UNIVERSITÉ DE CERGY PONTOISE
Laboratoire de Physique Théorique et Modélisation

THÈSE

présentée pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE CERGY PONTOISE

Spécialité : Physique Théorique

par

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sous la direction du Professeur Nikolai KITANINE

Titre :

Dynamical reflection algebras and associated boundary integrable models

soutenue publiquement le 12 décembre 2011

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Acknowledgment

In the first place, I am largely indebted to my PhD supervisor, Nikolai "Kolya" Kitanine. It was a wonderful opportunity to work besides Kolya on boundary spin chains. His patience and support enabled me to get all the necessary conditions for starting and completing a PhD project. I really appreciate his accurate sense of rigor and his very deep knowledge of quantum integrable systems theory. I would like to thank him very warmly for these last three years that I spent collaborating with him. Kolya, it was a great honor for me !

My taste for quantum integrable theory and algebraic technique comes from my Bachelor thesis, where I started to handle spectrum generating algebra. I was not aware of the link between this work and the algebraic Bethe ansatz. So I would like to thank Laurent Gallot for making me aware of this fascinating subject and directing my internship at LAPTH of Annecy in 2004.

During my Master year, I had the chance to follow the course on integrable systems by Jean Michel Maillet at Ecole Normale Supérieure of Lyon. I would like to show my gratitude to Jean Michel for this wonderful course, and his introduction to the algebraic Bethe ansatz, which has definitively had a great impact on my scientific life.

I owe my deepest gratitude to Veronique Terras from Ecole Normale Supérieure of Lyon, which has introduced me to the XYZ spin chain and to the wonderful subject of the vertex-face transformation during my Master thesis. I would like to thank her for her strong support during my PhD thesis, and the various discussions that we had and that we still have.

This thesis would not have been possible without the wonderful atmosphere of the LPTM, Cergy. It is a pleasure to thank Hung The Diep, the former laboratory director and Trong Tuong Truong the current one. Both of them have supported me a lot, and they make the PhD student conditions at Cergy to be the best one can imagine. I would like to thank Sylvie Villemin for her high professionalism and kindness, she helps me a lot to handle the administrative tasks. The LPTM laboratory is full of great personalities, it was a pleasure to collaborate on teaching lessons with Guy Trambly and Claire Pinette. It was great and we did a good job !

My collaboration with Kolya would not have been possible without the hospitality of the LPTHE laboratory of Paris VI. I would like to thank Olivier Babelon, the laboratory director, for easing our collaboration and the very frequent visits to his laboratory.

I am grateful to Olivier Babelon, Jean Avan, Vladimir Rubtsov, Christian Korff and Robert Weston for accepting to take part in my thesis jury. It is very challenging but also rewarding to have a so prestigious jury ! A special thanks to Vladimir Rubtsov and Robert Weston for their careful reporting of the manuscript.

I really appreciated to discuss with Jean Avan and Vladimir Rubtsov. I learnt at lot from their deep knowledge on dynamical and elliptic algebras. Our discussions were fruitful and very nice. Thank you for your support on my work and for the various discussion we had!

This thesis was financed by a special grant from the University of Cergy Pontoise and the LPTM laboratory. I owe my deepest gratitude to these two institutions for their financial support.

Abstract

This thesis is embedded in the general theory of quantum integrable models with boundaries, and the development of associated algebraic structures.

We first consider the question of the diagonalization of the XXZ hamiltonian with non-diagonal boundaries. We succeed to find the two sets of eigenstates and eigenvalues of the model if the boundaries parameters satisfy two conditions.

We introduce then a statistical physics model which we refer to be the face model with a reflecting end. Moreover, we compute exactly its partition function and show that it takes the form of a simple single matrix determinant.

We show that these two problems are related through the vertex-face transformation and are solved using a common algebraic structure, the dynamical reflection algebra and its dual. We focus from a mathematical perspective on this algebra in the general elliptic case. Both the co-module evaluation representation and its dual are introduced. We believe that these structures are the key ingredients for the analysis of face models with boundaries. In particular, using the concept of Drinfel'd twists, we show that the partition function of these models has a simple representation in the general case.

Finally, we attempt on a 'dynamization' of the Half-Turn-Symmetric vertex model. We describe its partition function in terms of the evaluation representation of the dynamical Yang-Baxter algebra, and find a set of conditions that uniquely determine it.

Resumé

Cette thèse s'inscrit dans le cadre général de la théorie des systèmes intégrables avec bords et le développement des structures algébriques associées.

D'une part, nous nous attaquons au problème de la diagonalisation de l'hamiltonien du modèle XXZ avec bords non diagonaux. Nous exhibons les deux ensembles d'états propres et valeurs propres du modèle si les paramètres de bords satisfont deux conditions.

D'autre part, nous introduisons un modèle de physique statistique que nous appelons le modèle face avec un bord réfléchissant. Nous calculons exactement sa fonction de partition et nous montrons que cette dernière se représente simplement sous la forme d'un unique déterminant matriciel.

Nous montrons que ces deux problèmes sont reliés par la transformation vertex-face et exhibent une structure algébrique commune, l'algèbre de réflexion dynamique. Nous nous intéressons aux aspects mathématiques de cette algèbre dans le cas elliptique général, et nous introduisons deux classes de ces représentations, la représentation de co-module d'évaluation et sa duale. Nous pensons que cette algèbre est la structure clef pour l'analyse des modèles faces avec bords. En particulier, nous montrons à l'aide de twists de Drinfel'd que leur fonction de partition se représente simplement dans le cas général.

Enfin, nous tentons une ’dynamisation’ du modéle à vertex ’Half-Turn-Symmetric’, et nous decrivons sa fonction de partitionen termes de représentation d’évaluation de l’algèbre de Yang-Baxter dynamique, et trouvons un ensemble de conditions la déterminant univoquement.

Notations

- General:
  The notation $X_{i_1\dots i_N}$ for any operator $X\in \mathrm{End}(\otimes_{j=1}^N V_{i_j})$, where $V_{i_{j=1,\dots,N}}$ is a linear space means that this operator acts as X in the space $\otimes_{j=1}^N V_{i_j}$ and trivially in any other space.
  $\lambda_i\in\mathbb{C}$: spectral parameter associated with the space V_i
  $\xi_j\in\mathbb{C}$: inhomogeneity parameter associated with the space V_j
  $\eta\in\mathbb{C}$: crossing parameter
  V: linear space.

- Pauli matrices:
  $\sigma^x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$
  $\sigma^y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$
  $\sigma^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

- Vertex models:
  R: Vertex R-matrix
  K_-: right boundary matrix
  K_+: left boundary matrix
  L: Quantum Lax operator
  T: monodromy matrix
  U_-: boundary (double rows) monodromy matrix
  U_+: dual boundary (double rows) monodromy matrix
  $\delta,\bar{\delta},\zeta,\bar{\zeta},\tau,\bar{\tau}\in\mathbb{C}$: spin chain boundary parameters

- Face (or dynamical) models:
  $\mathcal{R}$: face R-matrix
  $\mathcal{K}_-$: dynamical right boundary matrix
  $\mathcal{K}_+$: dynamical left boundary matrix
  $\mathcal{L}^t$: crossed Lax matrix
  $\mathcal{T}$: dynamical monodromy matrix
  $\mathcal{V}^t$: dual dynamical monodromy matrix
  $\mathcal{U}_-$: dynamical boundary monodromy matrix
  $\mathcal{U}_+$: dynamical dual boundary monodromy matrix

S: vertex-face transformation
$\theta \in \mathbb{C}$: dynamical or face parameter
$\zeta_-, \zeta_+ \in \mathbb{C}$: dynamical boundary parameters
$\mathcal{X}_{ij}(\lambda;\theta-\eta\sigma_k^z)$: mean that $\mathcal{X}$ act on $V_i \otimes V_j \otimes V_k$ as:

$$\mathcal{X}_{ij}(\lambda;\theta-\eta\sigma_k^z)|i\rangle \otimes |j\rangle \otimes |k\rangle = \{\mathcal{X}_{ij}(\lambda;\theta-\eta\mu)|i\rangle \otimes |j\rangle\} \otimes |k\rangle$$

with: $\sigma_k^z|k\rangle = \mu|k\rangle$
$[[1,N]]$: set of consecutive integers between 1 and N
Normal ordering: the $\sigma_{i,j}^z$ in the argument of any operator $X_{ij}(\lambda;\theta-\eta\sigma_{i,j}^z) \in \mathrm{End}(V_i \otimes V_j)$ (which does not necessary commute with it) is always on the right of all other operators involved in the definition of X.

- Elliptic functions:
  Let ε be a fixed complex parameter such that: $Im(\varepsilon) > 0$ and denotes $p = e^{2i\pi\varepsilon}$. We will use the following notation for elliptic functions [56, 119]:
  $h_1(\lambda;\varepsilon) = -2ip^{\frac{1}{8}}\sinh(\lambda)\prod_{n=1}^{\infty}(1-2p^n\cosh(2\lambda)+p^{2n})(1-p^n)$
  $h_2(\lambda;\varepsilon) = 2p^{\frac{1}{8}}\cosh(\lambda)\prod_{n=1}^{\infty}(1+2p^n\cosh(2\lambda)+p^{2n})(1-p^n)$
  $h_3(\lambda;\varepsilon) = \prod_{n=1}^{\infty}(1+2p^{n-\frac{1}{2}}\cosh(2\lambda)+p^{2n-1})(1-p^n)$
  $h_4(\lambda;\varepsilon) = \prod_{n=1}^{\infty}(1-2p^{n-\frac{1}{2}}\cosh(2\lambda)+p^{2n-1})(1-p^n)$
  $h(\lambda) = e^{\lambda}\prod_{n=0}^{\infty}(1-p^n e^{-2\lambda})(1-p^{n+1}e^{2\lambda})$

  Up to a multiplicative factor, $h(\lambda)$ equals the Jacobi theta function $\theta_1(i\lambda)$. This function is odd and satisfies the addition rule

$$\begin{aligned} h(x+u)h(x-u)h(y+v)h(y-v) - h(x+v)h(x-v)h(y+u)h(y-u) \\ = h(x+y)h(x-y)h(u+v)h(u-v). \end{aligned}$$

  In the degenerate case, we have: $\lim_{p\to 0} h(\lambda) = 2\sinh(\lambda)$.

Contents

Introduction

According to Galileo, "Nature's great book is written in mathematical language". It is a natural task in the physical sciences to explain observed phenomena and their various causes and consequences, trough the use of a mathematical framework. Nature possesses a huge amount of complex, highly non-trivial, and intricate phenomena. To understand them, physicist's approach consists of extracting the most relevant facts about these phenomena and building a theory around them. The aim is not to precisely describe the observed fact, but rather to focus on the few fundamental underlying phenomena. Following this tradition, Copernicus noticed that observed star motion takes the form of a circle. Although this is known to be non-exact, it allows one to highlight the fundamental features of star motion and the important role of this singular object which is the sun. Thus, a physics theory consists in full generality of a set of few, preferably simple, concepts which explain stylized facts and magnitude order. Then we use a number of sophisticated methods to describe the observed phenomena as perturbations of these theories. In this spirit, Kepler uses the Tyco Brahe data to refine Copernicus' theory for star motion, and he proposes that stars follow elliptical orbit.

The theory of integrable systems has its roots in theoretical physics, with the first attempt at mathematical formulation of physics laws by Newton. The integrable systems theory in classical mechanics consists of finding exact solutions to mechanic equations. Using his framework, Newton succeeded in finding an exact solution to Kepler's two-body problem. Liouville then proposes a generic framework where classical mechanics equations can be exactly solved by quadrature [79, 80]. In his work, the notion of integrals of motion (or conserved quantities) forming an involutive family is crucial. Apart from the Kepler problem, only a few examples were known before the second half of the twentieth century and the work of Gardner, Greene, Kruskal and Miura [50] on the conserved quantities of the Korteweg de Vries equation in fluid mechanics. Thereafter, the field developed quickly with the work of Lax [76] regarding his formulation of mechanic equations, which allows a strong framework for generating conserved integrals of motion for a classical system. All this research was put on a solid unified scheme in the work of Faddeev, Zakharov [127] and Gardner [51] where they built the link between Liouville integrability and Lax formulation. This method, known as the inverse scattering method, provides a generic mathematical theory which has enormous applications in classical mechanics, chaos theory, general relativity, gauge and string theory. This theoretical scheme uses a large range of interacting mathematical fields such as the theory of symplectic manifolds, and the theory of Poisson-Lie groups.

Alongside these recent developments is the emergence of quantum theory. The theory of quantum integrable systems has its roots in the very beginning of quantum theory itself. Using the Heiseinberg matrix formulation of quantum mechanics, the search for an exact solution to quantum problems was started by Heisenberg when he succeeded in developing a purely algebraic treatment of the harmonic oscillator. In the same spirit Pauli [95] succeeded in solving the hydrogen atom problem. In this framework, the concept of integrals of motion is also crucial, and the notion of spectrum generating algebra becomes particularly relevant.

Along with the development of quantum mechanics, the first quantum models for condensed matter appears, and among them the Heiseinberg model for ferromagnetism plays a central role. The key work of Bethe [10] for solving the one-dimensional periodic XXX Heisenberg model is often considered as the starting point of the modern theory of quantum integrable systems. In his work, the XXX Heisenberg hamiltonian eigenstates were constructed in terms of quasi-particles. We should mention that the Bethe method relies on an ansatz about the eigenstates: they should be eigenstates of both the translation operator and the total spin operator, both of which are integrals of motions. Translation invariance relies to the fact that the hamiltonian has a periodic boundary condition. Therefore, finding an exact solution implies that the rapidity of the scattering quasi-particles should satisfy a set of algebraic relations known as the Bethe equations. The primordial importance of the spin conservation is related to the conservation of the number of quasi-particles. This latter observation in connection with the integrability of the XXX Heisenberg hamiltonian was not apparent in Bethe's work.

Another key development in the theory of quantum integrable systems is the mathematical *tour de force* solution of the two-dimensional Ising model by Onsager [90], including the calculation of the magnetic order parameter. The Onsager solution makes use of the transfer matrix technology which now plays a central role in statistical physics and the use of a very important relation; the star triangle equation. At first sight, it may seem that the Onsager method is far from quantum theory since the Ising model is a classical model, but this is not the case. Indeed, equilibrium two-dimensional classical statistical mechanic is dual to one-dimensional quantum mechanics (and more generally any equilibrium (D+1)-dimensional classical statistical mechanic model is dual to a D-dimensional quantum mechanics model). In the framework of quantum integrable systems theory, this duality was highlighted and extensively used by Baxter in his analysis of vertex models. Moreover, he shows that Heisenberg spin chain hamiltonian are related to vertex model transfer matrices, establishing a first connection between Bethe solution for one-dimensional quantum models and the Onsager solution for two-dimensional classical statistical models. In addition, he uses the star triangle equation for finding solutions of vertex models.

A third achievement is the work of Yang [120] on the one-dimensional N-body problem with δ-repulsive interaction. He succeed in using Bethe ansatz, in the diagonalization of the quantum hamiltonian. The Yang method underlines another cubic relation involving the two-particle scattering matrix, which once again is the star triangle relation. This relation appeared as a consistency equation for the Bethe hypothesis.

A final step towards a theory of quantum integrable systems is the work of Zamolodchikov and Zamolodchikov [125] on integrable quantum field theory. They show that certain quantum relativistic theories are integrable if there is neither particle creation nor annihilation in the scattering process, or in others words if the number of particles is conserved. Another crucial condition for integrability is that multiple scattering processes should be factorizable into two particle scattering processes that obey the star triangle equation. In this bootstrap procedure we recover many of the stylized facts for a quantum integrable theory: the conservation of particles and the star triangle equation.

All these seemingly different methods were unified by the work of the Leningrad school around Faddeev [33]. It was shown that all these approaches have a common and unique algebraic version, which non-trivially take the form of a quantum version of the classical inverse scattering theory. In the quantum inverse scattering method, the star triangle equation, called now the Yang-Baxter equation, take the form of a consistency equation for a non-abelian spectrum generating algebra. This algebra, known as the Yang-Baxter algebra, leads very naturally to an involutive family of integrals of motion. This very powerful theory permits one to describe, in a unified scheme, the various quantum integrable models. It also provides a theoretical framework for both one-dimensional periodic quantum hamiltonians and for dual two-dimensional statistical physics models. It is at the heart of an important field of mathematics known as the theory of quantum groups.

The quantum inverse scattering method has two generalizations. The first one is contributed to Baxter [7–9] for handling models were the number of quasi-particles is not conserved. For this task he uses a crucial transformation, the vertex-face transformation, in order to diagonalize the eight-vertex model transfer matrix, or equivalently the XYZ spin chain hamiltonian. Felder [42] shows that this transformation leads to a new integrable structure known as dynamical Yang-Baxter algebra. This algebra is the key structure for the algebraic analysis of the eight-vertex model and also for another class of related, face models. The second generalization was developed by Sklyanin [110] for handling more general boundary conditions. He shows that another algebraic structure, the reflection algebra, is the key structure for the analysis of models with open boundaries, among them the XXZ model with diagonal boundary. This thesis is an attempt to describe in an unified scheme dynamical models with boundaries.

We make extensive use of the vertex-face transformation for the diagonalization of the XXZ hamiltonian model with the most general boundaries. This enables us to highlight the key ingredients for dynamical models with boundaries, which are the dynamical reflection equation and the associated algebra. We then generalize these structures and focus on their mathematical basis in a model-independent framework. We also present the application of this newly discovered structure in the field of statistical mechanics.

The thesis is organized into five chapters. The first of which is a theoretical background of the quantum inverse scattering method and the algebraic Bethe ansatz method for the resolution of the diagonalization problem of quantum spin chains hamiltonians with a conserved number of quasi-particle. In this chapter, we also describe the mathematics behind the Yang-Baxter equation and Yang-Baxter algebra. In the second chapter, we highlight the duality between quantum spin chains with a conserved number of particles and classical vertex models of statistical mechanics. In particular, we show how the quantum inverse scattering method for the quantum models can lead to the exact evaluation of partition functions of the associated vertex models. Then, in the third chapter, we focus on the Baxter's vertex-face transformation for the analysis of models with non-conservation of the third component of the total spin, or equivalently the number of quasi-particles. This will enable us to understand the vertex face technology in order to use it in another context, which is precisely the object of the fourth chapter. We uses the vertex face transformation in order to solve the XXZ model with the most general boundaries. This leads us to the discovery of a new dynamical algebra, the dynamical reflection algebra. We also introduce in this chapter a new face model with a boundary, which is canonically related to the XXZ model with general boundaries, and shows that the dynamical reflection structure enables us to exactly compute its partition function. Finally, in the fifth chapter, we generalize the newly discovered dynamical reflection algebra in a model-independent framework. We also show that it is the key structure for the general analysis of the face model with reflecting ends, and the exact computation of their partition functions.

Articles list

- Article 1
  Filali Ghali, Kitanine Nikolai : *The partition function of the trigonometric SOS model with a reflecting end*.
  J. Stat. Mech.: Theory Exp. (2010) L06001, erratum: J. Stat. Mech.: Theory and Exp.(2010) E07002

- Article 2
  Filali Ghali, Kitanine Nikolai : *Spin Chains with Non-Diagonal Boundaries and Trigonometric SOS Model with Reflecting End*.
  SIGMA 7 (2011) 012

- Article3
  Filali Ghali: *Elliptic dynamical reflection algebra and partition function of SOS model with reflecting end*. Journal of Geometry and Physics 61 (2011) 1789

Chapter 1

Algebraic framework for quantum integrable models

A quantum system is completely characterized by the data of a *hamiltonian*, which is a hermitian operator describing the time evolution. It acts on a specified Hilbert space describing the states of the system. By an integrable quantum model, we mean a hamiltonian for which it is possible to determine completely its spectrum, namely both eigenvalues and eigenstates. The main idea for the *quantum inverse scattering method* (QISM) [7, 34, 35, 53, 61, 65, 72, 107] is to find a set of *commuting quantum charges* forming with the hamiltonian an abelian subalgebra embedded into a bigger non abelian algebra. The *algebraic Bethe ansatz* technique allows one to represent the generators of this algebra as creation and annihilation operators. Their action on some *reference state* can thus generate hamiltonian eigenstates. The non abelian algebra is obtained using an auxiliary linear problem. Although this construction is available for *periodic boundary* hamiltonians and also for more general *open boundary conditions*, the existence of a reference state is a non trivial feature that only few hamiltonians share. Such a reference state may exist, but it can be troublesome to find it as it is mainly related to global symmetry of the hamiltonian. We will first present the general algebraic framework for the QISM and then show for third component of the total spin invariant hamiltonians how to construct eigenstates. Such hamiltonians conserve the number of (quasi-)particles and have two canonical eigenstates, the completely ferromagnetic states, which can be used as reference states.

1.1 Periodic systems and Yang-Baxter algebra

The key element of the QISM is a *quantum R-matrix* $R : \mathbb{C} \times \mathbb{C} \longrightarrow \mathrm{End}(V \otimes V)$, where V is a linear space, satisfying the Yang-Baxter equation:

$$R_{12}(\lambda_1,\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3) = R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2). \tag{1.1.1}$$

This equation in $V_1 \otimes V_2 \otimes V_3$, is a consistency condition for associativity of the algebra generated by the operator entries elements $T_{\alpha,\beta} \in End(\mathcal{H})$ of the *monodromy matrix* $T : \mathbb{C} \to V \otimes \mathcal{H}$, where $\mathcal{H}$ is the system Hilbert space. This algebra, known as the *Yang-Baxter algebra*, is written as an equation in $End(V_1 \otimes V_2 \otimes \mathcal{H})$:

$$R_{12}(\lambda_1,\lambda_2)T_1(\lambda_1)T_2(\lambda_2) = T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1,\lambda_2). \tag{1.1.2}$$

λ_i is referred to be the spectral parameter associated to the auxiliary space V_i. The remarkable point is that provided the elements of T satisfy the Yang-Baxter algebra relations it is possible to generate a family of commuting quantum charges, leading to an *involutive* set of operators, the transfer matrix $\mathbf{T}$:

$$\forall(\lambda,\mu) \in \mathbb{C}^2 \quad [\mathbf{T}(\lambda),\mathbf{T}(\mu)] = 0, \tag{1.1.3}$$

where:

$$\mathbf{T}(\lambda) = tr_0(T_0(\lambda)), \tag{1.1.4}$$

is the trace over the auxiliary space V_0. This transfer matrix is to be understood as a generating function for integrals of motions of a quantum system described by some representations of the Yang-Baxter algebra (1.1.2). The matrix elements $T_{\alpha,\beta}$ can generate by action on a *pseudo-vacuum* $|0\rangle$ or reference state the eigenstates of the commuting family and thus hamiltonian eigenstates if it belongs to this family. This construction is an algebraic achievement that arises from three *apparently* different fields of mathematical physics, namely the theory of factorized scattering matrix for (1+1)-integrable field theory [15, 120, 124–126], the mathematical analysis of exactly solvable statistical physics models in 2-dimensions developed mainly by Baxter [8, 9], and it is in some sense largely inspired from the quantized theory of non linear classical equation resolution that arises in classical integrable mechanics [31, 33, 108]. The relation to statistical physics will be extensively studied in this thesis within this modern framework. It is very instructive to note the physical picture underlying the QISM construction, especially in connection with non algebraic methods such as the coordinate Bethe ansatz (which is not the object of this thesis) [10, 57, 90, 91, 105, 116].

The R-matrix can be understood as a scattering matrix describing the underlying scattering in the system. Multi-particle process can factorized into two particle one and the

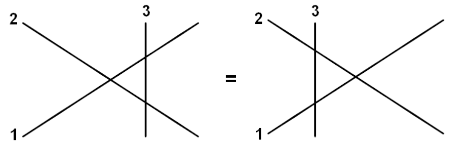


Figure 1.1: The Yang-Baxter equation (1.1.1)

Yang-Baxter equation (1.1.1) is the invariance of three-body scattering that should obey any (1+1) process to be integrable as illustrated in Figure 1.1. The monodromy matrix T_0 should be understood as the scattering matrix for any system excitation within the system: $1,\dots,N$ are indexes for the various points that we will associate to single quantum systems

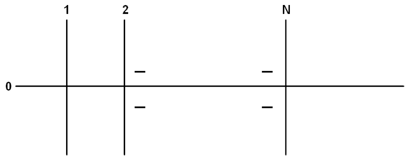


Figure 1.2: The monodromy operator T_0

with quantum space $V_{i,i=1,\dots,N}$ and 0 refer to an auxiliary space V_0. The Yang-Baxter algebra (1.1.1) reflects the invariance of the system scattering process for multiple scattering points:

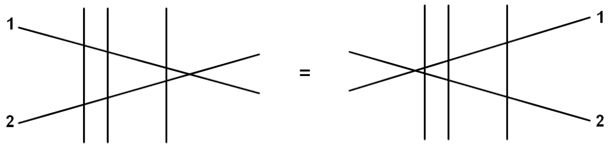


Figure 1.3: The Yang-Baxter algebra (1.1.2)

Finally, the periodic features of models that are described by the aforementioned framework are obvious from the trace formula (1.1.4) as any possible excitation starting in the system achieves a complete scattering and return to the same point:

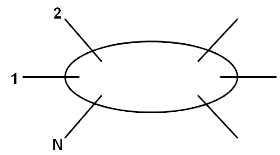


Figure 1.4: Trace formula (1.1.4) and periodicity

1.2 The first example: the periodic XXZ spin chains

For quantum spin chains, it is quite remarkable that starting from the local operators describing the quantum system we can find such quadratic algebra with an abelian subalgebra containing the hamiltonian. In this thesis, we focus on one-dimensional spin chains of size N where in each site $m = 1,..,N$ is a quantum system with Hilbert space $\mathcal{H}_m \sim \mathbb{C}^2$. Thus, the hamiltonian of the system acts in the Hilbert space $\mathcal{H} = \otimes_{m=1}^{N} \mathcal{H}_m$. In this section we are interested in the XXZ hamiltonian with periodic boundary condition:

$$\mathcal{H} = \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta(\sigma_i^z \sigma_{i+1}^z - 1) \tag{1.2.1}$$

$\Delta = \cosh\eta$ and $\eta \in \mathbb{C}$ is an anisotropy parameter. $\sigma_i^{x,y,z} \in End(\mathbb{C}^2)$ are the usual Pauli matrix. Since we assume periodic boundary condition, we use the following convention: $\sigma_{N+1}^{x,y,z} = \sigma_1^{x,y,z}$. The QISM framework for the XXZ spin chain requires the use of the *trigonometric solution* of the Yang-Baxter equation [33], a representation of the universal R-matrix of $\mathcal{U}_q(\widehat{sl}_2)$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$, known as the *six-vertex*[1] R-matrix, which depends on the difference of spectral parameters:

$$R(\lambda) = \begin{pmatrix} a^{6V}(\lambda) & 0 & 0 & 0 \\ 0 & b^{6V}(\lambda) & c^{6V}(\lambda) & 0 \\ 0 & c^{6V}(\lambda) & b^{6V}(\lambda) & 0 \\ 0 & 0 & 0 & a^{6V}(\lambda) \end{pmatrix}, \tag{1.2.2}$$

[1] The name six-vertex refers to the underlying statistical physics systems which is the object of the next chapter.

with:

$$a^{6V}(\lambda) = \sinh(\lambda+\eta), \quad c^{6V}(\lambda) = \sinh(\eta), \quad b^{6V}(\lambda) = \sinh(\lambda). \tag{1.2.3}$$

A canonical representation $L : \mathbb{C} \to End(V_0 \otimes \mathcal{H}_m), V_0 \sim \mathbb{C}^2$ of the algebra (1.1.2) in the quantum space $\mathcal{H}_m$, the Lax matrix, is given by the R-matrix itself. In this case, it can be rewritten in the more explicit form:

$$L(\lambda-\xi_m) = R(\lambda-\xi_m) = \begin{pmatrix} \sinh(\lambda-\xi_m+\eta\frac{1+\sigma_m^z}{2}) & \sinh\eta\sigma_m^- \\ \sinh\eta\sigma_m^+ & \sinh(\lambda-\xi_m+\eta\frac{1-\sigma_m^z}{2}) \end{pmatrix}, \tag{1.2.4}$$

$\xi_m \in \mathbb{C}, m = 1,\ldots,N$ are arbitrary parameters attached to each quantum space known as the inhomogeneity parameters, that we introduce here for convenience, and $\sigma_i^\pm$ are the usual creation and annihilation spin operators $\sigma_i^\pm = \frac{1}{2}(\sigma_i^x \pm i\sigma_i^y)$. Note that each single quantum system is a representation of the sl_2 algebra:

$$[\sigma_i^+, \sigma_j^-] = \delta_{ij}\sigma_i^z, \quad [\sigma_i^z, \sigma_j^\pm] = \pm\delta_{ij}\sigma_i^\pm, \tag{1.2.5}$$

and that it is embedded into a Yang-Baxter algebra representation. The bulk monodromy matrix $T_0(\lambda) \in \mathrm{End}(V_0 \otimes \mathcal{H})$, of the inhomogeneous system, a representation of the Yang-Baxter algebra on $\mathcal{H}$, is obtained as the following ordered product:

$$T_0(\lambda) = \prod_{i=1}^{N} L_{0i}(\lambda-\xi_i) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[0]} \tag{1.2.6}$$

In these last expressions, L_{0m} denotes the L-matrix in $\mathrm{End}(V_0 \otimes \mathcal{H}_m)$. It is easy to show that it satisfies the Yang-Baxter algebra relations (1.1.2). We would like to stress here that the hamiltonian (1.2.1) indeed belongs to the commuting family (1.1.4). In the homogenous limit,
$\forall m \in [[1,N]], \quad \xi_m = 0$:

$$\mathcal{H} = 2\sinh(\eta)\frac{d}{d\lambda}\ln \mathbf{T}(\lambda)\Big|_{\lambda=0} - 2N\cosh(\eta) \tag{1.2.7}$$

Once these algebraic tools are introduced, we can implement the algebraic Bethe ansatz scheme for the periodic XXZ hamiltonian diagonalization, by finding the transfer matrix spectrum. The system Hilbert space $\mathcal{H}$ is then a representation space of the algebra for $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ with a specific reference state which is an eigenstate of $A(\lambda)$ and $D(\lambda)$ and annihilated by $C(\lambda)$:

$$\begin{aligned} A(\lambda)|0\rangle &= a(\lambda)|0\rangle \\ D(\lambda)|0\rangle &= d(\lambda)|0\rangle \\ C(\lambda)|0\rangle &= 0 \end{aligned} \tag{1.2.8}$$

If such a state exists, the algebraic relations (1.1.2) ensure that we can construct eigenstates of the transfer matrix (1.1.4) in the form $\prod_{k=1}^{M} B(\lambda_k), M = 1,\dots,N$ provided the $\{\lambda_i\}$ satisfy some relations. For the XXZ hamiltonian, the completely ferromagnetic state with all spins up: $|0\rangle = \otimes_{i=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\xi_i} = \otimes_{i=1}^{N} |\uparrow\rangle_{\xi_i}$ is such a state. Indeed, due to the triangular structure of quantum matrix entries of L_{0i}, we easily check the following relations :

$$a(\lambda) = \prod_{i=1}^{N} a^{6V}(\lambda - \xi_i) \tag{1.2.9}$$

$$d(\lambda) = \prod_{i=1}^{N} b^{6V}(\lambda - \xi_i) \tag{1.2.10}$$

Due to the algebraic relations (1.1.2) and the corresponding relations for $A(\lambda), D(\lambda), B(\lambda)$:

$$[B(\lambda), B(\mu)] = 0 \tag{1.2.11}$$

$$B(\lambda)A(\mu) = \frac{c^{6V}(\lambda - \mu)}{a^{6V}(\lambda - \mu)} B(\mu)A(\lambda) + \frac{b^{6V}(\lambda - \mu)}{a^{6V}(\lambda - \mu)} A(\mu)B(\lambda) \tag{1.2.12}$$

$$B(\mu)D(\lambda) = \frac{b^{6V}(\lambda - \mu)}{a^{6V}(\lambda - \mu)} B(\lambda)D(\mu) + \frac{c^{6V}(\lambda - \mu)}{a^{6V}(\lambda - \mu)} D(\lambda)B(\mu) \tag{1.2.13}$$

the following theorem holds:

Theorem 1.2.1 (Faddeev-Sklyanin-Takhtajan)

$$\forall M \in [[1,N]] : \quad |\psi(\{\lambda_k\}_{k=1,\dots,M})\rangle = \prod_{k=1}^{M} B(\lambda_k)|0\rangle \tag{1.2.14}$$

is an eigenstate of the transfer matrix (1.1.4) $\mathbf{T}(\mu) = A(\mu) + D(\mu)$ *for any* μ *with eigenvalue* Λ*:*

$$\Lambda(\mu, \{\lambda_k\}_{k=1,\dots M}, \{\xi_j\}_{j=1,\dots,N}) = a(\mu) \prod_{i=1}^{M} \frac{a^{6V}(\lambda_i - \mu)}{b^{6V}(\lambda_i - \mu)} + d(\mu) \prod_{i=1}^{M} \frac{a^{6V}(\mu - \lambda_i)}{b^{6V}(\mu - \lambda_i)} \tag{1.2.15}$$

if the parameters $\{\lambda_k\}_{k=1\dots M}$ *satisfies the Bethe equations:*

$$\forall k \in [[1,M]] : \quad \frac{d(\lambda_k)}{a(\lambda_k)} \prod_{i=1, i\neq k}^{M} \frac{a^{6V}(\lambda_k - \lambda_i)}{a^{6V}(\lambda_i - \lambda_k)} = 1 \tag{1.2.16}$$

With the use if the QISM, we are able to find eigenvalues and eigenstates of the periodic XXZ hamiltonian.

Remark 1.2.1 *To be complete, we should address the question of the completeness of the Bethe states set. This is actually an open problem although Tarasov and Varchenko have shown completeness on some special points [111] .*

1.3 Open boundary and reflection algebra

The previous framework is very powerful for studying spin chains and more generally quantum systems with periodic (or twisted) boundary conditions. If we are interested in studying systems with open boundary conditions, then we should introduce another algebraic structure; known as the *reflection matrix* and associated algebra. The reflection matrix describes the reflection process at one system boundary and the associated algebra are the compatibility condition for this reflection with integrability. This algebra is naturally associated with bulk scattering. Since the bulk hamiltonian is the same, the very remarkable point is that we do not need to modify the previous construction. Indeed, this new structure is added to the latter as *co-module* over the previous structure, leading to a very intuitive algebraic scheme. We should stress that the construction that we shall present is an algebraic version of the scattering field theory on the half line developed by Cherednik [16] . Once again, we look for an algebra which leads to a family of commuting charges containing the hamiltonian. Following Sklyanin [110], we first restrict ourselves to the case where we assume a strong assumption for the R-matrix, it should depend on the difference of the spectral parameter. Taking into account the boundaries requires one to introduce the *reflection algebra* $B_-(R(\lambda))$ for elements $(U_-)_{\alpha,\beta}$ of $(U_-):\mathbb{C}\to\mathrm{End}(V\otimes\mathcal{H})$:

$$R_{12}(\lambda_1-\lambda_2)(U_-)_1(\lambda_1)R_{21}(\lambda_1+\lambda_2)(U_-)_2(\lambda_2) \\ = (U_-)_2(\lambda_2)R_{12}(\lambda_1+\lambda_2)(U_-)_1(\lambda_1)R_{21}(\lambda_1-\lambda_2), \tag{1.3.1}$$

and its *dual* $B_+(R(\lambda))$, the *dual reflection algebra* for elements $(U_+)_{\alpha,\beta}$ of $(U_+):\mathbb{C}\to\mathrm{End}(V\otimes\mathcal{H})$:

$$R_{12}(\lambda_2-\lambda_1)(U_+^{t_1})_1(\lambda_1)R_{21}(-\lambda_1-\lambda_2-2\eta)(U_+^{t_2})_2(\lambda_2) \\ = (U_+^{t_2})_2(\lambda_2)R_{12}(-\lambda_1-\lambda_2-2\eta)(U_+^{t_1})_1(\lambda_1)R_{21}(\lambda_2-\lambda_1). \tag{1.3.2}$$

Remark 1.3.1 *These two algebras are actually isomorphic. An obvious isomorphism* $\rho: B_-(R(\lambda))\longrightarrow B_+(R(\lambda))$ *is :*

$$\rho(U_-(\lambda)) = U_-^t(-\lambda-\eta) \tag{1.3.3}$$

The reflection process at one or other boundary (denoted by $\pm$) is described by boundary matrix $K_\pm$ which are scalar representations of the reflection algebra or its dual, $K_\pm:\mathbb{C}\to\mathrm{End}(V\otimes\mathbb{C})$:

$$R_{12}(\lambda_1-\lambda_2)K_-(\lambda_1)R_{21}(\lambda_2+\lambda_1)K_-(\lambda_2) \\ = K_-(\lambda_2)R_{12}(\lambda_1+\lambda_2)K_-(\lambda_1)R_{21}(\lambda_2-\lambda_1), \tag{1.3.4}$$

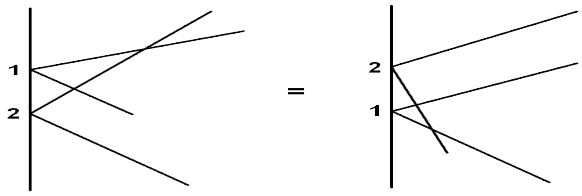


Figure 1.5: The reflection equation (1.3.4)

$$R_{12}(\lambda_2-\lambda_1)K_+^{t_1}(\lambda_1)R_{21}(-\lambda_1-\lambda_2-2\eta)K_+^{t_2}(\lambda_2) \\ = K_+^{t_2}(\lambda_2)R_{12}(-\lambda_1-\lambda_2-2\eta)K_+^{t_1}(\lambda_1)R_{21}(\lambda_2-\lambda_1). \tag{1.3.5}$$

The scattering-reflection at one boundary is described by higher dimensional representation of the reflection algebra as co-module over the Yang-Baxter algebra representation. This representation is the *boundary monodromy matrix*:

$$(U_-)_0(\lambda) = T_0(\lambda)(K_-)_0(\lambda)T_0^{-1}(-\lambda) \tag{1.3.6}$$

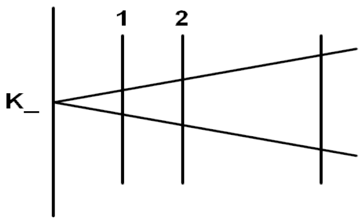


Figure 1.6: The boundary monodromy matrix $(U_-)_0$

Equivalently, the scattering-reflection process can be described starting with the second boundary and we shall use in this case the dual boundary monodromy matrix:

$$(U_+^{t_0})_0(\lambda) = T_0^{t_0}(\lambda)(K_+^{t_0})_0(\lambda)(T^{-1})_0^{t_0}(-\lambda). \tag{1.3.7}$$

Remark 1.3.2 *In the quantum group language,* $(T^{-1})_0^{t_0}(\lambda)$ *is the antipode of* $T(\lambda)$.

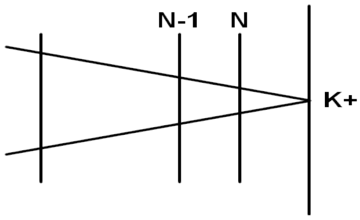


Figure 1.7: The boundary monodromy matrix $(U_+)_0$

Since the bulk monodromy matrix T satisfies the Yang-Baxter algebra relation (1.1.2), and the reflection matrix K_- the reflection equation (1.3.4), the boundary monodromy matrix satisfies the reflection algebra relation (1.3.1), and the corresponding relation for the dual double monodromy matrix is the dual reflection algebra (1.3.1). Although these algebraic relations are more involved than the usual Yang-Baxter algebra relation, they also provide a family of commuting charges describing the reflection-scattering-reflection process :

$$\forall(\mu,\lambda) \in \mathbb{C}^2 \quad [\mathbf{T}(\mu), \mathbf{T}(\lambda)] = 0, \tag{1.3.8}$$

where the integrals of motion generating function are:

$$\mathbf{T}(\lambda) = \mathrm{tr}_0\{(K_+)_0(\lambda)\,(U_-)_0(\lambda)\} = \mathrm{tr}_0\{(K_-)_0(\lambda)(U_+)_0(\lambda)\} \tag{1.3.9}$$

Thus this transfer matrix is an involutive family of quantum charges for any quantum systems described by some representations of the reflection algebras (1.3.1) or (1.3.2). The matrix elements $(U_\pm)_{\alpha,\beta}$ can again generate by action on a pseudo-vacuum $|0\rangle$ the commuting family eigenstates and thus hamiltonian eigenstates if it belong to this family.

1.4 The first example: the diagonal boundary XXZ spin chains

As in the periodic case, the boundary version of the QISM is not sufficient to construct a successful generic scheme for finding hamiltonian with open boundary eigenstates, as

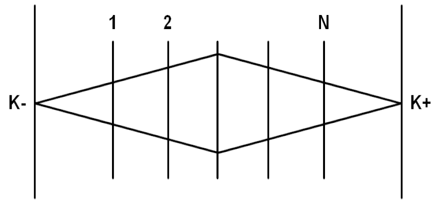


Figure 1.8: Trace formula (1.3.9) and open boundary

we need a triangular structure similar to the relations (1.2.8) for Bethe states construction. We present here a canonical example where such triangular structure is obvious, the XXZ model with diagonal boundary terms, which is the previous XXZ hamiltonian with magnetic fields at the boundary parallel to the (quantization) z-axis. The hamiltonian takes the following form:

$$\begin{aligned}\mathcal{H} = &\sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta(\sigma_i^z \sigma_{i+1}^z - 1) \\ &+ h_- \sigma_1^z + h_+ \sigma_N^z\end{aligned} \tag{1.4.1}$$

Here, the boundary magnetic fields take the form : $h_- = \sinh\eta \coth\zeta_-$ and $h_+ = \sinh\eta \coth\zeta_+$, which are parameterized by $\zeta_\pm \in \mathbb{C}$.

Even if this hamiltonian is no longer a translational invariant (such as the periodic hamiltonian (1.2.1)), it can be embedded in the boundary version of the QISM, thus leading to an integrable structure. We naturally consider the same six-vertex R-matrix and the same representation of the Yang-Baxter algebra as in the periodic case. However, we need to introduce two boundary matrices, which are $K_\pm : \mathbb{C} \times \mathbb{C} \to End(V), V \sim \mathbb{C}^2$:

$$K_-(\lambda, \zeta_-) = \begin{pmatrix} \sinh(\zeta_- + \lambda) & 0 \\ 0 & \sinh(\zeta_- - \lambda) \end{pmatrix} \tag{1.4.2}$$

and $K_+(\lambda, \zeta_+) = K_-(-\lambda - \eta, \zeta_+)$. So the boundary monodromy matrix reads:

$$\begin{aligned}(U_-)_0(\lambda) &= \widehat{\gamma}(\lambda) T_0(\lambda) (K_-)_0(\lambda) T_0^{-1}(-\lambda) \\ &= \begin{pmatrix} A_-(\lambda) & B_-(\lambda) \\ C_-(\lambda) & D_-(\lambda) \end{pmatrix}_{[0]},\end{aligned} \tag{1.4.3}$$

with $T_0(\lambda) = \prod_{i=1}^{N} L_{0i}(\lambda - \xi_i)$ and the dual boundary monodromy matrix is :

$$(U_+^{t_0})_0(\lambda) = \widehat{\gamma}(\lambda) T_0^{t_0}(\lambda)(K_+^{t_0})_0(\lambda)(T^{-1})_0^{t_0}(-\lambda) \tag{1.4.4}$$
$$= \begin{pmatrix} A_+(\lambda) & C_+(\lambda) \\ B_+(\lambda) & D_+(\lambda) \end{pmatrix}_{[0]}.$$

Remark 1.4.1 $\widehat{\gamma}(\lambda) = (-1)^N \prod_{i=1}^{N} \sinh(\lambda + \xi_i + \eta) \sinh(\lambda + \xi_i - \eta)$ *is a normalization factor that we introduce here for convenience.*

The hamiltonian (1.4.1) naturally belongs to the commuting family (1.3.9). In the homogeneous limit,
$\forall m[[1,N]], \quad \xi_m = 0$:

$$\mathcal{H} = c \frac{d}{d\lambda} \mathbf{T}(\lambda)\Big|_{\lambda=0} + constant, \tag{1.4.5}$$

with

$$c = \frac{2\sinh^{1-2N}(\eta)}{tr(K_-(0))tr((K_+(0))}.$$

The important point is that this hamiltonian is total spin invariant: $[\mathcal{H}, \sum_{i=1}^{N} \sigma_i^z] = 0$, and thus has a canonical completely ferromagnetic eigenstate with all spin up: $|0\rangle = \otimes_{i=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is a strong candidate for a reference state. Indeed, using the operators B_-, C_- or (B_+, C_+) we obtain all reference state requirements:

$$C_\pm(\lambda)|0\rangle = 0, \tag{1.4.6}$$
$$A_\pm(\lambda)|0\rangle = a(\lambda)|0\rangle, a(\lambda) \in \mathbb{C}$$
$$D_\pm(\lambda)|0\rangle = d(\lambda)|0\rangle, d(\lambda) \in \mathbb{C}.$$

Once again, the algebraic framework of section 1.3 and the existence of a reference state enable us to diagonalize the hamiltonian. Although the algebraic relations (1.3.1), (1.3.2) are more involved than for the periodic case, leading to a slightly more involved Bethe ansatz computation machinery, two important relations remain valid:

$$[B_-(\lambda), B_-(\mu)] = [C_-(\lambda), C_-(\mu)] = 0, \tag{1.4.7}$$

and

$$[B_+(\lambda), B_+(\mu)] = [C_+(\lambda), C_+(\mu)] = 0. \tag{1.4.8}$$

These allow us to use $B_\pm$ as creation operators and $C_\pm$ as annihilation operators.

Remark 1.4.2 *The operators $A_\pm$ and $D_\pm$ do no commute anymore:*

$$[A_\pm(\lambda), A_\pm(\mu)] \neq 0, \quad [D_\pm(\lambda), D_\pm(\mu)] \neq 0, \tag{1.4.9}$$

which means that they cannot be diagonalize simultaneously, although they still conserve the spin.

Due to the commutations relations for A_-, B_-, D_- (1.3.1) (or A_+, B_+, D_+ (1.3.2)), the following theorem holds:

Theorem 1.4.1 (Sklyanin [110])

$$\forall M \in [[1,N]] : \quad |\psi^-(\{\lambda_k\}_{k=1,\ldots,M})\rangle = \prod_{k=1}^{M} B_-(\lambda_k)|0\rangle \tag{1.4.10}$$

$$\text{or} \quad |\psi^+(\{\lambda_k\}_{k=1,\ldots,M})\rangle = \prod_{k=1}^{M} B_+(\lambda_k)|0\rangle \tag{1.4.11}$$

is an eigenstate of the transfer matrix (1.3.9)*:*

$$\mathbf{T}(\mu) = (K_+(\mu))_+^+ A_-(\mu) + (K_+(\mu))_-^- D_-(\mu) = (K_-(\mu))_+^+ A_+(\mu) + (K_-(\mu))_-^- D_+(\mu)$$

for any μ with eigenvalue Λ:

$$\begin{aligned} &\Lambda(\mu, \{\lambda_k\}_{k=1,\ldots M}, \{\xi_j\}_{j=1,\ldots,N}) \\ &= (-1)^N \{a(\mu)d(-\mu-\eta) \frac{\sinh(2\mu+2\eta)\sinh(\zeta_+ + \mu)\sinh(\zeta_- + \mu)}{\sinh(2\mu+\eta)\prod_{i=1}^M b(\lambda_i-\mu)b(\lambda_i+\mu+\eta)} \\ &+ a(-\mu-\eta)d(\mu) \frac{\sinh 2\mu \sinh(\mu-\zeta_+ +\eta)\sinh(\mu-\zeta_- +\eta)}{\sinh(2\mu+\eta)\prod_{i=1}^M b(\lambda_i+\mu)b(\lambda_i-\mu+\eta)}\} \end{aligned} \tag{1.4.12}$$

where:

$$b(\lambda) = \frac{b^{6V}(\lambda)}{a^{6V}(\lambda)}, \tag{1.4.13}$$

if the parameters $\{\lambda_k\}_{k=1,\ldots,M}$ satisfy the Bethe equations:

$$\begin{aligned} &y(\lambda_k, \{\lambda_i\}_{i\neq k, i=1,\ldots,M}, \{\xi_j\}_{j=1,\ldots,N}, \zeta_-, \zeta_+) \\ &= y(-\lambda_k-\eta, \{\lambda_i\}_{i\neq k, i=1,\ldots,M}, \{\xi_j\}_{j=1,\ldots,N}, \zeta_-, \zeta_+) \end{aligned} \tag{1.4.14}$$

with:

$$\begin{aligned} y(\lambda_k, \{\lambda_i\}_{i\neq k, i=1,\ldots,M}, \{\xi_j\}_{j=1,\ldots,N}, \zeta_-, \zeta_+) \\ = a(\lambda_k) d(-\lambda_k - \eta) \\ \times \sinh(\zeta_- + \mu) \sinh(\zeta_+ + \mu) \\ \times \prod_{i=1, i\neq k}^{M} \sinh(\lambda_k + \lambda_i) \sinh(\lambda_k - \lambda_i - \eta) \end{aligned} \tag{1.4.15}$$

Remark 1.4.3 *The Bethe construction is $\mathbb{Z}_2$ invariant due to the following involution:*

$$B_-(-\lambda-\eta) = -\frac{\sinh(2\lambda+2\eta)}{\sinh(2\lambda)} B_-(\lambda), \quad B_+(-\lambda-\eta) = -\frac{\sinh(2\lambda)}{\sinh(2\lambda+\eta)} B_+(\lambda) \tag{1.4.16}$$

1.5 Quantum integrability and quantum groups

The Yang-Baxter equation (1.1.1) and the associated Yang-Baxter algebra (1.1.2) are the heart of a well developed field of mathematics, the theory of quantum groups [13, 14, 24, 25, 32, 60, 109]. Quantum groups first arise in physics literature via Yang-Baxter type algebraic structure for solving integrable quantum systems. They find unexpected connections within a variety of mathematic domains, such as non-commutative geometry, the theory of knot invariants and low dimensional topology. They also have various physics applications in quantum random walk theory, low dimensional gravity, conformal field theory and of course, the theory of quantum integrable systems and classical exactly solvable systems of statistical mechanics [54, 64, 83, 94, 97, 113, 114, 117]. The name quantum group is somehow misleading because, as we will see, quantum groups are not groups, but rather algebra and co-algebra embedded together into a compatible structure known as Hopf algebra. The name quantum is also quite ambiguous, as it refers to the analogy between classical mechanics and quantum mechanics, where classical observable forming a commutative Hopf algebra, on a classical Poisson manifold with group structure, is replaced by a non-commutative Hopf algebra of operators on a Hilbert space. There are two *dual* approaches to Hopf algebra. Namely, Hopf algebra can be introduced via consistency conditions for co-algebraic properties of a given algebra. This is the original construction of Hopf algebra which was introduced by Dinfeld's and Jimbo. Such structure was in many cases related to deformations of universal enveloping algebra of classical Lie algebras and Kac-Moody algebras. Another dual presentation is the Faddeev-Reshetikhin-Takhtajan-Sklyanin (FRST formalism) or the RLL formalism, which is more natural in the framework of the QISM. This presentation is to be understood as a quantization, or rather deformation, of classical Lie group structure. We start by introducing the concept of a Hopf algebra following Drinfel'd and Jimbo's approach. Then we will turn to the very special case of quasi-triangular Hopf algebra, which is the algebraic framework for the Yang-Baxter equation. We continue with the dual approach of FRST and highlight the link with the QISM technology. Finally, we will shortly focus on the algebraic framework for the reflection equation.

1.5.1 Hopf algebra

Let us start by defining Hopf algebra via compatibility conditions between the algebra and the co-algebra structure.

Definition 1.5.1 *A $\mathbb{C}$-Hopf Algebra A is:*

i) an (associative unital) algebra over the field $\mathbb{C}$ with a product: $m : A \otimes A \to A$, and unit: $\imath : \mathbb{C} \to A$ which is a homomorphism of algebras

ii) a co-algebra over the field $\mathbb{C}$ with a co-product: $\Delta : A \to A \otimes A$ and a co-unit $\varepsilon : A \to \mathbb{C}$ which is a homomorphism of co-algebras

iii) the data of an anti-homomorphism of the algebra, the antipode $S : A \to A$ such that:

$$\Delta \circ (S \otimes id) \circ m = \imath \circ \varepsilon = \Delta \circ (id \otimes S) \circ m \tag{1.5.1}$$

Definition 1.5.2 *A Hopf algebra is said to be co-associative if the co-product satisfies the co-associativity condition:*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \tag{1.5.2}$$

Definition 1.5.3 *A Hopf algebra is said to be co-commutative if the co-product satisfies the co-commutativity condition:*

$$\Delta(X) = \Delta'(X) = P \circ \Delta(X), \tag{1.5.3}$$

where P is the usual permutation operator: $P(a_1 \otimes a_2) = a_2 \otimes a_1$.

Example 1.5.1 *To any Lie group $\mathcal{G}$ we can associate a natural Hopf algebra. Indeed, the universal enveloping algebra $U(\mathbf{g})$ of its Lie algebra $\mathbf{g}$ posses a trivial Hopf structure.*

The antipode and co-unit of the Hopf algebra follow from the inverse operation in the Lie group, they are given by:

$$\Delta(X) = X \otimes Id + Id \otimes X, \quad \text{for } X \in \mathbf{g} \tag{1.5.4}$$

$$S(X) = -X, \quad \text{for } X \in \mathbf{g} \tag{1.5.5}$$

$$S(Id) = Id \tag{1.5.6}$$

$$\varepsilon(X) = 0, \quad \text{for } X \in \mathbf{g} \tag{1.5.7}$$

$$\varepsilon(Id) = 1 \tag{1.5.8}$$

Take $\mathbf{g} = sl_2$, *and consider the universal enveloping algebra* $U(sl_2)$ *of the Lie algebra* sl_2 *with generator* $\sigma^z, \sigma^+, \sigma^-$ *and commutations relations:*

$$[\sigma^+, \sigma^-] = \sigma^z, \quad [\sigma^z, \sigma^\pm] = \pm\sigma^\pm. \tag{1.5.9}$$

In this way we recover the standard composition of intrinsic angular momentum in quantum mechanics.

This Hopf algebra is clearly co-commutative. Thus, classical Lie algebra can be embedded naturally into a co-commutative Hopf algebra. Construction of non co-commutative Hopf algebras, or quantum groups, was motivated by the QISM. It arose in this context in the work of Jimbo on the deformation of universal enveloping Lie algebra and in the work of Drinfel'd on quantization of Poisson-Lie structures.

Example 1.5.2 *A first simple example is the* $U_q(sl_2), q \in \mathbb{C}$ *algebra with commutations relations:*

$$[\sigma^+, \sigma^-] = \frac{q^{\sigma^z} - q^{-\sigma^z}}{q - q^{-1}}, \quad [\sigma^z, \sigma^\pm] = \pm\sigma^\pm. \tag{1.5.10}$$

The Hopf structure is given by:

$$\Delta(X) = X \otimes Id + Id \otimes X, \quad \text{for } X = \sigma^z, \sigma^\pm \tag{1.5.11}$$

$$S(X) = -q^\pm X, \quad \text{for } X = \sigma^z, \sigma^\pm \tag{1.5.12}$$

$$S(Id) = Id \tag{1.5.13}$$

$$\varepsilon(X) = 0, \quad \text{for } X = \sigma^z, \sigma^\pm \tag{1.5.14}$$

$$\varepsilon(Id) = 1 \tag{1.5.15}$$

Remark 1.5.1 *In the limit* $q \to 1$, *these commutations relations reduce to the one of* $U(sl_2)$ (1.5.9).

Definition 1.5.4 *A quasi-triangular Hopf algebra is a Hopf algebra A with an invertible elements $R \in A \otimes A$ such that:*

$$\Delta'(X) = R.\Delta(X).R^{-1}, \quad \text{for } X \in A \tag{1.5.16}$$

$$(\Delta \otimes id)(R) = R_{13}R_{23} \tag{1.5.17}$$

$$(id \otimes \Delta)(R) = R_{13}R_{12} \tag{1.5.18}$$

We say that R is the universal matrix of A.

Remark 1.5.2 *It is easy to show that for a quasi-triangular Hopf algebra, the R-matrix satisfies the Yang-Baxter equation:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.5.19}$$

Thus, the Yang-Baxter equation arises naturally as the key equation for quasi-triangular Hopf algebra.

Example 1.5.3 *$R = Id$ is a universal R-matrix for $U(sl_2)$ and the given Hopf structure* (1.5.4).

Example 1.5.4 *If we take the two-dimensional representation of $U_q(sl_2)$ which is given by the usual Pauli matrices, then*

$$R = \begin{pmatrix} q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 1 & q^{\frac{1}{2}} - q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix}. \tag{1.5.20}$$

is a two-dimensional representation of the universal R-matrix for $U_q(sl_2)$ and the Hopf structure (1.5.11).

Remark 1.5.3 *In all these examples, the R-matrix is constant, i.e. does not depend on any spectral parameter. It is possible to obtain a similar framework which leads to R-matrices with spectral parameters by replacing Lie algebra g by Kac-Moody algebra . Such structure naturally lead to the trigonometric six-vertex matrix* (1.2.2).

1.5.2 Yang-Baxter algebra as Hopf algebra

Finite dimensional case

Starting from an R-matrix : $R \in M_{\mathbb{C}^n \otimes \mathbb{C}^n}(\mathbb{C})$, we define the associative algebra $A(R)$ of functions over the formal quantum group associated to R, or matrix quantum group, which generators are the entries of $T \in M_{\mathbb{C}^n}(\mathbb{C} < T_{\alpha,\beta} >)$, where $\mathbb{C} < T_{\alpha,\beta} >$ is the non commutative algebra of polynomials over the field $\mathbb{C}$, where the fundamental commutation relations are the Yang-Baxter algebra relation:

$$R_{12}T_1T_2 = T_2T_1R_{12}. \tag{1.5.21}$$

Requiring associativity of the algebra $A(R)$ with respect to the matrix product $T_1T_2T_3$ simply lead to a consistency condition, the Yang-Baxter equation for R:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.5.22}$$

The algebra $A(R)$ can be embedded into a Hopf algebra structure if we define the co-product Δ, the co-unit ε and the antipode S as:

$$\begin{aligned} \Delta(T) &= T \otimes T \\ \varepsilon(T) &= Id \\ S(T) &= T^{-1} \end{aligned} \tag{1.5.23}$$

Such algebra is finite dimensional as it has n^2 generators $T_{\alpha,\beta}$.

Example 1.5.5 *Let us consider the simplest non-trivial example given by the 2×2 matrices :*

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.5.24}$$

For $a,b,c,d \in \mathbb{C}$ these matrices form the Lie group $GL_2(\mathbb{C})$. A trivial deformation of the Lie group into a commutative Hopf algebra is $A(R)$ with $R = Id_{2\times 2}$. In this situation, the a,b,c,d elements are given a trivial Hopf algebraic structure as $\mathbb{C}$-generators. The Yang-Baxter algebra relation reads trivially:

$$I_{12}T_1T_2 = T_2T_1I_{12} \Rightarrow T_1T_2 = T_2T_1 \tag{1.5.25}$$

or more explicitly:

$$\begin{aligned} ab = ba \quad ac = ca \quad ad = da \\ bc = cb \quad bd = db \\ cd = dc \end{aligned} \tag{1.5.26}$$

These relations are nothing but the $\mathbb{C}$ *commutativity. A consistent non commutative deformation of* $GL_2(\mathbb{C})$ *is* $(GL_2)_q(\mathbb{C}), q \in \mathbb{C}$*, or equivalently* $A(R)$ *with:*

$$R = \begin{pmatrix} q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 1 & q^{\frac{1}{2}} - q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix}. \tag{1.5.27}$$

The relations (1.5.21) *explicitly read:*

$$\begin{gathered} ab = q^{\frac{1}{2}} ba \quad ac = q^{\frac{1}{2}} ca \quad ad - da = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) bc \\ bc = cb \quad bd = q^{\frac{1}{2}} db \\ cd = q^{\frac{1}{2}} dc \end{gathered} \tag{1.5.28}$$

Note that the co-algebraic properties of $(GL_2(\mathbb{C}))_q$ *for* $q = 1$ *and* $q \neq 1$ *are identical although the algebraic properties are different.*

Remark 1.5.4 *The algebra* $(GL_2(\mathbb{C}))_q$ *is easily transformed into* $(SL_2(\mathbb{C}))_q$ *by imposing* $ad - q^{\frac{1}{2}} bc = 1$.

Infinite dimensional case

We turn now to the case of the Yang-Baxter algebra with spectral parameter (1.1.2). Following the same lines as in the previous section, the Yang-Baxter algebra (1.1.2) is simply the algebra $A(R(\lambda, \mu))$ of elements of $T(\lambda) \in M_{V \otimes V}(\mathbb{C} < T_{\alpha,\beta} >)$, where V is the $\mathbb{Z}$-graded vector space $V = \otimes_{m \in \mathbb{Z}} \lambda^m \mathbb{C}^n$ with fundamental commutations:

$$R_{12}(\lambda_1, \lambda_2) T_1(\lambda_1) T_2(\lambda_2) = T_2(\lambda_2) T_1(\lambda_1) R_{12}(\lambda_1, \lambda_2). \tag{1.5.29}$$

$T(\lambda)$ can be viewed as the formal Laurent series:

$$T(\lambda) = \sum_{\mathbb{Z}} \lambda^m T_m \tag{1.5.30}$$

A consistency condition for the associativity of the algebra product is the Yang-Baxter equation for R:

$$R_{12}(\lambda_1, \lambda_2) R_{13}(\lambda_1, \lambda_3) R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3) R_{13}(\lambda_1, \lambda_3) R_{12}(\lambda_1, \lambda_2). \tag{1.5.31}$$

The algebra $A(R(\lambda,\mu))$ can be embedded into a Hopf algebra structure if we define the co-product Δ, the co-unit ε and the antipode S as:

$$\begin{aligned}\Delta(T(\lambda)) &= T(\lambda)\otimes T(\mu) \\ \varepsilon(T(\lambda)) &= Id \\ S(T(\lambda)) &= (T^{-1})^t(\lambda)\end{aligned} \tag{1.5.32}$$

Such algebra is infinite-dimensional as it has an infinite number of generators $\lambda^m T_{\alpha,\beta}, m \in \mathbb{Z}$.

Remark 1.5.5 *If we choose $R(\lambda,\mu)$ to be the six-vertex R-matrix* (1.2.2)*, this is the algebraic structure underlying the XXZ model. As R is the R-matrix of $U_q(\widehat{sl}_2)$, the Yang-Baxter algebra underlying the integrability of the XXZ model is dual to $U_q(\widehat{sl}_2)$.*

1.5.3 Reflection algebra as co-module algebra

In this section, we present the algebraic formalism [5, 66, 86] of the reflection algebra (1.3.1). Our aim is to introduce some vocabulary that we will use in the next chapters.

Definition 1.5.5 *A co-module algebra B is an algebra together with an algebra with co-algebraic structure (co-product and co-unit) A, and a map (coaction) $\varphi : B \to A\otimes B$ which is:*

- *An algebra homomorphism:*

$$\varphi(B) \in A\otimes B \subset B \tag{1.5.33}$$

- *Consistent with the co-multiplication Δ of A:*

$$(\Delta\otimes Id)\circ\varphi = (Id\otimes\varphi)\circ\varphi \tag{1.5.34}$$

- *Consistent with the co-unit ε of A:*

$$(\varepsilon\otimes Id)\circ\varphi = Id \tag{1.5.35}$$

B is then an A-co-module algebra.

Given an R-matrix (we focus on the general case where the R-matrix is spectral parameterized), define the algebra $B(R(\lambda))$ generated by non commutative elements of $U_-(\lambda) \in M_{\mathbb{C}^n}(\mathbb{C} < (U_-)_{\alpha,\beta} >)$ satisfying the relations:

$$R_{12}(\lambda_1-\lambda_2)(U_-)_1(\lambda_1)R_{21}(\lambda_1+\lambda_2)(U_-)_2(\lambda_2) \\ = (U_-)_2(\lambda_2)R_{12}(\lambda_1+\lambda_2)(U_-)_1(\lambda_1)R_{21}(\lambda_1-\lambda_2). \tag{1.5.36}$$

The co-algebraic properties of such algebra are not clear, and Hopf algebraic structure is not the right framework for $B(R(\lambda))$. Rather, we have a coaction $\varphi : B(R(\lambda)) \to A(R(\lambda)) \otimes B(R(\lambda))$ which embed $B(R(\lambda))$ into an $A(R(\lambda))$-co-module algebra:

Theorem 1.5.1 (Kulish-Sklyanin [66]) *$B(R(\lambda))$ is an $A(R(\lambda))$-co-module algebra. Given $K_-(\lambda) \in B(R(\lambda))$ and $T(\lambda) \in A(R(\lambda))$, the coaction φ reads:*

$$\varphi(B(R(\lambda))) \in A(R(\lambda)) \otimes B(R(\lambda)) \subset B(R(\lambda)) \\ \varphi(K_-(\lambda)) = T(\lambda)K_-(\lambda)T^{-1}(-\lambda) \tag{1.5.37}$$

This theorem is the mathematical formulation of the co-module structure that is encountered in the boundary version of the QISM. It also provides a way to handle integrable reflection at a boundary, together with a bulk integrable scattering process.

Chapter 2

Quasi-particles invariant quantum hamiltonians and duality with classical vertex models

We turn to another aspect of the QISM which is at the cornerstone of the modern theory of quantum integrable systems. Namely, this algebraic framework enables us to highlight duality between one-dimensional quantum mechanics and two-dimensional statistical physics. This highly non trivial duality between the six-vertex model and the XXZ spin chain that is indeed generalizable to other models was first noticed by Lieb [78], Sutherland [105] and Baxter for the very general eight-vertex model [7,8] and then more formally put within the QISM by Faddeev-Takhtadzhan [36]. Indeed the algebraic Bethe ansatz technique of Chapter 1 establishes a clear relation between the quantum spin chains and two-dimensional models in statistical mechanics. The periodic XXZ spin chain was solved using diagonalization of the transfer matrix (1.1.4). This object is nothing but the statistical physics transfer matrix of the *six-vertex model*. Indeed, the partition function of the six-vertex model with periodic boundary conditions can be represented as the trace over the quantum spaces of the product of transfer matrix:

$$Z^{6V} = tr_{[V_0]} tr_{[\otimes_i V_{\xi_i}]} T_0^N = tr_{[\otimes_i V_{\xi_i}]} \mathbf{T}^N \tag{2.0.1}$$

This equivalence with two-dimensional models of statistical mechanics also turned out to be essential for the computation of scalar products and correlation functions of quantum integrable models. It was shown by Izergin and Korepin [58] that the partition function of the six-vertex model with *domain wall boundary conditions* (DWBC) is the key element for the study of the correlation functions of the periodic XXZ model. The elegant determinant representation for this partition function found by Izergin [59] is crucial for the computation of the correlation functions starting from the algebraic Bethe ansatz [67, 68].

For the open case the corresponding partition function was computed by Tsuchiya in 1998 [112]. His work shows that the transfer matrix of the open chain with diagonal boundary terms (1.3.9) is the *six-vertex model with reflecting ends* transfer matrix. Once again, the partition function has a determinant representation, and this was used first to compute the scalar products and norms of the Bethe vectors [115] and then to study the correlation functions of the open spin chains with external boundary magnetic fields parallel to the z axis [69, 70].

Such duality between integrable quantum spin chains and vertex models, or more generally for any system that is described by a R-matrix representation of the Yang-Baxter algebra or the reflection algebra, turns out to be very powerful for the third component of the total spin invariant hamiltonian. The importance of this duality is due to the underlying vertex model, which is *exactly solvable*, namely we can compute exactly its partition function. In most cases, the latter partition function has a simple and manageable representation.

In this chapter we would like to highlight such fundamental and highly non trivial duality, and show how the algebraic tools of Chapter 1 permit one to find an exact and manageable formula for the partition function of vertex models, opening the way to the exact evaluation of correlation functions of quantum integrable models. We focus our attention on hamiltonians with conserved number of quasi-particles or equivalently on models where the third component of the total spin is conserved. The case where this $U(1)$ symmetry is lost is the object of the next chapter.

2.1 A fundamental quantity

We learned from Chapter 1 that within the QISM framework the integrable hamiltonian eigenstates $|\Psi\rangle$ are constructed as the action of product of off-diagonal monodromy operators B, C on some reference state $|0\rangle$:

$$|\Psi\rangle = \prod_{i=1}^{M} B(\lambda_i)|0\rangle \tag{2.1.1}$$

where the spectral parameters $\lambda_{i=1,\dots,M}$ satisfy the Bethe equations. When $M = N$, the system's length, the Bethe vector is then proportional to the canonical orthogonal $\langle\bar{0}|$ of the pseudo-vacuum $|0\rangle$:

$$\prod_{j=1}^{N} B(\lambda_j)|0\rangle = Z_N|\bar{0}\rangle, Z_N \in \mathbb{C}. \tag{2.1.2}$$

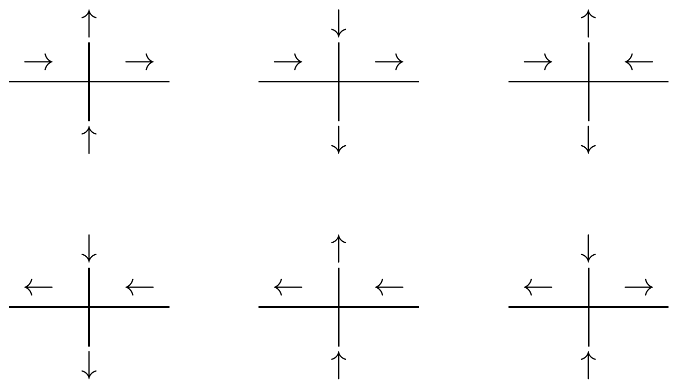
We are interested in the exact evaluation of the scalar Z_N, in order to push forward analysis of integrable models towards scalar products, form factors or correlations functions. This quantity has in many cases a clear statistical physics interpretation. We present these relations to statistical physics for the XXZ model with periodic boundary and diagonal open boundary conditions.

2.2 Periodic XXZ and square vertex model

Moving on from our previous construction, we turn now to a special case of two-dimensional statistical physics models defined on a square lattice of size $N \times N$. On each edge is attached a **classical** two-state variables, for instance, a canonical basis vector of $\mathbb{C}^2$,$\{|\uparrow\rangle, |\downarrow\rangle\}$, where:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.2.1}$$

To each vertex, we associate a statistical weight according to the adjacent edge configuration, and we allow only six non vanishing configurations. To these six configurations we give three statistical weights a^{6V}, b^{6V}, c^{6V}:



a^{6V} b^{6V} c^{6V}

Such vertex models are very universal and they have been extensively studied by statistical physicists first as a model for ferroelectricity [77, 104], and as an Ice model [81]. Vertex models then become very fundamental models for mathematical physics [6, 28, 78, 92, 106]. They are related to a large range of modern physics models such as the Toda model [73] and two dimensional gravity and random lattice [74, 128]. They are also linked to various mathematical methods such as KP and Toda tau functions [46–48], enumeration of alternating sign matrix [27, 75, 98, 98] and quantum groups [13]. A fundamental question in statistical physics is the computation of the partition function $Z^{6V}_{N,N}$

of this model. Since they arise as a necessary step towards computation of scalar products and correlation functions of quantum integrable models within the algebraic Bethe ansatz framework, the exact computation of their partition function is not only fundamental from a statistical physics perspective, but also from a quantum physics view point.

Following Korepin, we focus on *Domain Wall Boundary Condition* (DWBC), the arrows point inward along the left and right and outward along the top and bottom:

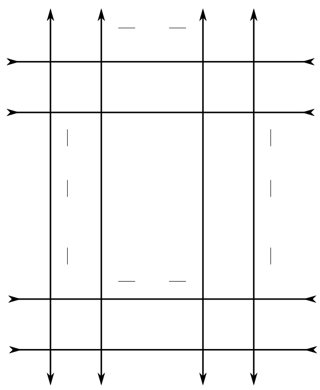


Figure 2.1: The vertex model with DWBC

The partition function of this vertex model is defined as:

$$Z^{6V}_{N,N}(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N}) = \sum_{\mathcal{C}\in\Omega}\prod_{i,j=1}^{N}\mathcal{W}(\mathcal{C}_{i,j})(\lambda_i-\xi_j). \qquad (2.2.2)$$

Here and in what follows, for any lattice model, $\lambda_{i,i=1,..,N}$ refers to the horizontal lines starting from the bottom, while $\xi_{j,j=1,...,N}$ refers to the vertical lines starting from the right. In this expression, Ω is the set of all possible configurations of the model, and any configurations can be decomposed into $N\times N$ local vertex configurations $\mathcal{C}_{i,j=1,\ldots,N}$:

$$\mathcal{C} = \prod_{i,j=1}^{N}\mathcal{C}_{i,j}. \qquad (2.2.3)$$

We consider an *inhomogeneous* model where the Boltzmann weight of each local configuration depends on the parameters $\lambda_{i=1,\ldots,N}$ and $\xi_{j=1,\ldots,N}$ indexing the lattice:

$$\mathcal{W}(\mathcal{C}_{i,j})(\lambda_i-\xi_j) = \varphi(\lambda_i-\xi_j), \quad \varphi = a^{6V}, b^{6V}, c^{6V}. \qquad (2.2.4)$$

If we collect the statistical weight a^{6V}, b^{6V}, c^{6V} into the entries of the six-vertex R-matrix (1.2.2), the partition function of the six-vertex model on a square lattice of size $N \times N$ with DWBC can be rewritten as:

$$Z^{6V}_{N,N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N}) = \langle 0|_\lambda \langle \bar{0}|_\xi \prod_{i,j=1}^{N} R_{ij}(\lambda_i-\xi_j)|0\rangle_\xi|\bar{0}\rangle_\lambda \tag{2.2.5}$$
$$= \langle 0|_\lambda \langle \bar{0}|_\xi \prod_{i=1}^{N} T(\lambda_i)|0\rangle_\xi|\bar{0}\rangle_\lambda$$
$$= \langle \bar{0}|_\xi \prod_{i=1}^{N} B(\lambda_i)|0\rangle_\xi$$

In this expression, $\langle \bar{0}| = \otimes_{i=1}^{N} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the orthogonal of $|0\rangle$ and the subscript $_\lambda, (_\xi)$ means that the vector lies in the auxiliary (quantum) space. Note that this formula shows that the quantity $Z^{6V}_{N,N}$ is nothing but the fundamental scalar object (2.1.2).

The partition function is then represented in terms of the XXZ spin chain's QISM ingredients. Its integrable structure leads to very deep insights regarding the analytical property of this partition function, enabling Korepin to find a set of conditions that uniquely determine the partition function.

Proposition 2.2.1 (Korepin [71]) *The partition function* (2.2.5) *satisfies the following properties:*

i) Initial condition
$Z^{6V}_{1,1}(\lambda_1,\xi_1) = \sinh\eta$

ii) Symmetry
$Z^{6V}_{N,N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N})$ *is a symmetric function of the* $\{\lambda_i\}_{i=1,\dots,N}$ *and the* $\{\xi_j\}_{j=1,\dots,N}$.

iii) Polynomiality

$$Z^{6V}_{N,N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N}) = \exp^{(N-1)\lambda_i} P_{N-1}(\exp^{2\lambda_i})$$

and similarly:

$$Z^{6V}_{N,N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N}) = \exp^{-(N-1)\xi_j} P_{N-1}(\exp^{-2\xi_j})$$

where P_{N-1} *is a polynomial of degree* $N-1$.

iv) Recursive relations

$$Z^{6V}_{N,N}\left(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N}\right)\Big|_{\lambda_i=\xi_j}$$
$$= \sinh(\eta)\prod_{k=1,k\neq i}^{N}\sinh(\lambda_i-\xi_k+\eta)\prod_{1\le k\le N,k\neq i}\sinh(\lambda_k-\xi_j+\eta)$$
$$Z^{6V}_{N-1,N-1}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j})$$

Lemma 2.2.1 *The set of conditions i)-iv) uniquely define the partition function* $Z^{6V}_{N,N}$.

These conditions enabled Izergin to propose a very simple formula for the partition function as a single determinant.

Theorem 2.2.1 (Izergin [59]) *The partition function of the trigonometric six-vertex model on a square lattice with DWBC is:*

$$Z^{6V}_{N,N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{i=1,\dots,N}) = (-1)^N det\mathcal{N}^{6V}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,..,N}) \tag{2.2.6}$$
$$\frac{\prod_{i=1}^N\prod_{j=1}^N\sinh(\lambda_j-\xi_j)\sinh(\lambda_j-\xi_j+\eta)}{\prod_{1\le k<j\le N}\sinh(\xi_j-\xi_k)\prod_{1\le k<j\le N}\sinh(\lambda_k-\lambda_j)} \tag{2.2.7}$$

where the $N\times N$ *matrix* $\mathcal{N}^{6V}$ *can be expressed as:*

$$\mathcal{N}^{6V}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,..,N})_{\alpha,\beta} = \frac{\sinh(\eta)}{\sinh(\lambda_\alpha-\xi_\beta+\eta)\sinh(\lambda_\alpha-\xi_\beta)} \tag{2.2.8}$$

The determinant form of the partition function is a compact and rather simple representation, and it is therefore a necessary representation to push forward quantum integrable models analysis towards computation of correlations functions in the QISM framework.

2.3 XXZ chain with diagonal boundaries and reflecting ends

Let us consider in this section a boundary variant of the previous vertex model. Namely, instead of considering the six-vertex model with the DWBC boundary condition on a square lattice, we allow two boundary Boltzmann configurations collected into the entries of the

Figure 2.2: The boundary statistical configurations (1.4.2)

boundary matrix K_- (1.4.2), and we consider the inhomogeneous six-vertex model with *reflecting end* and DWBC as illustrated in Figure 2.3.

The partition function of this model can be represented in terms if the boundary QISM ingredients of the XXZ spin chain with diagonal boundary:

$$Z^{6BV}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{i=1,\dots,N}) = \langle 0|_\lambda \langle \overline{0}|_\xi \prod_{i,j=1}^{N} R_{ij}(\lambda_i-\xi_j)K_-(\lambda_i)_i R_{ji}(\lambda_i+\xi_j)|\overline{0}\rangle_\lambda |0\rangle_\xi \tag{2.3.1}$$

$$= \langle 0|_\lambda \langle \overline{0}|_\xi \prod_{i=1}^{N} U_-(\lambda_i)|0\rangle_\xi |\overline{0}\rangle_\lambda$$

$$= \langle \overline{0}|_\xi \prod_{i=1}^{N} B_-(\lambda_i)|0\rangle_\xi$$

Within the framework of the QISM, even for the diagonal boundary case, it is again necessary to find a manageable, and preferably exact, formula for this partition function. Using the boundary QISM, Tsuchiya found a set of conditions that uniquely determine this partition function.

Proposition 2.3.1 (Tsuchiya [112]) *The partition function* (2.3.1) *satisfies the following properties:*

i) Initial condition
$Z^{6BV}_{1,2.1}(\lambda_1,\xi_1) = \sinh\eta(\sinh(\lambda_1-\xi_1)\sinh(\zeta_-+\lambda_1)+\sinh(\lambda_1+\xi_1)\sinh(\zeta_--\lambda_1))$

ii) Symmetry:
$Z^{6BV}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N})$ *is a symmetric function of the* $\{\lambda_i\}_{i=1,\dots,N}$ *and the* $\{\xi_j\}_{j=1,\dots,N}$.

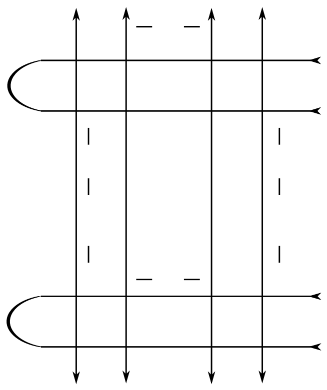


Figure 2.3: The vertex model with reflecting end and DWBC

iii) Polynomiality

$$Z^{6BV}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N}) = \exp^{-(2N+1)\lambda_i} P_{2N+1}(\exp^{2\lambda_i})$$

where P_{2N+1} *are polynomials of degree* $2N+1$.

iv) Recursive relations

$$Z^{6BV}_{N,2N}\left(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N}\right)\Big|_{\lambda_i=\xi_j} = \sinh(\eta)\sinh(\zeta_- + \lambda_i)\prod_{k=1}^{N}\sinh(\lambda_k+\xi_j)$$
$$\times \prod_{k=1,k\neq j}^{N}\sinh(\lambda_i+\xi_k+\eta)\sinh(\lambda_i-\xi_k+\eta)\sinh(\lambda_k-\xi_j+\eta)$$
$$\times Z^{6BV}_{N-1,2(N-1)}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j})$$

and:

$$Z^{6BV}_{N,2N}\left(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N}\right)\Big|_{\lambda_i=-\xi_j} = \sinh(\eta)\sinh(\zeta_- - \lambda_i)\prod_{k=1}^{N}\sinh(\lambda_k-\xi_j)$$
$$\times \prod_{k=1,k\neq k}^{N}\sinh(\lambda_i+\xi_k+\eta)\sinh(\lambda_i-\xi_k+\eta)\sinh(\lambda_{k-1}+\xi_j+\eta)$$
$$\times Z^{6BV}_{N-1,2(N-1)}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j})$$

v) Crossing

$$Z^{6BV}_{N,2N}(-\lambda_k-\eta,\{\lambda_i\}_{i\neq k,i=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N}) = -\frac{\sinh(2\lambda_k+2\eta)}{\sinh(2\lambda_k)} \times Z^{6BV}_{N,2N}(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N})$$

Lemma 2.3.1 *The set of conditions i)-iv) uniquely determines the partition function* $Z^{6BV}_{N,2N}$.

The remarkable point is that the only functions that satisfy these conditions have a nice representation, which takes the form of a slightly more complicated determinant than the square six-vertex partition function with DWBC.

Theorem 2.3.1 (Tsuchiya [112]) *The partition function of the trigonometric six-vertex model with reflecting end and DWBC is:*

$$Z^{6BV}_{N,2N}(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{i=1,\ldots,N}) = (-1)^N det\mathcal{N}^{6BV}(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{j=1,..,N}) \frac{\prod_{i,j=1}^N \sinh(\lambda_i-\xi_j+\eta)\sinh(\lambda_i-\xi_j)\sinh(\lambda_i+\xi_j+\eta)\sinh(\lambda_i+\xi_j)}{\prod_{1\le k<j\le N}\sinh(\xi_j-\xi_k)\sinh(\xi_j+\xi_k)\sinh(\lambda_j-\lambda_k)\sinh(\lambda_j+\lambda_k+\eta)} \tag{2.3.2}$$

where the $N\times N$ *matrix* $\mathcal{N}^{6BV}$ *can be expressed as:*

$$\mathcal{N}^{6BV}(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{j=1,..,N})_{\alpha,\beta} = \frac{\sinh(\eta)\sinh(2\lambda_\alpha)\sinh(\zeta_- -\xi_\beta)\sinh(\zeta_- +\xi_\beta)}{\sinh(\lambda_\alpha-\xi_\beta+\eta)\sinh(\lambda_\alpha-\xi_\beta)\sinh(\lambda_\alpha+\xi_\beta+\eta)\sinh(\lambda_\alpha+\xi_\beta)} \tag{2.3.3}$$

This expression permits one to compute scalar products and norms of Bethe states for the open XXZ spin chains with diagonal boundary conditions, and also to obtain an exact and manageable expression for form factors and correlations functions.

Chapter 3

Spin chain without S^z conservation: the vertex-face transformation

In this chapter, we are mainly concerned with quantum models that are completely embedded in the QISM framework, thus completely integrable, but where the algebraic Bethe ansatz scheme is elusive because these models do not possess a canonical reference state. It is actually possible to implement the algebraic Bethe ansatz machinery but only after a crucial transformation; the vertex-face transformation [11]. This transformation was first introduced by Baxter [8] in his analysis of the eight-vertex model. The "unusual difficulty of his work, based on deep technical intuition" [1], does not lead to a clear understanding of his method. However, the work of Felder and Varchenko [42] provided a comprehensive framework for the underlying algebraic structure.

In the first section, we present Faddeev-Takhtadzan's scheme [36, 54] to construct the eigenstates for a periodic XYZ hamiltonian, or equivalently, the eight-vertex transfer matrix eigenstates, using the modern language of Felder and Varchenko for the Baxter's vertex-face transformation [42]. Then we turn to a statistical physics description of this transformation. Indeed, vertex-face transformation is a mapping between the eight-vertex model configurations and the face model configurations [62]. We also present the result of Rosengren for the partition function of the face model on a square lattice with DWBC [100].

[1] according to Faddeev and Takhtadzan [36]

3.1 The first example: the periodic XYZ spin chain

We first recall the notations that we use for elliptic functions, where ε denote the elliptic modulus and $p = \exp^{2i\pi\varepsilon}$:

$$\begin{aligned}
h_1(\lambda;\varepsilon) &= -2ip^{\frac{1}{8}}\sinh(\lambda)\prod_{n=1}^{\infty}(1-2p^n\cosh(2\lambda)+p^{2n})(1-p^n)\\
h_2(\lambda;\varepsilon) &= 2p^{\frac{1}{8}}\cosh(\lambda)\prod_{n=1}^{\infty}(1+2p^n\cosh(2\lambda)+p^{2n})(1-p^n)\\
h_3(\lambda;\varepsilon) &= \prod_{n=1}^{\infty}(1+2p^{n-\frac{1}{2}}\cosh(2\lambda)+p^{2n-1})(1-p^n)\\
h_4(\lambda;\varepsilon) &= \prod_{n=1}^{\infty}(1-2p^{n-\frac{1}{2}}\cosh(2\lambda)+p^{2n-1})(1-p^n)\\
h(\lambda) &= e^{\lambda}\prod_{n=0}^{\infty}(1-p^ne^{-2\lambda})(1-p^{n+1}e^{2\lambda})
\end{aligned}$$

In this section we are interested in the very general XYZ hamiltonian with periodic boundary condition:

$$\mathcal{H} = \sum_{i=1}^{N}(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y + \Delta(\sigma_i^z\sigma_{i+1}^z - 1) \tag{3.1.1}$$

$\Delta,\gamma \in \mathbb{C}$ are anisotropy parameters that we choose as :

$$\Delta = \frac{h_4^2(0;2\varepsilon)h_2(\eta;2\varepsilon)h_3(\eta;2\varepsilon)}{h_4^2(\eta;2\varepsilon)h_2(0;2\varepsilon)h_3(0;2\varepsilon)}, \quad \gamma = \frac{h_1^2(\eta;2\varepsilon)}{h_4^2(\eta;2\varepsilon)}. \tag{3.1.2}$$

This hamiltonian is the most general Heisenberg type model with nearest neighbor spin interaction. Therefore, its resolution is of primary importance. The QISM framework for the XYZ spin chain requires one to take the most general *elliptic* solution of the Yang-Baxter equation (1.1.1) the eight vertex R-matrix, which depends on the difference of spectral parameters, $R : \mathbb{C} \to \mathrm{End}(V\otimes V), V \sim \mathbb{C}^2$:

$$R(\lambda) = \begin{pmatrix} a^{8V}(\lambda) & 0 & 0 & d^{8V}(\lambda)\\ 0 & b^{8V}(\lambda) & c^{8V}(\lambda) & 0\\ 0 & c^{8V}(\lambda) & b^{8V}(\lambda) & 0\\ d^{8V}(\lambda) & 0 & 0 & a^{8V}(\lambda)\end{pmatrix}. \tag{3.1.3}$$

The R-matrix entries are elliptic functions:

$$a^{8V}(\lambda) = \frac{h_4(\lambda;2\varepsilon)h_4(\eta;2\varepsilon)h_1(\lambda+\eta;2\varepsilon)}{h_4(\lambda+\eta;2\varepsilon)h_4(0;2\varepsilon)}, \quad c^{8V}(\lambda) = \frac{h_4(\lambda;2\varepsilon)h_1(\eta;2\varepsilon)}{h_4(0;2\varepsilon)}, \tag{3.1.4}$$

$$d^{8V}(\lambda) = \frac{h_1(\lambda;2\varepsilon)h_1(\eta;2\varepsilon)h_1(\lambda+\eta;2\varepsilon)}{h_4(\lambda+\eta;2\varepsilon)h_4(0;2\varepsilon)}, \quad b^{8V}(\lambda) = \frac{h_1(\lambda;2\varepsilon)h_4(\eta;2\varepsilon)}{h_4(0;2\varepsilon)}. \tag{3.1.5}$$

Once again, the monodromy matrix can be chosen as the ordered product of the eight-vertex Lax matrices L:

$$T_0(\lambda) = \prod_{i=1}^{N} L_{0i}(\lambda-\xi_i) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[0]}, \tag{3.1.6}$$

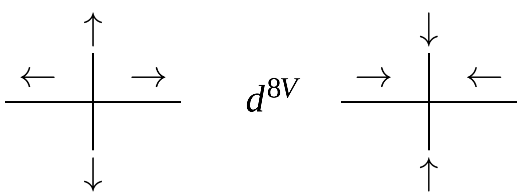
with: $L_{0m}(\lambda-\xi_m) = R_{0m}(\lambda-\xi_m)$, and so it satisfies the Yang-Baxter algebra relations (1.1.2) . In the homogeneous limit, the hamiltonian (3.1.1) also belongs to the commutative family (1.1.4):
$\forall m \in [[1,N]], \quad \xi_m = 0$:

$$\mathcal{H} = \frac{2h(\eta)}{h'(0)} \frac{d}{d\lambda} \ln \mathbf{T}(\lambda)\Big|_{\lambda=0} + constant \tag{3.1.7}$$

Although the periodic XYZ model is completely embedded within the QISM framework, the completely ferromagnetic state $|0\rangle = \otimes_{i=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is evidently no longer an eigenstate of the hamiltonian (3.1.1). Note that the latter is no longer spin invariant: $[\mathcal{H}, \sum_{i=1}^{N} \sigma_i^z] \neq 0$. This is explicit in the Lax matrix representation where the off diagonal elements L_-^+, L_+^- are no longer triangular, and thus:

$$C(\lambda)|0\rangle \neq 0. \tag{3.1.8}$$

Within the QISM duality with vertex models, this translates into a vertex model without charge conservation due to the non-vanishing statistical weight d^{8V} in the eight-vertex R-matrix. The corresponding statistical configuration can be pictured as:



The non-conservation of arrows flux (charge) at the vertex, or equivalently, the spin symmetry breaking is a serious difficulty that was beautifully solved by Baxter for the eight-vertex model, and then turned into a QISM framework by Faddeev and Takhtadzhan. This enabled us to find periodic XYZ Bethe states. Baxter was the first to notice that the eight-vertex model can be transformed into another statistical model similar to the six-vertex model, namely with six statistical configurations. This unexpected highly non-trivial transformation, known as the Baxter's *vertex-face* transformation, enables us to map the eight-vertex model into a *face* model. This model is defined on the *dual lattice*, *ie* where Boltzmann weights are associated to an adjacent face rather than a vertex, and the statistical variables lie onto the vertex. The remarkable point is that it is indeed possible to obtain a six-vertex like model, the price to pay is that such face model and associated Boltzmann weight will depend on an arbitrary continuous parameter, the *dynamical parameter*. Here we will use a modern presentation of Baxter's works by means of *dynamical Yang-Baxter algebra* which was mainly developed by Felder [42].

First we introduce the vertex-face matrix: $S : \mathbb{C}\times\mathbb{C} \to \mathrm{End}(V), V \sim \mathbb{C}^2$:

$$S(\lambda;\theta) = \begin{pmatrix} h_2(-\lambda-\theta;2\varepsilon) & h_2(\lambda-\theta;2\varepsilon) \\ h_3(-\lambda-\theta;2\varepsilon) & h_3(\lambda-\theta;2\varepsilon) \end{pmatrix}. \tag{3.1.9}$$

The new complex parameter θ is the dynamical parameter, or face parameter. This transformation map the vertex Boltzmann weights (3.1.3) into the face Boltzmann weights as follows:

$$\begin{aligned} &R_{12}(\lambda_1-\lambda_2)S_1(\lambda_1;\theta)S_2(\lambda_2;\theta-\eta\sigma_1^z) \\ &\qquad = S_2(\lambda_2;\theta)S_1(\lambda_1;\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta), \end{aligned} \tag{3.1.10}$$

where $\mathcal{R} : \mathbb{C}\times\mathbb{C} \to \mathrm{End}(V\otimes V), V\sim\mathbb{C}^2$:

$$\mathcal{R}(\lambda;\theta) = \begin{pmatrix} a^{face}(\lambda;\theta) & 0 & 0 & 0 \\ 0 & b^{face}(\lambda;\theta) & c^{face}(\lambda;\theta) & 0 \\ 0 & c^{face}(\lambda;-\theta) & b^{face}(\lambda;-\theta) & 0 \\ 0 & 0 & 0 & a^{face}(\lambda;\theta) \end{pmatrix}, \tag{3.1.11}$$

is the *dynamical $\mathcal{R}$-matrix*, with:

$$a^{face}(\lambda;\theta) = h(\lambda+\eta) \tag{3.1.12}$$

$$b^{face}(\lambda;\theta) = \frac{h(\lambda)h(\theta-\eta)}{h(\theta)} \tag{3.1.13}$$

$$c^{face}(\lambda;\theta) = \frac{h(\lambda-\theta)h(\eta)}{h(\theta)}. \tag{3.1.14}$$

Hence, we obtain a six-vertex R-matrix and thus a nilpotent structure for the one site off diagonal operators for the corresponding Lax operator, which is a good first step for the diagonalization of the hamiltonian (3.1.1). This nilpotent structure reflects a very important symmetry of the dynamical $\mathcal{R}$-matrix (3.1.11), the *weight zero* symmetry:

$$[\sigma_1^z + \sigma_2^z, \mathcal{R}_{12}(\lambda;\theta)] = 0. \tag{3.1.15}$$

We first show how to implement the Bethe ansatz construction using this transformation following Faddeev-Takhtadzhan and Felder-Varchenko work, and next we turn to the underlying statistical physics model.

3.2 Vertex-face correspondence: towards an underlying reference state

The strategy that we should follow is straightforward. As we succeed in finding a reference state using the transformation (3.1.10), the idea is to systematically translate all ingredients of the QISM (R,T,$\mathbf{T}$) and associated relations into the face QISM ingredients.

- Eight vertex R-matrix and Yang-Baxter equation (1.1.1):
  The eight-vertex R-matrix (3.1.3) translates into a dynamical face $\mathcal{R}$-matrix as in (3.1.10). The Yang-Baxter equation (1.1.1) for R translates into a *dynamical Yang-Baxter equation* for $\mathcal{R}$:

$$\begin{aligned}\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta-\eta\sigma_3^z)\mathcal{R}_{13}(\lambda_1-\lambda_3;\theta)\mathcal{R}_{23}(\lambda_2-\lambda_3;\theta-\eta\sigma_1^z)\\ =\mathcal{R}_{23}(\lambda_2-\lambda_3;\theta)\mathcal{R}_{13}(\lambda_1-\lambda_3;\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta).\end{aligned} \tag{3.2.1}$$

- Monodromy matrix T and Yang-Baxter algebra (1.1.2):
  To obtain a dynamical analog of the monodromy matrix T we should consider the following ordered product of the S matrices:

$$S_-(\{\xi\};\theta) = \prod_{i=N}^{1} S_i(\xi_i;\theta-\eta\sum_{k=i+1}^{N}\sigma_k^z). \tag{3.2.2}$$

  Using this higher dimensional vertex-face transformation, we obtain a *dynamical monodromy matrix* $\mathcal{T} : \mathbb{C}\times\mathbb{C} \to \mathrm{End}(V\otimes\mathcal{H}), V \sim \mathbb{C}^2$ as:

$$S_-(\{\xi\};\theta)S_0(\lambda;\theta-\eta\,\mathbf{S}^z)\mathcal{T}_0(\lambda;\theta) \tag{3.2.3}$$

$$= T_0(\lambda)S_0(\lambda;\theta)S_-(\{\xi\};\theta-\eta\sigma_0^z). \tag{3.2.4}$$

The dynamical monodromy matrix naturally takes the form of an ordered product of the dynamical $\mathcal{R}$-matrices:

$$\mathcal{T}_0(\lambda;\theta) = \prod_{i=1}^{N} \mathcal{R}_{0i}(\lambda-\xi_i;\theta-\eta\sum_{k=i+1}^{N}\sigma_i^z) \tag{3.2.5}$$

$$= \begin{pmatrix} \mathcal{A}(\lambda;\theta) & \mathcal{B}(\lambda;\theta) \\ \mathcal{C}(\lambda;\theta) & \mathcal{D}(\lambda;\theta) \end{pmatrix}_{[0]}. \tag{3.2.6}$$

The Yang-Baxter algebra relation (1.1.2) turns into a *dynamical Yang-Baxter algebra* relations for $\mathcal{T}$:

$$\begin{aligned}\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta-\eta\,\mathbf{S}^z)\mathcal{T}_1(\lambda_1;\theta)\mathcal{T}_2(\lambda_2;\theta-\eta\sigma_1^z)\\ =\mathcal{T}_2(\lambda_2;\theta)\mathcal{T}_1(\lambda_1;\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta),\end{aligned} \tag{3.2.7}$$

where $\mathbf{S}^z = \sum_{i=1}^N \sigma_i^z$. Note that the dynamical Yang-Baxter equation is an associativity compatibility condition for a dynamical algebra; but also a one site representation of a dynamical Lax matrix that we will choose as: $\mathcal{R}(\lambda;\theta)$.

We should stress at this point that the monodromy matrix $\mathcal{T}$ also satisfies the fundamental weight zero symmetry:

$$[\sigma_0^z+\mathbf{S}^z,\mathcal{T}_0(\lambda;\theta)] = 0. \tag{3.2.8}$$

More explicitly, this last equation read:

$$\begin{gathered}[\mathbf{S}^z,\mathcal{A}(\lambda;\theta)] = [\mathbf{S}^z,\mathcal{D}(\lambda;\theta)] = 0\\ [\mathcal{B}(\lambda;\theta),\mathbf{S}^z] = 2\mathcal{B}(\lambda;\theta), \quad [\mathcal{C}(\lambda;\theta),\mathbf{S}^z] = -2\mathcal{C}(\lambda;\theta)\end{gathered} \tag{3.2.9}$$

This is exactly the symmetry that is lost in the integrable structure of the periodic XYZ hamiltonian (3.1.1) and that is recovered here in dynamical context. Namely, if we consider a $\mathbf{S}^z$-diagonalize representation space, $\mathcal{A}(\lambda;\theta),\mathcal{D}(\lambda;\theta)$ are operators that conserve the spin, whereas $\mathcal{B}(\lambda;\theta),\mathcal{C}(\lambda;\theta)$ are creation and annihilation operators, and this is exactly what is needed for a successful implementation of the Bethe ansatz scheme. The only important modification comparing to the Yang-Baxter algebra (1.1.2) relation is that these relations involve commutations between monodromy operators $\mathcal{T}_{\alpha,\beta}$ valuated at different values of the dynamical parameter

θ, such as:

$$\begin{aligned}
\mathcal{B}(\lambda;\theta)\mathcal{B}(\mu;\theta+\eta) &= \mathcal{B}(\mu;\theta)\mathcal{B}(\lambda;\theta+\eta) \\
\mathcal{B}(\lambda;\theta)\mathcal{A}(\mu;\theta+\eta) &= \frac{c^{face}(\lambda-\mu;\theta)}{a^{face}(\lambda-\mu;\theta)}\mathcal{B}(\mu;\theta)\mathcal{A}(\lambda;\theta+\eta) \\
&+ \frac{b^{face}(\lambda-\mu;-\theta)}{a^{face}(\lambda-\mu;\theta)}\mathcal{A}(\mu;\theta)\mathcal{B}(\lambda;\theta+\eta) \\
\mathcal{D}(\mu;\theta)\mathcal{B}(\lambda;\theta+\eta) &= \frac{c^{face}(\lambda-\mu;-\theta+\eta\mathbf{S}^z)}{a^{face}(\lambda-\mu;\theta)}\mathcal{D}(\lambda;\theta)\mathcal{B}(\mu;\theta+\eta) \\
&+ \frac{b^{face}(\lambda-\mu;-\theta+\eta\mathbf{S}^z)}{a^{face}(\lambda-\mu;\theta)}\mathcal{B}(\lambda;\theta)\mathcal{D}(\mu;\theta+\eta)
\end{aligned} \tag{3.2.10}$$

Remark 3.2.1 *This dynamical monodromy matrix has a clear interpretation in statistical mechanics, it is an essential tool to study the elliptic face model on a square lattice. We will return to this correspondence later on.*

Remark 3.2.2 *The dynamical structure constants $a^{face}, b^{face}, c^{face}$ are rather operators, as it contain the operator $\mathbf{S}^z$.*

The relation (3.2.7) is actually the defining commutations relations of a fascinating object, the Felder's elliptic quantum group $E_{\tau,\eta}(sl_2)$. We will put to the side the details about this fundamental object for the moment, as it deserves a systematic analysis of later on.

- Commuting charges and trace formula:
  It turns out that the vertex-face transformation (3.1.10) is not enough to obtain a mapping between the integrals of motion of the periodic XYZ chain and their face counterpart. This is somehow misleading and we would like to stress that **no isomorphism between commuting charges for the periodic XYZ model and commuting charges for the face model exists**. Rather, a weaker relation is present. The crucial step is to restrict the analysis to a subspace with a fixed z component of the total spin **in the face picture** [2], and this **total spin should be zero**. This means that in the framework of the algebraic Bethe ansatz the number of creation operators,

[2] spins here have nothing to do with the XYZ spin, but rather refer to the canonical basis of $\mathbb{C}^2$

and hence the total spin, should be fixed to $\frac{N}{2}$. Thus we consider the action of the trace (1.1.4) on the space $V_{[0]} = \{|\psi\rangle\}$ with a fixed z component of the total spin:

$$\mathbf{S}^z |\psi\rangle = 0, \tag{3.2.11}$$

Theorem 3.2.1 (Felder-Varchenko [42]) *The action on this subspace leads to:*

$$\begin{aligned}\mathbf{T}(\lambda)S_-(\{\xi\};\theta)|\psi\rangle = [&S_-(\{\xi\};\theta+\eta)\mathcal{A}(\lambda;\theta+\eta)\\ &+S_-(\{\xi\};\theta-\eta)\mathcal{D}(\lambda;\theta-\eta)]|\psi\rangle.\end{aligned} \tag{3.2.12}$$

Therefore, finding eigenstates of $\mathbf{T}(\lambda)$ is equivalent to finding eigenstates for $\mathcal{A}(\lambda;\theta)$ and $\mathcal{D}(\lambda;\theta)$, and summing up all values of the dynamical parameter θ. This summation procedure is, in some sense, natural as the face parameter is absent in the XYZ model. This is actually possible as we know how to built a representation space for such a triangular structure.

- Existence of a reference state:
  Once we obtained these dynamical objects and their corresponding algebraic relations, we can easily check the Bethe ansatz requirement, namely the existence of a reference state $|0\rangle = \otimes_{i=1}^N \begin{pmatrix}1\\0\end{pmatrix}$ such that:

$$\begin{aligned}\mathcal{A}(\lambda;\theta)|0\rangle &= a(\lambda;\theta)|0\rangle\\ \mathcal{D}(\lambda;\theta)|0\rangle &= d(\lambda;\theta)|0\rangle\\ \mathcal{B}(\lambda;\theta)|0\rangle &\neq 0\\ \mathcal{C}(\lambda;\theta)|0\rangle &= 0\end{aligned} \tag{3.2.13}$$

In our case, the functions a and d are:

$$a(\lambda;\theta) = \prod_{i=1}^{N} a^{face}(\lambda-\xi_i;\theta), \quad d(\lambda;\theta) = \prod_{i=1}^{N} b^{face}(\lambda-\xi_i;-\theta+(i-1)\eta). \tag{3.2.14}$$

We say that $|0\rangle$ is a completely ferromagnetic state with all spin up in the face picture.

Remark 3.2.3 *A state satisfying the condition* (3.2.11) *can be obtained only with the action of $\frac{N}{2}$ operators $\mathcal{B}$ (or $\mathcal{C}$) on $|0\rangle$ (or $|\overline{0}\rangle$), so the spin chain length N should be **even**. The case of a chain with an odd number of sites is still an open problem.*

We have succeeded in translating all necessary ingredients of the algebraic Bethe ansatz scheme, and this should be implemented for the underlying face model using the dynamical structure $(\mathcal{R},\mathcal{T})$ rather than the vertex structure. The following theorem holds:

Theorem 3.2.2 (Faddeev-Takhtadjan [36], Felder-Varchenko [42])

$$\textbf{For } M=\frac{N}{2} \quad \textit{and} \quad \phi(\theta)=\omega^{\theta}\prod_{j=1}^{N}\frac{h(\eta)}{h(\theta+j\eta)},\omega\in\mathbb{C} \tag{3.2.15}$$

$$|\psi(\{\lambda_k\}_{k=1,\ldots,M})\rangle=\int\{S_-(\{\xi\};\theta)\prod_{k=1}^{M}\phi(\theta)\mathcal{B}(\lambda_k;\theta+(j-1)\eta)|0\rangle\}d\theta$$

is an eigenstate of the transfer matrix (1.1.4) $\mathbf{T}(\mu)=A(\mu)+D(\mu)$ *for any* μ *with eigenvalue* Λ*:*

$$\Lambda(\mu,\{\lambda_k\}_{k=1,\ldots M},\{\xi_j\}_{j=1,\ldots,N})=\omega a(\mu;\theta)\prod_{k=1}^{M}\frac{h(\lambda_k-\mu+\eta)}{h(\lambda_k-\mu)}+\omega^{-1}\widetilde{d}(\mu;\theta)\frac{h(\mu-\lambda_k+\eta)}{h(\mu-\lambda_k)} \tag{3.2.16}$$

where:

$$\widetilde{d}(\mu;\theta)=\prod_{i=1}^{N}h(\mu-\xi_i) \tag{3.2.17}$$

if the parameters $\{\lambda_k\}_{k=1\ldots M}$ *satisfy the Bethe equations:*

$$\frac{\widetilde{d}(\lambda_k;\theta)}{a(\lambda_k;\theta)}\prod_{l=1,l\neq k}^{M}\frac{h(\lambda_l-\lambda_k)}{h(\lambda_k-\lambda_l)}=\omega^2 \tag{3.2.18}$$

Remark 3.2.4 $\phi(\theta)$ *and* ω *are gauge parameters, other choices are possible.*

Remark 3.2.5 *The vector* $|\psi\rangle$ *is the Bethe vector of the face model [42], with eigenvalue* Λ *and Bethe equations* (3.2.18). *By linearity of* S_-, *and the functional* $f\to\int f$ *it is turned to a XYZ Bethe states with the same eigenvalue and Bethe equations. Neither the Bethe equations nor the eigenvalue* Λ *depend on the dynamical parameter* θ. *Although that is what we expect for the XYZ model, it is not obvious at all that this is the case for such face models. This is a very special feature of such face models and algebraic Bethe ansatz for dynamical Yang-Baxter algebra.*

3.3 Partition function of the elliptic face model on a square lattice with DWBC

As already mentioned in Chapter 2, we should compute in the QISM a partition function quantity in order to push forward analysis of the periodic XYZ spin, towards correlation functions of the model. This partition function takes the following form:

$$Z^{face}_{N,N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{i=1,\dots,N};\theta) = \langle 0|_\lambda \langle \bar{0}|_\xi \prod_{i,j=1}^{N} \mathcal{R}_{ij}(\lambda_i;\theta-\eta\sum_{k=1}^{i}\sigma^z_{\lambda_k}-\eta\sum_{k=1}^{j}\sigma^z_{\xi_k})|0\rangle_\xi|\bar{0}\rangle_\lambda \tag{3.3.1}$$

$$= \langle 0|_\lambda \langle \bar{0}|_\xi \prod_{i=1}^{N} \mathcal{T}(\lambda_i;\theta-\eta\sum_{k=1}^{i}\sigma^z_{\lambda_k})|0\rangle_\xi|\bar{0}\rangle_\lambda$$

$$= \langle \bar{0}|_\xi \prod_{i=1}^{N} \mathcal{B}(\lambda_i;\theta-\eta\sum_{k=1}^{i}\sigma^z_{\lambda_k})|0\rangle_\xi \tag{3.3.2}$$

Quite unexpectedly, this quantity has a clear interpretation in statistical physics, it is the partition function of the elliptic face model with DWBC. To see this, we should give the statistical physics model underlying the dynamical Yang-Baxter algebra (3.2.7), and especially the representation (3.2.5) for the dynamical monodromy matrix $\mathcal{T}$. As we mentioned at the beginning of this chapter, the transformation (3.1.10) can be interpreted as a mapping between the eight-vertex model statistical configurations to face configurations which can be defined in terms of a *height function* on a two-dimensional square lattice. Every square of the lattice is characterized by a height θ and its values for two adjacent squares differ by $\pm\eta$. There are six possible face configurations:

$\theta-\eta$	$\theta-2\eta$
θ	$\theta-\eta$

$\theta+\eta$	$\theta+2\eta$
θ	$\theta+\eta$

$\theta-\eta$	θ
θ	$\theta+\eta$

$\theta+\eta$	θ
θ	$\theta-\eta$

$\theta+\eta$	θ
θ	$\theta+\eta$

$\theta-\eta$	θ
θ	$\theta-\eta$

and the corresponding statistical weights $\mathcal{R}_{cd}^{ab}$ are collected into the dynamical $\mathcal{R}$-matrix (3.1.11).

Remark 3.3.1 *The presence of only six non-vanishing entries, and their specific distribution, within the dynamical $\mathcal{R}$-matrix* (3.1.11) *is here fundamental, due to the zero weight symmetry* (3.1.15)*. In this statistical mechanics context, this symmetry translates into a conservation rule, namely the height can differ only by* η*, and thus leading to only six possible configurations. This is known as the Ice Rule.*

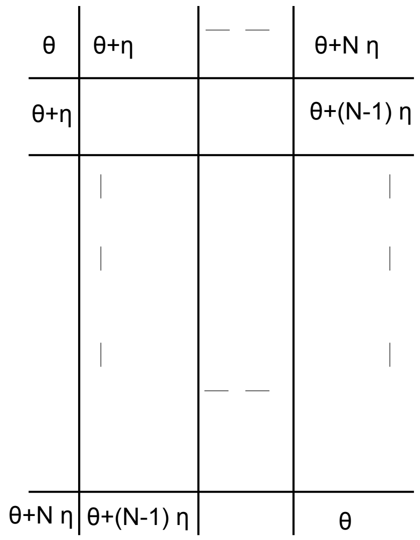


Figure 3.1: The face model with DWBC

We are interested in this face model with DWBC as illustrated in Figure 3.1. The partition function of this model is nothing but the quantity (3.3.1). The evaluation of this partition was a real challenge for almost thirty years. This is because the underlying algebraic structures were not clear when Baxter introduced this model. It is only after the breakthrough of Felder on the elliptic quantum group $E_{\tau,\eta}(sl_2)$ that the algebraic structure became clear. Furthermore, the elliptic parametrization makes the analytical property of the partition function difficult to handle. However, several works have led to a set of conditions that uniquely determine the partition function.

Proposition 3.3.1 (Baxter [8], Rosengren [100], Pakuliak-Rubtsov-Silantyev [93]) *The partition function* (3.3.1) *satisfies the following properties:*

i) Initial condition

$$Z^{face}_{1,1}(\lambda_1,\xi_1;\theta) = \frac{h(\eta)h(\lambda_1-\xi_1-\theta)}{h(\theta)}$$

ii) Symmetry
$Z^{face}_{N,N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta)$ *is a symmetric function of the* $\{\lambda_i\}_{i=1,\dots,N}$ *and the* $\{\xi_j\}_{j=1,\dots,N}$.

iii) Elliptic polynomiality
$Z^{face}_{N,N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta)$ *is an elliptic polynomial of each* $\{\lambda_i\}_{i=1,\dots,N}$ *of order N and norm* $\sum_{j=1}^N \xi_j-\theta$ *and of each* $\{\xi_j\}_{j=1,\dots,N}$ *of order N and norm* $\sum_{i=1}^N \lambda_i+\theta$.

iv) Recursive relations

$$Z^{face}_{N,N}\left(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta\right)\Big|_{\lambda_i=\xi_j} = \frac{\prod_{k=1,k\neq i}^N h(\xi_j-\xi_k+\eta)h(\lambda_k-\xi_j+\eta)}{h^{(N-2)}(\eta)} Z^{face}_{N-1,N-1}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j};\theta),$$

and

$$Z^{face}_{N,N}\left(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta\right)\Big|_{\lambda_i=-\xi_j} = \frac{h(\theta-N\eta)\prod_{k=1,k\neq i}^N h(\xi_j-\xi_k-\eta)h(\lambda_k-\lambda_i)}{h(\theta-(N-1)\eta)h^{(N-2)}(\eta)} Z^{face}_{N-1,N-1}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j};\theta).$$

Lemma 3.3.1 *The partition function* $Z^{face}_{N,N}$ *is uniquely determined by the set of conditions i)-iv).*

Once these conditions have been established, Rosengren also proposes an explicit, but complicated, formula for the partition function.

Theorem 3.3.1 (Rosengren [100]) *The partition function of the elliptic face model on the* *square lattice and DWBC is:*

$$Z^{face}_{N,N}(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N};\theta) \tag{3.3.3}$$

$$\begin{aligned} &= \frac{\prod_{i,j=1}^{N} h(\lambda_i-\xi_j)h(\lambda_i-\xi_j+\eta)}{\prod_{1\leq i<j\leq N} h(\lambda_i-\lambda_j)h(\xi_j-\xi_i)} \\ &\times \frac{h(\theta-N\eta)}{h^N(\gamma)h(\sum_{i=1}^{N}(\xi_i-\lambda_i)+\gamma+\theta-N\eta)} \\ &\times \sum_{S\subseteq\{1,\ldots,N\}} (-1)^{|S|}\frac{h(\theta+\gamma+|S|-N\eta)}{h(\theta+|S|-N\eta)} det\mathcal{N}^{face}(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N};\theta) \end{aligned} \tag{3.3.4}$$

where the $N\times N$ matrix $\mathcal{N}^{face}$ can be expressed as:

$$\mathcal{N}^{face}(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{j=1,..,N})_{\alpha,\beta} = \frac{h(\lambda^S_\alpha-\xi_\beta+\gamma)}{h(\lambda^S_\alpha-\xi_\beta)} \tag{3.3.5}$$

with the following convention: $\begin{cases} \lambda_i^S = \lambda_i+\eta, i\in S \\ \lambda_i^S=\lambda_i, i\notin S \end{cases}$

Remark 3.3.2 *Due to the theta functions' identities, the partition function does not depend on the complex parameter γ, which is a regularization parameter that is introduced here for convenience.*

This formula is definitively an Izergin-Korepin type formula for the partition function of the elliptic face model on a square lattice, but it is expressed as a sum of 2^N determinant instead of a single determinant. This makes considerably difficult to find a manageable formula for the scalar product and correlation functions for the periodic XYZ spin chain. A simpler formula for this partition function is still an open problem.

Chapter 4

Open boundary XXZ model

In the previous chapter, we apply the vertex-face transformation to the boundary case, namely we study the XXZ model with the most general boundary. The boundary part of the open XXZ hamiltonian leads to the same obstacle for finding Bethe states in the algebraic Bethe ansatz framework than the periodic XYZ model. Indeed, although the diagonal boundary XXZ spin chain analysis was pushed far forward by the exact computation of its correlations functions [69, 70], very few results are known about the general boundary case. Recently, the scientific community has shown a strong interest in the XXZ model with general boundaries because it is an exactly solvable model for non-equilibrium statistical physics. Namely, the *asymmetric simple exclusion process* (ASEP) which is the *default stochastic model for transport phenomena*[1] *[121]* is related by gauge transformation to the XXZ model with general boundary terms [30, 103]. The crucial point is that the general boundary XXZ model is integrable, or more correctly, possesses an integrable structure described by the boundary version of the QISM of Chapter 1. Therefore, exact results for the ASEP model could, in principle, be obtained by the analysis of the open XXZ model [18, 19]. From the algebraic Bethe ansatz point of view, the general boundary XXZ model eigenstates were found using various gauge transformation (resembling a component-wise vertex-face transformation) if the boundary parameters satisfy some conditions [12, 123]. In these previous works, the gauge transformation lead to various intricate exchange relations for Bethe states generating algebra. In our opinion, this approach has three main weaknesses. First, the underlying algebraic structure that arises from this gauge transformation is missing. Second, the Bethe states description is unclear. Third, the underlying two-dimensional statistical physics model has escaped description. We would like to tackle this problem using the algebraic version of the vertex-face transformation. We generalize the Baxter, Faddeev-Takhtadjan, Felder, and Rosengren's work

[1] according to H.T. Yau

to the boundary case for constructing open XXZ Bethe states and computing the partition function of the underlying face model. This will enable us to:

- highlight the integrable algebraic structure, which is a dynamical reflection algebra
- explicit description of the Bethe states, by establishing a clear relationship between the XXZ Bethe states and the underlying face Bethe states
- clearly describe the dual two-dimensional statistical physics model, by computing exactly its partition function

Although this program sounds natural, as the general boundary XXZ model is completely embedded into the boundary QISM, we will see that the algebraic Bethe ansatz in this context leads to some important limitations.

4.1 The hamiltonian: which reference state ?

In this section we are interested in the XXZ hamiltonian with general open boundary condition:

$$\mathcal{H} = \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z\right) + h_1 + h_N \tag{4.1.1}$$

Where the interaction with boundary magnetic fields is:

$$\begin{aligned} h_1 &= \frac{\sinh\eta}{\sinh\zeta\sinh\delta}\left[-\cosh\zeta\cosh\delta\sigma_1^z + \sinh\tau\sigma_1^x - i\cosh\tau\sigma_1^y\right] \\ h_N &= \frac{\sinh\eta}{\sinh\bar\zeta\sinh\bar\delta}\left[-\cosh\bar\zeta\cosh\bar\delta\sigma_N^z + \sinh\bar\tau\sigma_N^x - i\cosh\bar\tau\sigma_N^y\right] \end{aligned} \tag{4.1.2}$$

Any three components of the boundary magnetic fields can be expressed by six complex parameters δ, $\bar\delta$, ζ, $\bar\zeta$, τ and $\bar\tau$. The bulk part of this hamiltonian is the usual XXZ hamiltonian. As shown in Chapter 1, we only need to add to the QISM the boundary counterpart description. This means that starting with the six-vertex R-matrix (1.2.2), we should introduce the boundary monodromy matrices $K_\pm$ that solve the reflection equations. We consider here the most general solution [17, 55] K_- of the reflection equation (1.3.4):

$$\begin{aligned} K_-(\lambda) &\equiv K_-(\lambda;\delta,\zeta,\tau) \\ &= \begin{pmatrix} \frac{\cosh(\delta+\zeta)\,e^{-\lambda}-\cosh(\delta-\zeta)\,e^{\lambda}}{2\sinh(\delta+\lambda)\sinh(\lambda+\zeta)} & e^{-\tau}\,\frac{\sinh(2\lambda)}{2\sinh(\delta+\lambda)\sinh(\lambda+\zeta)} \\ -e^{\tau}\,\frac{\sinh(2\lambda)}{2\sinh(\delta+\lambda)\sinh(\lambda+\zeta)} & \frac{\cosh(\delta+\zeta)\,e^{\lambda}-\cosh(\delta-\zeta)\,e^{-\lambda}}{2\sinh(\delta+\lambda)\sinh(\lambda+\zeta)} \end{pmatrix} \end{aligned} \tag{4.1.3}$$

and its dual (1.3.5) $K_+(\lambda) = K_-(-\lambda-\eta;\overline{\delta},\overline{\zeta},\overline{\tau})$.

The boundary monodromy matrix that solves the reflection algebra is:

$$(U_-)_0(\lambda) = \widehat{\gamma}(\lambda)T_0(\lambda)(K_-)_0(\lambda)T_0^{-1}(-\lambda) = \begin{pmatrix} A_-(\lambda) & B_-(\lambda) \\ C_-(\lambda) & D_-(\lambda) \end{pmatrix}_{[0]}, \tag{4.1.4}$$

with $T_0(\lambda) = \prod_{i=1}^N L_{0i}(\lambda-\xi_i) = \prod_{i=1}^N R_{0i}(\lambda-\xi_i)$ and the dual boundary monodromy matrix is :

$$(U_+^{t_0})_0(\lambda) = \widehat{\gamma}(\lambda)T_0^{t_0}(\lambda)(K_+^{t_0})_0(\lambda)(T^{-1})_0^{t_0}(-\lambda) = \begin{pmatrix} A_+(\lambda) & C_+(\lambda) \\ B_+(\lambda) & D_+(\lambda) \end{pmatrix}_{[0]}. \tag{4.1.5}$$

Remark 4.1.1 $\widehat{\gamma}(\lambda) = (-1)^N \prod_{i=1}^N \sinh(\lambda+\xi_i+\eta)\sinh(\lambda+\xi_i-\eta)$ *is a normalization factor that we introduce here for convenience.*

The hamiltonian of the open XXZ spin chain with most general boundary fields (4.1.1) can be obtained in the homogeneous limit as the following derivative of the transfer matrix (1.3.9):
$\forall m \in [[1,N]], \quad \xi_m = 0$:

$$\mathcal{H} = c\frac{d}{d\lambda}\mathbf{T}(\lambda)\Big|_{\lambda=0} + constant, \tag{4.1.6}$$

where:

$$c = -8\coth(\delta)\coth(\overline{\delta}-\eta)\coth(\zeta)\coth(\overline{\zeta}-\eta). \tag{4.1.7}$$

Once again, we are in the same situation as for the periodic XYZ case. The general boundary XXZ model is completely embedded within the QISM framework, or more precisely its boundary version. However, the totally ferromagnetic state $|0\rangle = \otimes_{i=1}^N \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is evidently no longer an eigenstate of the hamiltonian (4.1.1) since it no longer conserves the third component of the total spin: $[\mathcal{H}, \sum_{i=1}^N \sigma_i^z] \neq 0$, due to the presence of its boundary terms. This is explicitly seen in the boundary matrix where the off-diagonal elements $K_\mp^\pm$ are non-vanishing, and thus:

$$C_-(\lambda)|0\rangle \neq 0, \quad C_+(\lambda)|0\rangle \neq 0. \tag{4.1.8}$$

Within the QISM duality with vertex model of statistical mechanics, this translates into a vertex model without charge conservation at the reflecting end due to the non-vanishing statistical weight $(K_\pm)_\pm^\pm$ in the boundary matrix. The corresponding statistical configuration are drawn as in Figure 4.1.

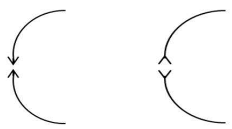


Figure 4.1: The off diagonal boundary statistical configurations (4.1.3)

Following Baxter's idea, we can use the vertex-face transformation in order to diagonalize the boundary matrix $K_{\pm}$ leading to a boundary model where the boundary satisfies a *conservation rule*. Once again, the trigonometric version of the transformation (3.1.9) will be very useful here.

Let us focus on the following gauge transformation:

$$S(\lambda;\theta) \equiv S(\lambda;\theta,\omega) = e^{\lambda/2} \begin{pmatrix} e^{-(\lambda+\theta+\omega)} & e^{-(\lambda-\theta+\omega)} \\ 1 & 1 \end{pmatrix}, \qquad (4.1.9)$$

which depends on two parameters, the previously defined dynamical parameter θ and an arbitrary complex parameter ω.

Remark 4.1.2 *This transformation* ***is not*** *the trigonometric limit of the elliptic vertex-face transformation* (3.1.9)*, but rather a gauge transformed limit [21].*

This trigonometric vertex-face matrix satisfies two important properties:

- It diagonalizes the boundary matrix K_-:

  $$\mathcal{K}_-(\lambda) = S^{-1}(\lambda;\delta-\zeta,\tau)K_-(\lambda)S(-\lambda;\delta-\zeta,\tau), \qquad (4.1.10)$$

  where the diagonal *dynamical boundary* matrix $\mathcal{K}_-(\lambda)$ is:

  $$\mathcal{K}_-(\lambda) \equiv \mathcal{K}_-(\lambda;\delta,\zeta) = \begin{pmatrix} \frac{\sinh(\delta-\lambda)}{\sinh(\delta+\lambda)} & 0 \\ 0 & \frac{\sinh(\zeta-\lambda)}{\sinh(\zeta+\lambda)} \end{pmatrix}. \qquad (4.1.11)$$

  This diagonal structure is the signature of a very important symmetry of the dynamical $\mathcal{K}$-matrix , the weight zero symmetry:

  $$[\sigma_0^z, (\mathcal{K}_-)_0(\lambda)] = 0. \qquad (4.1.12)$$

  This symmetry will once again lead to a fundamental conservation rule for statistical mechanics (see remark (3.3.1)).

- It is a trigonometric vertex-face transformation (3.1.10):

$$R_{12}(\lambda_1-\lambda_2)S_1(\lambda_1;\theta)S_2(\lambda_2;\theta-\eta\sigma_1^z)$$
$$=S_2(\lambda_2;\theta)S_1(\lambda_1;\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta), \quad (4.1.13)$$

  where the $\mathcal{R}$-matrix is the trigonometric limit (up to a irrelevant numerical factor) of the elliptic dynamical $\mathcal{R}$-matrix (3.1.11). $a^{face}, b^{face}, c^{face}$ are now trigonometric functions:

$$a^{face}(\lambda;\theta)=\sinh(\lambda+\eta) \quad (4.1.14)$$
$$b^{face}(\lambda;\theta)=\frac{\sinh(\lambda)\sinh(\theta-\eta)}{\sinh(\theta)}$$
$$c^{face}(\lambda;\theta)=\frac{\sinh(\lambda-\theta)\sinh(\eta)}{\sinh(\theta)}.$$

  Hence, we obtain a diagonal structure for the boundary matrix keeping the triangular structure for the off-diagonal one site Lax matrix. This means that this new structure is suitable to the Bethe ansatz requirement. The situation becomes now strictly similar to the diagonal boundary XXZ model of Chapter 1, with a dynamical object instead of the vertex model.

We will first show how to implement the Bethe ansatz construction using this transformation, and then we will turn to the underlying statistical physics models.

4.2 Vertex-face correspondence: towards an underlying reference state

The strategy that we should follow is very parallel to the one used in the next section for the periodic XYZ model. As we succeed in finding a reference state using the transformation (4.1.9), the idea is to systematically translate all ingredients of the QISM (R,U_-,$\mathbf{T}$) and associated relations onto the face QISM counterpart.

- Bulk part (six-vertex R-matrix and the bulk monodromy matrix T):
  Besides taking the trigonometric S-matrix (4.1.9), there are no modifications to the algebraic relations of the previous section. The only modification is to take the trigonometric dynamical $\mathcal{R}$-matrix (4.1.14) and the corresponding monodromy matrix that satisfy the dynamical Yang-Baxter algebra relations. Although this modification is not necessary (the six-vertex R-matrix (1.2.2) having the desired nilpotent structure), this enables us to work on the boundary K_--matrix.

- Boundary matrix K_- and the reflection equation (1.3.4):
  Using the transformation (4.1.10), and the zero-weight condition (3.1.15), (4.1.12), one can show that the reflection equation (1.3.4) turns into a *dynamical reflection equation* for $\mathcal{K}_-$, once we choose: $\theta = \delta - \zeta$:

$$\mathcal{R}_{12}(\lambda_1 - \lambda_2;\theta)(\mathcal{K}_-)_1(\lambda_1)\mathcal{R}_{21}(\lambda_1 + \lambda_2;\theta)(\mathcal{K}_-)_2(\lambda_2)$$
$$=(\mathcal{K}_-)_2(\lambda_2)\mathcal{R}_{12}(\lambda_1 + \lambda_2;\theta)(\mathcal{K}_-)_1(\lambda_1)\mathcal{R}_{21}(\lambda_1 - \lambda_2;\theta). \quad (4.2.1)$$

  Thus we succeed in obtaining a reflection equation for the face boundary.

  Remark 4.2.1 *This reflection equation is just the usual vertex reflection equation* (1.3.4) *with the use of the dynamical $\mathcal{R}$-matrix instead of the usual vertex R-matrix.*

- Boundary monodromy matrix U_- and reflection algebra (1.3.1):
  Using S_- (3.2.2) one can prove in a similar way that the dynamical boundary monodromy matrix defined as:

$$(\mathcal{U}_-)_0(\lambda;\theta) = \widehat{\gamma}(\lambda)\mathcal{T}_0(\lambda;\theta)(\mathcal{K}_-)_0(\lambda)\mathcal{T}_0^{-1}(-\lambda;\theta) = \begin{pmatrix} \mathcal{A}_-(\lambda;\theta) & \mathcal{B}_-(\lambda;\theta) \\ \mathcal{C}_-(\lambda;\theta) & \mathcal{D}_-(\lambda;\theta) \end{pmatrix}_{[0]},$$

  satisfies the *dynamical reflection algebra* relation:

$$\mathcal{R}_{12}(\lambda_1 - \lambda_2;\theta - \eta\,\mathbf{S}^z)(\mathcal{U}_-)_1(\lambda_1;\theta)\mathcal{R}_{21}(\lambda_1 - \lambda_2;\theta - \eta\,\mathbf{S}^z)(\mathcal{U}_-)_2(\lambda_2;\theta)$$
$$= (\mathcal{U}_-)_2(\lambda_2;\theta)\mathcal{R}_{12}(\lambda_1 + \lambda_2;\theta - \eta\,\mathbf{S}^z)(\mathcal{U}_-)_1(\lambda_1;\theta)\mathcal{R}_{21}(\lambda_1 - \lambda_2;\theta - \eta\,\mathbf{S}^z). \quad (4.2.2)$$

  Due to the zero-weight symmetry for the dynamical monodromy matrix $\mathcal{T}$ (3.2.8) and for the dynamical boundary matrix $\mathcal{K}_-$ (4.1.12), the dynamical boundary monodromy matrix also obeys to a zero weight condition:

$$[\sigma_0^z + \mathbf{S}^z, (\mathcal{U}_-)_0(\lambda;\theta)] = 0. \quad (4.2.3)$$

  This means that a suitable representation space for $\mathcal{U}_-$ elements algebra is an $\mathbf{S}^z$-diagonalizable module. In such a module, the elements $\mathcal{A}_-$ and $\mathcal{D}_-$ are operators that conserve the spin, whereas $\mathcal{B}_-$ and $\mathcal{C}_-$ are creation and annihilation operators:

$$[\mathbf{S}^z, \mathcal{A}_-(\lambda;\theta)] = [\mathbf{S}^z, \mathcal{D}_-(\lambda;\theta)] = 0 \quad (4.2.4)$$
$$(\mathbf{S}^z + 2Id)\mathcal{B}_-(\lambda;\theta) = \mathcal{B}_-(\lambda;\theta)\mathbf{S}^z, \quad (\mathbf{S}^z - 2Id)\mathcal{C}_-(\lambda;\theta) = \mathcal{C}_-(\lambda;\theta)\mathbf{S}^z$$

Remark 4.2.2 *This dynamical boundary monodromy matrix has a clear interpretation in statistical mechanics. It is an essential tool to study the trigonometric face model with a reflecting end. We will return to this correspondence later on.*

- Commuting charges **T** and trace formula:
Once again, the vertex-face transformation (4.1.9) is not enough to obtain a mapping between the integrals of motion generating function for the open XXZ chain and an equivalent trace like formula for this boundary face model. The crucial step is to restrict the analysis to a subspace with a fixed z component of the total spin **in the face picture**, but here we do not need any restriction on its value. The number of creation operators and hence the total spin is then still fixed. Thus, we consider the action of the trace (1.3.9) on the space $V_{[s]} = \{|\psi_s\rangle\}$ with a fixed z component of the total spin:
$$\mathbf{S}^z\,|\psi_s\rangle = s|\psi_s\rangle, \tag{4.2.5}$$
The action on this subspace leads to:
$$\begin{aligned}\mathbf{T}(\lambda)S_-(\{\xi\};\delta-\zeta)|\psi\rangle = S_-(\{\xi\};\delta-\zeta) \\ \times \mathrm{Tr}_0(\mathcal{K}_+(\lambda;\overline{\delta},\overline{\zeta})\frac{\sinh(\overline{\delta}-\overline{\zeta}-\eta\sigma_0^z)}{\sinh(\overline{\delta}-\overline{\zeta})}\mathcal{T}_-(\lambda;\delta-\zeta))|\psi\rangle,\end{aligned} \tag{4.2.6}$$
where:
$$\mathcal{K}_+(\lambda;\overline{\delta},\overline{\zeta}) = \mathcal{K}_-(-\lambda-\eta;\overline{\delta},\overline{\zeta}), \tag{4.2.7}$$
provided two constraints on the boundary parameter $(\delta,\zeta,\tau,\overline{\delta},\overline{\zeta},\overline{\tau})$:
$$\cosh(\overline{\delta}-\overline{\zeta}) = \cosh(\delta-\zeta-\eta s+\overline{\tau}-\tau-\eta), \tag{4.2.8}$$
$$\cosh(\overline{\delta}-\overline{\zeta}) = \cosh(\delta-\zeta-\eta s-\overline{\tau}+\tau+\eta). \tag{4.2.9}$$

Remark 4.2.3 *The constraints can be solved by imposing:*
$$\begin{aligned}\overline{\tau} =& \tau+\eta+i\pi n, \\ \overline{\delta}-\overline{\zeta} =& \delta-\zeta-\eta s+2i\pi m+i\pi n.\end{aligned} \tag{4.2.10}$$

Remark 4.2.4 *The conditions* (4.2.8) *and* (4.2.9) *explicitly depend on the subspace spin s (or equivalently its dimension). This means that the algebraic Bethe ansatz cannot lead to the complete description of the eigenstates.*

Remark 4.2.5 *If we choose to diagonalize the transfer matrix* (1.3.9) *in the dual space generated by subspace of type* $V^*_{[s]} = \{\langle\psi_s|\}$ *, then we need* ***two different conditions****:*

$$\cosh(\overline{\delta}-\overline{\zeta}) = \cosh(\delta-\zeta-\eta s+\overline{\tau}-\tau+\eta), \tag{4.2.11}$$
$$\cosh(\overline{\delta}-\overline{\zeta}) = \cosh(\delta-\zeta-\eta s-\overline{\tau}+\tau-\eta), \tag{4.2.12}$$

This means that right and left modules as constructed here ***do not correspond to the same open XXZ model****.*

- Existence of a reference state:
  Once we obtain these dynamical objects and corresponding algebraic relations, we can easily check the Bethe ansatz requirement, namely the existence of a reference state $|0\rangle = \otimes_{i=1}^{N}\begin{pmatrix}1\\0\end{pmatrix}$ such as:

$$\mathcal{A}_-(\lambda;\theta)|0\rangle = a(\lambda;\theta)|0\rangle \tag{4.2.13}$$
$$\mathcal{D}_-(\lambda;\theta)|0\rangle = d(\lambda;\theta)|0\rangle \tag{4.2.14}$$
$$\mathcal{B}_-(\lambda;\theta)|0\rangle \neq 0 \tag{4.2.15}$$
$$\mathcal{C}_-(\lambda;\theta)|0\rangle = 0 \tag{4.2.16}$$

  Some details concerning the computation of a and d are provided in the Appendix A.

Once we have collected all the boundary QISM ingredients into the face picture, application of the boundary Bethe ansatz leads to the following theorem.

Theorem 4.2.1 (Filali-Kitanine) *Let* δ, ζ, $\overline{\delta}$ *and* $\overline{\zeta}$ *satisfy the boundary constraints* (4.2.8) *and* (4.2.9) *with total spin s being even if N is even, and odd if N is odd,* $|s| < N$*. Then* $\forall M \in [[1,N]]$*:*

$$|\psi^1_-(\{\lambda_k\}_{k=1,\dots,M})\rangle = S_-(\{\xi\},\delta-\zeta)\prod_{k=1}^{M}\mathcal{B}_-(\lambda_k;\delta-\zeta)|0\rangle, \quad M=\frac{N-s}{2} \tag{4.2.17}$$

$$\textit{and}\quad |\psi^2_-(\{\lambda_k\}_{k=1,\dots,M})\rangle = S_-(\{\xi\},\delta-\zeta)\prod_{k=1}^{M}\mathcal{C}_-(\lambda_k;\delta-\zeta))|\overline{0}\rangle, \quad M=\frac{N+s}{2} \tag{4.2.18}$$

are eigenstates of the transfer matrix (1.3.9) *for any* μ *with eigenvalues* $\Lambda_{1,2}$*, where:*

$$\Lambda_1(\mu,\{\lambda_k\}_{k=1,\dots M},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta,\overline{\delta},\overline{\zeta}) = \frac{(-1)^N}{\sinh(\overline{\zeta}-\mu-\eta)\sinh(\delta+\mu)} \tag{4.2.19}$$
$$\times\{a(\mu)d(-\mu-\eta)\frac{\sinh(2\mu+2\eta)\sinh(\overline{\zeta}-\mu)\sinh(\overline{\delta}+\mu)\sinh(\delta-\mu)}{\sinh(\overline{\delta}-\mu-\eta)\sinh(2\mu+\eta)\prod_{i=1}^M b(\lambda_i-\mu)b(\lambda_i+\mu+\eta)}$$
$$+a(-\mu-\eta)d(\mu)\frac{\sinh(2\mu)\sinh(\overline{\zeta}+\mu+\eta)\sinh(\delta+\mu+\eta)\sinh(\zeta-\mu-\eta)}{\sinh(\zeta+\mu)\sinh(2\mu+\eta)\prod_{i=1}^M b(\lambda_i+\mu)b(\lambda_i-\mu+\eta)}\}$$

and:

$$\Lambda_2(\mu;\{\lambda_i\}_{i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta,\overline{\delta},\overline{\zeta}) \tag{4.2.20}$$
$$= \Lambda_1(\mu;\{\lambda_i\}_{i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\zeta,\delta,\overline{\zeta},\overline{\delta}),$$

if the parameters $\{\lambda_k\}_{k=1,\dots,M}$ *satisfy the Bethe equations* $y_{1,2}$*:*

$$y_{1,2}(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta,\overline{\delta},\overline{\zeta}) \tag{4.2.21}$$
$$= y_{1,2}(-\lambda_k-\eta,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta,\overline{\delta},\overline{\zeta})$$

Here we have:

$$y_1(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta,\overline{\delta},\overline{\zeta}) = a(\lambda_k)d(-\lambda_k-\eta) \tag{4.2.22}$$
$$\times\sinh(\delta-\mu)\sinh(\zeta+\mu)\sinh(\overline{\zeta}-\mu)\sinh(\overline{\delta}+\mu)$$
$$\times\prod_{i=1,i\neq k}^{M}\sinh(\lambda_k+\lambda_i)\sinh(\lambda_k-\lambda_i-\eta)$$

and:

$$y_2(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta,\overline{\delta},\overline{\zeta}) \tag{4.2.23}$$
$$= y_1(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\zeta,\delta,\overline{\zeta},\overline{\delta}),$$

And we use the short hand notation:

$$b(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda+\eta)}, \tag{4.2.24}$$

This theorem follows from usual Bethe ansatz computations. More details on the proof of this theorem are given in the appendix A.

Remark 4.2.6 *The open XXZ model has two sets of Bethe states. A more formal proof of this observation can be achieved using the $T-Q$ approach for integrable models [122].*

Remark 4.2.7 *The eigenvalues and eigenstates explicitly depend on the spin subspace once we solve the constraints* (4.2.8), (4.2.9).

Proposition 4.2.1 *The Bethe construction is symmetric, namely the two sets of states*

$$|\psi_{-}^{1,2}(\{\lambda_k\}_{k=1,\dots,M})\rangle$$

are isomorphic due to the following parity relation ($\delta \leftrightarrow \zeta$):

$$\mathcal{C}_{-}(\lambda;\delta-\zeta) = \Gamma_x \mathcal{B}_{-}(\lambda;\zeta-\delta)\Gamma_x, \tag{4.2.25}$$

where: $\Gamma_x = \prod_{i=1}^{N} \sigma_i^x$.

This symmetry follows from the parity symmetry of the dynamical $\mathcal{R}$-matrix. More details are given in the appendix B.

Proposition 4.2.2 *The Bethe construction is $\mathbb{Z}_2$ invariant due to the following involutions:*

$$\mathcal{B}_{-}(-\lambda-\eta;\delta-\zeta) = -(-1)^N \frac{\sinh(\lambda+\zeta)\sinh(2(\lambda+\eta))\sinh(\lambda+\delta)}{\sinh(2\lambda)\sinh(\lambda-\zeta+\eta)\sinh(\lambda-\delta+\eta)} \mathcal{B}_{-}(\lambda;\delta-\zeta) \tag{4.2.26}$$

and:

$$\mathcal{C}_{-}(-\lambda-\eta;\delta-\zeta) = -(-1)^N \frac{\sinh(\lambda+\delta)\sinh(2(\lambda+\eta))\sinh(\lambda+\zeta)}{\sinh(2\lambda)\sinh(\lambda-\delta+\eta)\sinh(\lambda-\zeta+\eta)} \mathcal{C}_{-}(\lambda;\delta-\zeta) \tag{4.2.27}$$

This proposition follows very naturally from a decomposition of the boundary operators $\mathcal{B}_{-}$ or $\mathcal{C}_{-}$ in terms of bulk operators element of $\mathcal{T}$. More details are given in the appendix B.

Remark 4.2.8 *Note that the boundary parameters $\tau, \bar{\tau}$ are absent from the Theorem* (4.2.1). *Indeed, they disappear from computations since the diagonalization of the vertex K_{-} matrix* (4.1.10). *This is the trace of the hermiticity of the open XXZ hamiltonian* (4.1.1) *which implies the $U(1)$ invariance for the boundary spin operators $\sigma_{1,N}^{x,y}$. So the eigenvalues of the hamiltonian should not depend on the boundary parameterization parameters $\tau, \bar{\tau}$.*

Remark 4.2.9 *At the free fermions point limit* $\eta = \frac{i\pi}{2}$*, the Bethe eigenvalues take the simpler form:*

$$\begin{aligned}
&\Lambda_1(\mu, \{\lambda_k\}_{k=1,\dots M}, \{\xi_j\}_{j=1,\dots,N}; \delta, \zeta, \overline{\delta}, \overline{\zeta}) \qquad (4.2.28)\\
&= (-1)^{M+1}\tanh(2\mu)\frac{\tanh(\overline{\zeta}-\mu)}{\tanh(\delta+\mu)}\prod_{i=1}^{M}\frac{\tanh(\mu+\lambda_i)}{\tanh(\mu-\lambda_i)}\\
&\times\prod_{i=1}^{N}\cosh(\mu+\xi_i)\cosh(\mu-\xi_i)\\
&\times\{\frac{\sinh(\delta-\mu)\sinh(\delta-\zeta-\eta s+\overline{\zeta}+\mu)}{\cosh(\delta+\mu)\sinh(\delta-\zeta-\eta s+\overline{\zeta}-\mu)}\\
&+\frac{\cosh(\overline{\zeta}+\mu)\cosh(\zeta-\mu)}{\sinh(\overline{\zeta}-\mu)\sinh(\zeta+\mu)}\prod_{i=1}^{N}\tanh(\mu+\xi_i)\tanh(\mu-\xi_i)\}
\end{aligned}$$

and the associated Bethe equation factorize into:

$$\begin{aligned}
&(-1)^M\prod_{j=1}^{N}\tanh(\lambda_i+\xi_j)\tanh(\lambda_i-\xi_j) \qquad (4.2.29)\\
&=\frac{\sinh(\zeta+\lambda_i)\sinh(\delta-\lambda_i)\sinh(\overline{\zeta}-\lambda_i)\sinh(\delta-\zeta-\eta s+\overline{\zeta}+\lambda_i)}{\sinh(\zeta-\lambda_i)\sinh(\delta+\lambda_i)\sinh(\overline{\zeta}+\lambda_i)\sinh(\delta-\zeta-\eta s+\overline{\zeta}-\lambda_i)}
\end{aligned}$$

4.3 Dual vertex-face correspondence: towards an equivalent model description

The boundary version of the QISM should be completely symmetric using the boundary monodromy matrix U_- around the K_- boundary or the dual boundary monodromy matrix U_+ around the K_+ boundary. As shown by Kitanine and his collaborators [69], it is also necessary to implement the Bethe ansatz construction using the dual boundary matrix U_+ for the model scalar product computation. To apply the Bethe ansatz technique with the use of U_+ we need to introduce a second vertex-face transformation for the construction of the boundary dynamical monodromy matrix (it will be necessary to construct a dynamical analog of the U_+ boundary monodromy matrix). As for the vertex case, dual elements and corresponding algebras in dynamical context are closely related to the antipode of the corresponding dynamical quantum group. Due to the complicated form of the crossing symmetries of the dynamical $\mathcal{R}$ matrix (see Appendix B), is not represented by the simple matrix transposition or inversion. To introduce dual dynamical object, we will need the *dual vertex-face* transformation:

$$\widetilde{S}(\lambda;\overline{\theta}) = \sigma^y S(\lambda;\overline{\theta})\sigma^y. \tag{4.3.1}$$

This transformation was found using various crossing symmetries of the R and $\mathcal{R}$ matrices. It has two important properties:

- It diagonalizes the boundary matrix K_+:

$$\begin{aligned}\mathcal{K}_+^t(\lambda) &= \widetilde{S}^{-1}(\lambda+\eta;\overline{\delta}-\overline{\zeta},\overline{\tau})K_+^t(\lambda)\widetilde{S}(-\lambda-\eta;\overline{\delta}-\overline{\zeta},\overline{\tau}) \\ &= \mathcal{K}_-^t(-\lambda-\eta,\overline{\delta},\overline{\zeta})\end{aligned} \tag{4.3.2}$$

  This diagonal structure is also the trace of a weight zero symmetry:

$$[\sigma_0^z,(\mathcal{K}_+)_0(\lambda)] = 0. \tag{4.3.3}$$

- It is a dual vertex-face transformation:

$$\begin{aligned}S_2(\lambda_2;\overline{\theta})\widetilde{S}_1(\lambda_1+\eta;\overline{\theta}+\eta\sigma_2^z)\mathcal{L}_{12}^{t_1}(\lambda_1-\lambda_2;\overline{\theta}) \\ = R_{12}^{t_1}(\lambda_1-\lambda_2)\widetilde{S}_1(\lambda_1+\eta;\overline{\theta})S_2(\lambda_2,\overline{\theta}-\eta\sigma_1^z),\end{aligned} \tag{4.3.4}$$

  where the "crossed" $\mathcal{L}$-operator will be used to construct the dynamical analog of the boundary monodromy matrix U_+:

$$\mathcal{L}_{12}^{t_1}(\lambda;\overline{\theta}) = \mathcal{R}_{12}^{t_1}(\lambda;\overline{\theta}+\eta\sigma_1^z)\frac{\sinh(\overline{\theta}-\eta\sigma_2^z)}{\sinh\overline{\theta}}. \tag{4.3.5}$$

  The crossed $\mathcal{L}$-operator also obeys a *transposed zero-weight* symmetry:

$$[\sigma_1^z-\sigma_2^z,\mathcal{L}_{12}^{t_1}(\lambda;\overline{\theta})] = 0. \tag{4.3.6}$$

  At this point we succeed in finding a dynamical analog to the transposed vertex R^t-matrix, which has nilpotent off-diagonal elements.

Once again, we will systematically translate all ingredients of the dual picture of the boundary QISM (R^t,U_+,$\mathbf{T}$) and associated relations into the dynamical counterpart to apply the Bethe ansatz machinery. Recall that the boundary QISM around K_+ need to work on the transposed monodromy matrix T^t (1.3.7).

- Bulk part (six-vertex R^t-matrix and transposed monodromy matrix T^t):
The dual vertex-face transformation (4.3.4) leads to the following dynamical Yang-Baxter equation for $\mathcal{L}^t$:

$$\mathcal{R}_{12}(\lambda_2-\lambda_1;\overline{\theta}+\eta\sigma_3^z)\mathcal{L}_{13}^{t_1}(\lambda_1-\lambda_3;\overline{\theta})\mathcal{L}_{23}^{t_2}(\lambda_2-\lambda_3;\overline{\theta}-\eta\sigma_1^z) \\ =\mathcal{L}_{23}^{t_2}(\lambda_2-\lambda_3;\overline{\theta})\mathcal{L}_{13}^{t_1}(\lambda_1-\lambda_3;\overline{\theta}-\eta\sigma_1^z)\mathcal{R}_{12}(\lambda_2-\lambda_1;\overline{\theta}). \quad (4.3.7)$$

Using the higher dimensional vertex-face transformation:

$$S_+(\{\xi\};\overline{\theta})=\prod_{i=1}^{N} S_i(\xi_i;\overline{\theta}+\eta\sum_{k=i+1}^{N}\sigma_k^z), \quad (4.3.8)$$

one can obtain a *dual dynamical monodromy matrix*:

$$\mathcal{V}_0^{t_0}(\lambda;\overline{\theta})=\prod_{i=N}^{1}\mathcal{L}_{0i}^{t_0}(\lambda-\xi_i;\overline{\theta}+\eta\sum_{i=1}^{i-1}\sigma_i^z), \quad (4.3.9)$$

The element $\mathcal{V}^t$ is then the dynamical analog of the transposed monodromy matrix T^t. The Yang-Baxter relation for this matrix can be written in the following form:

$$\mathcal{R}_{12}(\lambda_2-\lambda_1;\overline{\theta}+\eta\mathbf{S}^z)\mathcal{V}_1^{t_1}(\lambda_1;\overline{\theta})\mathcal{V}_2^{t_2}(\lambda_2;\overline{\theta}-\eta\sigma_1^z) \\ =\mathcal{V}_2^{t_2}(\lambda_2;\overline{\theta})\mathcal{V}_1^{t_1}(\lambda_1;\overline{\theta}-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_2-\lambda_1;\overline{\theta}). \quad (4.3.10)$$

It also obeys a transposed weight zero symmetry:

$$[\mathcal{V}_0^{t_0}(\lambda;\overline{\theta}),\sigma_0^z-\mathbf{S}^z]=0. \quad (4.3.11)$$

- Boundary matrix K_+ and the reflection equation (1.3.5):
Using transformation (4.3.2), and the weight zero condition (3.1.15), (4.3.3), one can show that the dual reflection equation (1.3.2) for K_+ turns into a *dual dynamical reflection equation* for $\mathcal{K}_+$, where $\overline{\theta}=\overline{\delta}-\overline{\zeta}$ as specified by (4.3.2):

$$\mathcal{R}_{12}(\lambda_2-\lambda_1;\overline{\theta})(\mathcal{K}_+^{t_1})_1(\lambda_1)\mathcal{R}_{21}(-\lambda_1-\lambda_2-2\eta;\overline{\theta})(\mathcal{K}_+^{t_2})_2(\lambda_2) \\ =(\mathcal{K}_+^{t_2})_2(\lambda_2)\mathcal{R}_{12}(-\lambda_1-\lambda_2-2\eta;\overline{\theta})(\mathcal{K}_+^{t_1})_1(\lambda_1)\mathcal{R}_{21}(\lambda_2-\lambda_1;\overline{\theta}). \quad (4.3.12)$$

Thus we succeed in obtaining a dual reflection equation for dual face type boundary matrix.

- Boundary monodromy matrix U_+ and dual reflection algebra (1.3.2):
  In a similar way, one can prove that the dual dynamical boundary monodromy matrix defined as:

$$(\mathcal{U}_+)_0^{t_0}(\lambda;\overline{\theta}) = \widetilde{\gamma}(\lambda)\mathcal{V}_0^t(\lambda;\overline{\theta})(\mathcal{K}_+^{t_0})_0(\lambda)(\mathcal{V}_0^{t_0})^{-1}(-\lambda-2\eta;\overline{\theta}) \tag{4.3.13}$$
$$= \begin{pmatrix} \mathcal{A}_+(\lambda;\overline{\theta}) & \mathcal{C}_+(\lambda;\overline{\theta}) \\ \mathcal{B}_+(\lambda;\overline{\theta}) & \mathcal{D}_+(\lambda;\overline{\theta}) \end{pmatrix}_{[0]},$$

  satisfies the *dual dynamical reflection algebra* relation:

$$\mathcal{R}_{12}(\lambda_2-\lambda_1;\overline{\theta}+\eta\,\mathbf{S}^z)(\mathcal{U}_+^{t_1})_1(\lambda_1;\overline{\theta})\mathcal{R}_{21}(-\lambda_1-\lambda_2-2\eta;\overline{\theta}+\eta\,\mathbf{S}^z))(\mathcal{U}_+^{t_2})_2(\lambda_2;\overline{\theta})$$
$$= (\mathcal{U}_+^{t_2})_2(\lambda_2;\overline{\theta})\mathcal{R}_{12}(-\lambda_1-\lambda_2-2\eta;\overline{\theta}+\eta\,\mathbf{S}^z)(\mathcal{U}_+^{t_1})_1(\lambda_1;\overline{\theta})\mathcal{R}_{21}(\lambda_2-\lambda_1;\overline{\theta}+\eta\,\mathbf{S}^z). \tag{4.3.14}$$

  Remark 4.3.1 $\widetilde{\gamma}(\lambda) = (-1)^N\prod_{i=1}^N \sinh(\lambda+\xi_i)\sinh(\lambda+\xi_i+2\eta)$ *is a normalization factor that we introduce here for convenience.*

- Commuting charges **T** and trace formula:
  Again, it is possible to write the trace in terms of the operator entries of the boundary monodromy matrix the $\mathcal{U}_+$. It is easy to check that if we consider the action of this trace on the states with z component of total spin s, and if the constraints (4.2.10) are satisfied, the non-diagonal terms in this expressions vanish. The trace formula (1.3.9) reads:

$$\mathbf{T}(\lambda)S_+(\{\xi\};\overline{\theta})|\psi\rangle = S_+(\{\xi\};\overline{\theta}) \tag{4.3.15}$$
$$\times\, Tr_0\left(\mathcal{U}_+^{t_0}(\lambda;\overline{\theta})\frac{\sinh(\delta-\zeta-\eta\sigma_0^z)}{\sinh(\delta-\zeta)}\mathcal{K}_-^{t_0}(\lambda;\delta,\zeta)\right)|\psi\rangle,$$

Theorem 4.3.1 (Filali-Kitanine) *Let* δ, ζ, $\overline{\delta}$ *and* $\overline{\zeta}$ *satisfy the boundary constraints* (4.2.8), *and* (4.2.9), *with total spin s being even if N is even, and odd if N is odd,* $|s|<N$. *Then* $\forall M\in[[1,N]]$*:*

$$|\psi_+^1(\{\lambda_k\}_{k=1,\ldots,M})\rangle = S_+(\{\xi\},\overline{\delta}-\overline{\zeta})\prod_{k=1}^M \mathcal{B}_+(\lambda_k;\overline{\delta}-\overline{\zeta})|0\rangle,\quad M=\frac{N-s}{2} \tag{4.3.16}$$

$$\textit{and}\quad |\psi_+^2(\{\lambda_k\}_{k=1,\ldots,M})\rangle = S_+(\{\xi\},\overline{\delta}-\overline{\zeta})\prod_{k=1}^M \mathcal{C}_+(\lambda_k;\overline{\delta}-\overline{\zeta}))|\overline{0}\rangle,\quad M=\frac{N+s}{2} \tag{4.3.17}$$

are eigenstates of the transfer matrix (1.3.9) *for any* μ *with eigenvalues*

$$\Lambda_{2,1}(\mu,\{\lambda_k\}_{k=1,\dots M},\{\xi_j\}_{j=1,\dots,N};\overline{\delta},\overline{\zeta},\delta,\zeta),$$

if the parameters $\{\lambda_k\}_{k=1,\dots,M}$ *satisfy the Bethe equations:*

$$\begin{aligned} & y_{2,1}(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\overline{\delta},\overline{\zeta},\delta,\zeta) \\ &\qquad = y_{2,1}(-\lambda_k-\eta,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\overline{\delta},\overline{\zeta},\delta,\zeta). \end{aligned} \tag{4.3.18}$$

4.4 Partition function of the trigonometric face model with reflecting end and DWBC

In the previous section, we have seen that the vertex boundary matrix $K_{\pm}$ describes a vertex boundary configuration without charge conservation. The vertex-face transformation enables us to work on the boundary by mapping it into a face type diagonal boundary. This reflects the weight zero symmetry which has a clear statistical mechanics interpretation: it is a symmetry rule for face statistical configuration. The whole face model is described by the representation $\mathcal{U}_-$ (4.2.2) of the dynamical reflection algebra (4.2.2). It is a trigonometric face model with one reflecting end (in the same way as usual reflection algebra with diagonal matrix K describes a six-vertex model with reflecting end (2.3.1)), as illustrated in Figure 4.2. This model is the dynamical or face counterpart of Tsuchiya's boundary vertex model of Chapter 2. The model is described in full in the more general (elliptic) case in the next chapter.

The algebra (4.2.2) makes this model exactly solvable, namely we can compute exactly its partition function. This partition function is (recall that $\theta=\delta-\zeta$):

$$\begin{aligned} & Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta) \\ &= \langle 0|_\lambda\langle\overline{0}|_\xi \prod_{i=1}^{N}\{\prod_{j=1}^{N}\{\mathcal{R}_{ij}(\lambda_i-\xi_j;\theta-\eta\sum_{k=j+1}^{N}\sigma^z_k)\}\mathcal{K}_-(\lambda_i;\theta)_i\prod_{j=N}^{1}\{\mathcal{R}_{ji}(\lambda_i+\xi_j;\theta-\eta\sum_{k=j+1}^{N}\sigma^z_k)\}\}|\overline{0}\rangle_\lambda|0\rangle_\xi \\ &= \langle 0|_\lambda\langle\overline{0}|_\xi\prod_{i=1}^{N}\mathcal{U}_-(\lambda_i;\theta)|0\rangle_\xi|\overline{0}\rangle_\lambda \\ &= \langle\overline{0}|_\xi\prod_{i=1}^{N}\mathcal{B}_-(\lambda_i;\theta)|0\rangle_\xi \end{aligned} \tag{4.4.1}$$

In this model, $\lambda_{i,i=1,\dots,N}$ refers to the horizontal lines starting from the bottom, while $\xi_{j,j=1,\dots,N}$ refers to the horizontal lines starting from the right.

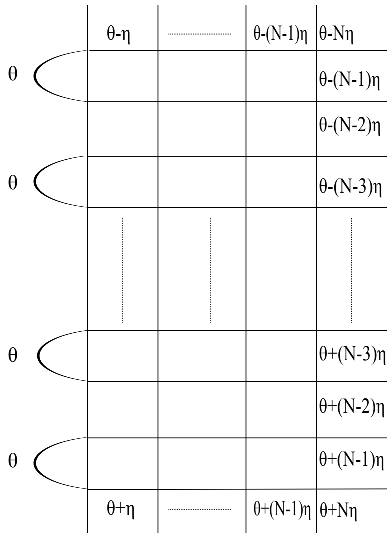


Figure 4.2: The face model with reflecting end and DWBC

Proposition 4.4.1 (Filali-Kitanine) *The partition function of the trigonometric face model with reflecting end and DWBC* (4.4.1) *satisfies the following property:*

i) Initial condition
For $N = 1$ the partition function is :

$$Z^{Bface}_{1,2}(\lambda_1,\xi_1;\delta,\zeta) = \frac{\sinh\eta\sinh(\delta-\zeta-\eta)}{\sinh^2(\delta-\zeta)} \times\left(\frac{\sinh(\delta-\lambda_1)}{\sinh(\delta+\lambda_1)}\sinh(\lambda_1-\xi_1)\sinh(\delta-\zeta+\lambda_1+\xi_1) + \frac{\sinh(\zeta-\lambda_1)}{\sinh(\zeta+\lambda_1)}\sinh(\lambda_1+\xi)\sinh(\delta-\zeta-\lambda_1+\xi_1)\right).$$

ii) Symmetry
$Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta)$ *is a symmetric function in the* $\{\lambda_i\}_{i=1,..,N}$ *and the* $\{\xi_j\}_{j=1,\dots,N}$.

iii) Polynomiality of the normalized partition function $\tilde{Z}$

$$\begin{aligned}\tilde{Z}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta) &= \exp\left((2N+2)\sum_{i=1}^{N}\lambda_i\right)\\ &\times \sinh(\delta+\lambda_i)\sinh(\zeta+\lambda_i)\\ &\times Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta)\\ &= P_{2N+2}(e^{2\lambda_i})\end{aligned}$$

where P_{2N+2} is a polynomial of degree $2N+2$.

iv) Recursive relations

$$\begin{aligned}Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta)\Big|_{\lambda_i=\xi_j} &= \frac{\sinh\eta\sinh(\zeta-\lambda_i)}{\sinh(\zeta+\lambda_i)}\\ \times\prod_{k=1}^{N}\sinh(\lambda_k+\xi_i)\frac{\sinh(\delta-\zeta+(N-2i)\eta)}{\sinh(\delta-\zeta+(N-2i+1)\eta)}&\\ \times\prod_{m=1,m\neq i}^{N}\sinh(\lambda_i-\xi_m+\eta)\sinh(\lambda_i+\xi_m+\eta)\sinh(\lambda_m-\xi_j+\eta)&\\ \times Z^{Bface}_{N-1,2(N-1)}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j};\delta,\zeta)&\end{aligned}$$

and:

$$\begin{aligned}Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta)\Big|_{\lambda_i=-\xi_j} &= \frac{\sinh\eta\sinh(\delta-\lambda_i)}{\sinh(\delta+\lambda_i)}\\ \times\prod_{k=1}^{N}\sinh(\lambda_k-\xi_j)\frac{\sinh(\delta-\zeta+(N-2i)\eta)}{\sinh(\delta-\zeta+(N-2i+1)\eta)}&\\ \times\prod_{m=1}^{N}\sinh(\lambda_i+\xi_m+\eta)\sinh(\lambda_i-\xi_m+\eta)\sinh(\lambda_{m-1}+\xi_j+\eta)&\\ \times Z^{Bface}_{N-1,2(N-1)}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j};\delta,\zeta)&\end{aligned}$$

v) Crossing symmetry

$$Z^{Bface}_{N,2N}(-\lambda_i-\eta,\{\lambda_m\}_{m\neq i,m=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta) = -\frac{\sinh(2(\lambda_i+\eta))\sinh(\lambda_i+\zeta)}{\sinh(2\lambda_i)\sinh(\lambda_i-\zeta+\eta)} \times\frac{\sinh(\lambda_i+\delta)}{\sinh(\lambda_i-\delta+\eta)}Z^{Bface}_{N.2N}(\lambda_i,\{\lambda_m\}_{m\neq i,m=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta).$$

Lemma 4.4.1 *The set of conditions i)-iv) uniquely determines the partition function $Z^{Bface}_{N,2N}$.*

From the trigonometric form of the $\mathcal{R}$-matrix ones can easily show the condition (iii), that the function $\tilde{Z}$ is a polynomial of degree at most $2N+2$ in each parameter $\exp^{2\lambda_i}, i = 1,\dots,N$. defined at $4N$ points. Due to the symmetries (ii), the recursion relations (iv) can be established for any points $\lambda_i = \pm\xi_j$, for $i, j = 1,\dots,N$. Due to the crossing symmetry (v), similar recursion relations can be established at the points $\lambda_i = \mp\xi_j-\eta$. Thus the normalized partition function $\tilde{Z}$ is defined at $4N$ different points. Hence we can prove by induction starting from the case $N = 2$ that the partition function is uniquely determined. This means that if we find a function satisfying the above conditions it is the partition function;

Theorem 4.4.1 (Filali-Kitanine) *The partition function of the trigonometric face model with a reflecting end and DWBC is:*

$$Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\delta,\zeta) \tag{4.4.2}$$

$$= (-1)^N\prod_{i=1}^{N}\left(\frac{\sinh(\delta-\zeta+\eta(N-2i))}{\sinh(\delta-\zeta+\eta(N-i))}\right)\det\mathcal{N}^{Bface}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,..,N})$$
$$\times\frac{\prod_{i,j=1}^{N}\sinh(\lambda_i+\xi_j)\sinh(\lambda_i-\xi_j)\sinh(\lambda_i+\xi_j+\eta)\sinh(\lambda_i-\xi_j+\eta)}{\prod_{1\le i<j\le N}\sinh(\xi_j+\xi_i)\sinh(\xi_j-\xi_i)\sinh(\lambda_j-\lambda_i)\sinh(\lambda_j+\lambda_i+\eta)}, \tag{4.4.3}$$

where the $N\times N$ matrix $\mathcal{N}^{Bface}$ can be expressed as:

$$\mathcal{N}^{Bface}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,..,N})_{\alpha,\beta} = \frac{\sinh(\delta+\xi_j)}{\sinh(\delta+\lambda_i)}\cdot\frac{\sinh(\zeta-\xi_j)}{\sinh(\zeta+\lambda_i)} \times\frac{\sinh(2\lambda_i)\sinh\eta}{\sinh(\lambda_i-\xi_j+\eta)\sinh(\lambda_i+\xi_j+\eta)\sinh(\lambda_i-\xi_j)\sinh(\lambda_i+\xi_j)}. \tag{4.4.4}$$

To prove the theorem, it is sufficient to check the properties (i) to (v).

This result is very important for the computation of the spin chains Bethe states scalar product and to further analysis regarding correlation functions of the XXZ model with general boundaries. The result is surprising for many reasons. First of all, it is very similar to the result for the partition function of Tsuchiya's vertex model (2.3.2). The dynamical nature of the model arise only as generic, rather simple, counting factor. Furthermore, this partition function takes the form of a single matrix determinant, which is not the case of the partition function for the face model on a square lattice. This means that adding boundaries to face models permits, in some sense, one to avoid inherent difficulties of handling dynamical objects. This result is not restricted to trigonometric face model, and we will generalize this result to the most general elliptic case in the next chapter. Another interesting feature of this partition function is its limit at the point $\eta = \frac{i\pi}{2}$, which is the XXZ chain's free fermions point.

Lemma 4.4.2 *The partition function at the point* $\eta = \frac{i\pi}{2}$ *takes the form of a Cauchy determinant:*

$$Z_{N,2N}^{Bface}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N},\eta = \frac{i\pi}{2}) \tag{4.4.5}$$

$$= (-8)^N \frac{\prod_{i=1}^N \prod_{j=1}^N \sinh(\lambda_i+\xi_j)\sinh(\lambda_i-\xi_j)\cosh(\lambda_i+\xi_j)\cosh(\lambda_i-\xi_j)}{\prod_{1\leq i<j\leq N}\sinh(\xi_j+\xi_i)\sinh(\xi_j-\xi_i)\sinh(\lambda_j-\lambda_i)\cosh(\lambda_j+\lambda_i)}$$

$$\times \tanh^{(-1)^N}(\delta-\zeta)\prod_{i=1}^N \frac{\sinh(2\lambda_i)\sinh(\delta-\zeta+\xi_i)\sinh(\zeta-\xi_i)}{\sinh(\lambda_i+\delta-\zeta)\sinh(\lambda_i+\zeta)}$$

$$\times \prod_{1\leq i<j\leq N} \frac{(\cosh(4\lambda_i)-\cosh(4\lambda_j))(\cosh(4\xi_j)-\cosh(4\xi_i))}{\cosh(4\lambda_i)-\cosh(4\xi_j)}$$

Chapter 5

Elliptic dynamical reflection algebras

The dynamical Yang-Baxter equation was first introduced by Gervais and Neveu [52] in their analysis of the Toda field theory. Later on, Babelon uses the same equation in his analysis of the Liouville theory [3] while Felder introduced it in the context of the quantization of the Knizhnik-Zamolodchikov-Bernard equation of conformal field theory on elliptic curves [38]. Afterward this equation appeared then in a variety of models such as the quantum Calogero-Moser model [1] or the relativistic Ruijsnaard-Schneider model [40]. The previous chapter also highlighted the relevance of face type models in statistical mechanics and associated dynamical Yang-Baxter algebra for spin chain analysis if the hamiltonian does not conserve the third component of the total spin, and as we have seen it is the underlying equation of the integrability of face type models. Unexpectedly, it also found applications in the field of combinatorics. Indeed, face type models, and hence description by the dynamical Yang-Baxter equation, are also related to dynamical enumeration of alternating sign matrices [100, 101].

Therefore, the construction of equivalent quantum group structure around solutions of the dynamical Yang-Baxter equation in the RLL framework became a real task for understanding the integrable structure of dynamical and elliptic models, and thus interest in dynamical Yang-Baxter algebra.

Although a quasi-Hopf interpretation of the dynamical Yang-Baxter equation has been discovered [4], the co-algebraic properties of dynamical Yang-Baxter algebra are not fully understood. From an algebraic point of view, they are intensively studied due to their relations to others quasi-hopf structures [29] and current algebras [63, 93], and the theory of representation of group, especially in the form of hypergeometric series [102].

The Sklyanin scheme for boundary integrable Yang-Baxter type models requires two reflection dual algebras. These are built-in as co-module over the Yang-Baxter algebra, and they lead to a commutative generating function family. This family is constructed as a suitable product between both algebra elements. A boundary integrable model is composed of a bulk scattering process whose integrability is insured by the Yang-Baxter algebra; with a reflection at the boundary whose integrable structure is given by the reflection algebra. The co-module construction insures the compatibility of both integrable process. A dynamical analog of this appears in the analysis of the non-diagonal boundary XXZ spin chain through the vertex-face transformation of Chapter 4. In this chapter, we would like to adopt a more generic framework for this task. Starting with the dynamical Yang-Baxter equation and its elliptic solution, we introduce the corresponding dynamical Yang-Baxter algebra, the Felder's elliptic quantum group $E_{\tau,\eta}(sl_2)$ [38, 39, 41]. Upon this algebra we built in by co-module construction a dynamical reflection algebra together with its associated dual and corresponding trace like generating function. The co-module evaluation representations of such algebras are introduced, which contain the boundary monodromy matrix which appears in the boundary spin chain analysis. This enables us to apply the Bethe ansatz technique in order to find the Bethe spectrum of the underlying physics model.

Our work is not the first attempt at developing a dynamical reflection algebra description. We should mention the work of Nagy-Avan-Rollet [87–89] on dynamical quadratic reflection algebra (and the particular case of their work [37]), where related, but different results can be found.

We believe the present work is more suitable for face model description. Unexpectedly, this algebraic construction enables us to show that underlying face models with reflecting ends are exactly solvable and we will compute exactly their partition functions. Quite surprisingly, they take the form of a single determinant, which is not the case of dynamical Yang-Baxter algebra face models on a square lattice.

Note that all the results presented in this section are valid in the trigonometric limit.

5.1 The elliptic quantum group $E_{\tau,\eta}(sl_2)$

The Felder's elliptic quantum group $E_{\tau,\eta}(sl_2)$ can be understood as the dynamical counterpart of the Yang-Baxter algebras. The main object for defining the elliptic quantum group $E_{\tau,\eta}(sl_2)$ is the dynamical $\mathcal{R}$-matrix (3.1.11), which satisfies the dynamical Yang-Baxter equation (3.2.1):

$$\begin{aligned}\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta-\eta\sigma_3^z)&\mathcal{R}_{13}(\lambda_1-\lambda_3;\theta)\mathcal{R}_{23}(\lambda_2-\lambda_3;\theta-\eta\sigma_1^z)\\ &=\mathcal{R}_{23}(\lambda_2-\lambda_3;\theta)\mathcal{R}_{13}(\lambda_1-\lambda_3;\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta).\end{aligned}$$

Unlike the Yang-Baxter equation, this equation is not algebraic but rather it is a difference equation, due to the shift in the auxiliary space. The $\mathcal{R}$-matrix possesses a fundamental weight zero symmetry:

$$[\mathcal{R}_{12}(\lambda;\theta),\sigma_1^z+\sigma_2^z]=0 \tag{5.1.1}$$

The elliptic $E_{\tau,\eta}(sl_2)$ quantum group is the algebra generated by meromorphic functions of the generator of $\mathfrak{h}$, the Cartan subalgebra of sl_2, that we denote as σ^z , and the matrix elements of $\mathcal{T}(\lambda;\theta)=\begin{pmatrix}\mathcal{A}(\lambda;\theta) & \mathcal{B}(\lambda;\theta)\\ \mathcal{C}(\lambda;\theta) & \mathcal{D}(\lambda;\theta)\end{pmatrix}\in \mathrm{End}(\mathbb{C}^2)$, with non-commutative entries, satisfying the dynamical Yang-Baxter algebra relations (3.2.7):

$$\begin{aligned}&\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta-\eta\sigma^z)\mathcal{T}_1(\lambda_1;\theta)\mathcal{T}_2(\lambda_2;\theta-\eta\sigma_1^z)\\ &\qquad=\mathcal{T}_2(\lambda_2;\theta)\mathcal{T}_1(\lambda_1;\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta).\end{aligned}$$

In this equation, σ^z is an non-evaluated abstract element.

Remark 5.1.1 *These commutation relations are also of difference type due to the shift in the auxiliary space.*

Remark 5.1.2 *The dynamical Yang-Baxter equation* (3.2.1) *is a consistency condition for the product associativity of $E_{\tau,\eta}(sl_2)$.*

We are only interested here in diagonalizable $\mathfrak{h}$-module V where the *weight zero* property holds:

$$[\mathcal{T}_0(\lambda;\theta),\sigma_0^z+\sigma_V^z]=0. \tag{5.1.2}$$

In this thesis, we are mostly interested in a particular representation of $E_{\tau,\eta}(sl_2)$, the well-known *evaluation representation* in the space $V=\otimes_{i=1}^N\mathbb{C}_i^2$. It is constructed from the dynamical $\mathcal{R}$-matrix (3.1.11):

$$\mathcal{T}_0(\lambda;\theta)=\prod_{i=1}^N \mathcal{R}_{0i}(\lambda-\xi_i;\theta-\eta\sum_{k=i+1}^N\sigma_i^z)=\begin{pmatrix}\mathcal{A}(\lambda;\theta) & \mathcal{B}(\lambda;\theta)\\ \mathcal{C}(\lambda;\theta) & \mathcal{D}(\lambda;\theta)\end{pmatrix} \tag{5.1.3}$$

This is precisely the representation that give rises to the elliptic face model on the square lattice which is related to periodic XYZ spin chain through the vertex-face transformation (3.1.10) and (3.2.3).

Note that another dual evaluation representation of $E_{\tau,\eta}(sl_2)$ in the space $V=\otimes_{i=1}^N\mathbb{C}_i^2$ is constructed from the *crossed Lax matrix*:

$$\mathcal{L}_{12}^{t_1}(\lambda;\theta)=\mathcal{R}_{12}^{t_1}(\lambda;\theta+\eta\sigma_1^z)\frac{h(\theta-\eta\sigma_2^z)}{h(\theta)}, \tag{5.1.4}$$

This new $\mathcal{L}$-operator also satisfies the dynamical Yang-Baxter equation:

$$\begin{aligned}\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta+\eta\sigma_3^z)\mathcal{L}_{13}^{t_1}(\lambda_3-\lambda_1;\theta)\mathcal{L}_{23}^{t_2}(\lambda_3-\lambda_2;\theta-\eta\sigma_1^z)\\ =\mathcal{L}_{23}^{t_2}(\lambda_3-\lambda_1;\theta)\mathcal{L}_{13}^{t_1}(\lambda_3-\lambda_1;\theta-\eta\sigma_1^z)\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta).\end{aligned} \tag{5.1.5}$$

Remark 5.1.3 *The introduction of this operator was initially motivated by the crossing symmetry of the dynamical $\mathcal{R}$-matrix:*

$$-\sigma_1^y\mathcal{R}_{12}^{t_1}(-\lambda-\eta;\theta+\eta\sigma_1^z)\sigma_1^y\frac{h(\theta-\eta\sigma_2^z)}{h(\theta)}=\mathcal{R}_{21}(\lambda;\theta), \tag{5.1.6}$$

Remark 5.1.4 *This operation is the right way to transpose dynamical objects. It is a way to reconcile matrix transposition operations to anti-homomorphisms of $E_{\tau,\eta}$.*

Remark 5.1.5 $\mathcal{L}_{12}^{t_1}(\lambda;\theta)$ *possesses a transposed weight zero symmetry:*

$$[\mathcal{L}_{12}^{t_1}(\lambda;\theta),\sigma_1^z-\sigma_2^z]=0. \tag{5.1.7}$$

Up to a central factor element, this representation is the corresponding *antipode* of $E_{\tau,\eta}(sl_2)$ and is represented as:

$$\mathcal{V}_0^{t_0}(\lambda;\theta)=\prod_{i=N}^{1}\mathcal{L}_{0i}^{t_0}(\lambda-\xi_i;\theta+\eta\sum_{k=1}^{i-1}\sigma_i^z). \tag{5.1.8}$$

This representation obeys to the transposed zero weight condition:

$$[\mathcal{V}_{12}^{t_1}(\lambda;\theta),\sigma_1^z-\sigma_2^z]=0. \tag{5.1.9}$$

The Yang-Baxter relation for this matrix can be written in the following form:

$$\begin{aligned}\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta+\eta\,\mathbf{S}^z)\mathcal{V}_1^{t_1}(-\lambda_1;\theta)\mathcal{V}_2^{t_2}(-\lambda_2;\theta-\eta\sigma_1^z)\\ =\mathcal{V}_2^{t_2}(-\lambda_2;\theta)\mathcal{V}_1^{t_1}(-\lambda_1;\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta).\end{aligned} \tag{5.1.10}$$

5.2 Elliptic dynamical reflection algebra

5.2.1 The algebra, its dual and the transfer matrix

In this section, we introduce an elliptic dynamical reflection algebra built in as co-module on $E_{\tau,\eta}(sl_2)$. The aim is to find a dynamical analog of Sklyanin algebraic framework for boundary models that are described by a dynamical integrable structure, rather than usual Yang-Baxter structure. We define $B_-(\mathcal{R}(\lambda;\theta))$ as the following dynamical reflection algebra generated by meromorphic functions of $\sigma^z \in \mathfrak{h}$ and the matrix element of $\mathcal{U}_-(\lambda;\theta) = \begin{pmatrix} \mathcal{A}_-(\lambda;\theta) & \mathcal{B}_-(\lambda;\theta) \\ \mathcal{C}_-(\lambda;\theta) & \mathcal{D}_-(\lambda;\theta) \end{pmatrix} \in End(\mathbb{C}^2)$ with non commutative entries subject to the relations:

$$\begin{aligned} &\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta-\eta\sigma^z)(\mathcal{U}_-)_1(\lambda_1;\theta)\mathcal{R}_{21}(\lambda_1+\lambda_2;\theta-\eta\sigma^z)(\mathcal{U}_-)_2(\lambda_2;\theta) \\ &= (\mathcal{U}_-)_2(\lambda_2;\theta)\mathcal{R}_{12}(\lambda_1+\lambda_2;\theta-\eta\sigma^z)(\mathcal{U}_-)_1(\lambda_1;\theta)\mathcal{R}_{21}(\lambda_1-\lambda_2;\theta-\eta\sigma^z). \end{aligned} \tag{5.2.1}$$

This algebra is a *minimal* dynamical generalization of the vertex type reflection algebra (1.3.1), as commutation relations hold for algebra elements evaluated at the same θ (there are no shifts in the auxiliary spaces). The only modification regarding the Sklyanin reflection algebra (1.3.1) is the structure constants which are now functions of θ and become non-evaluated operators.

Remark 5.2.1 *In this case, the commutation relations are algebraic equations rather than difference equations.*

We associate this algebra with its dual; the dual dynamical reflection algebra $B_+(\mathcal{R}(\lambda;\theta))$, which is defined in the same way. The relations for non-commutative elements $(\mathcal{U}_+)_{\alpha,\beta}$ takes the form:

$$\begin{aligned} &\mathcal{R}_{12}(\lambda_2-\lambda_1;\theta+\eta\sigma^z)(\mathcal{U}_+^{t_1})_1(\lambda_1;\theta)\mathcal{R}_{21}(-\lambda_1-\lambda_2-2\eta;\theta+\eta\sigma^z)(\mathcal{U}_+^{t_2})_2(\lambda_2;\theta) \\ &= (\mathcal{U}_+^{t_2})_2(\lambda_2;\theta)\mathcal{R}_{12}(-\lambda_1-\lambda_2-2\eta;\theta+\eta\sigma^z)(\mathcal{U}_+^{t_1})_1(\lambda_1;\theta)\mathcal{R}_{21}(\lambda_2-\lambda_1;\theta+\eta\sigma^z). \end{aligned} \tag{5.2.2}$$

Theorem 5.2.1 *The algebras $B_-(\mathcal{R}(\lambda;\theta))$ and $B_+(\mathcal{R}(\lambda;\theta))$ are isomorphic. An explicit isomorphism is given by:*

$$\rho : \mathcal{U}_-(\lambda;\theta) \longrightarrow \Gamma\mathcal{U}_-^t(-\lambda-\eta;\theta)\Gamma. \tag{5.2.3}$$

Γ is an involution operator that satisfy:

$$\begin{aligned} \Gamma^{-1} &= \Gamma, \\ \Gamma\sigma^z\Gamma &= -\sigma^z. \end{aligned} \tag{5.2.4}$$

This operator is non unique for sl_2. For two-dimensional representation of sl_2, it can be represented as σ^x or σ^y.

Motivated by the transfer matrix (1.3.9) formalism that we introduce through the various vertex-face transformation (4.2.6), (4.3.15), we look for a trace formula for quantum charges for any boundary models that are described by a representation of the dynamical reflection algebras (5.2.1),(5.2.2). We propose the following formal *dynamical transfer matrix*:

$$\mathbf{T}(\lambda;\theta) = \mathrm{tr}_0\{(\mathcal{U}_+^{t_0})_0(\lambda;\theta-\eta\sigma^z_{V_-})\frac{h(\theta-\eta\sigma^z_{V_-}-\eta\sigma^z_{V_+}-\eta\sigma^z_0)}{h(\theta-\eta\sigma^z_{V_-}-\eta\sigma^z_{V_+})}(\mathcal{U}_-)_0(\lambda;\theta-\sigma^z_{V_+})\} \tag{5.2.5}$$

Here $V_\pm$ are the representation space of $\mathcal{U}_\pm$. In the next section we propose a co-module evaluation representation of these algebras in a diagonalizable $\mathfrak{h}$-module V where the *weight zero* property holds for $\mathcal{U}_-(\lambda;\theta)$:

$$[(\mathcal{U}_-)_0(\lambda;\theta),\sigma^z_0+\sigma^z_V]=0. \tag{5.2.6}$$

We require $\mathcal{U}_+(\lambda;\theta)$ to satisfy the transposed weight zero condition:

$$[(\mathcal{U}_+^{t_0})_0(\lambda;\theta),\sigma^z_0-\sigma^z_V]=0. \tag{5.2.7}$$

5.2.2 Co-module evaluation representation

Let $\mathcal{K}_-:\mathbb{C}\times\mathbb{C}\longrightarrow End(\mathbb{C}^2)$ be a (scalar) representation of the reflection algebra $B_-(\mathcal{R}(\lambda;\theta))$ in $\mathbb{C}$ (*i.e* $\mathbb{C}$-number matrix), viewed as a one-dimensional $\mathfrak{h}$-module of sl_2 with the standard action on $v\in\mathbb{C}$, $\sigma^z.v=0$:

$$\begin{aligned}&\mathcal{R}_{12}(\lambda_1-\lambda_2;\theta)(\mathcal{K}_-)_1(\lambda_1;\theta)\mathcal{R}_{21}(\lambda_1+\lambda_2;\theta)(\mathcal{K}_-)_2(\lambda_2;\theta)\\&\quad=(\mathcal{K}_-)_2(\lambda_2;\theta)\mathcal{R}_{12}(\lambda_1+\lambda_2;\theta)(\mathcal{K}_-)_1(\lambda_1;\theta)\mathcal{R}_{21}(\lambda_1-\lambda_2;\theta).\end{aligned} \tag{5.2.8}$$

This is essentially the reflection equation (1.3.4) as introduced by Sklyanin [110], with the dynamical $\mathcal{R}$-matrix instead of the vertex R-matrix. A representation as above is said to be of weight zero if:

$$[(\mathcal{K}_-)_0(\lambda;\theta),\sigma^z_0]=0. \tag{5.2.9}$$

This implies that $\mathcal{K}_-$ is a diagonal solution of the above equation. Let $\mathcal{T}(\lambda;\theta)$ be a weight zero representation of $E_{\tau,\eta}(sl_2)$ in V and consider the specific diagonal solution of (5.2.8):

$$\mathcal{K}_-(\lambda;\theta)=\begin{pmatrix}\frac{h(\theta+\zeta_--\lambda)}{h(\theta+\zeta_-+\lambda)} & 0\\ 0 & \frac{h(\zeta_--\lambda)}{h(\zeta_-+\lambda)}\end{pmatrix}, \tag{5.2.10}$$

which depends on an arbitrary complex parameter ζ_-. Then:

$$\mathcal{U}_-(\lambda;\theta) = \mathcal{T}(\lambda,\theta)\mathcal{K}_-(\lambda;\theta)\mathcal{T}^{-1}(-\lambda;\theta), \tag{5.2.11}$$

is a weight zero representation of the dynamical reflection algebra in $\mathbb{C}\otimes V$.

Our main object of study is the dynamical boundary monodromy matrix, a representation of (5.2.1) in $\mathbb{C}\times V$, $V=\otimes_{i=1}^N \mathbb{C}_i^2$, with $\mathcal{T}$ as the evaluation representation of $E_{\tau,\eta}(sl_2)$ (5.1.3):

$$\begin{aligned}(\mathcal{U}_-)_0(\lambda;\theta) &= \widehat{\gamma}(\lambda)\mathcal{T}_0(\lambda;\theta)(\mathcal{K}_-)_0(\lambda;\theta)\mathcal{T}_0^{-1}(-\lambda;\theta) \\ &= \begin{pmatrix} \mathcal{A}_-(\lambda;\theta) & \mathcal{B}_-(\lambda;\theta) \\ \mathcal{C}_-(\lambda;\theta) & \mathcal{D}_-(\lambda;\theta)\end{pmatrix}_{[0]},\end{aligned} \tag{5.2.12}$$

with convenient normalization coefficients:

$$\widehat{\gamma}(\lambda) = (-1)^N \prod_{i=1}^N h(\lambda+\xi_i-\eta)h(\lambda+\xi_i+\eta). \tag{5.2.13}$$

5.2.3 Dual co-module representation

Using the dual representation of $E_{\tau,\eta}(sl_2)$ (5.1.8), it is possible to construct in a canonical way a dual co-module representation of the dual algebra $B_+(\mathcal{R}(\lambda;\theta))$. For this we need a $\mathbb{C}$-representation of the dual algebra, which is given by the dual boundary matrix $\mathcal{K}_+$ which satisfies the dual dynamical reflection equation:

$$\begin{aligned}&\mathcal{R}_{12}(\lambda_2-\lambda_1;\theta)(\mathcal{K}_+^{t_1})_1(\lambda_1;\theta)\mathcal{R}_{21}(-\lambda_1-\lambda_2-2\eta;\theta)(\mathcal{K}_+^{t_2})_2(\lambda_2;\theta) \\ &= (\mathcal{K}_+^{t_2})_2(\lambda_2;\theta)\mathcal{R}_{12}(-\lambda_1-\lambda_2-2\eta;\theta)(\mathcal{K}_+^{t_1})_1(\lambda_1;\theta)\mathcal{R}_{21}(\lambda_2-\lambda_1;\theta).\end{aligned} \tag{5.2.14}$$

$\mathcal{K}_+ : \mathbb{C}\times\mathbb{C}\to \mathrm{End}(\mathbb{C}^2)$ is a dynamical *left* boundary matrix which we take as:

$$\mathcal{K}_+(\lambda;\theta) \equiv \mathcal{K}_+(\lambda;\theta,\xi_+) = \mathcal{K}_-(-\lambda-\eta;\theta,\xi_+). \tag{5.2.15}$$

Here ξ_+ is an arbitrary complex parameter.

A co-module evaluation representation takes the form:

$$\begin{aligned}(\mathcal{U}_+^{t_0})_0 &= \widetilde{\gamma}(\lambda)\mathcal{V}_0^{t_0}(\lambda;\theta)(\mathcal{K}_+^{t_0})_0(\lambda;\theta)(\mathcal{V}_0^{t_0})(-\lambda-2\eta;\theta) \\ &= \begin{pmatrix} \mathcal{A}_+(\lambda;\theta) & \mathcal{C}_+(\lambda;\theta) \\ \mathcal{B}_+(\lambda;\theta) & \mathcal{D}_+(\lambda;\theta)\end{pmatrix}_{[0]},\end{aligned} \tag{5.2.16}$$

with convenient normalization coefficients:

$$\widetilde{\gamma}(\lambda) = (-1)^N \prod_{i=1}^N h(\lambda+\xi_i)h(\lambda+\xi_i+2\eta). \tag{5.2.17}$$

5.2.4 The Bethe Ansatz

In the previous section, we introduced a dynamical reflection structure, together with its dual structure leading to quantum charges defined by (5.2.5). Using the representation of $\sigma^z = \mathbf{S}^z$ in $V = \otimes_{i=1}^N \mathbb{C}_i^2$, it is possible to construct two transfer matrices. The first is based on the co-module evaluation representation of $B_-(\mathcal{R}(\mu;\theta))$ (5.2.12) and the scalar representation of its dual (5.2.15):

$$\mathbf{T}_1(\mu;\theta) = \mathrm{tr}_0\{(\mathcal{K}_+^{t_0})_0(\mu;\theta-\eta\mathbf{S}^z)\frac{h(\theta-\eta\mathbf{S}^z-\eta\sigma_0^z)}{h(\theta-\eta\mathbf{S}^z)}(\mathcal{U}_-)_0(\mu;\theta)\}. \tag{5.2.18}$$

The second matrix is based on the dual co-module evaluation representation of $B_+(\mathcal{R}(\mu;\theta))$ (5.2.16) and the scalar representation of $B_-(\mathcal{R}(\lambda;\theta))$ (5.2.10):

$$\mathbf{T}_2(\mu;\theta) = \mathrm{tr}_0\{(\mathcal{U}_+^{t_0})_0(\mu;\theta)\frac{h(\theta-\eta\mathbf{S}^z-\eta\sigma_0^z)}{h(\theta-\eta\mathbf{S}^z)}(\mathcal{K}_-^{t_0})_0(\mu;\theta-\eta\mathbf{S}^z)\}. \tag{5.2.19}$$

Remark 5.2.2 *Besides their similarity, these two transfer matrices are different. They are related to the same algebra but to different representations.*

Thus, the next question is to find their spectrum. Bethe ansatz scheme leads to the following theorem:

Theorem 5.2.2

$$\forall M \in [[1,N]]: \quad |\psi_-^1(\{\lambda_k\}_{k=1,\dots,M})\rangle = \prod_{k=1}^{M}\mathcal{B}_-(\lambda_k;\theta)|0\rangle, M = \frac{N-s}{2} \tag{5.2.20}$$

$$\textit{and} \quad |\psi_-^2(\{\lambda_k\}_{k=1,\dots,M})\rangle = \prod_{k=1}^{M}\mathcal{C}_-(\lambda_k;\theta)|\bar{0}\rangle, M = \frac{N+s}{2} \tag{5.2.21}$$

belonging to the subspace with a fixed z-component of the total spin:

$$\mathbf{S}^z|\psi_-^{1,2}(\{\lambda_k\}_{k=1,\dots,M})\rangle = s|\psi_-^{1,2}(\{\lambda_k\}_{k=1,\dots,M})\rangle, \tag{5.2.22}$$

are eigenstates of the transfer matrix (5.2.18) *for any μ with eigenvalues $\Lambda_{1,2}$ if the parameters $\{\lambda_k\}_{k=1,\dots,M}$ satisfy the Bethe equations $y_{1,2}$:*

$$\begin{aligned} &y_{1,2}(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\theta,s) \\ &\qquad = y_{1,2}(-\lambda_k-\eta,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\theta,s) \end{aligned} \tag{5.2.23}$$

Here, the eigenvalues are:

$$\begin{aligned}
&\Lambda_1(\mu,\{\lambda_k\}_{k=1,\dots M},\{\xi_j\}_{j=1,\dots,N};\theta,s) \\
&\quad = \frac{(-1)^N}{h(\zeta_+-\mu-\eta)h(\theta+\zeta_-+\mu)} \\
&\quad \times\{a(\mu)d(-\mu-\eta)\frac{h(2\mu+2\eta)h(\zeta_+-\mu)h(\theta+\zeta_+-s+\mu)h(\theta+\zeta_--\mu)}{h(\theta+\zeta_--\mu-\eta)h(2\mu+\eta)\prod_{i=1}^M b(\lambda_i-\mu)b(\lambda_i+\mu+\eta)} \\
&\quad +a(-\mu-\eta)d(\mu)\frac{h(2\mu)h(\zeta_++\mu+\eta)h(\theta+\zeta_-+\mu+\eta)h(\zeta_--\mu-\eta)}{h(\zeta_-+\mu)h(2\mu+\eta)\prod_{i=1}^M b(\lambda_i+\mu)b(\lambda_i-\mu+\eta)}\}
\end{aligned} \tag{5.2.24}$$

and:

$$\begin{aligned}
&\Lambda_2(\mu;\{\lambda_i\}_{i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\theta,s) \\
&\quad = \Lambda_1(\mu;\{\lambda_i\}_{i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};-\theta,-s)|_{\zeta_-=\zeta_-+\theta,\zeta_+=\zeta_++\theta},
\end{aligned} \tag{5.2.25}$$

The y_1 function is:

$$\begin{aligned}
&y_1(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\theta,s) \\
&\quad = \times h(\theta+\zeta_--\mu)h(\zeta_-+\mu)h(\theta+\zeta_+-s+\mu)h(\zeta_+-\mu) \\
&\quad \times a(\lambda_k)d(-\lambda_k-\eta)\prod_{i=1,i\neq k}^{M} h(\lambda_k+\lambda_i)h(\lambda_k-\lambda_i-\eta),
\end{aligned} \tag{5.2.26}$$

whereas the y_2 function is:

$$\begin{aligned}
&y_2(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};\theta,s) \\
&\quad = y_1(\lambda_k,\{\lambda_i\}_{i\neq k,i=1,\dots,M},\{\xi_j\}_{j=1,\dots,N};-\theta,-s)|_{\zeta_-=\zeta_-+\theta,\zeta_+=\zeta_++\theta}
\end{aligned} \tag{5.2.27}$$

where we use the shorthand notations:

$$a(\lambda)=\prod_{i=1}^N a^{face}(\lambda-\xi_i;\theta), d(\lambda)=\prod_{i=1}^N \frac{h(\lambda-\xi_i)}{h(\lambda-\xi_i+\eta)}, b(\lambda)=\frac{h(\lambda)}{h(\lambda+\eta)}, \tag{5.2.28}$$

This theorem follows from standard Bethe ansatz computations. More details are given on the appendix A.

Remark 5.2.3 *Note the very strong similitude between the Bethe construction (eigenvalues and Bethe equations) for this boundary dynamical model and the corresponding theorem for the boundary* ***diagonal*** *spin chains* (1.4.1).

Proposition 5.2.1 *The sets* $|\psi_-^{1,2}(\{\lambda_k\}_{k=1,\dots,M})\rangle$ *are isomorphic to each other. The isomorphism is a consequence of the following parity symmetry of the boundary monodromy matrix* $\mathcal{U}_-$*:*

$$\sigma_0^x \mathcal{U}_-(\lambda;\theta,\zeta_-)\sigma_0^x = \Gamma_x \mathcal{U}_-(\lambda;-\theta,\zeta_-+\theta)\Gamma_x \tag{5.2.29}$$

and the equivalent symmetry for the transfer matrix:

$$\mathbf{T}_1(\lambda;\theta,\zeta_-,\zeta_+) = \Gamma_x \mathbf{T}_1(\lambda;-\theta,\zeta_-+\theta,\zeta_++\theta)\Gamma_x \tag{5.2.30}$$

Thus the symmetric form of the Bethe equations and eigenvalues remains clear.

More details on the proof of this proposition and the parity symmetry are given in the appendix B.

Proposition 5.2.2 *The Bethe construction is* $\mathbb{Z}_2$ *invariant due to the following involution:*

$$\mathcal{B}_-(-\lambda-\eta;\theta) = -(-1)^N \frac{h(\lambda+\zeta_-)h(2(\lambda+\eta))h(\lambda+\zeta_-+\theta)}{h(2\lambda)h(\lambda-\zeta_-+\eta)h(\lambda-\theta-\zeta_-+\eta)} \mathcal{B}_-(\lambda;\theta), \tag{5.2.31}$$

and:

$$\mathcal{C}_-(-\lambda-\eta;\theta) = -(-1)^N) \frac{h(\lambda+\theta+\zeta_-)h(2(\lambda+\eta))h(\lambda+\zeta_-)}{h(2\lambda)h(\lambda-\theta-\zeta_-+\eta)h(\lambda-\zeta_-+\eta)} \mathcal{C}_-(\lambda;\theta). \tag{5.2.32}$$

This symmetry is given by a decomposition of the boundary operators $\mathcal{B}_-$ and $\mathcal{C}_-$ into the bulk operators $\mathcal{A},\mathcal{B},\mathcal{C}$ and $\mathcal{D}$. More details on this decomposition are given in the appendix B.

Theorem 5.2.3

$$\forall M \in [[1,N]] : \quad |\psi_+^1(\{\lambda_k\}_{k=1,\dots,M})\rangle = \prod_{k=1}^{M} \mathcal{B}_+(\lambda_k;\theta)|0\rangle \tag{5.2.33}$$

$$\textit{and} \quad |\psi_+^2(\{\lambda_k\}_{k=1,\dots,M})\rangle = \prod_{k=1}^{M} \mathcal{C}_+(\lambda_k;\theta)|\bar{0}\rangle \tag{5.2.34}$$

with subspace total spin: (5.2.35)

$$\mathbf{S}^z|\psi_+^{1,2}(\{\lambda_k\}_{k=1,\dots,M})\rangle = s|\psi_+^{1,2}(\{\lambda_k\}_{k=1,\dots,M})\rangle \tag{5.2.36}$$

are eigenstates of the transfer matrix $\mathbf{T}_2$ (5.2.19) *for any* μ *with eigenvalues* $\Lambda_{1,2}$ *if the parameters* $\{\lambda_k\}_{k=1,\dots,M}$ *satisfy the Bethe equations* (5.2.23)*.*

Remark 5.2.4 *Bethe equations and eigenvalues explicitly depend on the eigenspace spin s (or equivalently its dimension d as $d = 2s+1$).*

Remark 5.2.5 *The transfer matrix* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are different, and their Bethe states are not the same. They do, however , share the same Bethe eigenvalues and Bethe equations. This suggests that such transfer matrix and the underlying dynamical model should be isomorphic. This isomorphism is nothing but the isomorphism between* $B_-(R(\lambda;\theta))$ *and* $B_+(R(\lambda;\theta))$ (5.2.1).

5.3 Elliptic face model with reflecting end

5.3.1 The model

Let us now introduce the face model underlying the representation (5.2.12) of the dynamical reflection algebra (5.2.1). For this, recall that the face model introduced in Chapter 3 is a two-dimensional statistical mechanics lattice model, where Boltzmann weights are attached to each face, with six possible face configurations where each height θ can differ only by $\pm\eta$ for adjacent sides. The corresponding statistical weights, $\mathcal{R}^{ab}_{cd}$, are collected into the dynamical $\mathcal{R}$-matrix (3.1.11); defined as the bulk part representation of (5.2.12). We consider this model with a reflecting end, which means that each horizontal line makes a U-turn on the left side of the lattice. Since we choose a diagonal solution (5.2.10) of the dynamical reflection equation (5.2.8), it produces **two** configurations characterized by the weights $(\mathcal{K}_-)^{\pm}_{\pm}(\lambda;\theta)$:

$(\mathcal{K}_-)^{+}_{+}(\lambda;\theta)$ $\qquad$ $(\mathcal{K}_-)^{-}_{-}(\lambda;\theta)$

It is important to note that such a reflecting end imposes a constant external height θ for the left side of the lattice. We impose DWBC, such that the heights decrease from left to right on the upper boundary, the heights grow from left to right on the lower boundary. Since the left external height is fixed, these two conditions determine completely the configuration on the right boundary (heights decreasing in the upward direction).

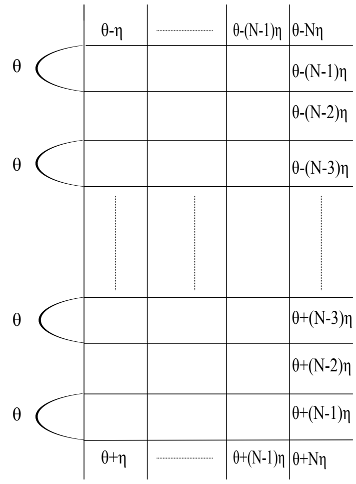


Figure 5.1: The face model with reflecting end and DWBC

5.3.2 Partition function

The partition function of the face model introduced in the previous section can be written in terms of the boundary monodromy matrix (5.2.12):

$$
\begin{aligned}
&Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta) \qquad (5.3.1)\\
&= \langle 0|_\lambda \langle \bar{0}|_\xi \prod_{i=1}^{N}\{\prod_{j=1}^{N}\{\mathcal{R}_{ij}(\lambda_i-\xi_j;\theta-\eta\sum_{k=j+1}^{N}\sigma_k^z)\}\mathcal{K}_-(\lambda_i;\theta)_i\prod_{j=N}^{1}\{\mathcal{R}_{ji}(\lambda_i+\xi_j;\theta-\eta\sum_{k=j+1}^{N}\sigma_k^z)\}\}|\bar{0}\rangle_\lambda|0\rangle_\xi\\
&= \langle 0|_\lambda \langle \bar{0}|_\xi \prod_{i=1}^{N}\mathcal{U}_-(\lambda_i;\theta)|0\rangle_\xi|\bar{0}\rangle_\lambda\\
&= \langle \bar{0}|_\xi \prod_{i=1}^{N}\mathcal{B}_-(\lambda_i;\theta)|0\rangle_\xi
\end{aligned}
$$

The dynamical reflection algebra introduced previously enables one to establish a set of properties, univocally defining the partition function.

Proposition 5.3.1 (Filali) *The partition function* (5.3.1) *satisfies the following properties:*

i) Initial condition

$$Z_{1,2}^{Bface}(\lambda_1,\xi_1;\theta) = \frac{h(\eta)h(\theta-\eta)}{h^2(\theta)} \times \Big(\frac{h(\theta+\zeta_- -\lambda_1)}{h(\theta+\zeta_- +\lambda_1)} h(\lambda_1-\xi_1)h(\theta+\lambda_1+\xi_1) + \frac{h(\zeta_- -\lambda_1)}{h(\zeta_- +\lambda_1)} h(\lambda_1+\xi_1)h(\theta-\lambda_1+\xi_1)\Big)$$

ii) Symmetry
$Z_{N,2N}^{Bface}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{i=1,\dots,N};\theta)$ *is a symmetric function of the* $\{\lambda_i\}_{i=1,\dots,N}$ *and the* $\{\xi_j\}_{j=1,\dots,N}$.

iii) Elliptic polynomiality of the normalized partition function $\tilde{Z}$
For each parameter $\{\lambda_i\}_{i=1,\dots,N}$ *the normalized partition function*

$$\tilde{Z}_{N,2N}^{Bface}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{i=1,\dots,N};\theta) = \prod_{i=1}^{N} \frac{h(\theta+\zeta_- +\lambda_i)h(\zeta_- +\lambda_i)}{h(2\lambda_i)} Z_{N,2N}^{Bface}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta),$$

is a theta function of order $2N-2$ *and norm* $(N-1)\eta$ *with respect to the variable* λ_i.

iv) Recursive relations

$$Z_{N,2N}^{Bface}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta)\Big|_{\lambda_i=\xi_j} = \frac{h(\eta)h(\zeta_- -\lambda_i)}{h(\zeta_- +\lambda_i)} \times \prod_{k=1}^{N} h(\lambda_k+\xi_j)\frac{h(\theta+(N-2i)\eta)}{h(\theta+(N-2i+1)\eta)} \times \prod_{k=1,k\neq i}^{N} h(\lambda_i-\xi_k+\eta)h(\lambda_i+\xi_k+\eta)h(\lambda_k-\xi_j+\eta) \times Z_{(N-1),2(N-1)}^{Bface}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j};\theta)$$

and:

$$Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta)\Big|_{\lambda_i=-\xi_j} = \frac{h(\eta)h(\theta+\zeta_- -\lambda_i)}{h(\theta+\zeta_- +\lambda_i)}$$
$$\times\prod_{k=1}^{N} h(\lambda_k-\xi_j)\frac{h(\theta+(N-2i)\eta)}{h(\theta+(N-2i+1)\eta)}$$
$$\times\prod_{k=1,k\neq j}^{N} h(\lambda_i+\xi_k+\eta)h(\lambda_i-\xi_k+\eta)h(\lambda_{k-1}+\xi_j+\eta)$$
$$\times Z^{Bface}_{(N-1),2(N-1)}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j};\theta)$$

Lemma 5.3.1 *The partition function $Z^{Bface}_{N,2N}$ that satisfy the set of conditions i)-iv) is unique.*

Indeed, it is sufficient to observe that the normalized partition function $\tilde{Z}$ is a theta function of order $2N-2$ and norm $(N-1)\eta$ in each parameter $\lambda_{i,i=1,\dots,N}$. So we need $2N-1$ independent conditions to uniquely determine it. Using the symmetry (ii) the recursion relations (iv) can be established for any point $\lambda_i=\xi_j$, or $\lambda_i=-\xi_j$ for $i,j=1,\dots,N$. Hence we can prove by induction starting from the case $N=2$ that the partition function is uniquely determined as we need.

Theorem 5.3.1 (Filali) *The partition function of the elliptic face model with reflecting ends and DWBC is:*

$$Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta) \tag{5.3.2}$$
$$=(-1)^N\prod_{i=1}^{N}\left(\frac{h(\theta+\eta(N-2i))}{h(\theta+\eta(N-i))}\right)\det\mathcal{N}^{Bface}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta)$$
$$\times\frac{\prod_{i,j=1}^{N} h(\lambda_i+\xi_j)h(\lambda_i-\xi_j)h(\lambda_i+\xi_j+\eta)h(\lambda_i-\xi_j+\eta)}{\prod_{1\le i<j\le N} h(\xi_j+\xi_i)h(\xi_j-\xi_i)h(\lambda_j-\lambda_i)h(\lambda_j+\lambda_i+\eta)}$$

where the $N\times N$ matrix M_{ij} can be expressed as:

$$\mathcal{N}^{Bface}_{\alpha,\beta}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta)=\frac{h(\theta+\zeta_- +\xi_\beta)h(\zeta_- -\xi_\beta)}{h(\theta+\zeta_- +\lambda_\alpha)h(\zeta_- +\lambda_\alpha)} \tag{5.3.3}$$
$$\times\frac{h(2\lambda_\alpha)h(\eta)}{h(\lambda_\alpha-\xi_\beta+\eta)h(\lambda_\alpha+\xi_\beta+\eta)h(\lambda_\alpha-\xi_\beta)h(\lambda_\alpha+\xi_\beta)}$$

To prove the theorem, it is sufficient to check the properties (i) to (iv).

Remark 5.3.1 *Notice the strong similitude between this partition function for this diagonal boundary face model and the equivalent formula for the diagonal boundary vertex model* (2.3.2).

Remark 5.3.2 *The partition function for diagonal boundary face model is expressed as a single determinant. This crucial result shows that adding a reflecting end of the Tsuchiya's type to face models leads to a simpler model.*

The main result of this chapter is that dynamical reflection algebras, although more involved than $E_{\tau,\eta}(sl_2)$, lead to a simpler Bethe ansatz and describe very convenient statistical physics models. Indeed, one can construct eigenstates with spin values belonging to the total range $(-\frac{N}{2} < s < \frac{N}{2})$. Most importantly, this structure leads to an explicit and simple formula for the partition function of the corresponding face model.

5.4 A dynamical generalization of the Kuperberg HTS model

As we already mentioned, vertex and face models are related to combinatorics. This quite unexpected and intriguing relation was noticed by Kuperberg [75]. He found bijections between the six-vertex (or the face) model configurations with enumeration of Alternating Sign Matrix (ASM) [84, 85], *ie* square matrices with entries 1, -1 and 0 such that each row and column sums to 1, and 1 and -1 alternate along rows and columns. This bijection is illustrated in Figure 5.2. To turn a state of the vertex model on an $N \times N$ grid with domain-wall boundary conditions into an alternating-sign matrix of order N, replace each vertex by $-1, +1$ or 0 according to the following marking:

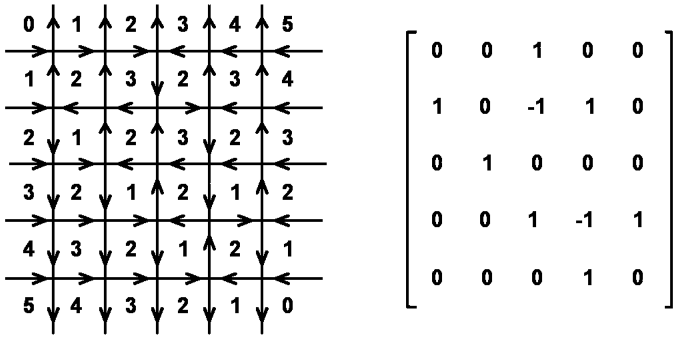
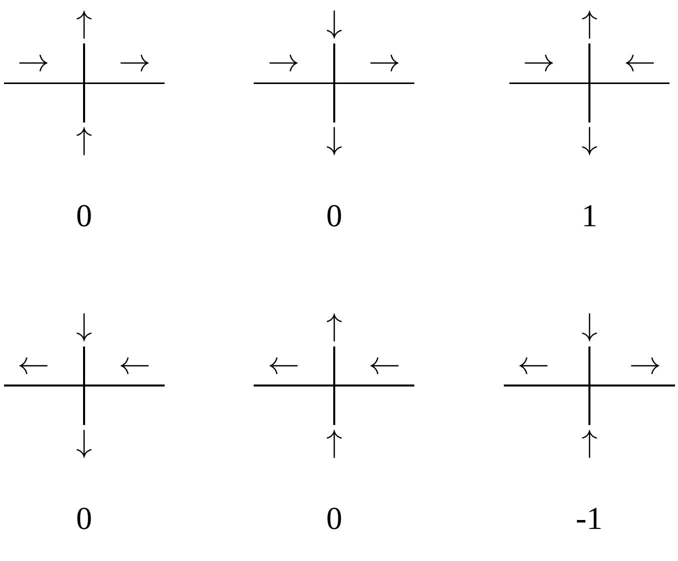


Figure 5.2: Vertex, face and ASM



This deep relation permits one to apply all the quantum integrability technology (such as the q-deformed Knizhnik-Zamolodchikov equation [20, 22, 23, 49]) to new problems in the field of combinatorics.

Kuperberg also noticed that enumeration of various other symmetry classes of alternating-sign matrices are related to partition functions of vertex models with various special boundaries. We are interested here in the case of Half-Turn-Symmetric boundaries which enumerate vertically and horizontally ASM as illustrated in Figure 5.3. The HTS conjugacy classe of ASM [99] is the set of all ASM such that for any element $a_{i,j}$ of the (square)

$N \times N$ matrix, the following relation hold:

$$a_{i,j} = a_{N-1-i,N-1-j} \tag{5.4.1}$$

Here is an example of such matrix for $N = 6$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{5.4.2}$$

Our aim is to generalize this model to the dynamical case, with respect to the more general elliptic parametrization, and to compute its partition function. The choice of this class of boundary (or as shown by Kuperberg the choice of this symmetry class of ASM) is mainly due to its *apriori* simplicity. Our long term goal is to achieve a complete catalog of face models with Kuperberg-type boundaries to study dynamical enumeration (in the sense of Rosengren [100]) of all symmetry classes of ASM.

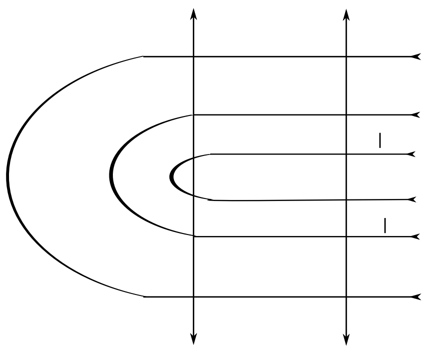


Figure 5.3: The vertex model with HTS boundary and DWBC

5.4.1 The model

The model that we propose is the dynamical analog of the Kuperberg HTS model with DWBC, which can be pictured as in the Figure 5.4.

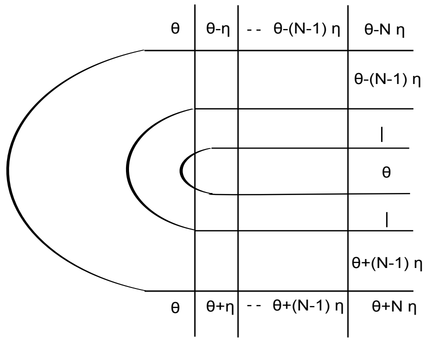


Figure 5.4: The face model with a HTS boundary and DWBC

Statistical configurations and Boltzmann weights are the same that for the previous face models on a square lattice and with reflecting ends, except that the we do not allow extra Boltzmann weights for the boundary. This means that we take the boundary matrix $\mathcal{K}_-$ to be the identity:

$$\mathcal{K}_-(\lambda;\theta) \equiv \mathcal{K}_-(\lambda;\theta,\infty) = Id. \tag{5.4.3}$$

5.4.2 Partition function

The partition function of the HTS face model introduced in the previous section can be written in terms of the monodromy matrix (5.1.3):

$$Z^{DHTS}_{N,2N}(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N};\theta) \tag{5.4.4}$$

$$= \Lambda\langle 0|_\lambda\langle\bar{0}|_\xi \prod_{i=1}^{N}\{\mathcal{T}_i(\lambda_i;\theta-\eta\sum_{k=1}^{i-1}\sigma_i^z)\}\prod_{i=N}^{1}\{\mathcal{T}_i^{-1}(-\lambda_i;\theta-\eta\sum_{k=1}^{i-1}\sigma_i^z)\}$$

$$= \Lambda_{-1}\langle 0|_\lambda\langle\bar{0}|_\xi \prod_{i=1}^{N-1}\{\mathcal{T}_i(\lambda_i;\theta-\eta\sum_{k=1}^{i-1}\sigma_i^z)\}\mathcal{B}_-(\lambda_N;\theta-\eta\sum_{k=1}^{N-1}\sigma_i^z)\prod_{i=N-1}^{1}\{\mathcal{T}_i^{-1}(-\lambda_i;\theta-\eta\sum_{k=1}^{i-1}\sigma_i^z)\}$$

$$= \langle 0|_\lambda\langle\bar{0}|_\xi \prod_{i=1}^{N}\{\prod_{j=1}^{N}\{\mathcal{R}_{ij}(\lambda_i-\xi_j;\theta-\eta\sum_{k=1}^{i-1}\sigma^z_{\lambda_k}-\eta\sum_{k=j+1}^{N}\sigma^z_{\xi_k})\}$$

$$\times\prod_{j=N}^{1}\{\mathcal{R}_{ji}(\lambda_i+\xi_j;\theta-\eta\sum_{k=1}^{i-1}\sigma^z_{\lambda_k}-\eta\sum_{k=j+1}^{N}\sigma^z_{\xi_k})\}\}|\bar{0}\rangle_\lambda|0\rangle_\xi \tag{5.4.5}$$

where: $\Lambda = \prod_{i=1}^{N} \widehat{\gamma}(\lambda_i)$, $\Lambda_{-1} = \prod_{i=1}^{N-1} \widehat{\gamma}(\lambda_i)$. The dynamical Yang-Baxter algebra together with the reflection structure enables one to establish a set of properties defining it in an unique way.

Proposition 5.4.1 *The partition function* (5.3.1) *satisfies the following properties.*

i) Initial condition

$$Z_{1,2}^{DHTS}(\lambda_1,\xi_1;\theta) = \frac{h(\eta)h(\theta-\eta)}{h^2(\theta)} \times (h(\lambda_1-\xi_1)h(\theta+\lambda_1+\xi_1) + h(\lambda_1+\xi_1)h(\theta-\lambda_1+\xi_1))$$

Note that the face model with a reflecting end partition function and the face HTS model partition function are equal for $N=1$. This should be obvious from the picture 5.3.

ii) Symmetry
$Z_{N,2N}^{DHTS}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{i=1,\dots,N};\theta)$ *is a symmetric function of the* $\{\lambda_i\}_{i=1,\dots,N}$ *and the* $\{\xi_j\}_{j=1,\dots,N}$.

iii) Elliptic polynomiality
For each parameter $\{\lambda_i\}_{i=1,\dots,N}$ *the partition function* $Z_{N,2N}^{DHTS}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta)$ *is an elliptic polynomial of order* $2N$ *and norm* θ.

iv) Recursive relations

$$\left. Z_{N,2N}^{DHTS}(\{\lambda_i\}_{i=1,\dots,N},\{\xi_j\}_{j=1,\dots,N};\theta)\right|_{\lambda_i=\xi_j} = -\frac{h(\theta-(N-1)\eta)h(\theta-N\eta)}{h^2(\theta)} \times \prod_{k=1}^{N} h(\lambda_i-\xi_k+\eta)h(\lambda_i+\xi_k) \times \prod_{k=1,k\neq i}^{N} h(\lambda_k-\xi_j+\eta)h(\lambda_k+\xi_j) \times Z_{(N-1),2(N-1)}^{DHTS}(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j};\theta+\eta)$$

and:

$$Z^{DHTS}_{N,2N}\left(\{\lambda_i\}_{i=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N};\theta\right)\Big|_{\lambda_i=-\xi_j} = \frac{h^2(\theta-\eta)}{h(\theta+(N-1)\eta)h(\theta+(N-2)\eta)}$$
$$\times\prod_{k=1}^{N} h(\lambda_k+\xi_j+\eta)h(\lambda_k-\xi_j)$$
$$\times\prod_{k=1,k\neq i}^{N} h(\lambda_i-\xi_k)h(\lambda_i+\xi_k+\eta)$$
$$\times Z^{DHTS}_{(N-1),2(N-1)}\left(\{\lambda_m\}_{m\neq i},\{\xi_n\}_{n\neq j};\theta-\eta\right)$$

v) Crossing symmetry

$$Z^{DHTS}_{N,2N}(-\lambda_i-\eta,\{\lambda_m\}_{m\neq i,m=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N};\theta) = -\frac{h(2(\lambda_i+\eta))}{h(2\lambda_i)}$$
$$\times Z^{DHTS}_{N.2N}(\lambda_i,\{\lambda_m\}_{m\neq i,m=1,\ldots,N},\{\xi_j\}_{j=1,\ldots,N};\theta).$$

Lemma 5.4.1 *The partition function $Z^{DHTS}_{N,2N}$ is uniquely defined by the set of conditions $i)-v)$.*

Indeed, it is sufficient to observe that, according to the condition iii), the partition function is a theta function of order $2N$ and norm θ in each parameter $\lambda_{i,i=1,\ldots,N}$. So we need $2N+1$ independent conditions to uniquely determine it. Using the symmetry (ii) the recursion relations (iv) can be established for any point $\lambda_i=\xi_j$, or $\lambda_i=-\xi_j$ for $i,j=1,\ldots,N$. Due to the crossing symmetry (v), similar recursion can be established at the points $\lambda_i=\mp\xi_j-\eta$. Thus the partition function is defined at $4N$ different points. Hence we can prove by induction starting from the case $N=2$ that the partition function is uniquely determined.

At this point, we should propose a simple and manageable formula for $Z^{DHTS}_{N,2N}$. This is still an open problem.

Chapter 6

Conclusions and perspectives

In this thesis, we tackle the problem of boundary integrable models without quasi-particles conservation through the analysis of the XXZ spin chain with boundaries. Our main tools are the vertex-face transformation and the algebraic Bethe ansatz technique, which are implemented in a very algebraic and simple form. Our method works provided two strong conditions on the boundary parameters. These enable us to find some eigenstates of the model and the associated eigenvalues. It turns out that the vertex-face transformation in this context highlights a new integrable structure, the dynamical reflection algebra, which can describe a new face model with reflecting end. We generalize this structure to the elliptic case, and we show that the underlying face model is exactly solvable. The very important point is that its partition function takes the form of a single determinant in the general case.

This work should be continued, and leaves several open questions:

- Boundary XXZ spin chains:
  As we have seen, the diagonalization of the XXZ hamiltonian through our method requires two conditions on the boundary parameters (4.2.8) (4.2.9). This very special case has the advantage to lead to a simple Bethe ansatz, and the underlying face model with reflecting end is very convenient as its partition function is rather simple. Unfortunately, this case is too degenerated, and it is impossible to describe the full set of eigenstates. Most importantly, the dual states are unaccessible through our method, so it seems very difficult to push forward the analysis towards correlation functions. It turns out that at least one condition is absolutely necessary to diagonalize the XXZ hamiltonian [12, 122], so it is very important to drop out at least one condition, and to look for a simple Bethe ansatz for the XXZ spin chain with general boundary. The search for the underlying dynamical model should be also a very interesting point. The very curious fact is that it is indeed possible to recover the only

required conditions, but this leads to a dynamical like model with a triangular (rather than diagonal) boundary matrix. This model is not clearly described at the moment, and the underlying algebra is unclear. This work should be easily generalized to the elliptic case, which corresponds to the XYZ hamiltonian with general boundary. We believe that the same obstacle must be encountered in this case, and we believe that we should tackle the question of the conditions on the boundary parameters for the trigonometric case before this. If this is achieved within a simple form, then we can go through the computation of the scalar product and the correlation functions of boundary spins chains. This is very important also for out of equilibrium model.

- Dynamical reflection algebra
  In connection with the first point, we found triangular solution to our dynamical reflection equation. The very important point is that this solution is related to general boundary spin chain matrix trough the vertex-face transformation. The next point is to look for a dressing procedure for the construction of a co-module representation of the dynamical reflection algebra upon this solution. In other words, we look for a weight zero representation of the dynamical reflection algebra without the (too restrictive) weight zero condition on the dynamical boundary matrix. We believe that such representations should exist, and a deeper analysis of the dynamical symmetries of the dynamical $\mathcal{R}$-matrix can lead to a solution to this problem.

- ASM and the three-coloring model $\eta = i\frac{\pi}{3}$
  We already mentioned the link of face models and combinatorics, in particular for enumeration of alternating sign matrix. Namely, the partition function for the square face model and the Rosengren's formula (Theorem (3.3.1)) lead to very interesting combinatorics at the point $\eta = i\frac{\pi}{3}$ [96]. We also already mentioned the main inconvenience of the Rosengren's formula, which is not represented as a single determinant. We believe that our result (Theorem (5.3.1)) can also lead to interesting combinatorics, and this should be easier in our case as our formula is represented as a single determinant.

- Boundary face model
  We believe that our dynamical reflection algebra is the right framework for the analysis of face models with boundary, at least for the various integrable aspects. We started the analysis of other models, among them the DHTS model. It should be interesting to compute the partition function of various other boundary face models, especially in connection with the previous point for the enumeration of alternating sign matrix.

Appendix A

Proof of the theorem 5.2.2 (4.2.1)

In this appendix we give the derivation of the algebraic Bethe ansatz for co-module evaluation representation of the dynamical reflection algebra which associated to the transfer matrix (5.2.18):

$$\mathbf{T}_1(\lambda;\theta) = \mathrm{tr}_0\{(\mathcal{K}_+^{t_0})_0(\lambda;\theta-\eta\mathbf{S}^z)\frac{h(\theta-\eta\mathbf{S}^z-\eta\sigma_0^z)}{h(\theta-\eta\mathbf{S}^z)}(\mathcal{U}_-)_0(\lambda;\theta)\}.$$

This enables us to prove the theorem (5.2.2). The theorem (5.2.3) can be proved along the same lines. Note that in the trigonometric limit, the Bethe ansatz theorems for the open XXZ spin chains (4.2.1),(4.3.1) follow directly with the convenient restriction.

We start by introducing a modified operator $\widetilde{\mathcal{D}}_-(\lambda;\theta)$:

$$\begin{aligned}\widetilde{\mathcal{D}}_-(\lambda;\theta) = \frac{h(\theta-\eta\,\mathbf{S}^z+\eta)}{h(\theta-\eta\,\mathbf{S}^z)}\{&\mathcal{D}_-(\lambda;\theta)\\ &-\frac{h(\theta-\eta\,\mathbf{S}^z+2\lambda+\eta)h(\eta)}{h(2\lambda+\eta)h(\theta-\eta\,\mathbf{S}^z+\eta)}\mathcal{A}_-(\lambda;\theta)\}.\end{aligned} \tag{A.0.1}$$

The transfer matrix can be expressed in terms of the operators $\mathcal{A}_-(\lambda;\theta)$, $\widetilde{\mathcal{D}}_-(\lambda;\theta)$ as:

$$\begin{aligned}\mathbf{T}_1(\lambda;\theta) &= \frac{h(\zeta_++\lambda+\eta)}{h(\zeta_+-\lambda-\eta)}\widetilde{\mathcal{D}}_-(\lambda;\theta)\\ &+\frac{h(\zeta_+-\lambda)h(\zeta_++\theta-\eta\,\mathbf{S}^z+\lambda)h(2\lambda+2\eta)}{h(\zeta_+-\lambda-\eta)h(\zeta_++\theta-\eta\,\mathbf{S}^z-\lambda-\eta)h(2\lambda+\eta)}\mathcal{A}_-(\lambda;\theta).\end{aligned} \tag{A.0.2}$$

The action of the operators $\mathcal{A}_-(\lambda;\theta)$, $\widetilde{\mathcal{D}}_-(\lambda;\theta)$ on the reference state $|0\rangle = \otimes_{i=1}^N \begin{pmatrix}1\\0\end{pmatrix}$

is:

$$\mathcal{A}_-(\lambda;\theta)|0\rangle = \frac{h(\theta+\zeta_- - \lambda)}{h(\theta+\zeta_- + \lambda)} \prod_{i=1}^{N} h(\lambda-\xi_i+\eta)h(\lambda+\xi_i+\eta)|0\rangle,$$
$$\widetilde{\mathcal{D}}_-(\lambda;\theta)|0\rangle = \frac{h(2\lambda)h(\zeta_- - \lambda-\eta)h(\theta+\zeta_- +\lambda+\eta)}{h(2\lambda+\eta)h(\zeta_- +\lambda)h(\theta+\zeta_- +\lambda)} \times \prod_{i=1}^{N} h(\lambda-\xi_i)h(\lambda+\xi_i)|0\rangle. \tag{A.0.3}$$

The dynamical reflection relation (5.2.1), gives the following commutation rules for the operators $\mathcal{A}_-$, $\widetilde{\mathcal{D}}_-$ and $\mathcal{B}_-$:

$$\begin{aligned}\mathcal{A}_-(\lambda_1;\theta)\mathcal{B}_-(\lambda_2;\theta) = &- \frac{h(\eta)h(\theta-\eta\mathbf{S}^z-2\eta-\lambda_1-\lambda_2)}{h(\theta-\eta\mathbf{S}^z-\eta)h(\lambda_1+\lambda_2+\eta)}\mathcal{B}_-(\lambda_1;\theta)\widetilde{\mathcal{D}}_-(\lambda_2;\theta)\\ &+ \frac{h(\lambda_1+\lambda_2)h(\lambda_1-\lambda_2-\eta)}{h(\lambda_1-\lambda_2)h(\lambda_1+\lambda_2+\eta)}\mathcal{B}_-(\lambda_2;\theta)\mathcal{A}_-(\lambda_1;\theta)\\ &- \frac{h(\eta)h(2\lambda_2)h(\lambda_1-\lambda_2-\theta+\mathbf{S}^z+\eta)}{h(\theta-\eta\mathbf{S}^z-\eta)h(\lambda_1-\lambda_2)h(2\lambda_2+\eta)}\mathcal{B}_-(\lambda_1;\theta)\mathcal{A}_-(\lambda_2;\theta),\end{aligned} \tag{A.0.4}$$

$$\begin{aligned}\widetilde{\mathcal{D}}_-(\lambda_1;\theta)\mathcal{B}_-(\lambda_2;\theta) = &\frac{h(\lambda_1+\lambda_2+\theta-\eta\mathbf{S}^z)}{h(\theta-\eta\mathbf{S}^z-\eta)}\\ &\times \frac{h(\eta)h(2\lambda_2)h(2\lambda_1+2\eta)}{h(\lambda_1+\lambda_2+\eta)h(2\lambda_1+\eta)h(2\lambda_2+\eta)}\mathcal{B}_-(\lambda_1;\theta)\mathcal{A}_-(\lambda_2;\theta)\\ &+ \frac{h(\lambda_1-\lambda_2+\eta)h(\lambda_1+\lambda_2+2\eta)}{h(\lambda_1-\lambda_2)h(\lambda_1+\lambda_2+\eta)}\mathcal{B}_-(\lambda_2;\theta)\widetilde{\mathcal{D}}_-(\lambda_1;\theta)\\ &- \frac{h(\eta)h(2\lambda_1+2\eta)h(\lambda_1-\lambda_2+\theta-\eta\mathbf{S}^z-\eta)}{h(\lambda_1-\lambda_2)h(2\lambda_1+\eta)h(\theta-\eta\mathbf{S}^z-\eta)}\mathcal{B}_-(\lambda_1;\theta)\widetilde{\mathcal{D}}_-(\lambda_2;\theta).\end{aligned} \tag{A.0.5}$$

Now one can easily show, using the usual algebraic Bethe ansatz, that a state constructed by the action of operators $\mathcal{B}_-$:

$$\forall M \in [[0,N]] : |\psi^1_-(\{\lambda_k\}_{k=1,\dots,M})\rangle = \prod_{k=1}^{M} \mathcal{B}_-(\lambda_k;\theta)|0\rangle, \tag{A.0.6}$$

is an eigenstate of the transfer matrix $\mathbf{T}_1(\mu;\theta)$ provided the spectral parameters satisfy the Bethe equations (5.2.23). Using a similar computation one can show that :

$$\forall M \in [[0,N]] : |\psi^2_-(\{\lambda_k\}_{k=1,\dots,M})\rangle = \prod_{k=1}^{M} \mathcal{C}_-(\lambda_k;\theta)|\bar{0}\rangle \tag{A.0.7}$$

is an eigenstate of the transfer matrix $\mathbf{T}_1(\mu;\theta)$ provided the spectral parameters satisfy the Bethe equations (5.2.27). Note however that this follows directly from the isomorphism (5.2.1).

Appendix B

Proof of the proposition 5.2.1 (4.2.1) and 5.2.2 (4.2.2)

In this appendix we prove various symmetries of the boundary operators. We first proceed to a boundary-bulk decomposition of the boundary operators which enables us to understand the $\mathbb{Z}_2$ symmetry of the Bethe construction and the crossing symmetry of the partition function. We then prove the parity symmetry of the boundary operators, which enable us to prove the proposition (5.2.1).

B.1 Boundary-bulk decomposition

First of all, we will need the following fundamental symmetries of the dynamical $\mathcal{R}$-matrix:

- Weight zero:

$$[\sigma_1^z + \sigma_2^z, \mathcal{R}_{12}(\lambda;\theta)] = 0. \tag{B.1.1}$$

  It is easy to see that this relation induces a similar relation for the transposed $\mathcal{R}$-matrix:

$$[\sigma_1^z - \sigma_2^z, \mathcal{R}_{12}^{t_1}(\lambda;\theta)] = 0. \tag{B.1.2}$$

- Unitarity:

$$\mathcal{R}_{12}(\lambda;\theta)\mathcal{R}_{21}(-\lambda;\theta) = -h(\lambda-\eta)h(\lambda+\eta)\,\mathrm{Id}. \tag{B.1.3}$$

- Crossing Symmetry:

$$-\sigma_1^y \mathcal{R}_{12}^{t_1}(-\lambda-\eta;\theta+\eta\sigma_1^z)\sigma_1^y \frac{h(\theta-\eta\sigma_2^z)}{h(\theta)} = \mathcal{R}_{21}(\lambda;\theta). \tag{B.1.4}$$

Using these symmetries, we can rewrite the inverse bulk monodromy matrix (5.1.3) as:

$$\begin{aligned}
\mathcal{T}_0^{-1}(-\lambda;\theta) &= \prod_{i=N}^{1} \mathcal{R}_{0i}^{-1}(-\lambda-\xi_i;\theta-\eta\sum_{k=i+1}^{N}\sigma_k^z) \qquad (B.1.5)\\
&= \widehat{\gamma}^{-1}(\lambda)\prod_{i=N}^{1}\mathcal{R}_{i0}(\lambda+\xi_i;\theta-\eta\sum_{k=i+1}^{N}\sigma_k^z)\\
&= (-1)^N\widehat{\gamma}^{-1}(\lambda)\sigma_0^y\{\prod_{i=N}^{1}\mathcal{R}_{0i}^{t_0}(-\lambda-\eta-\xi_i;\theta+\sigma_0^z-\eta\sum_{k=i+1}^{N}\sigma_k^z)\frac{h(\theta-\eta\sum_{k=i}^{N}\sigma_k^z)}{h(\theta)}\}\sigma_0^y\\
&= (-1)^N\widehat{\gamma}^{-1}(\lambda)\sigma_0^y\mathcal{T}^{t_0}(-\lambda-\eta;\theta+\sigma_0^z)\sigma_0^y\frac{h(\theta-\eta\mathbf{S}^z)}{h(\theta)}
\end{aligned}$$

The last lines follows because of (B.1.2). Using the representation (5.2.12) of the double monodromy matrix, we can then rewrite it as:

$$\begin{aligned}
(\mathcal{U}_-)_0(\lambda;\theta) &= (-1)^N\mathcal{T}_0(\lambda;\theta)(\mathcal{K}_-)_0(\lambda;\theta) \qquad (B.1.6)\\
&\times\sigma_0^y\mathcal{T}_0^{t_0}(-\lambda-\eta;\theta+\sigma_0^z)\sigma_0^y\frac{h(\theta-\eta\mathbf{S}^z)}{h(\theta)}
\end{aligned} \tag{B.1.7}$$

This enables us to decompose the boundary operators $\mathcal{B}_-$ and $\mathcal{C}_-$ (5.2.12) in terms of the bulk operators $\mathcal{A}$ and $\mathcal{B}$ (5.1.3):

$$\begin{aligned}
\mathcal{B}_-(\lambda;\theta) =&(-1)^N\Big((\mathcal{K}_-)_-^-\mathcal{B}(\lambda;\theta)\mathcal{A}(-\lambda-\eta;\theta+\eta)-(\mathcal{K}_-)_+^+\mathcal{A}(\lambda;\theta)\mathcal{B}(-\lambda-\eta;\theta-\eta)\Big)\\
&\times\frac{h(\theta-\eta\mathbf{S}^z)}{h(\theta)},
\end{aligned} \tag{B.1.8}$$

and:

$$\begin{aligned}
\mathcal{C}_-(\lambda;\theta) =&(-1)^N\Big(-(\mathcal{K}_-)_+^+\mathcal{C}(\lambda;\theta)\mathcal{D}(-\lambda-\eta;\theta-\eta)+(\mathcal{K}_-)_-^-\mathcal{D}(\lambda;\theta)\mathcal{C}(-\lambda-\eta;\theta+\eta)\Big)\\
&\times\frac{h(\theta-\eta\mathbf{S}^z)}{h(\theta)},
\end{aligned} \tag{B.1.9}$$

Using the dynamical Yang-Baxter algebra for the bulk monodromy matrix (3.2.7), this leads to the following symmetry of the $\mathcal{B}_-,\mathcal{C}_-$ operators:

$$\mathcal{B}_-(-\lambda-\eta;\theta) = -(-1)^N \frac{h(\lambda+\zeta_-)h(2(\lambda+\eta))h(\lambda+\zeta_-+\theta)}{h(2\lambda)h(\lambda-\zeta_-+\eta)h(\lambda-\theta-\zeta_-+\eta)}\mathcal{B}_-(\lambda;\theta), \quad \text{(B.1.10)}$$

and:

$$\mathcal{C}_-(-\lambda-\eta;\theta) = -(-1)^N \frac{h(\lambda+\theta+\zeta_-)h(2(\lambda+\eta))h(\lambda+\zeta_-)}{h(2\lambda)h(\lambda-\theta-\zeta_-+\eta)h(\lambda-\zeta_-+\eta)}\mathcal{C}_-(\lambda;\theta). \quad \text{(B.1.11)}$$

Using such symmetries, the $\mathbb{Z}_2$ construction proposition (5.2.2) of the Bethe theorem remains clear.

B.2 Parity symmetry

The proposition (5.2.1) highlights a simple relation between the two sets of Bethe states. This is because the boundary operators enjoy a generalized parity symmetry. First, we note that the dynamical $\mathcal{R}$-matrix satisfies the following parity symmetry:

$$\mathcal{R}_{21}(\lambda;\theta) = \sigma_1^{x,y}\sigma_2^{x,y}\mathcal{R}_{12}(\lambda;\theta)\sigma_1^{x,y}\sigma_2^{x,y} = \mathcal{R}_{12}(\lambda;-\theta). \quad \text{(B.2.1)}$$

Using this symmetry, we easily find the corresponding symmetry for the dynamical monodromy matrix (5.1.3):

$$\sigma_0^x\mathcal{T}_0(\lambda;\theta)\sigma_0^x = \Gamma_x\mathcal{T}(\lambda;-\theta)\Gamma_x, \quad \text{(B.2.2)}$$

A similar relation exist for the chosen $\mathcal{K}_-$ solution (5.2.10):

$$\sigma_0^x(\mathcal{K}_-)_0(\lambda;\theta,\zeta_-)\sigma_0^x = (\mathcal{K}_-)_0(\lambda;-\theta,\zeta_-+\theta), \quad \text{(B.2.3)}$$

leading to the parity symmetry for $\mathcal{U}_-(\lambda;\theta)$:

$$\sigma_0^x(\mathcal{U}_-)_0(\lambda;\theta,\zeta_-)\sigma_0^x = \Gamma_x(\mathcal{U}_-)_0(\lambda;-\theta,\zeta_-+\theta)\Gamma_x, \quad \text{(B.2.4)}$$

which implies the following relation between $\mathcal{B}_-$ and $\mathcal{C}_-$:

$$\mathcal{C}_-(\lambda;\theta,\zeta) = \Gamma_x\mathcal{B}_-(-\theta,\zeta_-+\theta)\Gamma_x. \quad \text{(B.2.5)}$$

Finally, using the solution $\mathcal{K}_+$ (5.2.15) and the relation (B.2.4), it is obvious that:

$$\mathbf{T}_1(\lambda;\theta,\zeta_-,\zeta_+) = \Gamma_x\mathbf{T}_1(\lambda;-\theta,\zeta_-+\theta,\zeta_++\theta)\Gamma_x \quad \text{(B.2.6)}$$

This last relation gives a clear understanding of the relation (5.2.1) between the two sets of Bethe states, the symmetric form of their eigenvalues and Bethe equations.

Note that similar relations can be established for $\mathcal{B}_+, \mathcal{C}_+$ (5.2.16) and $\mathbf{T}_2(\lambda;\theta,\zeta_-,\zeta_+)$ (5.2.19) using the parity symmetry for the crossed Lax matrix (5.1.4):

$$\sigma_1^{x,y}\sigma_2^{x,y}\mathcal{L}_{12}^{t_1}(\lambda;\theta)\sigma_1^{x,y}\sigma_2^{x,y} = \mathcal{L}_{12}^{t_1}(\lambda;-\theta), \tag{B.2.7}$$

or more directly using the isomorphism theorem (5.2.1):

$$\mathcal{C}_+(-\lambda-\eta;\theta) = (-1)^N \frac{h(\lambda+\eta-\zeta_+)h(2\lambda)h(\lambda+\eta-\zeta_+-\theta)}{h(\lambda+\zeta_++\theta)h(\lambda+\zeta_+)h(2(\lambda+\eta))}\mathcal{C}_+(\lambda;\theta), \tag{B.2.8}$$

and:

$$\mathcal{B}_+(-\lambda-\eta;\theta) = (-1)^N \frac{h(\lambda+\eta-\theta-\zeta_+)h(2\lambda)h(\lambda+\eta-\zeta_+)}{h(\lambda+\zeta_+)h(\lambda+\theta+\zeta_+)h(2(\lambda+\eta))}\mathcal{B}_+(\lambda;\theta). \tag{B.2.9}$$

Appendix C

Proof of the property iii) of the proposition 5.3.1

In this section, we discuss the proof of the property *iii*) of the proposition (5.3.1) regarding to the elliptic polynomiality of the partition function of the face model with reflecting ends. Two main methods exist for finding determinant formulas for partition functions of vertex models which are described by the QISM duality (see Chapter 2). The original method of Izergin-Korepin of Chapter 2 consists of a three step process:

i) Find a set of conditions that uniquely determine the partition function

ii) Propose a formula for the partition function

iii) Prove that this formula satisfies the set of previously established conditions

This method was successfully used for finding a convenient formula for the partition function of the trigonometric vertex model with DWBC (theorem (2.2.1)) and the trigonometric vertex model with reflecting end theorem (2.3.1). Moreover, this method is still successful for the face model with reflecting end (theorem (4.4.1)). Particularly, the step *i*) contains a strong condition regarding the polynomiality of the partition function in some variables (the spectral parameter). This condition is easily derived from the vertex R-matrix or the face $\mathcal{R}$-matrix in the trigonometric case. Step *ii*) is particularly difficult. The great achievement of Izergin regards the partition function of the trigonometric vertex model with DWBC, which enables us to look for similar Izergin type formula if one applies this method to others models.

For elliptic models, this last polynomiality argument no longer holds as the $\mathcal{R}$-matrix is parameterized in terms of elliptic functions. As shown by Rosengren [100], Pakuliak, Sylantyev and Rubtsov [93] in their analysis of the elliptic face model with DWBC, such polynomiality argument should be replaced by a generalized elliptic polynomiality condition. However, the elliptic polynomiality form of the partition function is not directly derived from the $\mathcal{R}$-matrix description of the partition function, and we need to know the structure of the general statistical configuration of the face model. This is possible for the elliptic face model with DWBC. As one can see from the figure 3.1, for **each possible configurations** there exists a k, with $1 \leq k \leq N$, such that the second row is:

$$\theta+\eta|\theta+2\eta|\ldots|\theta+k\eta|\theta+(k-1)\eta|\theta+k\eta|\theta+(k+1)\eta|\ldots|\theta+(N-1)\eta$$

This configuration is an elliptic polynomial with orders and norms **which do not depend on** k, thus we can access to the elliptic polynomiality of the partition function, therefore achieving the step $i)$. Unfortunately, for face models with reflecting ends, it seems very difficult to accomplish this task. As shown in figure 4.1, we did not find such a generic configuration.

A second method for finding partition functions of such models was found by Kitanine, Maillet and Terras [67, 69] with the use of the concept of Drinfel'd twist. Drinfel'd twists were introduced [26] in order to relate Hopf algebra to quasi-Hopf algebra structure in a consistent way. Representation of Drinfel'd twists was first applied by Maillet and Sanchez de Santos [82] in order to obtain a completely symmetric representation of the bulk monodromy operators for Yang-Baxter type algebra, which are highly non local in terms of the quantum local operators. These permitted them to reduce drastically the combinatorial difficulty of handling highly non-local representation. The idea is to perform a change of basis in the space of states where the bulk monodromy operators remains completely symmetric. Using the representation of the partition function in this new basis, Kitanine-Maillet-Terras succeed, by an iteration procedure, to compute a determinant formula for the partition function of the trigonometric vertex model with DWBC, with an without reflecting end. The very nice point is that they succeed in proving (using determinant properties and the Liouville theorem) that such a determinant formula reduces to the Izergin and Tsuchiya determinant. This method permits one to avoid the difficulty of the step $iii)$ of the first method, but requires one to handle singular and asymptotic features of trigonometric functions. We believe that this last point should be very difficult for elliptic functions.

We have chosen to apply the Izergin original method, and use the Drinfel'd twist representation for finding elliptic polynomiality of the partition function of the elliptic face model with reflecting ends. For the evaluation representation of dynamical Yang-Baxter algebras, factorizing Drinfel'd twists' representations were discovered by Albert and collaborators [2]. This construction is based on a dynamical $\mathcal{F}$-matrix which factorizes the dynamical $\mathcal{R}$-matrix in the following way:

$$\mathcal{F}_{21}(-\lambda;\theta)\mathcal{R}_{12}(\lambda;\theta) = \mathcal{F}_{12}(\lambda;\theta). \tag{C.0.1}$$

After a suitable co-product over all quantum spaces it leads to a change of basis $\mathcal{F}_{\{\xi\}}$ where the bulk operators $\mathcal{A}(\lambda;\theta), \mathcal{B}(\lambda;\theta)$ (5.1.3) have symmetric expressions:

$$\begin{aligned}\overline{\mathcal{A}}(\lambda;\theta) &= \mathcal{F}_{\{\xi\}}(\theta)\mathcal{A}(\lambda;\theta)\mathcal{F}^{-1}_{\{\xi\}}(\theta-\eta) \\ &= \frac{h(\theta-\eta)}{h\left(\theta+\eta(\frac{N-\mathbf{S}^z}{2}-1)\right)} \otimes_{i=1}^N \begin{pmatrix} h(\lambda-\xi_i+\eta) & 0 \\ 0 & h(\lambda-\xi_i)\end{pmatrix}_i,\end{aligned} \tag{C.0.2}$$

and:

$$\begin{aligned}\overline{\mathcal{B}}(\lambda;\theta) &= \mathcal{F}_{\{\xi\}}(\theta)\mathcal{B}(\lambda;\theta)\mathcal{F}^{-1}_{\{\xi\}}(\theta+\eta) \\ &= \frac{h(\eta)}{h(\theta)}\sum_{i=1}^N h(\theta-\lambda+\xi_i)\sigma_i^- \otimes_{j\neq i}^N \begin{pmatrix} h(\lambda-\xi_j+\eta) & 0 \\ 0 & \frac{h(\lambda-\xi_i)h(\xi_i-\xi_j+\eta)}{h(\xi_i-\xi_j)}\end{pmatrix}_j.\end{aligned} \tag{C.0.3}$$

Using the decomposition (B.1.8) of the boundary operator $\mathcal{B}_-(\lambda;\theta)$, it is easy to compute its expression in this new basis:

$$\begin{aligned}\overline{\mathcal{B}}_-(\lambda;\theta) &= \mathcal{F}_{\{\xi\}}(\theta)\mathcal{B}_-(\lambda;\theta)\mathcal{F}^{-1}_{\{\xi\}}(\theta) \\ &= \gamma(\lambda)\sum_{i=1}^N \Bigg\{ \frac{h(\theta+\zeta_-+\xi_i)}{h(\theta+\zeta_-+\lambda)}\frac{h(\zeta_--\xi_i)}{h(\zeta_-+\lambda)} h(2\lambda)h(\eta) \\ &\quad \sigma_i^- \otimes_{j\neq i}^N \begin{pmatrix} h(\lambda+\xi_j)h(\lambda-\xi_j+\eta) & 0 \\ 0 & \frac{h(\lambda-\xi_j)h(\lambda+\xi_j+\eta)h(\xi_i-\xi_j+\eta)}{h(\xi_i-\xi_j)}\end{pmatrix}_j \Bigg\} \\ &\quad \times \frac{h(\theta-\eta\mathbf{S}^z)}{h(\theta+\eta\frac{N-\mathbf{S}^z}{2})}.\end{aligned} \tag{C.0.4}$$

A very important property is that the reference states $|0\rangle = \prod_{i=1}^N \uparrow_{\xi_i}$ and $|\bar{0}\rangle = \prod_{i=1}^N \downarrow_{\xi_i}$ are left and right invariant under the action of $\mathcal{F}_{\{\xi\}}$:

$$\mathcal{F}_{\{\xi\}}(\theta)|0\rangle = \mathcal{F}^{-1}_{\{\xi\}}(\theta)|0\rangle = |0\rangle, \langle 0|\mathcal{F}_{\{\xi\}}(\theta) = \langle 0|\mathcal{F}^{-1}_{\{\xi\}}(\theta) = \langle 0|, \tag{C.0.5}$$

and

$$\mathcal{F}_{\{\xi\}}(\theta)|\bar{0}\rangle = \mathcal{F}^{-1}_{\{\xi\}}(\theta)|\bar{0}\rangle = \langle\bar{0}|, \langle\bar{0}|\mathcal{F}_{\{\xi\}}(\theta) = \langle\bar{0}|\mathcal{F}^{-1}_{\{\xi\}}(\theta) = \langle\bar{0}|. \tag{C.0.6}$$

And thus the partition function (5.3.1) takes the following form:

$$Z^{Bface}_{N,2N}(\{\lambda_i\}_{i=1,\dots,N}, \{\xi_j\}_{j=1,\dots,N}; \theta) = \langle\bar{0}|_\xi \prod_{i=1}^{N} \overline{\mathcal{B}}_-(\lambda_i;\theta)|0\rangle_\xi \tag{C.0.7}$$

The action of the $\overline{\mathcal{B}}_-(\lambda_N;\theta)$ operator is then easily computed due to the symmetric representation of $\overline{\mathcal{B}}_-(\lambda_N;\theta)$ in the $\mathcal{F}$-basis. The operator $\frac{h(\theta+\zeta_-+\lambda_N)h(\zeta_-+\lambda_N)}{h(2\lambda_N)}\overline{\mathcal{B}}(\lambda_N;\theta)$ acts due to (C.0.4) as $\sum_{i=1}^{N}\prod_{k=1,k\neq i}^{N} h(\lambda_N+\xi_k)h(\lambda_N-\xi_k+\eta)$, which is a theta function of order $2N-2$ and norm $(N-1)\eta$ with respect to the variable λ_N. The proof the step *iii*) of the proposition (5.3.1) is then achieved.

Bibliography

[1] Avan J., Babelon O., Billey E. : *The Gervais-Neveu-Felder Equation and the Quantum Calogero-Moser Systems*. Commun. Math. Phys. 178 (1996) 281

[2] Albert T.D., Boos H. , Flume R., Poghossian R.H., Ruhlig K. : *An F-twisted XYZ model*. Lett.Math.Phys. 53 (2000) 201

[3] Babelon O.: *Universal exchange algebra for Bloch waves and Liouville theory*. Commun. Math. Phys. 139 (1991) 619

[4] Babelon O. ,Berbard D., Billey E. : *A quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations.* Phys.Lett. B 375 (1996) 89

[5] Ballesteros A.,Musso F., Ragnisco O. : *Comodule algebras and integrable systems.* J.Phys.A: 35 (2002) 8197

[6] Batchelor M.T., Baxter R.J., O'Rourke M.J.,Yung C.M. : *Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions* . J. Phys. A 28 (1995) 2759

[7] Baxter, R.J. : *Exactly solved models in statistical mechanics*. London, Academic Press (1982)

[8] Baxter, R.J.: *Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg spin chains I,II,III*. Ann.Phys 76 (1973), 1, 25, 48

[9] Baxter, R.J.: *One dimensional anisotropic Heisenberg chains*. Ann.Phys 70 (1972) 323

[10] Bethe,H. : *Zur Theorie Der Metalle I*.Zeitschrift fur physiks. 71 (1931) 205

[11] Buffenoir E.,Roche Ph., Terras V. : *Universal vertex-IRF Transformation for Quantum Affine Algebras.* arxiv:[math-ph] 0707.0955

[12] Cao J., Lin H., Shi K., Wang Y. : *Exact solution of XXZ spin chain with unparallel boundary fields.* Nuclear Physics B 663 (2003) 487

[13] Chari V., Pressley A. : *A guide to Quantum groups*. CUP (1984)

[14] Chari V.,Pressley A. : *Quantum affine algebras.* Commun. Math. Phys. 142 (1991) 261

[15] Cherednik I.V. : *On a method of constructing factorized S-matrices in elementary functions*. Theor.mathFiz. 43 (1980) 117

[16] Cherednik I.V. : *Factorizing particles on a half-line and root systems* . Theor.Math.Phys : 61 (1984) 977

[17] de Vega H., Gonzalez R. : *Boundary K-matrices for the six-vertex and the n(2n-1)* A_{n-1} *vertex models.* Journal of Phys.A : vol. 26, no.12 (1993) L519

[18] de Gier J.,Essler F.H.L : *Exact spectral gaps of the asymmetric exclusion process with open boundary*. J.Stat.Mech. (2006) P12011

[19] de Gier J.,Essler F.H.L : *Slowest relaxation mode of the partially asymmetric exclusion process with open boundary.* J.Phys.A 41 (2008) 48

[20] de Gier J.,Pyatov P.,Zinn-Justin P. : *Punctured plane partitions and the q-deformed Knizhnik-Zamolodchikov and Hirota equations.* J. Combin. Theory Ser. A 116 (2009) 772

[21] C.Destri, H.J. de Vega, H.J. Giacomini: *The Six-Vertex Model Eigenvectors as Critical Limit of the Eight-Vertex Model Bethe Ansatz.* J. Stat. Phys. 56 (1989) 2/4

[22] Di Francesco P.,Zinn-Justin P. : *Quantum Knizhnik-Zamolodchikov equation: reflecting boundary conditions and combinatorics.* J. Stat. Mech. Theory Exp. (2007) P12009

[23] Di Francesco P.,Zinn-Justin P. : *Quantum Knizhnik-Zamolodchikov Equation, Totally Symmetric Self-Complementary Plane Partitions and Alternating Sign Matrices.* Theor. Math. Phys. 154 (2008) 331

[24] Drinfel'd V. : *Hopf algebras and the quantum Yang-Baxter equation.* Soviet. Math. Dokl. Vol. 32 (1985) 1060

[25] DrinfelŠd, V.G. : *Quantum Groups.* in Proc. of the I. C. M. (Berkeley, 1986)

[26] Drinfel'd V.: *Quasi-Hopf algebra*. Len.Math. J. 1 (1990) 1419

[27] Elkies N., Kuperberg G., Larsen M., Propp J.: *Alternating-sign matrices and domino tilings. II*. J. Algebraic Combin. 1 (1992) 219Ű234

[28] Eloranta K. :*Diamond Ice*. J. Stat. Phys. 96 (1999) 1091

[29] Enriquez B., Felder G. : *Elliptic quantum group and quasi-Hopf algebra*. Comm.Math.Phys 195 (1998) 651

[30] Essler F. H. L., Rittenberg V. : *Representations of the quadratic algebra and partially asymmetric diffusion with open boundary*. J. Phys. A: Math. Gen. 29 (1996) 3375

[31] Faddeev L.D. , Takhtadzhan L.A. : *Quantum inverse scattering method*. Russ. Math. Surv. 34 (1979) 11

[32] Faddeev, L.D., Reshetikhin N., and Takhtadzhyan, L. : *Quantization of Lie groups and Lie algebras*. Leningrad Math. J. 1 (1990) 193

[33] Faddeev L.D, Sklyanin E.K. , Takhtajan L.A : *Quantum inverse problem method I* Theor. Math. Phys. 40 (1979) 688

[34] Faddeev, L.D. : *Integrable models in 1+1 dimensional quantum field theory*. In Les Houches 561 (1982) 561 Elsevier Science Publ. (1984)

[35] Faddeev, L.D., Sklyanin, E.K., and Takhtajan, L. : *The quantum inverse scattering method*. Theor. Math. Phys. 40 (1979) 688

[36] Faddeev, L.D., Sklyanin, E.K., and Takhtajan, L. : *The quantum method of the inverse problem and the XYZ model*.Russ. Math. Surv. 34 (1979) 11

[37] Fan H., Hou B., Shi K. : *Representation of the boundary elliptic quantum group $BE_{\tau,\eta}(sl_2)$ and the Bethe ansatz*. Nuclear Physics B 496 (1997) 551

[38] Felder G. :*Conformal field theory and integrable systems associated to elliptic curves*. In Proceedings of the International Congress of Mathematicians, Vol.1, 2 (Zürich, 1994) (1995), pp. 1247-1255.

[39] Felder G. : *Elliptic quantum groups*. Proceedings of the International Congress of Mathematicians,Vol.1, 2 (Zurich, 1994), pp. 1247–1255, Basel, 1995, Birkhauser.

[40] Felder G., Varchenko A.: *Elliptic Quantum Groups and Ruijsenaars Models*. Journal of Stat.Phys. 89 (1997)

[41] Felder G., Varchenko A. : *On representation of the elliptic quantum group* $E_{\tau,\eta}(sl_2)$. Comm. Math. Phys. 181 (1996) 741

[42] Felder G., Varchenko A. : *Algebraic Bethe ansatz for the elliptic quantum group* $E_{\tau,\eta}$. Nuclear Phys. B 480 (1996) 485

[43] Filali G., Kitanine N : *The partition function of the trigonometric SOS model with a reflecting end*. J. Stat. Mech.: Theory Exp. (2010) L06001, erratum: J. Stat. Mech.: Theory and Exp.(2010) E07002

[44] Filali G., Kitanine N. : *Spin Chains with Non-Diagonal Boundaries and Trigonometric SOS Model with Reflecting End*. SIGMA 7 (2011) 012

[45] Filali G.: *Elliptic dynamical reflection algebra and partition function of SOS model with reflecting end*. Journal of Geometry and Physics 61 (2011) 1789

[46] Foda O.,Wheeler M., Zuparic M. : *Domain wall partition functions and KP*. J. Stat. Mech. (2009) P03017

[47] Foda O.,Wheeler M., Zuparic M. : *XXZ scalar products and KP*. Nucl. Phys. B 820 [FS] (2009) 649

[48] Foda O., Schrader G. : *XXZ scalar products, Miwa variables and discrete KP*, arXiv:1003.2524.

[49] Fonseca T.,Zinn-Justin P. : *On the Doubly Refined Enumeration of Alternating Sign Matrices and Totally Symmetric Self-Complementary Plane Partitions.* Electron. J. Combin. 15 (2008)

[50] Gardner C.S, Greene J.M., Kruskal M.D., Miura R.M.: *Method for solving the Korteweg-de Vries equation* Phys. Rev. Lett. 19 (1967) 1095

[51] Gardner C.S.: *Korteweg de Vries equation and generalization. IV, The Korteweg de Vries equation as a Hamiltonian system.* J. Math. Phys. 12 (1971) 1548

[52] Gervais, J.L., Neveu, A. : *Novel triangle relation and absence of tachyons in Liouville string field theory.* Nucl. Phys. B 238 (1984) 125

[53] Gaudin, M. : *La fonction d ' onde de Bethe.* Masson, Paris (1983)

[54] Gomez C., Ruiz-Altaba M., Sierra G. : *Quantum groups in two dimensional physics*. CUP (1996)

[55] Ghoshal S., Zamolodchikov A. : *Boundary S matrix and boundary state in two-dimensional integrable quan- tum field theory*. Internat. J. Modern Phys. A 9 (1994) 3841

[56] Gradstein I.S., Ryzik I.M.: *Table of integrals, series and products*. Fizmatgiz, Moscow (1963)

[57] Heisenberg, W.: *Zur theorie der Ferromagnetismus.* Zeitschrift fur physik, 49 (1928) 619

[58] Izergin A.G., Korepin V.E.: *The quantum inverse scattering method approach to correlation functions*. Comm. Math. Phys. 94 (1984) 67

[59] Izergin A.G.: *Partition function if the six-vertex model in a finite volume*.Sov. Phys. Dokl. 32 (1987) 878

[60] Jimbo, M.: *A q-difference analogue of U(G) and the Yang-Baxter equation.* Lett. Math.Phys. 10 (1985) 63

[61] Jimbo, M. and Miwa, T.: *Algebraic analysis of solvable lattice models.* AMS (1995)

[62] M. Jimbo, A. Kuniba, T. Miwa, M. Okado: *The A_n face model*.Commun. Math. Phys. 119 (1988) 543

[63] Jimbo M., Konno H., Odake S., Shiraishi J. : *Elliptic Algebra $U_{q,p}(\widehat{sl}_2)$: Drinfeld Currents and vertex Operators* .Commun. Math. Phys. 199 (1999) 605

[64] Jones V.F.R. : *Hecke algebra, representations of braid groups and links polynomi-als*.Annals of Math. (1987) 126

[65] Kulish, P.P, and Sklyanin, E.K. : *Quantum spectral transform method. Recent devel-opments.* In Lect. Notes in Phys. vol. 151, 61, Springer verlag (1982)

[66] Kulish, P.P, and Sklyanin, E.K. : *Algebraic structures related to reflection equations*. Journal of Phys. A : 25 (1992) 5963

[67] Kitanine N., Maillet J.M., Terras V. : *Form factors of the XXZ Heisenberg spin-1/2 finite chain*. Nucl. Phys. B 554 [FS] (1999) 647

[68] Kitanine N., Maillet J.M., Terras V. : *Correlation functions of the XXZ Heisenberg spin-1/2 chain in a mag- netic field.* Nucl. Phys. B 567 [FS] (2000) 554

[69] Kitanine N., Kozlowski K.K., Maillet J.M., Niccoli G., Slavnov N.A, Terras V. : *Correlation functions of the open XXZ chain. I.*J. Stat. Mech. Theory Exp. (2007) P10009

[70] Kitanine N., Kozlowski K.K., Maillet J.M., Niccoli G., Slavnov N.A, Terras V. : *Correlation functions of the open XXZ chain. II* . J. Stat. Mech.: Theory Exp. (2007) P07010

[71] Korepin V. : *Calculation of norms of Bethe wave functions*. Comm. Math. Phys. 86 (1982) 391

[72] Korepin, V. E., Bogoliubov, N.M. and Izergin, A.G. : *Quantum Inverse scattering method and correlation functions.* Cambridge University Press, Cambridge (1993)

[73] Korepin V. E., Zinn-Justin P. : *Thermodynamic limit of the six-vertex model with domain wall boundary conditions.* J. Phys. A 33 (2000) 7053

[74] Kostov I : *Exact Solution of the Six-vertex Model on a Random Lattice*.Nuclear Physics B 3 (1999) 575

[75] Kuperberg G.: *Another proof of the alternating-sign matrix conjecture*. Internat. Math. Res. Notices 3 (1996) 1390

[76] Lax P.D.:*Integrals of nonlinear equations of evolutions and solitary waves* Comm. Pure and Appl. Math. 21 (1968) 467

[77] Lieb E.H., Wu F.Y. : *Two Dimensional Ferroelectric Models*, in C. Domb, M. S. Green, J. Lebowitz, *Phase Transitions and Critical Phenomena* New York: Academic Press,1 (1972) 331

[78] Lieb E.H. : *Residual Entropy of Square Ice.* Phys. Rev. 162 (1967) 162

[79] Louiville J.: *Solution nouvelle d'un problème d'analyse, relatif aux phénoménes thermo-mécaniques* J.Math.Pure.Appl 2 (1837) 439

[80] Sturm C., Louiville J.: *Extrait d'un Mémoire sur le développement des fonctions en séries dont les différents termes sont assujettis à satisfaire à une même équation différentielle linéaire, contenant un paramétre variable*. J.Math.Pure.Appl 2 (1837) 220

[81] Linus P. : *The Structure and Entropy of Ice and of Other Crystals with Some Randomness of Atomic Arrangement.* J. Am. Chem. Soc. 57 (1935) 2680

[82] Maillet J.M., Sanchez de Santos J. : *Drinfel'd twist and the algebraic Bethe ansatz*. In L. D. Faddeev's Seminar on Mathematical Physics, 137Ű178, Amer. Math. Soc. Transl. Ser. 2, 201, Amer. Math. Soc., Providence, RI, (2000).

[83] Majid S. : *Foundations of quantum groups*. CUP (1995)

[84] Mills W.H, Robbins D.P., Rumsey H. Jr. : *Proof of the Macdonald conjecture*. Inventiones Mathematicae. 66 (1982) 73

[85] Mills W.H, Robbins D.P.,Rumsey H. Jr. : *Alternating sign matrices and descending plane partitions*. Journ. of Comb. Theory A 34 (1983) 340

[86] Molev A:, Ragoucy E. : *Representations of reflections algebras*. Rev. Math. Phys. 14 (2002) 317

[87] Nagy Z., Avan J., Rollet G. : *Construction of dynamical quadratic algebras*. Lett.Math.Phys 67 (2004) 1

[88] Nagy Z., Avan J., Doikou A., Rollet G. : *Commuting quantum traces for quadratic algebras*. J.Math.Phys 46 (2005) 083516

[89] Nagy Z., Avan J. : *Spin chain from dynamical quadratic algebras*. J.Stat.Mech (2005) P03005

[90] Onsager,L. : *Cristal statistics I. A two-dimensional model with an order-disorder transition.* Phys.Rev 65 (1944) 117

[91] Orbach, R.: *Linear antiferromagnetic chain with anistropic coupling.* Phys.Rev 112 (1958) 309

[92] Owczarek A.L., Baxter R.J. : *Surface free energy of the critical six-vertex model with free boundary.* J. Phys. A 22 (1989) 1141

[93] Pakuliak S., Rubtsov V., Silantyev A. : *The SOS model partition function and the elliptic weight functions.* J. Phys. A 41 (2008)

[94] Pasquier V., Saleur H. : *Common strucutures between finite systems and conformal fields theory through quantum groups.* Nuclear Physics B 330 (1990) 523

[95] Pauli W.:*On the hydrogen spectrum from the standpoint of the new quantum mechanics*. Z. Physik 36 (1926) 336

[96] Razumov A. V., Stroganov Yu. G.: *Three-coloring statistical model with domain wall boundary conditions: Trigonometric limit*. Theor.Math.Phys 161 (2009) 2

[97] Reshetikhin N., Turaev V.G. : *Ribbon graphs and their invariants derived from quantum groups*. Comm.Math.Phys 127 (1990) 1

[98] Robbins D., Rumsey H. : *Determinants and alternating sign matrices*. Adv. in Math. 62 (1986) 169

[99] Robbins D.: *Symmetry Classes of Alternating Sign Matrices*. arxiv math.CO-0008045

[100] Rosengren H. : *An Izergin-Korepin type identity for the 8VSOS model, with applications to alternating sign matrices* .Adv. in Appl. Math. 43 (2009) 137

[101] Rosengren H. : *The three-colour model with domain wall boundary conditions*. Adv. Appl. Math 46 (2011) 481

[102] Rosengren H. : *Felder's elliptic quantum group and elliptic hypergeometric series on the root system A_n*. Int. Math. Res. Not. 2011 (2011) 2861

[103] Sandow S. : *Partially asymmetric exclusion process with open boundary.* Phys. Rev. E 50 (1994) 2660

[104] Slater J.C. : *Theory of the Transition in KH2PO4.* Journ. Chem. Phys. 9 (1941) 16

[105] Sutherland B., Yang C.N., Yang C.P. : *Exact solution of a model of two-dimensional ferroelectrics in an arbitrary external electric field.* Phys.Rev.Lett 19 (1967) 586

[106] Sutherland B. : *Exact Solution of a Two-Dimensional Model for Hydrogen-Bonded Crystals.* Phys. Rev. Lett. 19 (1967) 103

[107] Sklyanin, E.K. : *Quantum version of the method of inverse scattering problem*. Zap. Nauchn. Semin. LOMI 95 (1980) 55

[108] Sklyanin E.K. : *Quantum version of the inverse scattering problem method*. J.Sov.Math 19 (1982) 1546

[109] Sklyanin, E.K. : *On some algebraic structures connected with the Yang-Baxter equation.* Funct. Anal. Appl. 16 (1983) 263

[110] Sklyanin E.K. : *Boundary conditions for integrable quantum systems*. J. Phys. A 21 (1988) 2375

[111] Tarasov V., Varchenko A. : *Completeness of Bethe vectors and Difference Equations with Regular Singular Points*.Int. Math. Res. Notices 13 (1995) 637

[112] Tsuchiya O. : *Determinant formula for the six-vertex model with reflecting end*. J. Math. Phys. 39 (1998) 5946

[113] Turaev V. : *The Yang-Baxter equation and invariants of links*. Invent.Math 92 (1988) 527

[114] Turaev V. : *State sum invariants of 3-manifolds and quantum 6j-symbols*. Topology 31 (1992) 865

[115] Wang Y.-S. : *The scalar products and the norm of Bethe eigenstates for the boundary XXX Heisenberg spin-1/2 finite chain*. Nuclear Phys. B 622 (2002) 633

[116] Walker, L.R. : *Antiferromagnetic linear chain*. Phys.Rev 116 (1959) 1089

[117] Witten E. : *Quantum field theory and the Jones polynomial*.Comm.Math.Phys: 121 (1989) 351

[118] Weber H. : *Elliptische Functionen und algebraische Zahlen*, Friedrich Vieweg und Sohn, Braunschweig, (1891)

[119] Whittaker E.T., Watson G.N.: *A Course of Modern Analysis*, fourth ed., Cambridge University Press, Cambridge, (1927)

[120] Yang C.N. : *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction.* Phys.Rev.Lett. 19 (1967) 1312

[121] Yau H.T.: $(logt)2/3$ *law of the two dimensional asymmetric simple exclusion process.* Ann. Math. 159 (2004) 377

[122] Yang W.L., Nepomechie R., Zhang Y.Z. : *Q-operator and T-Q relation from the fusion hierarchy.* Phys.Lett. B 633 (2006) 4

[123] Yang W.L.,Zhang Y.Z. : *On the second reference state and complete eigenstates of the open XXZ chain.* JHEP 04 (2007) 044

[124] Zamolodchikov A.B. : *An exact two particle S-matrix for quantum solitons of the sine-Gordon model*. Pisma ZhETF 25 (1977) 499

[125] Zamolodchikov A.B., Zamolodchikov Al.B. : *Factorized S-matrices in two dimensions as the exact solution of certain relativistic quantum field theory models*. Ann. Phys. 120 (1979) 253

[126] Zamolodchikov A.B. : *Factorized S-matrices in lattice statistical systems.* Sov.Sci.Rev. A 2 (1980) 1

[127] Zakharov V.A, Faddeev L.D.: *The Korteweg de Vries equation is a completely integrable Hamiltonian system*. Funktsional. Anal, i Prilozhen. 5 (1971) 18

[128] Zinn-Justin P. : *The six-vertex model on random lattices*. Europhys. Lett. (2000) 50