in one-to-one correspondence with the subsemigroup $Y^{++} = Y \cap C_f^v$ of Y (where C_f^v is the fundamental Weyl chamber). Now, to get a well defined convolution product, we have to ask (as in [BrK10]) the support $supp(\varphi)$ of a $\varphi \in \mathcal{H}_R$ to be almost finite: $supp(\varphi) \subset \bigcup_{i=1}^n (\lambda_i - Q_+^\vee) \cap Y^{++}$, where $\lambda_i \in Y^{++}$ and $Q_+^\vee$ is the subsemigroup of Y generated by the fundamental coroots. Note that $(\lambda - Q_+^\vee) \cap Y^{++}$ is infinite except when G is reductive.

With this definition we are able to prove that $\mathcal{H}_R$ is really an algebra, which generalizes the known spherical Hecke algebras in the finite or affine split case (§2). In the split case, we prove that $\mathcal{H}_R$ is commutative (§3).

The structure constants of $\mathcal{H}_R$ are the non-negative integers $m_{\lambda,\mu}(\nu)$ (for $\lambda, \mu, \nu \in Y^{++}$) such that $c_\lambda * c_\mu = \sum_{\nu\in Y^{++}} m_{\lambda,\mu}(\nu)c_\nu$, where c_λ is the characteristic function of $K\lambda K$. Each chamber (= alcove) in $\mathscr{I}$ has only a finite number of adjacent chambers along a given panel. These numbers are called parameters of $\mathscr{I}$ and their set $\mathcal{Q}$ is finite; in the split case, there is only one parameter q: the number of elements of the residue field κ of $\mathcal{K}$. In §4 we show that the structure constants are polynomials in these parameters with integral coefficients depending only on the geometry of an apartment.

Actually this article is written in a more general framework (explained in §1): we ask $\mathscr{I}$ to be an abstract ordered hovel (as defined in [Ro11]) and G a strongly transitive group of automorphisms.

The general definition and study of Hecke algebras for split Kac-Moody groups over local fields was also undertaken by Alexander Braverman, David Kazhdan and Manish Patnaik (as we knew from [P10]). A preliminary draft appeared recently [BrKP12]. Their arguments are algebraic without use of a geometric object as a hovel, and the proofs seem complete (temporarily?) only for the untwisted affine case. In addition to the construction of the spherical Hecke algebra (as here), they prove the Satake isomorphism, they give a formula for spherical functions and they build the Iwahori-Hecke algebra. We hope to generalize, in a near future, these results to our general framework.

One should notice that these authors use, instead of our group K, a smaller K_1, a priori slightly different. With the notations of 3.4 below, K_1 is generated by T_0, U_0^+ and U_0^-, hence $K = U_0^{nm-}.K_1 = U_0^{pm+}.K_1$, with $U_0^- \subset U_0^{nm-} \subset U^-$ and $U_0^+ \subset U_0^{pm+} \subset U^+$. But they prove, at least in the untwisted affine case, that $U^- \cap U^+.K_1 \subset K_1$ [BrKP12, proof of 6.4.3];

so $U_0^{nm-} \subset U^- \cap K \subset U^- \cap U^+.K_1 \subset K_1$ and $K = K_1$. This result answers positively a question in [Ro13, 5.4], at least for points of type 0 and in the untwisted affine split case.

1 General framework

1.1 Vectorial data

We consider a quadruple $(V, W^v, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ where V is a finite dimensional real vector space, W^v a subgroup of $GL(V)$ (the vectorial Weyl group), I a finite set, $(\alpha_i^\vee)_{i\in I}$ a family in V and $(\alpha_i)_{i\in I}$ a free family in the dual V^*. We ask these data to verify the conditions of [Ro11, 1.1]. In particular, the formula $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ defines in V a linear involution which is in W^v and $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system.

To be more concrete we consider the Kac-Moody case of [*l.c.* ; 1.2]: the matrix $\mathbb{M} = (\alpha_j(\alpha_i^\vee))_{i,j\in I}$ is a generalized Cartan matrix. Then W^v is the Weyl group of the corresponding Kac-Moody Lie algebra $\mathfrak{g}_{\mathbb{M}}$ and the associated real root system is $\Phi = \{w(\alpha_i) \mid w \in W^v, i \in I\} \subset Q = \bigoplus_{i\in I} \mathbb{Z}.\alpha_i$. We set $\Phi^\pm = \Phi \cap Q^\pm$ where $Q^\pm = \pm(\bigoplus_{i\in I}(\mathbb{Z}_{\geq 0}).\alpha_i)$ and $Q^\vee = (\bigoplus_{i\in I} \mathbb{Z}.\alpha_i^\vee)$, $Q^\vee_\pm = \pm(\bigoplus_{i\in I}(\mathbb{Z}_{\geq 0}).\alpha_i^\vee)$. We have $\Phi = \Phi^+ \cup \Phi^-$ and, for $\alpha = w(\alpha_i) \in \Phi$, $r_\alpha = w.r_i.w^{-1}$ and $\alpha^\vee = w(\alpha_i^\vee)$ depend only on α, and $r_\alpha(v) = v - \alpha(v).\alpha^\vee$.

The set Φ is an (abstract reduced) real root system in the sense of [MoP89], [MoP95] or [Ba96]. We shall sometimes also use the set $\Delta = \Phi \cup \Delta^+_{im} \cup \Delta^-_{im}$ of real or imaginary roots (with $-\Delta^-_{im} = \Delta^+_{im} \subset Q^+$, W^v-stable) defined in [Ka90]. It is an (abstract reduced) root system in the sense of [Ba96].

The *fundamental positive chamber* is $C^v_f = \{v \in V \mid \alpha_i(v) > 0, \forall i \in I\}$. Its closure $\overline{C^v_f}$ is the disjoint union of the vectorial facets $F^v(J) = \{v \in V \mid \alpha_i(v) > 0, \forall i \in J, \alpha_i(v) > 0, \forall i \in I \setminus J\}$ for $J \subset I$. The positive (resp. negative) vectorial facets are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$.

The *Tits cone* $\mathcal{T}$ is the (disjoint) union of the positive vectorial facets. It is a W^v-stable convex cone in V.

1.2 The model apartment

As in [Ro11, 1.4] the model apartment $\mathbb{A}$ is V considered as an affine space and endowed with a family $\mathcal{M}$ of walls. These walls are affine hyperplanes directed by Kerα for $\alpha \in \Phi$.

We ask this apartment to be **semi-discrete** and the origin 0 to be **special**. This means that these walls are defined as follows: $\mathcal{M} = \{M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\} \mid \alpha \in \Phi, k \in \Lambda_\alpha\}$ (with $\Lambda_\alpha = k_\alpha.\mathbb{Z}$ a non trivial discrete subgroup of $\mathbb{R}$). Using the following lemma (*i.e.* replacing Φ by $\widetilde{\Phi}$) we shall assume $\Lambda_\alpha = \mathbb{Z}, \forall \alpha \in \Phi$.

For $\alpha = w(\alpha_i) \in \Phi$, $k \in \Lambda_\alpha (= \mathbb{Z})$ and $M = M(\alpha, k)$, the reflection $r_{\alpha,k} = r_M$ with respect to M is the affine involution of $\mathbb{A}$ with fixed point set the wall M and associated linear involution r_α. The affine Weyl group W^a is the group generated by the reflections r_M for $M \in \mathcal{M}$; it has to stabilize $\mathcal{M}$.

With these data we can define in $\mathbb{A}$ half-apartments, sectors, sector-faces, facets, enclosures... as in [GR08] or [Ro11]; see also [Ro12], [Ro13].

There is a preorder on $\mathbb{A}$ defined by $x \leq y \iff y - x \in \mathcal{T}$.

Lemma 1.3. *For all* $\alpha \in \Phi$ *we choose* $k_\alpha > 0$ *and define* $\widetilde{\alpha} = \alpha/k_\alpha$, $\widetilde{\alpha}^\vee = k_\alpha.\alpha^\vee$. *Then* $\widetilde{\Phi} = \{\widetilde{\alpha} \mid \alpha \in \Phi\}$ *is the (abstract reduced) real root system (in the sense of [MoP89], [MoP95] or*

[Ba96]) associated to $(V, W^v, (k_{\alpha_i}^{-1}.\alpha_i)_{i\in I}, (k_{\alpha_i}.\alpha_i^\vee)_{i\in I})$ *hence to the generalized Cartan matrix* $\widetilde{\mathbb{M}} = (k_{\alpha_j}^{-1}.\alpha_j(k_{\alpha_i}.\alpha_i^\vee))_{i,j\in I}$*. Moreover with* $\widetilde{\Phi}$*, the walls are described using the subgroups* $\widetilde{\Lambda}_\alpha = \mathbb{Z}$*.*

Proof. For $\alpha, \beta \in \Phi$, the group W^a contains the translation τ by $k_\alpha.\alpha^\vee$ and $\tau(M(\beta, 0)) = M(\beta, -\beta(k_\alpha.\alpha^\vee))$. So $k_\alpha.\beta(\alpha^\vee) \in \Lambda_\beta$ *i.e.* $\widetilde{\beta}(\widetilde{\alpha}^\vee) = k_\beta^{-1}.k_\alpha.\beta(\alpha^\vee) \in \mathbb{Z}$. Hence $\widetilde{\mathbb{M}} = (k_{\alpha_j}^{-1}.\alpha_j(k_{\alpha_i}.\alpha_i^\vee))_{i,j\in I}$ is a generalized Cartan matrix and the lemma is clear, as $k_{w\alpha} = k_\alpha$. □

1.4 The hovel

We consider a thick ordered affine hovel $\mathscr{I}$ of type $\mathbb{A}$ as in [Ro11] with a strongly transitive group of automorphisms G (*i.e.* all isomorphisms involved in the axioms (MAi) of [Ro11] are induced by elements of G, *cf.* [Ro13, 4.10]). We choose in $\mathscr{I}$ a fundamental apartment which we identify with $\mathbb{A}$. As G is strongly transitive the apartments of $\mathscr{I}$ are the sets $g.\mathbb{A}$ for $g \in G$ and the stabilizer N of $\mathbb{A}$ in G induces in $\mathbb{A}$ a group $\nu(N)$ of affine automorphisms which permutes the walls, sectors, sector-faces... and contains the affine Weyl group W^a [Ro13, 4.13.1]. We denote G_0 the fixator of $0 \in \mathbb{A}$ in G by K.

We ask $\mathscr{I}$ to be of **finite thickness**: the number of chambers (=alcoves) containing a given panel has to be finite (≥ 3). This number is the same for any panel in a given wall M [Ro11, 2.9]; we denote it by $1 + q_M$.

We ask $\nu(N)$ to be **type-preserving** for its action on the vectorial facets. This means that the associated linear map $\overrightarrow{w}$ of any $w \in \nu(N)$ is in W^v. As $\nu(N)$ contains W^a and stabilizes $\mathcal{M}$, we have $\nu(N) = W^v \ltimes Y$, where W^v fixes the origin 0 of $\mathbb{A}$ and Y is a group of translations such that: $Q^\vee \subset Y \subset P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z}, \forall \alpha \in \Phi\}$.

We ask Y to be **discrete** in V. This is clearly satisfied if Φ generates V^* *i.e.* $(\alpha_i)_{i\in I}$ is a basis of V^*.

Examples: The main examples of all the above situation are provided by the hovels of almost split Kac-Moody groups over fields complete for a discrete valuation and with a finite residue field, see [Ro12], [Ch10], [Ch11] or [Ro13]. Some details in the split case can be found in Section 3.

Remarks: a) In the following we often refer to [GR08] which deals with split Kac-Moody groups and residue fields containing $\mathbb{C}$. But the results cited are easily generalized to our present framework, using the above references.

b) For an almost split Kac-Moody group over a local field $\mathcal{K}$, the set of roots Φ is ${}^{\mathcal{K}}\Phi_{red} = \{{}^{\mathcal{K}}\alpha \in {}^{\mathcal{K}}\Phi \mid \frac{1}{2}.{}^{\mathcal{K}}\alpha \notin {}^{\mathcal{K}}\Phi\}$ where the relative root system ${}^{\mathcal{K}}\Phi$ describes well the commuting relations between the root subgroups. Unfortunately $\widetilde{\Phi}$ gives a worst description of these relations.

1.5 Type 0 vertices

The elements of Y considered as the subset $Y = N.0$ of $V = \mathbb{A}$ are called *vertices of type* 0 in $\mathbb{A}$; they are special vertices. We note $Y^+ = Y \cap \mathcal{T}$ and $Y^{++} = Y \cap \overline{C_f^v}$. The type 0 vertices in $\mathscr{I}$ are the points on the orbit $\mathscr{I}_0$ of 0 by G. This set $\mathscr{I}_0$ is often called the affine Grassmannian as it is equal to G/K.

In general, G is not equal to $KYK = KNK$ [GR08, 6.10] *i.e.* $\mathscr{I}_0 \neq K.Y$.

We know that $\mathscr{I}$ is endowed with a $G-$invariant preorder $\leq$ which induces the known one on $\mathbb{A}$ [Ro11, 5.9]. We set $\mathscr{I}^+ = \{x \in \mathscr{I} \mid 0 \leq x\}$, $\mathscr{I}_0^+ = \mathscr{I}_0 \cap \mathscr{I}^+$ and $G^+ = \{g \in G \mid 0 \leq g.0\}$;

so $\mathscr{I}_0^+ = G^+.0 = G^+/K$. As $\leq$ is a $G-$invariant preorder, G^+ is a semigroup.

If $x \in \mathscr{I}_0^+$ there is an apartment A containing 0 and x (by definition of $\leq$) and all apartments containing 0 are conjugated to $\mathbb{A}$ by K (axiom (MA2) of [Ro11]); so $x \in K.Y^+$ as $\mathscr{I}_0^+ \cap \mathbb{A} = Y^+$. But $\nu(N \cap K) = W^v$ and $Y^+ = W^v.Y^{++}$ (with unicity of the element in Y^{++}); so $\mathscr{I}_0^+ = K.Y^{++}$, more precisely $\mathscr{I}_0^+ = G^+/K$ is the disjoint union of the KyK/K for $y \in Y^{++}$.

We have proved that the map $Y^{++} \to K\backslash G^+/K$ is one-to-one and onto.

1.6 Vectorial distance and $Q^\vee-$order

For $x \in \mathcal{T}$, we note x^{++} the unique element in $\overline{C_f^v}$ conjugated by W^v to x.

Let $\mathscr{I} \times_\leq \mathscr{I} = \{(x, y) \in \mathscr{I} \times \mathscr{I} \mid x \leq y\}$ be the set of increasing pairs in $\mathscr{I}$. Such a pair (x, y) is always in a same apartment $g\mathbb{A}$; so $g^{-1}y - g^{-1}x \in \mathcal{T}$ and we define the *vectorial distance* $d^v(x, y) \in \overline{C_f^v}$ by $d^v(x, y) = (g^{-1}y - g^{-1}x)^{++}$. It does not depend on the choices made.

For $(x, y) \in \mathscr{I}_0 \times_\leq \mathscr{I}_0 = \{(x, y) \in \mathscr{I}_0 \times \mathscr{I}_0 \mid x \leq y\}$, $d^v(x, y) \in Y^{++}$. Actually, as $\mathscr{I}_0 = G.0$, K is the fixator of 0 and $\mathscr{I}_0^+ = K.Y^{++}$ (with unicity of the element in Y^{++}), the map d^v induces a bijection between the set $\mathscr{I}_0 \times_\leq \mathscr{I}_0/G$ of orbits of G in $\mathscr{I}_0 \times_\leq \mathscr{I}_0$ and Y^{++}.

Any $g \in G^+$ is in $K.d^v(0, g0).K$.

For $x, y \in \mathbb{A}$, we say that $x \leq_{Q^\vee} y$ (resp. $x \leq_{Q^\vee_\mathbb{R}} y$) when $y - x \in Q^\vee_+$ (resp. $y - x \in Q^\vee_{\mathbb{R}+} = \sum_{i \in I} \mathbb{R}_{\geq 0}.\alpha_i^\vee$). We get thus a preorder which is an order at least when $(\alpha_i^\vee)_{i\in I}$ is free or $\mathbb{R}_+-$free (*i.e.* $\sum a_i\alpha_i^\vee = 0, a_i \geq 0 \Rightarrow a_i = 0, \forall i$).

1.7 Paths

We consider piecewise linear continuous paths $\pi : [0, 1] \to \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ is in an orbit $W^v.\lambda$ of some $\lambda \in \overline{C_f^v}$ under the vectorial Weyl group W^v. Such a path is called a $\lambda-$*path*; it is increasing with respect to the preorder relation on $\mathbb{A}$.

For any $t \neq 0$ (resp. $t \neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π at t from the left (resp. from the right). Further, we define $w_\pm(t) \in W^v$ to be the smallest element in its $(W^v)_\lambda-$class such that $\pi'_\pm(t) = w_\pm(t).\lambda$ (where $(W^v)_\lambda$ is the fixator in W^v of λ). Moreover, we denote by $\pi_-(t) = \pi(t) - [0, 1)\pi'_-(t) = [\pi(t), \pi(t - \varepsilon))$ (resp. $\pi_+(t) = \pi(t) + [0, 1)\pi'_+(t) = [\pi(t), \pi(t + \varepsilon))$ (for $\varepsilon > 0$ small) the positive (resp. negative) segment-germ of π at t.

The reverse path $\overline{\pi}$ defined by $\overline{\pi} = \pi(1 - t)$ has symmetric properties, it is a $(-\lambda)-$path.

For any choices of $\lambda \in \overline{C_f^v}$, $\pi_0 \in \mathbb{A}$, $r \in \mathbb{N} \setminus \{0\}$ and sequences $\underline{\tau} = (\tau_1, \tau_2, \ldots, \tau_r)$ of elements in $W^v/(W^v)_\lambda$ and $\underline{a} = (a_0 = 0 < a_1 < a_2 < \cdots < a_r = 1)$ of elements in $\mathbb{R}$, we define a $\lambda-$path $\pi = \pi(\lambda, \pi_0, \underline{\tau}, \underline{a})$ by the formula:

$$\pi(t) = \pi_0 + \sum_{i=1}^{j-1} (a_i - a_{i-1})\tau_i(\lambda) + (t - a_{j-1})\tau_j(\lambda) \quad \text{for} \quad a_{j-1} \leq t \leq a_j.$$

Any $\lambda-$path may be defined in this way (and we may assume $\tau_j \neq \tau_{j+1}$).

Definition 1.8. [KM08, 3.27] A *Hecke path* of shape λ with respect to $-C_f^v$ is a $\lambda-$path such that, for all $t \in [0, 1]\setminus\{0, 1\}$, $\pi'_+(t) \leq_{W^v_{\pi(t)}} \pi'_-(t)$, which means that there exists a $W^v_{\pi(t)}-$chain from $\pi'_-(t)$ to $\pi'_+(t)$, *i.e.* finite sequences $(\xi_0 = \pi'_-(t), \xi_1, \ldots, \xi_s = \pi'_+(t))$ of vectors in V and $(\beta_1, \ldots, \beta_s)$ of real roots such that, for all $i = 1, \ldots, s$:

i) $r_{\beta_i}(\xi_{i-1}) = \xi_i$,

ii) $\beta_i(\xi_{i-1}) < 0$,

iii) $r_{\beta_i} \in W^v_{\pi(t)}$ *i.e.* $\beta_i(\pi(t)) \in \mathbb{Z}$: $\pi(t)$ is in a wall of direction $\mathrm{Ker}(\beta_i)$.

iv) each β_i is positive with respect to $-C^v_f$ *i.e.* $\beta_i(C^v_f) > 0$.

Remarks 1.9. 1) The path is folded at $\pi(t)$ by applying successive reflections along the walls $M(\beta_i, -\beta_i(\pi(t)))$. Moreover conditions ii) and iv) tell us that the path is "positively folded" (cf. [GL05]) *i.e.* centrifugally folded with respect to the sector germ $\mathfrak{S}_{-\infty} = germ_\infty(-C^v_f)$.

2) Let $\mathfrak{c}_- = germ_0(-C^v_f)$ be the negative fundamental chamber (= alcove). A *Hecke path* of shape λ with respect to $\mathfrak{c}_-$ [BCGR11] is a $\lambda-$path in the Tits cone $\mathcal{T}$ satisfying the above conditions except that we replace iv) by :

iv') each β_i is positive with respect to $\mathfrak{c}_-$ *i.e.* $\beta_i(\pi(t) - \mathfrak{c}_-) > 0$.

Then ii) and iv') tell us that the path is centrifugally folded with respect to the center $\mathfrak{c}_-$.

2 Convolution algebras

2.1 Wanted

We consider the space $\widehat{\mathcal{H}}^{\mathscr{I}}_R = \widehat{\mathcal{H}}_R(\mathscr{I}, G) = \{\varphi : \mathscr{I}_0 \times_\leq \mathscr{I}_0 \to R \mid \varphi(gx, gy) = \varphi(x, y), \forall g \in G\}$ of $G-$invariant functions on $\mathscr{I}_0 \times_\leq \mathscr{I}_0$ with values in a ring R (essentially $\mathbb{C}$ or $\mathbb{Z}$). We want to make $\widehat{\mathcal{H}}^{\mathscr{I}}_R$ (or some large subspace) an algebra for the following convolution product:

$(\varphi * \psi)(x, y) = \sum_{x\leq z\leq y} \varphi(x, z)\psi(z, y)$

It is clear that this product is associative and $R-$bilinear if it exists.

Via d^v, $\widehat{\mathcal{H}}^{\mathscr{I}}_R$ is linearly isomorphic to the space $\widehat{\mathcal{H}}_R = \{\varphi : Y^{++} = K\backslash G^+/K \to R\}$, which can be interpreted as the space of $K-$bi-invariant functions on G^+. We consider the subspace $\mathcal{H}^f_R$ of functions with finite support in $Y^{++} = K\backslash G^+/K$; its natural basis is $(c_\lambda)_{\lambda\in Y^{++}}$ where c_λ sends λ to 1 and $\mu \neq \lambda$ to 0. Clearly c_0 is a unit for $*$.

In this setting the convolution product should be: $(\varphi * \psi)(g) = \sum_{h\in G^+/K} \varphi(h)\psi(h^{-1}g)$, where we consider φ and ψ as trivial on $G \setminus G^+$.

In $\widehat{\mathcal{H}}^{\mathscr{I}}_R$, $(c_\lambda * c_\mu)(x, y)$ is the number of triangles $[x, z, y]$ with $d^v(x, z) = \lambda$ and $d^v(z, y) = \mu$.

As suggested by [BrK10] and lemma 2.4, we consider also the subspace $\mathcal{H}_R$ of $\widehat{\mathcal{H}}_R$ of functions φ with *almost finite* support *i.e.* $supp(\varphi) \subset \cup_{i=1}^n (\lambda_i - Q^\vee_+) \cap Y^{++}$ where $\lambda_i \in Y^{++}$.

2.2 Retractions onto Y^+

For all $x \in \mathscr{I}^+$ there is an apartment containing x and $\mathfrak{c}_-$ [Ro11, 5.1] and this apartment is conjugated to $\mathbb{A}$ by an element of K fixing $\mathfrak{c}_-$ (axiom (MA2) of *l.c.*). So, by the usual arguments and [*l.c.* , 5.5] we can define a retraction $\rho_{\mathfrak{c}_-}$ of $\mathscr{I}^+$ into $\mathbb{A}$ with center $\mathfrak{c}_-$; its image is $\rho_{\mathfrak{c}_-}(\mathscr{I}^+) = \mathcal{T} = \mathscr{I}^+ \cap \mathbb{A}$ and $\rho_{\mathfrak{c}_-}(\mathscr{I}^+_0) = Y^+$.

There is also a retraction $\rho_{-\infty}$ of $\mathscr{I}$ onto $\mathbb{A}$ with center the sector-germ $\mathfrak{S}_{-\infty}$ [GR08, 4.4].

For $\rho = \rho_{\mathfrak{c}_-}$ or $\rho_{-\infty}$ the image of a segment $[x, y]$ with $(x, y) \in \mathscr{I} \times_\leq \mathscr{I}$ and $d^v(x, y) = \lambda \in \overline{C^v_f}$ is a $\lambda-$path [GR08, 4.4]. In particular, $\rho(x) \leq \rho(y)$.

2.3 Convolution product

The convolution product in $\widehat{\mathcal{H}}_R$ should be $(\varphi*\psi)(y)=\sum\varphi(z)\psi(d^v(z,y))$ where the sum runs over the $z\in\mathscr{I}_0^+$ such that $0\le z\le y$.

1) Using $\rho_{\mathfrak{c}_-}$ we have, for $\lambda,\mu,y\in Y^{++}$, $(c_\lambda*c_\mu)(y)=\sum_{w\in W^v/(W^v)_\lambda}N_{\mathfrak{c}_-}(\mu,w.\lambda,y)$ where $N_{\mathfrak{c}_-}(\mu,w.\lambda,y)$ is the number of $z\in\mathscr{I}_0^+$ with $d^v(z,y)=\mu$ and $\rho_{\mathfrak{c}_-}(z)=w.\lambda\in Y^+$. Note that, if $N_{\mathfrak{c}_-}(\mu,w\lambda,y)>0$, there exists a $\mu-$path from $w\lambda$ to y, hence $y\in w\lambda+Y^+$.

So $c_\lambda*c_\mu$ is the formal sum $c_\lambda*c_\mu=\sum_{\nu\in Y^{++}}m_{\lambda,\mu}(\nu)c_\nu$ where the structure constant $m_{\lambda,\mu}(\nu)=\sum_{w\in W^v/(W^v)_\lambda}N_{\mathfrak{c}_-}(\mu,w.\lambda,\nu)\in\mathbb{Z}_{\ge0}\cup\{+\infty\}$ is also equal to the number of triangles $[x,z,y]$ with $d^v(x,z)=\lambda$ and $d^v(z,y)=\mu$, for any fixed pair $(x,y)\in\mathscr{I}_0\times_\le\mathscr{I}_0$ with $d^v(x,y)=\nu$ (*e.g.* $(x,y)=(0,\nu)$).

2) Using $\rho_{-\infty}$ we have $m_{\lambda,\mu}(\nu)=\sum_{z'}N_{-\infty}(\mu,z',\nu)$ where the sum runs over the z' in $Y^+(\lambda)=\rho_{-\infty}(\{z\in\mathscr{I}_0^+\mid d^v(0,z)=\lambda\})$ and $N_{-\infty}(\mu,z',\nu)\in\mathbb{Z}_{\ge0}\cup\{+\infty\}$ is the number of $z\in\mathscr{I}_0^+$ with $d^v(0,z)=\lambda$, $d^v(z,y)=\mu$ (for any $y\in\mathscr{I}_0^+$ with $d^v(0,y)=\nu$ *e.g.* $y=\nu$) and $\rho_{-\infty}(z)=z'$. But $\rho_{-\infty}([0,z])$ is a $\lambda-$path hence increasing with respect to $\le$, so $Y^+(\lambda)\subset Y^+$. Moreover, $\rho_{-\infty}([z,\nu])$ is a $\mu-$path, so z' has to be in $\nu-Y^+$. Hence, z' has to run over the set $Y^+(\lambda)\cap(\nu-Y^+)\subset Y^+\cap(\nu-Y^+)$.

Actually, the image by $\rho_{-\infty}$ of a segment $[x,y]$ with $(x,y)\in\mathscr{I}\times_\le\mathscr{I}$ and $d^v(x,y)=\lambda\in Y^{++}$ is a Hecke path of shape λ with respect to $-C_f^v$ [GR08, th. 6.2]. Hence the following results:

Lemma 2.4. *a) For $\lambda\in Y^{++}$ and $w\in W^v$, $w\lambda\in\lambda-Q_+^\vee$, i.e. $w\lambda\le_{Q^\vee}\lambda$.*

b) Let π be a Hecke path of shape $\lambda\in Y^{++}$ with respect to $-C_f^v$, from $y_0\in Y$ to $y_1\in Y$. Then $\lambda=\pi'(0)^{++}=\pi'(1)^{++}$, $\pi'(0)\le_{Q^\vee}\lambda$, $\pi'(0)\le_{Q^\vee_\mathbb{R}}(y_1-y_0)\le_{Q^\vee_\mathbb{R}}\pi'(1)\le_{Q^\vee}\lambda$ and $y_1-y_0\le_{Q^\vee}\lambda$.

c) If moreover $(\alpha_i^\vee)_{i\in I}$ is free, we may replace above $\le_{Q^\vee_\mathbb{R}}$ by $\le_{Q^\vee}$.

d) For $\lambda,\mu,\nu\in Y^{++}$, if $m_{\lambda,\mu}(\nu)>0$, then $\nu\in\lambda+\mu-Q_+^\vee$ i.e. $\nu\le_{Q^\vee}\lambda+\mu$.

N.B. By d) above, if $x\le z\le y$ in $\mathscr{I}_0$, then $d^v(x,y)\le_{Q^\vee}d^v(x,z)+d^v(z,y)$.

Proof. a) By definition, for $\lambda\in Y$, $w\lambda\in\lambda+Q^\vee$, hence a) follows from [Ka90, 3.12d] used in a realization where $(\alpha_i^\vee)_{i\in I}$ is free.

b) By definition of Hecke paths in 1.8, $\lambda=\pi'(0)^{++}=\pi'(1)^{++}$. Moreover, $\forall t\in[0,1]$, $\lambda=\pi'_-(t)^{++}=\pi'_+(t)^{++}$ and we know how to get $\pi'_+(t)$ from $\pi'_-(t)$ by successive reflections; this proves that $\pi'_+(t)\in\pi'_-(t)+Q^\vee_{\mathbb{R}+}$. By integrating the locally constant function $\pi'(t)$, we get $\pi'(0)\le_{Q^\vee_\mathbb{R}}(y_1-y_0)\le_{Q^\vee_\mathbb{R}}\pi'(1)\le_{Q^\vee_\mathbb{R}}\lambda$.

It is proved (but not stated) in [GR08, 5.3.3] that any Hecke path of shape λ starting in $y_0\in Y$ can be transformed in the path $\pi_\lambda(t)=y_0+\lambda t$ by applying successively the operators e_{α_i} or $\widetilde{e}_{\alpha_i}$ for $i\in I$; moreover $e_{\alpha_i}(\pi)(1)=\pi(1)+\alpha_i^\vee$ and $\widetilde{e}_{\alpha_i}(\pi)(1)=\pi(1)$, hence $y_1-y_0\le_{Q^\vee}\lambda$.

c) By b) $y_1-y_0-\pi'(0)\in Q^\vee_{\mathbb{R}+}\cap Q^\vee=Q^\vee_+$, so $\pi'(0)\le_{Q^\vee}(y_1-y_0)$. Idem for $y_1-y_0\le_{Q^\vee}\pi'(1)$.

d) If $m_{\lambda,\mu}(\nu)>0$ we have an Hecke path of shape λ (resp. μ) from 0 to z' (resp. from z' to ν). So c) follows from b). □

Proposition 2.5. *Suppose $(\alpha_i^\vee)_{i\in I}$ free in V. Then for all $\lambda,\mu,\nu\in Y^{++}$, $m_{\lambda,\mu}(\nu)$ is finite.*

N.B. Actually in the following, we may replace the condition $(\alpha_i^\vee)_{i\in I}$ free by $(\alpha_i^\vee)_{i\in I}$ $\mathbb{R}^+-$free.

Proof. We have to count the $z \in \mathscr{I}_0^+$ such that $d^v(0,z) = \lambda$ and $d^v(z,\nu) = \mu$. We set $z' = \rho_{-\infty}(z)$. By lemma 2.4b, $z' \in \lambda - Q_+^\vee$ and $\nu \in z' + \mu - Q_+^\vee$, hence z' is in $(\lambda - Q_+^\vee) \cap (\nu - \mu + Q_+^\vee)$ which is finite as $(\alpha_i^\vee)_{i\in I}$ is free or $\mathbb{R}^+-$free. So, we fix now z'. By [GR08, cor. 5.9] there is a finite number of Hecke paths π' of shape μ from z' to ν. So, we fix now π'. And by [*l.c.* th. 6.3] (see also 4.10, 4.11) there is a finite number of segments $[z,\nu]$ retracting to π'; hence the number of z is finite. □

Theorem 2.6. *Suppose $(\alpha_i^\vee)_{i\in I}$ free or $\mathbb{R}^+-$free, then $\mathcal{H}_R$ is an algebra.*

Proof. We saw that for $\lambda, \mu, \nu \in Y^{++}$, $m_{\lambda,\mu}(\nu)$ is finite; hence $c_\lambda * c_\mu$ is well defined (eventually as an infinite formal sum). Let us consider $\varphi, \psi \in \mathcal{H}_R$: $supp(\varphi) \subset \cup_{i=1}^m (\lambda_i - Q_+^\vee)$, $supp(\psi) \subset \cup_{j=1}^n (\mu_j - Q_+^\vee)$. Let $\nu \in Y^{++}$. If $m_{\lambda,\mu}(\nu) > 0$ with $\lambda \in supp(\varphi)$, $\mu \in supp(\psi)$ (hence $\lambda \in \lambda_i - Q_+^\vee$, $\mu \in \mu_j - Q_+^\vee$ for some i,j), we have $\lambda + \mu \in \nu + Q_+^\vee$ by lemma 2.4d. So $\lambda \in (\nu - \mu + Q_+^\vee) \cap (\lambda_i - Q_+^\vee) \subset (\nu - \mu_j + Q_+^\vee) \cap (\lambda_i - Q_+^\vee)$, a finite set. For the same reasons μ is in a finite set, so $\varphi * \psi$ is well defined.

With the above notations $\nu \in (\lambda + \mu - Q_+^\vee) \subset \cup_{i,j} (\lambda_i + \mu_j - Q_+^\vee)$, so $\varphi * \psi \in \mathcal{H}_R$. □

Definition 2.7. $\mathcal{H}_R = \mathcal{H}_R(\mathscr{I}, G)$ is the *spherical Hecke algebra* (with coefficients in R) associated to the hovel $\mathscr{I}$ and its strongly transitive automorphism group G.

Remark. We shall now investigate $\mathcal{H}_R$ and some other possible convolution algebras in $\widehat{\mathcal{H}}_R$ by separating the cases: finite, indefinite and affine.

2.8 Finite case

In this case Φ and W^v are finite, $(\alpha_i^\vee)_{i\in I}$ is free, $\mathcal{T} = V$ and the relation $\leq$ is trivial. The hovel $\mathscr{I} = \mathscr{I}^+$ is a locally finite Bruhat-Tits building.

Let ρ be the half sum of positive roots. As $2\rho \in Q$ and $\rho(\alpha_i^\vee) = 1$, $\forall i \in I$, we see that an almost finite set in Y^{++} is always finite. So $\mathcal{H}_R$ and $\mathcal{H}_R^f$ are equal.

The algebra $\mathcal{H}_\mathbb{C}$ was already studied by I. Satake in [Sa63]. Its close link with buildings is explained in [P06]. The algebra $\mathcal{H}_\mathbb{Z}$ is the spherical Hecke ring of [KLM08], where the interpretation of $m_{\lambda,\mu}(\nu)$ as a number of triangles in $\mathscr{I}$ is already given.

$\widehat{\mathcal{H}}_R$ is not an algebra as *e.g.* $m_{\lambda,(-w_0)\lambda}(0) \neq 0$ $\forall \lambda \in Y^{++}$ (where w_0 is the greatest element in W^v).

2.9 Indefinite case

Lemma. *Suppose now Φ associated to an indefinite indecomposable generalized Cartan matrix. Then there is in Δ_{im}^+ an element δ (of support I) such that $\delta(\alpha_i^\vee) < 0$, $\forall i \in I$ and a basis $(\delta_i)_{i\in I}$ of the real vector space $Q_\mathbb{R}$ spanned by Φ such that $\delta_i(\mathcal{T}) \geq 0$, $\forall i \in I$.*

Proof. Any $\delta \in \Delta_{im}^+$ takes positive values on $\mathcal{T}$ [Ka90, 5.8]. Now, in the indefinite case, there is $\delta \in \Delta_{im}^+ \cap (\oplus_{i\in I} \mathbb{R}_{>0}.\alpha_i)$ such that $\delta(\alpha_i^\vee) < 0$, $\forall i \in I$ [*l.c.* 4.3], hence $\delta + \alpha_i \in \Delta^+$, $\forall i \in I$. Replacing eventually δ by 3δ [*l.c.* 5.5], we have $(\delta + \alpha_i)(\alpha_j^\vee) < 0$, $\forall i,j \in I$, hence $\delta + \alpha_i \in \Delta_{im}^+$. The wanted basis is inside $\{\delta\} \cup \{\delta_0 + \alpha_i \mid i \in I\}$. □

The existence of $\delta \in \Delta_{im}^+$ as in the lemma proves that $(\alpha_i^\vee)_{i\in I}$ is $\mathbb{R}^+-$free. So $\mathcal{H}_R$ is an algebra. The following example 2.10 proves that $\mathcal{H}_R^f$ is in general not a subalgebra.

If $(\alpha_i)_{i\in I}$ generates (*i.e.* is a basis of) V^*, $\widehat{\mathcal{H}}_R$ is also an algebra (the *formal spherical Hecke algebra*): Let $\nu \in Y^{++}$, we have to prove that there is only a finite number of pairs $(\lambda,\mu)\in (Y^{++})^2$ such that $m_{\lambda,\mu}(\nu)>0$. Let z' be as in the proof of 2.5. We saw in 2.3 that $z'\in Y^+\cap(\nu-Y^+)=Y\cap\mathcal{T}\cap(\nu-\mathcal{T})$. By the lemma, $\mathcal{T}\cap(\nu-\mathcal{T})$ is bounded, hence $Y\cap\mathcal{T}\cap(\nu-\mathcal{T})$ is finite. So we may fix z'. Now $\lambda\in z'+Q^\vee_+$ hence (for δ as in the lemma) $\delta(\lambda)\le\delta(z')$; as $\alpha_i(\lambda)\in\mathbb{Z}_{>0}$ $\forall i\in I$ and $\delta\in\oplus_{i\in I}\mathbb{R}_{>0}.\alpha_i$ this gives only a finite number of possibilities for λ. Similarly $\mu\in\nu-z'+Q^\vee_+$ has to be in a finite set.

Actually $\widehat{\mathcal{H}}_R$ is often equal to $\mathcal{H}_R$ when $(\alpha_i^\vee)_{i\in I}$ is free and $(\alpha_i)_{i\in I}$ generates V^* (hence the matrix $\mathbb{M}=(\alpha_j(\alpha_i^\vee))$ is invertible), see the following example 2.10.

2.10 An indefinite rank 2 example

Let us consider the Kac-Moody matrix $\mathbb{M}=\begin{pmatrix}2&-3\\-3&2\end{pmatrix}$. The basis of Φ and V^* is $\{\alpha_1,\alpha_2\}$ and we consider the dual basis $(\varpi_1^\vee,\varpi_2^\vee)$ of V. In this basis $\alpha_1^\vee=\begin{pmatrix}2\\-3\end{pmatrix}$, $\alpha_2^\vee=\begin{pmatrix}-3\\2\end{pmatrix}$ and the matrices of r_1, r_2, r_2r_1 and r_1r_2 are respectively $\begin{pmatrix}-1&0\\3&1\end{pmatrix}$, $\begin{pmatrix}1&3\\0&-1\end{pmatrix}$, $M=\begin{pmatrix}8&3\\-3&-1\end{pmatrix}$ and $M^{-1}=\begin{pmatrix}-1&-3\\3&8\end{pmatrix}$. The eigenvalues of M or M^{-1} are $a_\pm=(7\pm\sqrt{45})/2$. In a basis diagonalizing M and M^{-1} we see easily that $(r_2r_1)^n+(r_1r_2)^n=a_n.Id_V$ where $a_n=a_+^n+a_-^n$ is in $\mathbb{N}$ and increasing up to infinity ($a_0=2$, $a_1=7$, $a_2=47$, $a_3=322$,...).

Consider now $\lambda=\mu=-\alpha_1^\vee-\alpha_2^\vee=\begin{pmatrix}1\\1\end{pmatrix}$ in $Y^{++}\subset\mathbb{Z}_{\ge0}.\varpi_1^\vee\oplus\mathbb{Z}_{\ge0}.\varpi_2^\vee$. We have $(r_2r_1)^n.\lambda+(r_1r_2)^n.\lambda=a_n.\lambda$. This means that $m_{\lambda,\lambda}(a_n.\lambda)\ge N_{\mathfrak{c}_-}(\lambda,(r_2r_1)^n\lambda,a_n.\lambda)\ge1$, for all positive n (and the same thing for $N_{-\infty}$). So $c_\lambda * c_\lambda$ is an infinite formal sum.

Actually $(-Q^\vee_+)\cap Y^{++}\supset\mathbb{Z}_{\ge0}.5\varpi_1^\vee\oplus\mathbb{Z}_{\ge0}.5\varpi_2^\vee$, hence Y^{++} itself is almost finite!

2.11 An affine rank 2 example

Let us consider the Kac-Moody matrix $\mathbb{M}=\begin{pmatrix}2&-2\\-2&2\end{pmatrix}$. The basis of Φ is $\{\alpha_1,\alpha_2\}$ but we consider a realization V of dimension 3 for which $\{\alpha_1^\vee,\alpha_2^\vee\}$ is free and with basis of V^* $\{\alpha_o=-\rho,\alpha_1,\alpha_2\}$. More precisely, if $(\varpi_0^\vee,\varpi_1^\vee,\varpi_2^\vee)$ is the dual basis of V, we have $\alpha_1^\vee=\begin{pmatrix}-1\\2\\-2\end{pmatrix}$, $\alpha_2^\vee=\begin{pmatrix}-1\\-2\\2\end{pmatrix}$ and the matrices of r_1, r_2, r_1r_2 and r_2r_1 are respectively $\begin{pmatrix}1&1&0\\0&-1&0\\0&2&1\end{pmatrix}$, $\begin{pmatrix}1&0&1\\0&1&2\\0&0&-1\end{pmatrix}$, $M=\begin{pmatrix}1&1&3\\0&-1&-2\\0&2&3\end{pmatrix}$ and $M^{-1}=\begin{pmatrix}1&3&1\\0&3&2\\0&-2&-1\end{pmatrix}$. A classical calculus using triangulation tells us that $(r_2r_1)^n+(r_1r_2)^n=\begin{pmatrix}2&4n^2&4n^2\\0&1&0\\0&0&1\end{pmatrix}$. Actually $c=\alpha_1^\vee+\alpha_2^\vee=-2\varpi_0^\vee\in Q^\vee_+$ is the canonical central element [Ka90, § 6.2] and the above calculations are peculiar cases of [*l.c.* § 6.5].

Let's consider now $\lambda=\mu=\sum_{i=1}^2 a_i\varpi_i^\vee\in Y^{++}\subset\oplus_{i=1}^2\mathbb{Z}_{\ge0}\varpi_i^\vee$. We have $(r_2r_1)^n(\lambda)+(r_1r_2)^n(\lambda)=\lambda-2n^2|\lambda|c$ with $|\lambda|=a_1+a_2$. This means that $m_{\lambda,\lambda}(\lambda-2n^2|\lambda|c)\ge$

$N_{\mathfrak{c}_-}(\lambda,(r_2r_1)^n(\lambda),\lambda-2n^2|\lambda|c)\geq 1$, $\forall n\in\mathbb{Z}$ (and the same thing for $N_{-\infty}$). So $c_\lambda * c_\lambda$ is an infinite formal sum.

Moreover as c is fixed by r_1 and r_2, $(r_2r_1)^n(\lambda+2n^2|\lambda|c)+(r_1r_2)^n(\lambda)=\lambda$, so $m_{\lambda+2n^2|\lambda|c,\lambda}(\lambda)\geq 1$, $\forall n\in\mathbb{Z}$, and $\widehat{\mathcal{H}}_R$ is not an algebra.

Remark also that, if we consider the essential quotient $V^e=V/\mathbb{R}c$, the above calculus tells that $m_{\lambda,\lambda}(\lambda)\geq\sum_{n\in\mathbb{Z}}N_{\mathfrak{c}_-}(\lambda,(r_2r_1)^n(\lambda),\lambda)$ is infinite if $|\lambda|>0$.

2.12 Affine indecomposable case

We saw in the example 2.11 above that $m_{\lambda,\lambda}(\lambda)$ may be infinite, $\forall\lambda\in Y^{++}$ when $(\alpha_i^\vee)_{i\in I}$ is not free. So, in this case, $\widehat{\mathcal{H}}_R$ seems to contain no algebra except $R.c_0$.

Remark also that $(\alpha_i^\vee)_{i\in I}$ free is equivalent to $(\alpha_i^\vee)_{i\in I}$ $\mathbb{R}^+-$free in the affine indecomposable case as the only possible relation between the $\alpha_i^\vee$ is $c=0$ where $c=\sum_{i\in I}a_i^\vee.\alpha_i^\vee$ (with $a_i^\vee\in\mathbb{Z}_{>0}$ $\forall i\in I$) is the canonical central element.

An almost finite subset in Y^{++} is a finite union of subsets like $Y_\lambda=(\lambda-Q_+^\vee)\cap Y^{++}$. Let δ be the smallest positive imaginary root in Δ. Then $\delta(Q_+^\vee)=0$ so $Y_\lambda\subset\{y\in Y^{++}\mid\delta(y)=\delta(\lambda)\}=Y'_\lambda$. But $\delta=\sum_{i\in I}a_i.\alpha_i$ with $a_i\in\mathbb{Z}_{>0}$ $\forall i\in I$, so the image of Y'_λ in $V^e=V/\mathbb{R}c$ (where $\mathbb{R}c=\cap_{i\in I}Ker(\alpha_i)$) is finite. It is now clear that Y_λ is a finite union of sets like $\mu-\mathbb{Z}_{\geq 0}.c$ with $\mu\in Y^{++}$. Hence an almost finite subset as defined above is the same as an almost finite union (of double cosets) as defined in [BrK10].

The algebra $\mathcal{H}_\mathbb{C}$ is the Hecke algebra introduced by A. Braverman and D. Kazhdan in [BrK10]. We gave above a combinatorial proof that it is an algebra, without algebraic geometry.

3 The split Kac-Moody case

3.1 Situation

As in [Ro12] or [Ro13], we consider a split Kac-Moody group $\mathfrak{G}$ associated to a root generating system (RGS) $\mathcal{S}=(\mathbb{M},Y_\mathcal{S},(\overline{\alpha}_i)_{i\in I},(\alpha_i^\vee)_{i\in I})$ over a field $\mathcal{K}$ endowed with a discrete valuation ω (with value group $\Lambda=\mathbb{Z}$ and ring of integers $\mathcal{O}=\omega^{-1}([0,+\infty]))$ and finite residue field $\kappa=\mathbb{F}_q$. So, $\mathbb{M}=(a_{i,j})_{i,j\in I}$ is a Kac-Moody matrix, $Y_\mathcal{S}$ a free $\mathbb{Z}-$module, $(\alpha_i^\vee)_{i\in I}$ a family in $Y_\mathcal{S}$, $(\overline{\alpha}_i)_{i\in I}$ a family in the dual $X=Y_\mathcal{S}^*$ of $Y_\mathcal{S}$ and $\overline{\alpha}_j(\alpha_i^\vee)=a_{i,j}$.

If $(\overline{\alpha}_i)_{i\in I}$ is free in X, we consider $V=V_Y=Y_\mathcal{S}\otimes_\mathbb{Z}\mathbb{R}$ and the clear quadruple $(V,W^v,(\alpha_i=\overline{\alpha}_i)_{i\in I},(\alpha_i^\vee)_{i\in I})$. In general, we may define $Q=\mathbb{Z}^I$ with canonical basis $(\alpha_i)_{i\in I}$, then $V=V_Q=Hom_\mathbb{Z}(Q,\mathbb{R})$ is also in a quadruple as in 1.1. A third example V^{xl} of choice for V is explained in [Ro13]. We always denote by $bar:Q\to X$ the linear map sending α_i to $\overline{\alpha}_i$.

With these vectorial data we may define what was considered in 1.1 and 1.2 (we choose $\Lambda_\alpha=\Lambda=\mathbb{Z}$, $\forall\alpha\in\Phi$).

Now the hovel $\mathscr{I}$ in 1.4 is as defined in [Ro12] or [Ro13] and the strongly transitive group is $G=\mathfrak{G}(\mathcal{K})$. By [Ro11, 6.11] or [Ro12, 5.16] we have $q_M=q$ for any wall M.

3.2 Generators for G

The Kac-Moody group $\mathfrak{G}$ contains a split maximal torus $\mathfrak{T}$ with character group X and cocharacter group $Y_\mathcal{S}$. We note $T=\mathfrak{T}(\mathcal{K})$. For each $\alpha\in\Phi\subset Q$ there is a group homomorphism $x_\alpha:\mathcal{K}\to G$ which is one-to-one; its image is the subgroup U_α. Now G is generated by T and the subgroups U_α for $\alpha \in \Phi$, submitted to some relations given by Tits [T87], also available in [Re02] or [Ro12]. We set $U^\pm$ the subgroup generated by the subgroups U_α for $\alpha \in \Phi^\pm$.

We shall explain now only a few of the relations. For $u \in \mathcal{K}$, $t \in T$ and $\alpha \in \Phi$ one has:

(KMT4) $t.x_\alpha(u).t^{-1} = x_\alpha(\overline{\alpha}(t).u)$ (where $\overline{\alpha} = bar(\alpha)$)

For $u \neq 0$, we note $\widetilde{s}_\alpha(u) = x_\alpha(u).x_{-\alpha}(u^{-1}).x_\alpha(u)$ and $\widetilde{s}_\alpha = \widetilde{s}_\alpha(1)$.

(KMT5) $\widetilde{s}_\alpha(u).t.\widetilde{s}_\alpha(u)^{-1} = r_\alpha(t)$ (W^v acts on V, $Y_\mathcal{S}$, X hence on T)

3.3 Weyl groups

Actually the stabilizer N of $\mathbb{A} \subset \mathscr{I}$ is the normalizer of $\mathfrak{T}$ in G. The image $\nu(N)$ of N in $Aut(\mathbb{A})$ is a semi-direct product $\nu(N) = \nu(N_0) \ltimes \nu(T)$ with:

N_0 is the fixator of 0 in N and $\nu(N_0)$ is isomorphic to W^v acting linearly on $\mathbb{A} = V$. Actually $\nu(N_0)$ is generated by the elements $\nu(\widetilde{s}_\alpha)$ which act as r_α (for $\alpha \in \Phi$).

$t \in T$ acts on $\mathbb{A}$ by a translation of vector $\nu(t) \in V$ such that $\overline{\chi}(\nu(t)) = -\omega(\chi(t))$ for any $\overline{\chi} \in X = Y_\mathcal{S}^*$ and $\chi \in X$ or Q which are related by $\overline{\chi} = \chi$ if $V = V_Y$ or $\overline{\chi} = bar(\chi)$ if $V = V_Q$.

So, $\nu(N) = W^v \ltimes Y$ where Y is closely related to $Y_\mathcal{S} \simeq T/\mathfrak{T}(\mathcal{O})$: as $\Lambda = \omega(\mathcal{K}) = \mathbb{Z}$, they are equal if $V = V_Y$ and, if $V = V_Q$, $Y = bar^*(Y_\mathcal{S})$ is the image of $Y_\mathcal{S}$ by the map $bar^* : Y_\mathcal{S} \to Hom_\mathbb{Z}(Q, \mathbb{Z})$ dual to bar.

So, the choice $V = V_Y$ is more pleasant. The choice $V = V_Q$ is made *e.g.* in [Ch10], [Ch11] or [Re02] and has good properties in the indefinite case, *cf.* 2.9. They coincide both when $(\overline{\alpha}_i)_{i\in I}$ is a basis of $X \otimes \mathbb{R} = V_Y^*$. This assumption generalizes semi-simplicity, in particular the center of $\mathfrak{G}$ is then finite [Re02, 9.6.2].

3.4 The group K

The group $K = G_0$ should be equal to $\mathfrak{G}(\mathcal{O})$ for some integral structure of $\mathfrak{G}$ over $\mathcal{O}$ *cf.* [GR08, 3.14]. But the appropriate integral structure is difficult to define in general. So we define K by its generators:

The group N_0 is generated by $T_0 = \mathfrak{T}(\mathcal{O}) = T \cap K$ and the elements $\widetilde{s}_\alpha$ for $\alpha \in \Phi$ (this is clear by 3.3). The group U_0, generated by the groups $U_{\alpha,0} = x_\alpha(\mathcal{O})$ for $\alpha \in \Phi$, is in K. We note $U_0^\pm = U_0 \cap U^\pm$. In general $U_0^\pm$ is not generated by the groups $U_{\alpha,0}$ for $\alpha \in \Phi^\pm$ [Ro12, 4.12.3a].

It is likely that K is greater than the group generated by N_0 and U_0 (*i.e.* by U_0 and T_0). We have to define groups $U_0^{pm+} \supset U_0^+$ and $U_0^{nm-} \supset U_0^-$ as follows. In a formal positive completion $\widehat{G}^+$ of G, we can define a subgroup $U_0^{ma+} = \prod_{\alpha\in\Delta^+} U_{\alpha,0}$ of the subgroup $U^{ma+} = \prod_{\alpha\in\Delta^+} U_\alpha$ of $\widehat{G}^+$, with $U^+ \subset U^{ma+}$ (where $U_{\alpha,0}$ and U_α are suitably defined for α imaginary). Then $U_0^{pm+} = U_0^{ma+} \cap G = U_0^{ma+} \cap U^+$. The group U_0^{nm-} is defined similarly with Δ^- using a group $U_0^{ma-} \subset U^{ma-}$ in a formal negative completion $\widehat{G}^-$ of G.

Now $K = G_0 = U_0^{nm-}.U_0^+.N_0 = U_0^{pm+}.U_0^-.N_0$ [Ro12, 4.14, 5.1]

Proposition 3.5. *There is an involution θ (called Chevalley involution) of the group G such that $\theta(t) = t^{-1}$ for all $t \in T$ and $\theta(x_\alpha(u)) = x_{-\alpha}(u)$ for all $\alpha \in \Phi$ and $u \in \mathcal{K}$. Moreover K is $\theta-$stable and θ induces the identity on $W^v = N/T$.*

Proof. This involution is well known on the corresponding complex Lie algebra, see [Ka90, 1.3.4] where one uses for the generators e_α a convention different from ours ($[e_\alpha, e_{-\alpha}] = -\alpha^\vee$ as in [T87] or [Re02]). Hence the proposition follows when κ contains $\mathbb{C}$ or is at least of characteristic 0. But here we have to use the definition of G by generators and relations.

We see in [Ro12, 1.5, 1.7.5] that $\widetilde{s}_\alpha(-u) = \widetilde{s}_\alpha(u)^{-1}$ and $\widetilde{s}_\alpha(u) = \widetilde{s}_{-\alpha}(u^{-1})$. So for the wanted involution θ we have $\theta(\widetilde{s}_\alpha(u)) = \widetilde{s}_{-\alpha}(u) = \widetilde{s}_\alpha(u^{-1})$. We have now to verify the relations between the $\theta(x_\alpha(u)) = x_{-\alpha}(u)$, $\theta(t) = t^{-1}$ and $\theta(\widetilde{s}_\alpha(u)) = \widetilde{s}_\alpha(u^{-1})$. This is clear for (KMT4) and (KMT5) (as $r_\alpha = r_{-\alpha}$). The three other relations are:

(KMT3) $(x_\alpha(u), x_\beta(v)) = \prod x_\gamma(C^{\alpha,\beta}_{p,q}.u^p v^q)$ for $(\alpha,\beta) \in \Phi^2$ prenilpotent and, for the product, $\gamma = p\alpha + q\beta$ runs in $(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi$. But the integers $C^{\alpha,\beta}_{p,q}$ are picked up from the corresponding formula between exponentials in the automorphism group of the corresponding complex Lie algebra. As we know that θ is defined in this Lie algebra, we have $C^{-\alpha,-\beta}_{p,q} = C^{\alpha,\beta}_{p,q}$ and (KMT3) is still true for the images by θ.

(KMT6) $\widetilde{s}_\alpha(u^{-1}) = \widetilde{s}_\alpha.\alpha^\vee(u)$ for α simple and $u \in \mathcal{K} \setminus \{0\}$.
This is still true after a change by θ as $\theta(\widetilde{s}_\alpha(u^{-1})) = \widetilde{s}_\alpha(u)$ and $(-\alpha)^\vee(u) = \alpha^\vee(u^{-1})$.

(KMT7) $\widetilde{s}_\alpha.x_\beta(u).\widetilde{s}_\alpha^{-1} = x_\gamma(\varepsilon.u)$ if $\gamma = r_\alpha(\beta)$ and $\widetilde{s}_\alpha(e_\beta) = \varepsilon.e_\gamma$ in the Lie algebra (with $\varepsilon = \pm 1$). This is still true after a change by θ because $\widetilde{s}_\alpha(e_\beta) = \varepsilon.e_\gamma \Rightarrow \widetilde{s}_\alpha(e_{-\beta}) = \varepsilon.e_{-\gamma}$ (as $r_\alpha(\beta^\vee) = \gamma^\vee$).

So θ is a well defined involution of G, $\theta(U_0) = U_0$, $\theta(N_0) = N_0$ and $\theta(U_0^\pm) = U_0^\mp$. But the isomorphism θ of U^+ onto U^- can clearly be extended to an isomorphism θ from U^{ma+} onto U^{ma-} sending U_0^{ma+} onto U_0^{ma-}. So $\theta(U_0^{pm+}) = U_0^{nm-}$ and $\theta(K) = K$. As $\theta(\widetilde{s}_\alpha) = \widetilde{s}_\alpha$, θ induces the identity on $W^v = N/T$. □

Theorem 3.6. *The algebra $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$ is commutative, when it exists.*

Notation: To be clearer we shall sometimes write $\widehat{\mathcal{H}}_R(\mathfrak{G},\mathcal{K})$ or $\mathcal{H}_R(\mathfrak{G},\mathcal{K})$ instead of $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$.

Proof. The formula $\theta^\#(g) = \theta(g^{-1})$ defines an anti-involution ($\theta^\#(gh) = \theta^\#(h).\theta^\#(g)$) of G which induces the identity on T and stabilizes K. In particular $\theta^\#(G^+) = \theta^\#(KY^{++}K) = G^+$ and $\theta^\#(K\lambda K) = K\lambda K$, $\forall \lambda \in Y^{++}$. For $\varphi, \psi \in \widehat{\mathcal{H}}_R$ and $g \in G^+$, one has: $(\varphi * \psi)(g) = (\varphi * \psi)(\theta^\#(g)) = \sum_{h \in G^+/K} \varphi(h)\psi(h^{-1}\theta^\#(g))$. The map $h \mapsto h' = \theta^\#(h^{-1}\theta^\#(g)) = g\theta^\#(h^{-1})$ is one-to-one from G^+/K onto G^+/K. So, $(\varphi * \psi)(g) = \sum_{h' \in G^+/K} \varphi(\theta^\#(h'^{-1}g))\psi(\theta^\#(h')) = \sum_{h' \in G^+/K} \varphi(h'^{-1}g)\psi(h') = (\psi * \varphi)(g)$. □

Remarks 3.7. 1) When $\mathfrak{G}$ is an almost split Kac-Moody group over the field $\mathcal{K}$ (supposed complete or henselian) which splits over a finite Galois extension $\mathcal{L}$, the hovel ${}^{\mathcal{K}}\mathscr{I}$ over $\mathcal{K}$ exists and embeds in the hovel $\mathscr{I}$ over $\mathcal{L}$ [Ro13, § 6]. Suppose that ${}^{\mathcal{K}}\mathscr{I}$ intersects $\mathscr{I}_0$, then we may choose $0 \in {}^{\mathcal{K}}\mathscr{I}$ and the fundamental apartments ${}^{\mathcal{K}}\mathbb{A} \subset \mathbb{A}$ with the same origin 0. Suppose moreover $\mathbb{A}$ stable under $\Gamma = Gal(\mathcal{L}/\mathcal{K})$ *i.e.* the corresponding torus defined over $\mathcal{K}$. Then it seems clear that θ commutes to Γ, so its restriction ${}^{\mathcal{K}}\theta$ to $\mathfrak{G}(\mathcal{K}) = \mathfrak{G}(\mathcal{L})^\Gamma$ has good properties and the above properties tell that $\widehat{\mathcal{H}}_R(\mathfrak{G},\mathcal{K})$ or $\mathcal{H}_R(\mathfrak{G},\mathcal{K})$ is commutative.

2) The commutativity of $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$ is linked to the choice of a special vertex for the origin 0. Even in the semi-simple case, other choices may give non commutative convolution algebras, see [Sa63] and [KeR07].

4 Structure constants

We come back to the general framework of § 1. We shall compute the structure constants of $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$ by formulas depending on $\mathbb{A}$ and the numbers q_M of 1.4. Note that there are only a finite number of them: as $q_{wM} = q_M$, $\forall w \in \nu(N)$ and $wM(\alpha,k) = M(w\alpha,k)$, we may suppose $M = M(\alpha_i, k)$ with $i \in I$ and $k \in \mathbb{Z}$. Now $\alpha_i^\vee \in Q^\vee \subset Y$; as $\alpha_i(\alpha_i^\vee) = 2$ the translation by $\alpha_i^\vee$ permutes the walls $M = M(\alpha_i, k)$ (for $k \in \mathbb{Z}$) with two orbits. So Y has at most two orbits in the set of the constants $q_{M(\alpha_i,k)}$, those of $q_i = q_{M(\alpha_i,0)}$ and $q'_i = q_{M(\alpha_i,\pm 1)}$. Hence the number of (possibly) different parameters is at most $2.|I|$. We denote by $\mathcal{Q} = \{q_1, \cdots, q_l, q'_1 = q_{l+1}, \cdots, q'_l = q_{2l}\}$ this set of parameters.

4.1 Centrifugally folded galleries of chambers

Let x be a point in the standard apartment $\mathbb{A}$. Let Φ_x be the set of all roots α such that $\alpha(x) \in \mathbb{Z}$. It is a closed subsystem of roots. Its associated Weyl group W_x^v is a Coxeter group.

We have twinned buildings $\mathscr{I}_x^+$ (resp. $\mathscr{I}_x^-$) whose elements are segment germs $[x, y) = germ_x([x, y])$ for $y \in \mathscr{I}$, $y \neq x$, $y \geq x$ (resp. $y \leq x$). We consider their unrestricted structure, so the associated Weyl group is W^v and the chambers (resp. closed chambers) are the local chambers $C = germ_x(x + C^v)$ (resp. local closed chambers $\overline{C} = germ_x(x + \overline{C^v})$), where C^v is a vectorial chamber, *cf.* [GR08, 4.5] or [Ro11, § 5]. To $\mathbb{A}$ is associated a twin system of apartments $\mathbb{A}_x = (\mathbb{A}_x^-, \mathbb{A}_x^+)$.

We choose in $\mathbb{A}_x^-$ a negative (local) chamber C_x^- and denote C_x^+ its opposite in $\mathbb{A}_x^+$. We consider the system of positive roots Φ^+ associated to C_x^+ (*i.e.* $\Phi^+ = w\Phi_f^+$, if Φ_f^+ is the system Φ^+ defined in 1.1 and $C_x^+ = germ_x(x + wC_f^v)$). We note $(\alpha_i)_{i\in I}$ the corresponding basis of Φ and $(r_i)_{i\in I}$ the corresponding generators of W^v.

Fix a reduced decomposition of an element $w \in W^v$, $w = r_{i_1} \cdots r_{i_r}$ and let $\mathbf{i} = (i_1, ..., i_r)$ be the type of the decomposition. We consider now galleries of (local) chambers $\mathbf{c} = (C_x^-, C_1, ..., C_r)$ in the apartment $\mathbb{A}_x^-$ starting at C_x^- and of type $\mathbf{i}$. The set of all these galleries is in bijection with the set $\Gamma(\mathbf{i}) = \{1, r_{i_1}\} \times \cdots \times \{1, r_{i_r}\}$ via the map $(c_1, ..., c_r) \mapsto (C_x^-, c_1C_x^-, ..., c_1 \cdots c_rC_x^-)$. Let $\beta_j = -c_1 \cdots c_j(\alpha_{i_j})$, then β_j is the root corresponding to the common limit hyperplane $M_j = M_{\beta_j}$ of $C_{j-1} = c_1 \cdots c_{j-1}C_x^-$ and $C_j = c_1 \cdots c_jC_x^-$ and satisfying to $\beta_j(C_j) \geq \beta_j(x)$ (actually M_j is a wall $\iff \beta_j \in \Phi_x$). In the following, we shall identify a sequence $(c_1, ..., c_r)$ and the corresponding gallery.

Definition 4.2. Let $\mathfrak{Q}$ be a chamber in $\mathbb{A}_x^+$. A gallery $\mathbf{c} = (c_1, ..., c_r) \in \Gamma(\mathbf{i})$ is said to be centrifugally folded with respect to $\mathfrak{Q}$ if $c_j = 1$ implies $\beta_j \in \Phi_x$ and $w_{\mathfrak{Q}}^{-1}\beta_j < 0$, where $w_{\mathfrak{Q}} = w(C_x^+, \mathfrak{Q}) \in W^v$ (*i.e.* $\mathfrak{Q} = w_{\mathfrak{Q}}C_x^+$). We denote this set of centrifugally folded galleries by $\Gamma_{\mathfrak{Q}}^+(\mathbf{i})$.

Proposition 4.3. *A gallery* $\mathbf{c} = (C_x^-, C_1, ..., C_r) \in \Gamma(\mathbf{i})$ *belongs to* $\Gamma_{\mathfrak{Q}}^+(\mathbf{i})$ *if, and only if,* $C_j = C_{j-1}$ *implies that* $M_j = M_{\beta_j}$ *is a wall and separates* $\mathfrak{Q}$ *from* $C_j = C_{j-1}$.

Proof. We saw that M_j is a wall $\iff \beta_j \in \Phi_x$. We have the following equivalences:
(M_j separates $\mathfrak{Q}$ from $C_j = C_{j-1}$) $\iff$ ($w_{\mathfrak{Q}}^{-1}M_j$ separates C_x^+ from $w_{\mathfrak{Q}}^{-1}C_j = w_{\mathfrak{Q}}^{-1}C_{j-1}$) $\iff$ ($w_{\mathfrak{Q}}^{-1}\beta_j$ is a negative root). □

The group $\overline{G}_x = G_x/G_{\mathscr{I}_x}$ acts strongly transitively on $\mathscr{I}_x^+$ and $\mathscr{I}_x^-$. For any root $\alpha \in \Phi_x$ with $\alpha(x) = k \in \mathbb{Z}$, the group $\overline{U}_\alpha = U_{\alpha,k}/U_{\alpha,k+1}$ is a finite subgroup of $\overline{G}_x$ of cardinality $q_{x,\alpha} = q_{M(\alpha,-\alpha(x))} \in \mathcal{Q}$. We denote by u_α the elements of this group.

Next, let $\rho_{\mathfrak{Q}} : \mathscr{I}_x \to \mathbb{A}_x$ be the retraction centered at $\mathfrak{Q}$. To a gallery of chambers $\mathbf{c} = (c_1, ..., c_r) = (C_x^-, C_1, ..., C_r)$ in $\Gamma(\mathbf{i})$, one can associate the set of all galleries of type $\mathbf{i}$

starting at C_x^- in $\mathscr{I}_x^-$ that retract onto $\mathbf{c}$, we denote this set by $\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$. We denote the set of minimal galleries in $\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$ by $\mathcal{C}^m_{\mathfrak{Q}}(\mathbf{c})$. Set

$$g_j = \begin{cases} c_j & \text{if } w_{\mathfrak{Q}}^{-1}\beta_j > 0 \text{ or } \beta_j \notin \Phi_x \\ u_{c_j(\alpha_{i_j})}c_j & \text{if } w_{\mathfrak{Q}}^{-1}\beta_j < 0 \text{ and } \beta_j \in \Phi_x. \end{cases} \tag{1}$$

Proposition 4.4. $\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$ *is the non empty set of all galleries* $(C_x^- = C'_0, C'_1, ..., C'_r)$ *where* $\forall j:\ C'_j = g_1\cdots g_jC_x^-$ *with each* g_j *chosen as in (1) above. For all* j *the local chambers* $\mathfrak{Q}$ *and* C'_j *are in the apartment* $g_1\cdots g_j\mathbb{A}_x$.

The set $\mathcal{C}^m_{\mathfrak{Q}}(\mathbf{c})$ *is empty if, and only if, the gallery* $\mathbf{c}$ *is not centrifugally folded with respect to* $\mathfrak{Q}$. *The gallery* $(C_x^- = C'_0, C'_1, ..., C'_r)$ *is minimal if, and only if,* $c_j \neq 1$ *for any* j *with* $w_{\mathfrak{Q}}^{-1}\beta_j > 0$ *or* $\beta_j \notin \Phi_x$ *and* $u_{c_j(\alpha_{i_j})} \neq 1$ *for any* j *with* $c_j = 1$ *and* $w_{\mathfrak{Q}}^{-1}\beta_j < 0$.

Remark. For g_j as in equation (1) we may write $g_j = u_{c_j(\alpha_{i_j})}c_j$ (with $u_{c_j(\alpha_{i_j})} = 1$ if $w_{\mathfrak{Q}}^{-1}\beta_j > 0$ or $\beta_j \notin \Phi_x$). Then in the product $g_1\cdots g_j$ we may gather the c_k on the right and, as $c_1\cdots c_k(\alpha_{i_k}) = -\beta_k$, we may write $g_1\cdots g_j = u_{-\beta_1}\cdots u_{-\beta_j}.c_1\cdots c_j$. Hence $C'_j := g_1\cdots g_jC_x^- = u_{-\beta_1}\cdots u_{-\beta_j}C_j$. When $u_{-\beta_k} \neq 1$ we have $\beta_k \in \Phi_x$ and $w_{\mathfrak{Q}}^{-1}\beta_k < 0$; so it is clear that $\rho_{\mathfrak{Q}}(C'_j) = C_j$.

The gallery $(C_x^- = C'_0, C'_1, ..., C'_r)$ (of type $\mathbf{i}$) is minimal if, and only if, we may also write (uniquely) $C'_j = u_{-\alpha_{i_1}}.u_{r_{i_1}(-\alpha_{i_2})}\cdots u_{r_{i_1}\cdots r_{i_{j-1}}(-\alpha_{i_j})}.r_{i_1}\cdots r_{i_j}(C_x^-) = h_1\cdots h_j.r_{i_1}\cdots r_{i_j}(C_x^-)$ with $h_k = u_{r_{i_1}\cdots r_{i_{k-1}}(-\alpha_{i_k})} \in \overline{U}_{r_{i_1}\cdots r_{i_{k-1}}(-\alpha_{i_k})}$ (which fixes C_x^-). In particular, $C'_j \in h_1\cdots h_j\mathbb{A}_x$. But this formula gives no way to know when $\rho_{\mathfrak{Q}}(C'_j) = C_j$. We know only that, when $\beta_k \notin \Phi_x$ *i.e.* $r_{i_1}\cdots r_{i_{k-1}}(-\alpha_{i_k}) \notin \Phi_x$, we have necessarily $h_k = 1$.

Proof. As the type $\mathbf{i}$ of $(C_x^- = C'_0, C'_1, ..., C'_r)$ is the type of a minimal decomposition, this gallery is minimal if, and only if, two consecutive chambers are different. So the last assertion is a consequence of the first ones. We prove these properties for $(C_x^- = C'_0, C'_1, ..., C'_j)$ by induction on j. We write in the following just H_j for the common limit hyperplane H_{β_j} of C_{j-1} and C_j of type i_j.

There are five possible relative positions of $\mathfrak{Q}$, C_x^- and C_1 with respect to H_1 and we seek C'_1 with $\rho_{\mathfrak{Q}}(C'_1) = C_1$ and $\overline{C'_1} \supset \overline{C_x^-} \cap H_1$.

0) $\beta_1 = -c_1\alpha_{i_1} \notin \Phi_x$, then H_1 is not a wall, each C'_1 with $\overline{C'_1} \supset \overline{C_x^-} \cap H_1$ is equal to C_x^- or $r_{i_1}C_x^-$ and C'_1 or C_x^- are contained in the same apartments. So $C'_1 = C_1 = c_1C_x^-$; C_1 and $\mathfrak{Q}$ are in $g_1\mathbb{A}_x = \mathbb{A}_x$ with $g_1 = c_1$. When $C'_1 = C_x^-$, we have $c_1 = 1$ and $\mathbf{c}$ is not centrifugally folded.

We suppose now $\beta_1 \in \Phi_x$, so H_1 is a wall.

1) C_x^- is on the same side of H_1 as $\mathfrak{Q}$ and C_1 not, then $c_1 = r_{i_1}$, $\beta_1 = \alpha_{i_1}$, $w_{\mathfrak{Q}}^{-1}\beta_1 < 0$, $C'_1 = g_1C_x^- = u_{-\alpha_{i_1}}r_{i_1}C_x^- = u_{-\alpha_{i_1}}C_1$. But $u_{-\alpha_{i_1}}$ pointwise stabilizes the halfspace bounded by H_1 containing C_x^-, hence $u_{-\alpha_{i_1}}(\mathfrak{Q}) = \mathfrak{Q}$ and C'_1 are in the apartment $g_1\mathbb{A}_x$.

2) $\mathfrak{Q}$ and $C_x^- = C_1$ are separated by H_1, then $c_1 = 1$, $\beta_1 = -\alpha_{i_1}$, $w_{\mathfrak{Q}}^{-1}\beta_1 < 0$, $C'_1 = g_1C_x^- = u_{\alpha_{i_1}}C_x^-$ but $u_{\alpha_{i_1}}$ pointwise stabilizes the halfspace bounded by H_1 not containing C_x^-, hence $\mathfrak{Q}$ and C'_1 are in the apartment $g_1\mathbb{A}_x$.

3) C_1 is on the same side of H_1 as $\mathfrak{Q}$ and C_x^- not, then $c_1 = r_{i_1}$, $\beta_1 = \alpha_{i_1}$, $w_{\mathfrak{Q}}^{-1}\beta_1 > 0$ and C'_1 has to be C_1 so $g_1 = c_1 = r_{i_1}$, $w_{\mathfrak{Q}}^{-1}(\alpha_{i_1}) > 0$, moreover $\mathfrak{Q}$ and $C'_1 = r_{i_1}C_x^- = C_1$ are in the apartment $g_1\mathbb{A}_x$.

4) $\mathfrak{Q}$ and $C_x^- = C_1$ are on the same side of H_1. Then $c_1 = 1$ and $w_{\mathfrak{Q}}^{-1}\beta_1 > 0$; the gallery $\mathbf{c}$ is not centrifugally folded. So $\rho_{\mathfrak{Q}}(C_1') = C_1$ implies $C_1' = C_x^- = g_1 C_x^-$ with $g_1 = c_1 = 1$ as in (1). But the gallery $(C_x^- = C_0', C_1', ..., C_j')$ cannot be minimal.

By induction we assume now that the chambers $\mathfrak{Q}$ and $C_{j-1}' = g_1 \cdots g_{j-1} C_x^-$ are in the apartment $A_{j-1} = g_1 \cdots g_{j-1}\mathbb{A}_x$. Again, we have five possible relative positions for $\mathfrak{Q}, C_{j-1}$ and C_j with respect to H_j. We seek C_j' with $\rho_{\mathfrak{Q}}(C_j') = C_j$ and $\overline{C_j'} \supset \overline{C_{j-1}'} \cap g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$.

0) $\beta_j = -c_1 \cdots c_j \alpha_{i_j} \notin \Phi_x$, then H_j is not a wall, each C_j' with $\overline{C_j'} \supset \overline{C_{j-1}'} \cap g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ is equal to $C_{j-1}' = g_1 \cdots g_{j-1} C_x^-$ or $g_1 \cdots g_{j-1} r_{i_j} C_x^-$; moreover C_j' or C_{j-1}' are contained in the same apartments. So $C_j' = g_1 \cdots g_{j-1} c_j C_x^-$ and $\mathfrak{Q}$ are in $g_1 \cdots g_j \mathbb{A}_x = g_1 \cdots g_{j-1}\mathbb{A}_x$ with $g_j = c_j$. When $C_j' = C_{j-1}'$, we have $c_j = 1$ and $\mathbf{c}$ is not centrifugally folded.

We suppose now $\beta_j \in \Phi_x$, so H_j is a wall.

1) C_{j-1} is on the same side of $H_j = c_1 \cdots c_{j-1} H_{\alpha_{i_j}}$ as $\mathfrak{Q}$ and C_j not, then $c_j = r_{i_j}$, $\beta_j = c_1 \cdots c_{j-1}\alpha_{i_j}$, $w_{\mathfrak{Q}}^{-1}\beta_j < 0$. Moreover $\mathfrak{Q}$ and C_{j-1}' are on the same side of $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ in A_{j-1}, and

$$\begin{aligned} C_j' &= g_1 \cdots g_{j-1} u_{-\alpha_{i_j}} r_{i_j} C_x^- \\ &= g_1 \cdots g_{j-1} u_{-\alpha_{i_j}} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}' \\ &= g_1 \cdots g_{j-1} u_{-\alpha_{i_j}} (g_1 \cdots g_{j-1})^{-1} g_1 \cdots g_{j-1} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}', \end{aligned}$$

where $g_1 \cdots g_{j-1} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}'$ is the chamber adjacent to C_j' along $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ in A_{j-1}. Moreover, $g_1 \cdots g_{j-1} u_{-\alpha_{i_j}} (g_1 \cdots g_{j-1})^{-1}$ pointwise stabilizes the halfspace bounded by $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ containing C_{j-1}' and $\mathfrak{Q}$. So $\mathfrak{Q}$ and C_j' are in the apartment $g_1 \cdots g_j \mathbb{A}_x$.

2) $C_{j-1} = C_j$ and $\mathfrak{Q}$ are separated by H_j, then $c_j = 1$, $\beta_j = -c_1 \cdots c_{j-1}\alpha_{i_j}$, $w_{\mathfrak{Q}}^{-1}\beta_j < 0$. Moreover C_{j-1}' and $\mathfrak{Q}$ are separated by $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ in A_{j-1}, and $\mathfrak{Q}$ and the chamber

$$g_1 \cdots g_{j-1} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}'$$

are on the same side of this wall. For $u_{\alpha_{i_j}} \neq 1$

$$C_j' = g_1 \cdots g_{j-1} u_{\alpha_{i_j}} C_x^- = g_1 \cdots g_{j-1} u_{\alpha_{i_j}} (g_1 \cdots g_{j-1})^{-1} C_{j-1}'$$

is a chamber adjacent (or equal) to C_{j-1}' along $g_1 \cdots g_{j-1} H_{\alpha_{i_j}} = g_1 \cdots g_{j-1} u_{\alpha_{i_j}} H_{\alpha_{i_j}}$ in $g_1 \cdots g_j \mathbb{A}_x$ (with $g_j = u_{\alpha_{i_j}}$).

The root-subgroup $g_1 \cdots g_{j-1} U_{\alpha_{i_j}} (g_1 \cdots g_{j-1})^{-1}$ pointwise stabilizes the halfspace bounded by $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ and containing the chamber $g_1 \cdots g_{j-1} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}'$. So $\mathfrak{Q}$ and C_j' are in the apartment $g_1 \cdots g_j \mathbb{A}_x$.

3) C_j is on the same side of $H_j = c_1 \cdots c_{j-1} H_{\alpha_{i_j}}$ as $\mathfrak{Q}$ and C_{j-1} not, then $c_j = r_{i_j}$, $\beta_j = c_1 \cdots c_{j-1}\alpha_{i_j}$, $w_{\mathfrak{Q}}^{-1}\beta_j > 0$. and so $C_j' = g_1 \cdots g_{j-1} r_{i_j} C_x^-$. Whence $\mathfrak{Q}$ and C_j' are in the apartment $g_1 \cdots g_j \mathbb{A}_x$.

4) $C_{j-1} = C_j$ and $\mathfrak{Q}$ are on the same side of $H_j = c_1 \cdots c_{j-1} H_{\alpha_{i_j}}$, then $c_j = 1$, $\beta_j = -c_1 \cdots c_{j-1}\alpha_{i_j}$ and $w_{\mathfrak{Q}}^{-1}\beta_j > 0$. The gallery $\mathbf{c}$ is not centrifugally folded. So $\rho_{\mathfrak{Q}}(C_j') = C_j$ implies $C_j' = C_{j-1}' = g_1 \cdots g_j C_x^-$ with $g_j = c_j = 1$ as in (1). But the gallery $(C_x^- = C_0', C_1', ..., C_j')$ cannot be minimal. □

Corollary 4.5. *If* $\mathbf{c} \in \Gamma^+_{\mathfrak{Q}}(\mathbf{i})$*, then the number of elements in* $\mathcal{C}^m_{\mathfrak{Q}}(\mathbf{c})$ *is:*

$$\sharp \mathcal{C}^m_{\mathfrak{Q}}(\mathbf{c}) = \prod_{k=1}^{t(\mathbf{c})} q_{j_k} \times \prod_{l=1}^{r(\mathbf{c})} (q_{j_l} - 1)$$

where $q_j = q_{x,\beta_j} = q_{x,\alpha_{i_j}} \in \mathcal{Q}$*,* $t(\mathbf{c}) = \sharp\{j \mid c_j = r_{i_j}, \beta_j \in \Phi_x$ *and* $w_{\mathfrak{Q}}^{-1}\beta_j < 0\}$ *and* $r(\mathbf{c}) = \sharp\{j \mid c_j = 1, \beta_j \in \Phi_x$ *and* $w_{\mathfrak{Q}}^{-1}\beta_j < 0\}$.

4.6 Galleries and opposite segment germs

Suppose now $x \in \mathbb{A} \cap \mathscr{I}^+$. Let ξ and η be two segment germs in $\mathbb{A}^+_x$. Let $-\eta$ and $-\xi$ opposite respectively η and ξ in $\mathbb{A}^-_x$. Let $\mathbf{i}$ be the type of a minimal gallery between C^-_x and $C_{-\xi}$, where $C_{-\xi}$ is the negative (local) chamber containing $-\xi$ such that $w(C^-_x, C_{-\xi})$ is of minimal length. Let $\mathfrak{Q}$ be a chamber of $\mathbb{A}^+_x$ containing η. We suppose ξ and η conjugated by W^v_x.

Lemma. *The following conditions are equivalent:*

(i) There exists an opposite ζ *to* η *in* $\mathscr{I}^-_x$ *such that* $\rho_{\mathbb{A}_x, C^-_x}(\zeta) = -\xi$.

(ii) There exists a gallery $\mathbf{c} \in \Gamma^+_{\mathfrak{Q}}(\mathbf{i})$ *ending in* $-\eta$.

(iii) $\xi \leq_{W^v_x} \eta$ *(in the sense of 1.8, with* Φ^+ *defined as in 4.1 using* C^-_x*).*

Moreover the possible ζ *are in one-to-one correspondence with the disjoint union of the sets* $\mathcal{C}^m_{\mathfrak{Q}}(\mathbf{c})$ *for* $\mathbf{c}$ *in the set* $\Gamma^+_{\mathfrak{Q}}(\mathbf{i}, -\eta)$ *of galleries in* $\Gamma^+_{\mathfrak{Q}}(\mathbf{i})$ *ending in* $-\eta$*. More precisely, if* $\mathbf{m} \in \mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$ *is associated to* $(h_1, \cdots, h_r)$ *as in remark 4.4, then* $\zeta = h_1 \cdots h_r(-\xi)$.

Proof. If $\zeta \in \mathscr{I}^-_x$ opposites η and if $\rho_{\mathbb{A}_x, C^-_x}(\zeta) = -\xi$, then any minimal gallery $\mathbf{m} = (C^-_x, M_1, ..., M_r \ni \zeta)$ retracts onto a minimal gallery between C^-_x and $C_{-\xi}$. So we can as well assume that $\mathbf{m}$ has type $\mathbf{i} = (i_1, ..., i_r)$ and then ζ determines $\mathbf{m}$. Now, if we retract $\mathbf{m}$ from $\mathfrak{Q}$, we get a gallery $\mathbf{c} = \rho_{\mathbb{A}_x, \mathfrak{Q}}(\mathbf{m})$ in $\mathbb{A}^-_x$ starting at C^-_x, ending in $-\eta$ and centrifugally folded with respect to $\mathfrak{Q}$.

Reciprocally, let $\mathbf{c} = (C^-_x, C_1, ..., C_r) \in \Gamma^+_{\mathfrak{Q}}(\mathbf{i})$, such that $-\eta \in C_r$. According to proposition and remark 4.4, there exists a minimal gallery $\mathbf{m} = (C^-_x, C'_1, ..., C'_r)$ in the set $\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$, and the chambers C'_j can be described by $C'_j = g_1 \cdots g_j C^-_x = h_1 \cdots h_j . r_{i_1} \cdots r_{i_j} C^-_x$ where each h_k fixes C^-_x, hence $\rho_{\mathbb{A}_x, C^-_x}$ restricts on C'_j to the action of $(h_1 \cdots h_j)^{-1}$.

Let $\zeta \in C'_r$ opposite η in any apartment containing those two. The minimality of the gallery $\mathbf{m} = (C^-_x, C'_1, ..., C'_r)$ ensures that $\rho_{\mathbb{A}_x, C^-_x}(\zeta) \in C_{-\xi}$; hence $\rho_{\mathbb{A}_x, C^-_x}(\zeta) = -\xi$ as they are both opposite η up to conjugation by W^v_x.

So we proved the equivalence $(i) \iff (ii)$ and the last two assertions.

Now the equivalence $(i) \iff (iii)$ is proved in [GR08, Prop. 6.1 and th. 6.3]: in this reference we speak of Hecke paths with respect to $-C^v_f$, but the essential part is a local discussion in $\mathscr{I}_x$ (using only C^-_x and the twin building structure of $\mathscr{I}^\pm_x$) which gives this equivalence. □

4.7 Liftings of Hecke paths

Let π be a $\lambda-$path from $z' \in Y^+$ to $y \in Y^+$ entirely contained in the Tits cone $\mathcal{T}$, hence in a finite union of closed sectors $w\overline{C^v_f}$ with $w \in W^v$. By [GR08, 5.2.1], for each $w \in W^v$ there is only a finite number of $s \in]0, 1]$ such that the reverse path $\bar{\pi}(t) = \pi(1-t)$ leaves, in $\pi(s)$, a wall positively with respect to $-w\overline{C^v_f}$, *i.e.* this wall separates $\pi_-(s)$ from $-w\overline{C^v_f}$.

Therefore, we are able to define $\ell\in\mathbb{N}$ and $0<t_1<t_2<\cdots<t_\ell\leq 1$ such that the $z_k=\pi(t_k)$, $k\in\{1,...,\ell\}$ are the only points in the path where at least one wall containing z_k separates $\pi_-(t_k)$ and the local chamber $\mathfrak{c}_-$ of 1.9.2.

For each $k\in\{1,...,\ell\}$ we choose for $C^-_{z_k}$ (as in 4.1) the germ in z_k of the sector of vertex z_k containing $\mathfrak{c}_-$. Let $\mathbf{i}_k$ be a fixed reduced decomposition of the element $w_-(t_k)\in W^v$ and let $\mathfrak{Q}_k$ be a fixed chamber in $\mathscr{I}^+_{z_k}$ containing $\eta_k=\pi_+(t_k)$. We note $-\xi_k=\pi_-(t_k)$. When π is a Hecke path (or a billiard path as in [GR08]), ξ_k and η_k are conjugated by $W^v_{z_k}$.

When π is a Hecke path with respect to $\mathfrak{c}_-$, $\{z_1,\cdots,z_\ell\}$ includes all points where the piecewise linear path π is folded and, in the other points, all galleries in $\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)$ are unfolded.

Let $S_{\mathfrak{c}_-}(\pi,y)$ be the set of all segments $[z,y]$ such that $\rho_{\mathfrak{c}_-}([z,y])=\pi$.

Theorem 4.8. *$S_{\mathfrak{c}_-}(\pi,y)$ is non empty if, and only if, π is a Hecke path with respect to $\mathfrak{c}_-$. Then, we have a bijection*

$$S_{\mathfrak{c}_-}(\pi,y)\simeq\prod_{k=1}^{\ell}\coprod_{\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)}\mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$$

In particular the number of elements in this set is a polynomial in the numbers $q\in\mathcal{Q}$ with coefficients in $\mathbb{Z}_{\geq 0}$ depending only on $\mathbb{A}$.

N.B. So the image by $\rho_{\mathfrak{c}_-}$ of a segment in $\mathscr{I}^+$ is a Hecke path with respect to $\mathfrak{c}_-$.

Proof. The restriction of $\rho_{\mathfrak{c}_-}$ to $\mathscr{I}_{z_k}$ is clearly equal to $\rho_{\mathbb{A}_{z_k},C^-_{z_k}}$; so the lemma 4.6 tells that π is a Hecke path with respect to $\mathfrak{c}_-$ if, and only if, each $\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)$ is non empty.

We set $t_0=0$ and $t_{\ell+1}=1$. We shall build a bijection from $S_{\mathfrak{c}_-}(\pi_{|[t_{n-1},1]},y)$ onto $\prod_{k=n}^{\ell}\coprod_{\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)}\mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$ by decreasing induction on $n\in\{1,\cdots,\ell+1\}$. For $n=\ell+1$ and if $t_\ell\neq 1$, no wall cutting $\pi([t_\ell,1])$ separates $y=\pi(1)$ from $\mathfrak{c}_-$; so a segment s in $\mathscr{I}$ with $s(1)=y$ and $\rho_{\mathfrak{c}_-}\circ s=\pi$ has to coincide with π on $[t_\ell,1]$.

Suppose now that $s\in S_{\mathfrak{c}_-}(\pi_{|[t_n,1]},y)$ is determined, in the following way, by a unique element in $\prod_{k=n+1}^{\ell}\coprod_{\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)}\mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$: For an element $(\mathbf{m}_{n+1},\mathbf{m}_{n+2},...,\mathbf{m}_\ell)$ in this last set, each $\mathbf{m}_k=(C^-_{z_k},C^k_1,...,C^k_{r_k})$ is a minimal gallery given by a sequence of elements $(h^k_1,...,h^k_{r_k})\in(\overline{G}_{z_k})^{r_k}$, as in remark 4.4 and, for $t\in[t_n,t_{n+1}]$, we have $s(t)=(h^\ell_1...h^\ell_{r_\ell})\cdots(h^{n+1}_1...h^{n+1}_{r_{n+1}})\pi(t)$ where actually each h^k_j is a chosen element of $U_{-r_{i_1}\cdots r_{i_{j-1}}(\alpha_{i_j})}$ whose class in $\overline{U}_{-r_{i_1}\cdots r_{i_{j-1}}(\alpha_{i_j})}$ is the h^k_j defined above; in particular each h^k_j fixes $\mathfrak{c}_-$.

We note $g=(h^\ell_1...h^\ell_{r_\ell})\cdots(h^{n+1}_1...h^{n+1}_{r_{n+1}})\in G_{\mathfrak{c}_-}$. Then $g^{-1}s(t_n)=\pi(t_n)=z_n$.

If $s\in S_{\mathfrak{c}_-}(\pi_{|[t_{n-1},1]},y)$ and $s_{|[t_n,1]}$ is as above, then $g^{-1}s_-(t_n)$ is a segment germ in $\mathscr{I}^-_{z_n}$ opposite $g^{-1}s_+(t_n)=\pi_+(t_n)=\eta_n$ and retracting to $\pi_-(t_n)$ by $\rho_{\mathfrak{c}_-}$. By lemma 4.6 and the above remark, this segment germ determines uniquely a minimal gallery $\mathbf{m}_n\in\mathcal{C}^m_{\mathfrak{Q}_n}(\mathbf{c})$ with $\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_n}(\mathbf{i}_n,-\eta_n)$.

Conversely such a minimal gallery $\mathbf{m}_n$ determines a segment germ $\zeta\in\mathscr{I}^-_{z_n}$, opposite $\pi_+(t_n)=\eta_n$ such that $\rho_{\mathbb{A}_{z_n},C^-_{z_n}}(\zeta)=\pi_-(t_n)$. By lemma 4.6, $\zeta=(h^n_1...h^n_{r_n})\pi_-(t_n)$ for some well defined $(h^n_1,\cdots,h^n_{r_n})\in(\overline{G}_{z_n})^{r_n}$. As above we replace each g^n_j by a chosen element of $G_{(z_n\cup\mathfrak{c}_-)}$ whose class in $\overline{G}_{z_n}$ is this g^n_j. As no wall cutting $[z_{n-1},z_n]$ separates $z_n=\pi(t_n)$ from $\mathfrak{c}_-$, any segment retracting by $\rho_{\mathfrak{c}_-}$ onto $[z_{n-1},z_n]$ and with $[z_n,x)=\pi_-(t_n)$ (resp.

$=\zeta$, $=g\zeta$) is equal to $[z_{n-1},z_n]$ (resp. $(h_1^n...h_{r_n}^n)[z_{n-1},z_n]$, $g(h_1^n...h_{r_n}^n)[z_{n-1},z_n]$). We set $s(t)=(h_1^\ell...h_{r_\ell}^\ell)\cdots(h_1^{n+1}...h_{r_{n+1}}^{n+1})(h_1^n...h_{r_n}^n)\pi(t)$ for $t\in[t_{n-1},t_n]$.

With this inductive definition, s is a $\lambda-$path, $s(1)=y$, $\rho_{\mathfrak{c}_-}\circ s=\pi$ and $s_{|[t_{k-1},t_k]}$ is a segment $\forall k\in\{1,...,\ell+1\}$. Moreover, for $k\in\{1,...,\ell\}$, the segment germs $[s(t_k),s(t_{k+1}))$ and $[s(t_k),s(t_{k-1}))$ are opposite. By the following lemma this proves that s itself is a segment. □

Lemma 4.9. *Let x,y,z be three points in an ordered hovel $\mathscr{I}$, with $x\le y\le z$ and suppose the segment germs $[y,z)$, $[y,x)$ opposite in the twin buildings $\mathscr{I}_y$. Then $[x,y]\cup[y,z]$ is the segment $[x,z]$.*

Proof. For any $u\in[y,z]$, we have $x\le y\le u\le z$, hence x and $[u,y)$ or $[u,z)$ are in a same apartment [Ro11, 5.1]. As $[y,z]$ is compact we deduce that there are points $u_0=y,u_1,\cdots,u_\ell=z$ such that x and $[u_{i-1},u_i]$ are in a same apartment A_i, for $1\le i\le\ell$. Now A_1 contains x and $[y,u_1]$, hence also $[x,y]$ (axiom (MAO) of *l.c.*). But $[y,x)$ and $[y,u_1)=[y,z)$ are opposite, so $[x,y]\cup[y,u_1]=[x,u_1]$. The lemma follows by induction. □

Remark 4.10. The same things as above may be done for the retraction $\rho_{-\infty}$ instead of $\rho_{\mathfrak{c}_-}$: for all x we choose $C_x^-=germ_x(x-C_f^v)$. For a $\lambda-$path π in $\mathbb{A}$ from z' to y, [GR08, 5.2.1] tells that we have a finite number of points $z_k=\pi(t_k)$ where at least a wall is left positively by the path $\bar\pi(t)=\pi(1-t)$. We define as above $\mathbf{i}_k$, $\mathfrak{Q}_k$, η_k and ξ_k. Now $S_{-\infty}(\pi,y)$ is the set of all segments $[z,y]$ such that $\rho_{-\infty}([z,y])=\pi$.

In [GR08, Theorems 6.2 and 6.3], we have proven that $S_{-\infty}(\pi,y)$ is nonempty if, and only if, π is a Hecke path with respect to $-C_f^v$. Moreover, we have shown that, for $\mathscr{I}$ associated to a split Kac-Moody group over $\mathbb{C}((t))$, $S_{-\infty}(\pi,y)$ is isomorphic to a quasi-affine toric complex variety. The arguments above prove that, with our choice for $\mathscr{I}$, $S_{-\infty}(\pi,y)$ is finite, with the following precision (which generalizes to the Kac-Moody case some formulae of [GL11]):

Proposition 4.11. *Let π be a Hecke path with respect to $-C_f^v$ from z' to y. Then we have a bijection:*

$$S_{-\infty}(\pi,y)\simeq\prod_{k=1}^{\ell}\coprod_{\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)}\mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$$

In particular the number of elements in this set is a polynomial in the numbers $q\in\mathcal{Q}$ with coefficients in $\mathbb{Z}_{\ge0}$ depending only on $\mathbb{A}$.

Theorem 4.12. *Let $\lambda,\mu,\nu\in Y^{++}$, $\mathfrak{c}_-$ the negative fundamental alcove and suppose $(\alpha_i^\vee)_{i\in I}$ $\mathbb{R}^+-$free. Then*

a) The number of Hecke paths of shape μ with respect to $\mathfrak{c}_-$ starting in $z'=w\lambda$ (for some $w\in W^v$ fixing 0) and ending in $y=\nu$ is finite.

b) The structure constant $m_{\lambda,\mu}(\nu)$ i.e. the number of triangles $[0,z,\nu]$ in $\mathscr{I}$ with $d_v(0,z)=\lambda$ and $d_v(z,\nu)=\mu$ is equal to:

$$m_{\lambda,\mu}(\nu)=\sum_{w\in W^v/(W^v)_\lambda}\sum_{\pi}\prod_{k=1}^{\ell_\pi}\sum_{\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)}\sharp\mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})\qquad(2)$$

where π runs over the set of Hecke paths of shape μ with respect to $\mathfrak{c}_-$ from $w\lambda$ to ν and ℓ_π, $\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)$ and $\mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$ are defined as above for each such π.

c) In particular the structure constants of the Hecke algebra $\mathcal{H}_R$ are polynomials in the numbers $q\in\mathcal{Q}$ with coefficients in $\mathbb{Z}_{\ge0}$ depending only on $\mathbb{A}$.

Proof. We saw in 2.3.1 that $m_{\lambda,\mu}(\nu)$ is the number of $z \in \mathscr{I}_0^+$ such that $d_v(0,z) = \lambda$ and $d_v(z,\nu) = \mu$. Such a z determines uniquely a Hecke path $\pi = \rho_{\mathfrak{c}_-}([z,\nu])$ of shape μ with respect to $\mathfrak{c}_-$ from $z' = \rho_{\mathfrak{c}_-}(z)$ to ν. But $d_v(0,z) = \lambda$ and $0 \in \mathfrak{c}_-$, so $d_v(0,z') = \lambda$ *i.e.* $z' = w\lambda$ with $w \in W^v$. So the formula (2) follows from theorem 4.8.

We know already that $m_{\lambda,\mu}(\nu)$ is finite (2.5) and $S_{\mathfrak{c}_-}(\pi,y) \neq \emptyset$ (theorem 4.8), hence a) is clear. Now c) follows from corollary 4.5 □

Remark 4.13. The commutativity of $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$ corresponds to polynomials identities (depending on $\mathbb{A}$) in the variables $q \in \mathcal{Q}$. In the homogeneous case (where all $q \in \mathcal{Q}$ are equal) and for $\mathbb{A}$ associated to a RGS as in 3.1 we saw that these identities are verified for q a power of a prime number; hence for any q.

So, for any choice of a homogeneous hovel with the same $\mathbb{A}$ and Y, $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$ are still commutative.

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