AN INTERACTING PARTICLE MODEL AND A PIERI-TYPE FORMULA FOR THE ORTHOGONAL GROUP

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Abstract. We introduce a new interacting particle model with blocking and pushing interactions. Particles evolve on $\mathbb{Z}_+$ jumping on their own volition rightwards or leftwards according to geometric jumps with parameter $q \in (0,1)$. We show that the model involves a Pieri-type formula for the orthogonal group. We prove that the two extreme cases - $q = 0$ and $q = 1$ - lead respectively to a random tiling model studied in [1] and a random matrix model considered in [4].

1. introduction

In [1] A. Borodin and J. Kuan consider a random tiling model with a wall which is related to the Plancherel measure for the orthogonal group and thus to representation theory of this group. Similar connection holds for the interacting particle model and the random matrix model considered in [4]. The aim of this paper is to establish a direct link between the random tiling model on one side and the interacting particle model or the random matrix model on the other side. For this we consider an interacting particle model depending on a parameter and show that these models correspond to different parameter values.

The paper is organized as follows. Definition of the set of Gelfand-Tsetlin patterns for the orthogonal group is recalled in section 2. Section 3 is devoted to the description of the particle model. We recall in section 4 the description of an interacting particle model equivalent to the random tiling model studied in [1]. Models considered in that paper involve Markov kernels which can be obtained with the help of a Pieri-type formula for the orthogonal group. These Markov kernels are constructed in section 5 after recalling some elements of representation theory. We describe the matrix model related to the particle model in section 6. Results are stated in section 7 and proved in section 8.

Acknowlegments: The author would like to thank Alexei Borodin for his suggestions and helpful explanations.

2. Gelfand-Tsetlin patterns

Let n be a positive integer. For $x, y \in \mathbb{R}^n$ such that $x_n \le \dots \le x_1$ and $y_n \le \dots \le y_1$, we write $x \preceq y$ if x and y are interlaced, i.e.

$$x_n \le y_n \le x_{n-1} \le \dots \le x_1 \le y_1.$$

When $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{n+1}$ we add the relation $y_{n+1} \le x_n$. We denote by $|x|$ the vector of $\mathbb{R}^n$ whose components are the absolute values of those of x.

Definition 2.1. *Let k be a positive integer.*

(1) *We denote by GT_k the set of Gelfand-Tsetlin patterns defined by*

$$\begin{aligned} GT_k &= \{(x^1, \cdots, x^k) : x^i \in \mathbb{N}^{j-1} \times \mathbb{Z} \text{ when } i = 2j-1, \\ &\qquad x^i \in \mathbb{N}^j \text{ when } i = 2j, \text{ and } |x^{i-1}| \preceq |x^i|, 1 \leq i \leq k\}. \end{aligned}$$

(2) *If $x = (x^1, \ldots, x^k)$ is a Gelfand-Tsetlin pattern, x^i is called the i^{th} row of x for $i \in \{1, \ldots, k\}$.*

(3) *For $\lambda \in \mathbb{Z}^{[\frac{k+1}{2}]}$ the subset of Gelfand-Tsetlin patterns having a k^{th} row equal to λ is denoted by $GT_k(\lambda)$ and its cardinality is denoted by $s_k(\lambda)$.*

Usually, a Gelfand Tsetlin pattern is represented by a triangular array as indicated at figure 1 for $k = 2r$.

$$\begin{array}{ccccccccccc}
 & & & & & x_1^1 & & & & & \\
 & & & & -x_1^2 & & x_1^2 & & & & \\
 & & & -x_1^3 & & x_2^3 & & x_1^3 & & & \\
 & & -x_1^4 & & -x_2^4 & & x_2^4 & & x_1^4 & & \\
 & & & & \cdots & & \cdots & & & & \\
 & -x_1^{2r-1} \cdots & & -x_{r-1}^{2r-1} & & x_r^{2r-1} & & x_{r-1}^{2r-1} & & \cdots & x_1^{2r-1} \\
-x_1^{2r} & & \cdots & & -x_r^{2r} & & x_r^{2r} & & \cdots & & x_1^{2r}
\end{array}$$

FIGURE 1. A Gelfand–Tsetlin pattern of GT_{2r}

3. An interacting particle model with exponential jumps

In this section we construct a random process $(X(t))_{t\geq 0}$ evolving on the subset of GT_k of Gelfand-Tsetlin patterns with non negative valued components. This process can be viewed as an interacting particle model. For this, we associate to a Gelfand-Tsetlin pattern $x = (x^1, \ldots, x^k)$, a configuration of particles on the integer lattice $\mathbb{Z}^2$ putting one particle labeled by (i, j) at point $(k - i, x_j^i)$ of $\mathbb{Z}^2$ for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, [\frac{i+1}{2}]\}$. Several particles can be located at the same point. In the sequel we will say "particle x_j^i" instead of saying "particle labeled by (i, j) located at point $(x_j^i, k-i)$". Let $q \in (0, 1)$. Consider two independent families

$$(\xi_j^i(n + \frac{1}{2}))_{i=1,\ldots,k, j=1,\ldots,[\frac{i+1}{2}];\, n\geq 0}, \quad \text{and} \quad (\xi_j^i(n))_{i=1,\ldots k, j=1,\ldots,[\frac{i+1}{2}];\, n\geq 1},$$

of identically distributed independent random variables such that

$$\mathbb{P}(\xi_1^1(\frac{1}{2}) = x) = \mathbb{P}(\xi_1^1(1) = x) = q^x(1 - q), \quad x \in \mathbb{N},$$

and the Markov kernel R on $\mathbb{N}$ defined by

$$R(x, y) = \begin{cases} \frac{1-q}{1+q}(q^{|x-y|} + q^{x+y}) & \text{if } y \in \mathbb{N}^* \\ \\ \frac{1-q}{1+q} q^x & \text{otherwise,} \end{cases}$$

for $x \in \mathbb{N}$. Actually the probability measure $R(x, .)$ on $\mathbb{N}$ is the law of the random variable $|x + \xi_1^1(1) - \xi_1^1(\frac{1}{2})|$.

Particles evolve as follows. At time 0 all particles are at zero, i.e. $X(0) = 0$. All particles, except those labeled by $(2l-1, l)$ for $l \in \{1, \ldots, [\frac{k+1}{2}]\}$ (i.e. particles near the wall), try to jump to the left at times $n+\frac{1}{2}$ and to the right at times n, $n \in \mathbb{N}$. Particles labeled by $(2l-1,l)$, for $l \in \{1,\dots,[\frac{k+1}{2}]\}$, jump on their own volition at integer times only. Notice that these particles can eventually move at half-integer times if they are pushed by another particle. Suppose that at time n there is one particle at point $(k-i, X^i_j(n))$ of $\mathbb{Z}^2$, for $i=1,\dots,k$, $j=1,\dots,[\frac{i+1}{2}]$. Positions of particles are updated downward as follows. Figure 2 gives an example of an evolution of a pattern between times n and $n+1$. In that example, the particles $(3,2)$ and $(4,2)$ are respectively pushed by the particles $(2,1)$ and $(3,1)$ and the particles $(3,1)$ and $(4,2)$ are respectively blocked by the particles $(2,1)$ and $(3,2)$ between times n and $n+\frac{1}{2}$. The particle $(3,1)$ is pushed by the particle $(2,1)$ and the particles $(3,2)$ and $(4,2)$ are respectively blocked by the particles $(2,1)$ and $(3,1)$ between times $n+\frac{1}{2}$ and $n+1$.

At time $n+1/2$: All particles except particles $X^{2l-1}_l(n)$ for $l \in \{1,\dots,[\frac{k+1}{2}]\}$, try to jump to the left one after another in the lexicographic order pushing the particles in order to stay in the set of Gelfand-Tsetlin patterns and being blocked by the initial configuration $X(n)$ of the particles. Let us indicate how the first three rows of a pattern are updated at time $n+\frac{1}{2}$.

- Particle $X^1_1(n)$ doesn't move. We let

$$X^1_1(n+\frac{1}{2}) = X^1_1(n).$$

- Particle $X^2_1(n)$ tries to jump to the left according to a geometric jump. It is blocked by $X^1_1(n)$. If it is necessary it pushes $X^3_2(n)$ to an intermediate position denoted by $\tilde{X}^3_2(n)$, i.e.

$$X^2_1(n+\frac{1}{2}) = \max\big(X^1_1(n), X^2_1(n) - \xi^2_1(n+\frac{1}{2})\big),$$
$$\tilde{X}^3_2(n) = \min\big(X^3_2(n), X^2_1(n+\frac{1}{2})\big).$$

- Particle $X^3_1(n)$ tries to move to the left according to a geometric jump being blocked by $X^2_1(n)$:

$$X^3_1(n+\frac{1}{2}) = \max\big(X^2_1(n), X^3_1(n) - \xi^3_1(n+\frac{1}{2})\big).$$

  Particle $\tilde{X}^3_2(n)$ doesn't move. We let

$$X^3_2(n+\frac{1}{2}) = \tilde{X}^3_2(n).$$

Suppose now that rows 1 through $l-1$ have been updated for some $l>1$. Then the particles $X^l_1(n),\dots,X^l_{[\frac{l+1}{2}]}(n)$ of row l are pushed to intermediate positions

$$\tilde{X}^l_i(n) = \min\big(X^l_i(n), X^{l-1}_{i-1}(n+\frac{1}{2})\big),\ i \in \{1,\dots,[\frac{l+1}{2}]\},$$

whit the convention $X^{l-1}_0(n+\frac{1}{2}) = +\infty$. Then particles $\tilde{X}^l_1(n),\dots,\tilde{X}^l_{[\frac{l}{2}]}(n)$ try to jump to the left according to geometric jump being blocked as follows by the initial position $X(n)$ of the particles. For $i=1,\dots,[\frac{l}{2}]$,

$$X^l_i(n+\frac{1}{2}) = \max\big(X^{l-1}_i(n), \tilde{X}^l_i(n) - \xi^l_i(n+\frac{1}{2})\big).$$

When l is odd, particle $\tilde{X}^l_{\frac{l+1}{2}}(n)$ doesn't move and we let

$$X^l_{\frac{l+1}{2}}(n+\frac{1}{2}) = \tilde{X}^l_{\frac{l+1}{2}}(n).$$

At time $n+1$: All particles except particles $X^{2l-1}_l(n+\frac{1}{2})$ for $l \in \{1,\dots,[\frac{k+1}{2}]\}$, try to jump to the right one after another in the lexicographic order pushing particles in order to stay in the set of Gelfand-Tsetlin patterns and being blocked by the initial configuration $X(n+\frac{1}{2})$ of the particles. Particles $X^{2l-1}_l(n+\frac{1}{2})$, for $l \in \{1,\dots,[\frac{k+1}{2}]\}$, try to move on their own volition according to the law $R(X^{2l-1}_l(n+\frac{1}{2}),.)$. The first three rows are updated as follows.

- Particle $X^1_1(n+\frac{1}{2})$ moves according to the law $R(X^1_1(n+\frac{1}{2}),.)$ pushing $X^2_1(n+\frac{1}{2})$ to an intermediate position $\tilde{X}^2_1(n+\frac{1}{2})$:

$$X^1_1(n+1) = \big|X^1_1(n+\frac{1}{2}) + \xi^1_1(n+1) - \xi^1_1(n+\frac{1}{2})\big|,$$
$$\tilde{X}^2_1(n+\frac{1}{2}) = \max\big(X^2_1(n+\frac{1}{2}), X^1_1(n+1)\big).$$

- Particle $\tilde{X}^2_1(n+\frac{1}{2})$ jumps to the right according to a geometric jump pushing $X^3_1(n+\frac{1}{2})$ to an intermediate position $\tilde{X}^3_1(n+\frac{1}{2})$, i.e.

$$X^2_1(n+1) = \tilde{X}^2_1(n+\frac{1}{2}) + \xi^2_1(n+1),$$
$$\tilde{X}^3_1(n+\frac{1}{2}) = \max\big(X^3_1(n+\frac{1}{2}), X^2_1(n+1)\big).$$

- Particle $X^3_2(n+\frac{1}{2})$ tries to move according to the law $R(X^1_1(n+\frac{1}{2}),.)$. It is blocked by $X^2_1(n+\frac{1}{2})$. Particle $\tilde{X}^3_1(n+\frac{1}{2})$ moves to the right according to a geometric jump. That is

$$X^3_2(n+1) = \max(\big|X^3_2(n+\frac{1}{2}) + \xi^3_2(n+1) - \xi^3_2(n+\frac{1}{2})\big|, X^2_1(n+\frac{1}{2})),$$
$$X^3_1(n+1) = \tilde{X}^3_1(n+\frac{1}{2}) + \xi^3_1(n+1).$$

Suppose rows 1 through $l-1$ have been updated for some $l>1$. Then particles of row l are pushed to intermediate positions

$$\tilde{X}^l_i(n+\frac{1}{2}) = \max\big(X^{l-1}_i(n+1), X^l_i(n+\frac{1}{2})\big),\ i \in \{1,\dots,[\frac{l+1}{2}]\},$$

with the convention $X^{l-1}_{\frac{l+1}{2}}(n+1) = 0$ when l is odd. Then particles $\tilde{X}^l_1(n+\frac{1}{2}),\dots,\tilde{X}^l_{[\frac{l}{2}]}(n+\frac{1}{2})$ try to jump to the right according to geometric jump being blocked by the initial position of the particles as follows. For $i=1,\dots,[\frac{l}{2}]$,

$$X^l_i(n+1) = \min\big(X^{l-1}_{i-1}(n+\frac{1}{2}), \tilde{X}^l_i(n+\frac{1}{2}) + \xi^l_i(n+1)\big).$$

When l is odd, particle $X^l_{\frac{l+1}{2}}(n+\frac{1}{2})$ is updated as follows.

$$X^l_{\frac{l+1}{2}}(n+1) = \min(\big|X^l_{\frac{l+1}{2}}(n+\frac{1}{2}) + \xi^l_{\frac{l+1}{2}}(n+1) - \xi^l_{\frac{l+1}{2}}(n+\frac{1}{2})\big|, X^{l-1}_{\frac{l-1}{2}}(n+\frac{1}{2})).$$

4. An interacting particle model with exponential waiting times

In this section we describe an interacting particle model on $\mathbb{Z}^2$ where particles try to jump by one rightwards or leftwards after exponentially distributed waiting times. The evolution of the particles is described by a random process $(Y(t))_{t\geq 0}$ on the subset of GT_k of Gelfand-Tsetlin patterns with non negative valued components. As in the previous model, at time $t \geq 0$ there is one particle labeled by (i,j) at point $(k-i, Y_j^i(t))$ of the integer lattice, for $i = 1,\dots,k$, $j = 1,\dots,[\frac{i+1}{2}]$. Every particle tries to jump to the left or to the right by one after independent exponentially distributed waiting time with mean 1. Particles are pushed and blocked according to the same rules as previously. That is when particle labeled by (i,j) wants to jump to the right at time $t \geq 0$ then

(1) if $i, j \geq 2$ and $Y_j^i(t^-) = Y_{j-1}^{i-1}(t^-)$ then the particles don't move and $Y(t) = Y(t^-)$,
(2) else particles $(i,j),(i+1,j),\dots,(i+l,j)$ jump to the right by one for l the largest integer such that $Y_j^{i+l}(t^-) = Y_j^i(t^-)$ i.e.

$$Y_j^i(t) = Y_j^i(t^-) + 1, \dots, Y_j^{i+l}(t) = Y_j^{i+l}(t^-) + 1.$$

When particle labeled by (i,j) wants to jump to the left at time $t \geq 0$ then

(1) if i is odd, $j = (i+1)/2$ and $Y_j^i(t^-) = 0$ then particle labeled by (i,j) is reflected by 0 and everything happens as described above when this particle try to jump to the right by one,
(2) if i is odd, $j = (i+1)/2$ and $Y_j^i(t^-) \geq 1$ then $Y_j^i(t) = Y_j^i(t^-) - 1$,
(3) if i is even or $j \neq (i+1)/2$, and $Y_j^i(t^-) = Y_j^{i-1}(t^-)$ then particles don't move,
(4) if i is even or $j \neq (i+1)/2$, and $Y_j^i(t^-) > Y_j^{i-1}(t^-)$ then particles $(i,j),(i+1,j+1),\dots,(i+l,j+l)$ jump to the left by one for l the largest integer such that $Y_{j+l}^{i+l}(t^-) = Y_j^i(t^-)$. Thus

$$Y_j^i(t) = Y_j^i(t^-) - 1, \dots, Y_{j+l}^{i+l}(t) = Y_{j+l}^{i+l}(t^-) - 1.$$

This random particle model is equivalent to a random tiling model with a wall, as it is explained in detail in [1].

5. Markov Kernel on the set of irreducible representations of the orthogonal group

When a finite dimensionnal representation V of a group G is completely reducible, there is a natural way that we'll recall later in our particular case to associate to this decomposition a probability measure on the set of irreducible representations of G. The transition probabilities of the random process $(X^k(t), t \geq 0)$ which will be proved to be Markovian are obtained in that manner. Actually we recover them considering decomposition into irreducible components of tensor products of particular irreducible representations of the special orthogonal group.

Let d be an integer greater than 2. Let us recall some usual properties of the finite dimensional representations of the compact group $SO(d)$ of $d \times d$ orthogonal matrices with determinant equal to 1 (see for instance [5] for more details). The set of finite dimensional representations of $SO(d)$ is indexed by the set

$$\{\lambda \in \mathbb{R}^r : 2\lambda_r \in \mathbb{N}, \lambda_i - \lambda_{i+1} \in \mathbb{N}, i = 1,\dots,r-1\},$$

when $d=2r+1$ and by the set

$$\{\lambda\in\mathbb{R}^r:\lambda_{r-1}+\lambda_r\in\mathbb{N},\,\lambda_i-\lambda_{i+1}\in\mathbb{N},\,i=1,\dots,r-1\},$$

when $d=2r$. Actually we are only interested with representations indexed by a subset $\mathcal{W}_d$ of these sets defined by

$$\mathcal{W}_d=\{\lambda\in\mathbb{R}^r:\lambda_r\in\mathbb{N},\lambda_i-\lambda_{i+1}\in\mathbb{N},\,i=1,\dots,r-1\},$$

when $d=2r+1$ and

$$\mathcal{W}_d=\{\lambda\in\mathbb{R}^r:\lambda_r\in\mathbb{Z},\lambda_{r-1}+\lambda_r\in\mathbb{N},\,\lambda_i-\lambda_{i+1}\in\mathbb{N},\,i=1,\dots,r-1\},$$

when $d=2r$. For $\lambda\in\mathcal{W}_d$, using standard notations, we denote by V_λ the so called irreducible representation with highest weight λ of $SO(d)$. The subset of $\mathcal{W}_d$ of elements having non-negative components is denoted by $\mathcal{W}_d^+$.

Let m be an integer and λ an element of $\mathcal{W}_d$. Consider the irreducible representations V_λ and V_{γ_m} of $SO(d)$, with $\gamma_m=(m,0,\cdots,0)$. The decomposition of the tensor product $V_\lambda\otimes V_{\gamma_m}$ into irreducible components is given by a Pieri-type formula for the orthogonal group. As it has been explained in [3], it can be deduced from [6]. We have

$$V_\lambda\otimes V_{\gamma_m}=\oplus_\beta M_{\lambda,\gamma_m}(\beta)V_\beta, \tag{1}$$

where the direct sum is over all $\beta\in\mathcal{W}_d$ such that

- when $d=2r+1$, there exists an integer $s\in\{0,1\}$ and $c\in\mathbb{N}^r$ which satisfy

$$\begin{cases} c\preceq\lambda, \quad c\preceq\beta \\ \sum_{i=1}^r(\lambda_i-c_i+\beta_i-c_i)+s=m, \end{cases}$$

  s being equal to 0 if $c_r=0$. In addition, the multiplicity $M_{\lambda,\gamma_m}(\beta)$ of the irreducible representation with highest weight β is the number of $(c,s)\in\mathbb{N}^r\times\{0,1\}$ satisfying these relations.
- when $d=2r$, there exists $c\in\mathbb{N}^{r-1}$ which verifies

$$\begin{cases} c\preceq|\lambda|, \quad c\preceq|\beta| \\ \sum_{k=1}^{r-1}(\lambda_k-c_k+\beta_k-c_k)+|\lambda_r-\mu_r|=m. \end{cases}$$

  In addition, the multiplicity $M_{\lambda,\gamma_m}(\beta)$ of the irreducible representation with highest weight β is the number of $c\in\mathbb{N}^{r-1}$ satisfying these relations.

Let us consider a family $(\mu_m)_{m\geq0}$ of Markov kernels on $\mathcal{W}_d$ defined by

$$\mu_m(\lambda,\beta)=\frac{\dim(V_\beta)}{\dim(V_\lambda)\dim(V_{\gamma_m})}M_{\lambda,\gamma_m}(\beta),$$

for $\lambda,\beta\in\mathcal{W}_d$ and $m\geq0$. It is known that for $\lambda\in\mathcal{W}_d$ the dimension of V_λ is given by the number $s_{d-1}(\lambda)$ defined in definition 2.1. Thus

$$\mu_m(\lambda,\beta)=\frac{s_{d-1}(\beta)}{s_{d-1}(\lambda)s_{d-1}(\gamma_m)}M_{\lambda,\gamma_m}(\beta).$$

Let $\xi_1,\dots,\xi_d$ be independent geometric random variables with parameter q and ϵ a Bernoulli random variable such that

$$\mathbb{P}(\epsilon=1)=1-\mathbb{P}(\epsilon=0)=\frac{q}{1+q}.$$

Consider a random variable T on $\mathbb{N}$ defined by

$$T = \sum_{i=1}^{d-1} \xi_i + \epsilon,$$

when $d = 2r + 1$ and

$$T = |\xi_1 - \xi_2| + \sum_{i=3}^{d} \xi_i,$$

when $d = 2r$.

Lemma 5.1. *The law of T is a measure ν on $\mathbb{N}$ defined by*

$$\nu(m) = \frac{1}{1+q}(1-q)^{d-1} q^m s_{d-1}(\gamma_m), \quad m \in \mathbb{N}.$$

Proof. When $d = 2r + 1$, for $m = 0$ the property is true. For $m \geq 1$

$$\begin{aligned}
\mathbb{P}(T = m) &= \frac{q}{1+q}\mathbb{P}(\sum_{i=1}^{d-1} \xi_i = m-1) + \frac{1}{1+q}\mathbb{P}(\sum_{i=1}^{d-1} \xi_i = m) \\
&= \frac{1}{1+q}(1-q)^{d-1} q^m \operatorname{Card}\{(k_1, \ldots, k_{d-1}) \in \mathbb{N}^{d-1} : \sum_{i=1}^{d-1} k_i \in \{m-1, m\}\} \\
&= \frac{1}{1+q}(1-q)^{d-1} q^m \sum_{(k_1,\ldots,k_{d-1}) \in \mathbb{N}^{d-1} : \sum_{i=1}^{d-1} k_i = m} (2 1_{k_1 \geq 1} + 1_{k_1 = 0}) \\
&= \frac{1}{1+q}(1-q)^{d-1} q^m s_{d-1}(\gamma_m).
\end{aligned}$$

So the lemma is proved in the odd case. Moreover

$$\mathbb{P}(|\xi_1 - \xi_2| = k) = \begin{cases} 2\frac{1-q}{1+q} q^k & \text{if } k \geq 1, \\ \frac{1-q}{1+q} & \text{otherwise.} \end{cases}$$

Thus when $d = 2r$,

$$\begin{aligned}
\mathbb{P}(T = m) &= \frac{1}{1+q}(1-q)^{d-1} q^m \sum_{(k_1,\ldots,k_{d-1}) \in \mathbb{N}^{d-1} : \sum_{i=1}^{d-1} k_i = m} (2 1_{k_1 \geq 1} + 1_{k_1 = 0}) \\
&= \frac{1}{1+q}(1-q)^{d-1} q^m s_{d-1}(\gamma_m).
\end{aligned}$$

□

Lemma 5.1 implies in particular that the measure ν is a probability measure. Thus one defines a Markov kernel P_d on $\mathcal{W}_d$ by letting

$$P_d(\lambda, \beta) = \sum_{m=0}^{+\infty} \mu_m(\lambda, \beta)\nu(m), \tag{2}$$

for $\lambda, \beta \in \mathcal{W}_d$. We'll see that the kernel P_d describes the evolution of the $(d-1)^{\text{th}}$ row of the random process on the set of Gelfand-Tsetlin patterns observed at integer times.

Proposition 5.2. *For $\lambda, \beta \in \mathcal{W}_d$,*

$$P_d(\lambda,\beta) = \sum_{c\in\mathbb{N}^r : c\preceq \lambda,\beta} (1-q)^{d-1}\frac{s_{d-1}(\beta)}{s_{d-1}(\lambda)} q^{\sum_{i=1}^r(\lambda_i+\beta_i-2c_i)}(1_{c_r>0}+\frac{1_{c_r=0}}{1+q})$$

when $d=2r+1$ and

$$P_d(\lambda,\beta) = \sum_{c\in\mathbb{N}^{r-1} : c\preceq |\lambda|,|\beta|} \frac{(1-q)^{d-1}}{q+1}\frac{s_{d-1}(\beta)}{s_{d-1}(\lambda)} q^{\sum_{i=1}^{r-1}(\lambda_i+\beta_i-2c_i)+|\lambda_r-\beta_r|}$$

when $d=2r$.

Proof. The proposition follows immediately from the tensor product rules recalled for the decomposition (1). □

6. Random matrices

Let us denote by $\mathcal{M}_{d,d'}$ the set of $d\times d'$ real matrices. A standard Gaussian variable on $\mathcal{M}_{d,d'}$ is a random variable having a density with respect to the Lebesgue measure on $\mathcal{M}_{d,d'}$ equal to

$$M\in\mathcal{M}_{d,d'} \mapsto \frac{1}{\sqrt[dd']{2\pi}}\exp(-\frac{1}{2}\mathrm{tr}(MM^*)).$$

We write $\mathcal{A}_d$ for the set $\{M\in\mathcal{M}_{d,d} : M+M^*=0\}$ of antisymmetric $d\times d$ real matrices, and $i\mathcal{A}_d$ for the set $\{iM : M\in\mathcal{A}_d\}$. Since a matrix in $i\mathcal{A}_d$ is Hermitian, it has real eigenvalues $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_d$. Morever, antisymmetry implies that $\lambda_{d-i+1}=-\lambda_i$, for $i=1,\cdots,[d/2]+1$, in particular $\lambda_{[d/2]+1}=0$ when d is odd. Consider the subset $\mathcal{C}_d$ of $\mathbb{R}_+^{[\frac{d}{2}]}$ defined by

$$\mathcal{C}_d=\{x\in\mathbb{R}^{[\frac{d}{2}]} : x_1>\cdots>x_{[\frac{d}{2}]}>0\},$$

and its closure

$$\bar{\mathcal{C}}_d=\{x\in\mathbb{R}^{[\frac{d}{2}]} : x_1\geq\cdots\geq x_{[\frac{d}{2}]}\geq 0\}.$$

Definition 6.1. *We define the function h_d on $\mathcal{C}_d$ by*

$$h_d(\lambda)=c_d(\lambda)^{-1}V_d(\lambda),\quad \lambda\in\mathcal{C}_d,$$

where the functions V_d and c_d are given by :

$$V_n(\lambda)=\prod_{1\leq i<j\leq[\frac{d}{2}]}(\lambda_i-\lambda_j)\prod_{1\leq i<j\leq[\frac{d}{2}]}(\lambda_i+\lambda_j)\prod_{1\leq i\leq[\frac{d}{2}]}\lambda_i^{\varepsilon},$$

$$c_n(\lambda)=\prod_{1\leq i<j\leq[\frac{d}{2}]}(j-i)\prod_{1\leq i<j\leq[\frac{d}{2}]}(d-j-i)\prod_{1\leq i\leq[\frac{d}{2}]}([\frac{d}{2}]+\frac{1}{2}-i)^{\varepsilon},$$

whit ε equal to 1 when $d\notin 2\mathbb{N}$ and 0 otherwise.

The next proposition is a consequence of Propositions 4.8 and 5.1 of [3].

Proposition 6.2. *Let $(M(n), n\geq 0)$, be a random process on $i\mathcal{A}_d$ defined by*

$$M(n)=\sum_{l=1}^n Y_l\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}Y_l^*,$$

where the Y_l's are independent standard Gaussian variables on $\mathcal{M}_{d,2}$. If $\Lambda(n)$ is the vector of $\bar{\mathcal{C}}_d$ whose components are the $[\frac{d}{2}]$ largest eigenvalues of $M(n)$, $n\in\mathbb{N}$, *then the random process* $(\Lambda(n), n \geq 0)$ *is a Markov chain on* $\bar{\mathcal{C}}_d$ *with transition probabilities*

$$p_d(x, dy) = \frac{h_d(y)}{h_d(x)} m_d(x,y)\, dy,$$

for $x, y \in \mathcal{C}_d$*, where* dy *is the Lebesgue measure on* $\mathbb{R}_+^{[\frac{d}{2}]}$ *and*

$$m_d(x,y) = \int_{\mathbb{R}_+^r} 1_{\{z \preceq x,y\}} e^{-\sum_{i=1}^m (y_i + x_i - 2z_i)}\, dz$$

when $d = 2r+1$ *and*

$$m_d(x,y) = \int_{\mathbb{R}_+^{r-1}} 1_{\{z \preceq |x|,|y|\}} e^{-\sum_{i=1}^{r-1} (x_i + y_i - 2z_i)} (e^{-|x_r - y_r|} + e^{-(x_r + y_r)})\, dz$$

when $d = 2r$.

7. Results

The main result of our paper states in particular that if only one row of the patterns $(X(t), t \geq 0)$ is considered by itself, it found to be a Markov process too. Actually we state the result for the process observed at integer times, even if the process observed at the whole time is also Markovian as we'll see in section 8.

Theorem 7.1. *The random process* $(X^k(n))_{n \geq 0}$ *is a Markov process on* $\mathcal{W}_{k+1}^+$*. If we denote by* R_k *its transition kernel then*

- $R_1 = R$.
- *when* k *is even* $R_k = P_{k+1}$*,*
- *when* k *is an odd integer greater than 2*

$$R_k(x,y) = \begin{cases} P_{k+1}(x,y) + P_{k+1}(x,\tilde{y}) & \text{if } y_{\frac{k+1}{2}} \neq 0 \\ P_{k+1}(x,y) & \text{otherwise,} \end{cases}$$

for $x, y \in \mathcal{W}_{k+1}^+$*, where* $\tilde{y} = (y_1, \ldots, y_{\frac{k-1}{2}}, -y_{\frac{k+1}{2}})$.

Corollary 7.2. *Let* $(e_1, \ldots, e_{[\frac{k+1}{2}]})$ *be the canonical basis of* $\mathbb{R}^{[\frac{k+1}{2}]}$*. The random process* $(Y^k(t), t \geq 0)$ *is a Markov process with infinitesimal generator defined by*

$$A_k(\lambda, \beta) = \begin{cases} 2\frac{s_k(\beta)}{s_k(\lambda)} 1_{\beta \in \mathcal{W}_k} & \text{if } k \text{ is odd, } \lambda_{\frac{k+1}{2}} = 0 \text{ and } \beta_{\frac{k+1}{2}} = 1 \\ \frac{s_k(\beta)}{s_k(\lambda)} 1_{\beta \in \mathcal{W}_k} & \text{otherwise,} \end{cases}$$

for $\lambda \in \mathcal{W}_k$*, and* $\beta \in \{\lambda + e_1, \ldots, \lambda + e_{[\frac{k+1}{2}]}, \lambda - e_1, \ldots, \lambda - e_{[\frac{k+1}{2}]}\}$.

If $(\Lambda(n), n \geq 0)$ is the process of eigenvalues considered in Proposition 6.2 with $d = k+1$ then the following theorem holds.

Theorem 7.3. *Letting* $q = 1 - \frac{1}{N}$*, the process* $(\frac{X^k(n)}{N}, n \geq 1)$ *converges in distribution towards the process of eigenvalues* $(\Lambda(n), n \geq 1)$ *as* N *goes to infinity.*

8. PROOFS

8.1. **Proof of Theorem 7.1.** For $k = 1$, Theorem 7.1 is clearly true. The proof of the theorem for $k \geq 2$ rests on an intertwining property and an application of a Pitman and Rogers criterion given in [7].

Notation 8.1. *Let ξ_1 and ξ_2 be two independent geometric random variables. For $x, a \in \mathbb{N}$ such that $x \geq a$, the law of the random variable*

$$\max(a, x - \xi_1),$$

is denoted by $\overset{a\leftarrow}{P}(x, .)$. For $x, b \in \mathbb{N}$ such that $x \leq b$ we denote by $\overset{\rightarrow b}{P}(x, .)$ and $\overset{\rightarrow b}{R}(x, .)$ the laws of the random variables

$$\min(b, x + \xi_1) \quad \textit{and} \quad \min(b, |x + \xi_1 - \xi_2|).$$

For $x, y \in \mathbb{R}^2$ such that $x \leq y$ we let

$$P(x, y) = (1 - q)q^{y-x}.$$

The two following lemmas are proved by straightforward computations.

Lemma 8.2. *For $a, x, y \in \mathbb{N}$ such that $a \leq y \leq x$*

$$\overset{a\leftarrow}{P}(x, y) = \begin{cases} (1-q)q^{x-y} & \text{if } a + 1 \leq y \\ q^{x-a} & \text{if } y = a. \end{cases}$$

For $b, x, y \in \mathbb{N}$ such that $b \geq y \geq x$

$$\overset{\rightarrow b}{P}(x, y) = \begin{cases} (1-q)q^{y-x} & \text{if } y \leq b - 1 \\ q^{b-x} & \text{if } y = b. \end{cases}$$

For $b, x, y \in \mathbb{N}$ such that $b \geq y, x$

$$\overset{\rightarrow b}{R}(x, y) = \begin{cases} \frac{1-q}{1+q}(q^{|y-x|} + q^{x+y}) & \text{if } y \leq b - 1, y > 0 \\ \frac{1-q}{1+q}q^{x} & \text{if } y \leq b - 1, y = 0 \\ \frac{1}{1+q}q^{b}(q^{-x} + q^{x}) & \text{if } y = b, y > 0 \\ 1 & \text{if } y = b, y = 0. \end{cases}$$

Lemma 8.3. *For $(x, y, z) \in \mathbb{N}^3$ such that $0 < z \leq y$*

$$\sum_{u=0}^{z}(1_{u=0} + 2\,1_{u>0})R(u, x)\overset{u\leftarrow}{P}(y, z) = (1 - q)(1_{x=0} + 2\,1_{x>0})q^{x\vee z+y-2z}. \tag{3}$$

For $(x, y, a) \in \mathbb{N}^3$ such that $a \leq y$ and $y \leq x$

$$\sum_{u=a}^{y} q^{u}\overset{u\leftarrow}{P}(x, y) = q^{x-y}q^{a}. \tag{4}$$

For $(x, y, a) \in \mathbb{N}^3$ such that $y \leq a$ and $x \leq y$

$$\sum_{v=y}^{a} q^{-v}\overset{\rightarrow v}{P}(x, y) = q^{y-x}q^{-a}. \tag{5}$$

For $y \in \mathbb{N}, y' \in \mathbb{N}^*$ *such that* $y' \leq a$

$$\sum_{v=y'}^{a} q^{v \vee y - 2v} \overset{\to v}{R}(y \wedge v, y') = \frac{1}{1-q} q^{-a} R(y, y'). \tag{6}$$

Let us first prove Theorem 7.1 for $k = 2$. Consider the set

$$\mathcal{W}_{2,3}^{+} = \{(z, y) \in \mathbb{N}^2 : z \leq y\}.$$

Define a Markov kernel S_2 on $\mathcal{W}_{2,3}^{+}$ by letting

$$S_2((z_0, y_0), (z, y)) = \begin{cases} (1-q)^2 \frac{s_2(y)}{s_2(y_0)} q^{y_0 + y - 2z} 1_{z \leq y_0 \wedge y} & \text{if } z > 0 \\ \frac{(1-q)^2}{1+q} \frac{s_2(y)}{s_2(y_0)} q^{y_0 + y} & \text{if } z = 0. \end{cases}$$

for $(z_0, y_0), (z, y) \in \mathcal{W}_{2,3}^{+}$ and another one L_2 from $\mathcal{W}_{2,3}^{+}$ to $\mathbb{N} \times \mathcal{W}_{2,3}^{+}$ by letting

$$L_2((z_0, y_0), (x, z, y)) = (1_{x=0} + 2\, 1_{x>0}) \frac{1}{s_2(y)} 1_{x \leq y} 1_{(z_0, y_0) = (z, y)},$$

for $(z_0, y_0), (z, y) \in \mathcal{W}_{2,3}^{+}$ and $x \in \mathbb{N}$. The fact that S_2 is a Markov kernel follows from Proposition 5.2 with $d = 3$. The random process

$$(X_1^1(n), X_1^2(n - \frac{1}{2}), X_1^2(n))_{n \geq 1},$$

is clearly Markovian. Let us denote by Q_2 its transition kernel. Then Q_2, L_2 and S_2 satisfy the following intertwining property.

Lemma 8.4.

$$L_2 Q_2 = S_2 L_2.$$

Proof. For $(x, z, y), (x', z', y') \in \mathbb{N} \times \mathcal{W}_{2,3}^{+}$ such that $x \leq y$ and $x' \leq y'$

$$Q_2((x, z, y), (x', z', y')) = R(x, x') \overset{x \leftarrow}{P}(y, z') P(x' \vee z', y').$$

Thus

$$L_2 Q_2((z, y), (x', z', y')) = \sum_{x=0}^{z'} \frac{s_1(x)}{s_2(y)} R(x, x') \overset{x \leftarrow}{P}(y, z') P(x' \vee z', y')$$

As L_2, S_2 and Q_2 are Markov kernels, it is sufficient to prove the identity for $z' > 0$. In that case the identity (3) of Lemma 8.3 implies that

$$\sum_{x=0}^{z'} \frac{(1_{x=0} + 2\, 1_{x>0})}{s_2(y)} R(x, x') \overset{x \leftarrow}{P}(y, z') = (1-q)(1_{x'=0} + 2\, 1_{x'>0}) q^{x' \vee z' + y - 2z'}.$$

Thus

$$L_2 Q_2((z, y), (x', z', y')) = \frac{(1_{x'=0} + 2\, 1_{x'>0})}{s_2(y)} (1-q)^2 q^{y + y' - 2z'},$$

which proves that

$$L_2 Q_2 = S_2 L_2.$$

□

Since the random process

$$(X_1^1(n), X_1^2(n-\frac{1}{2}), X_1^2(n))_{n\geq 1}$$

is Markovian with transition kernel Q_2, the intertwining property stated in Lemma 8.4 and the criterion of Pitman and Rogers given in [7] imply the following proposition. It states that the second row of the random process on the set of Gelfand-Tsetlin patterns is Markovian and gives its transition kernel.

Proposition 8.5. *We let $X_1^2(-\frac{1}{2}) = X_1^2(1) = 0$. The random process*

$$(X_1^2(n-\frac{1}{2}), X_1^2(n))_{n\geq 0}$$

is a Markov process on $\mathcal{W}_{2,3}^+$ with transition kernel S_2.

As for $(z,y) \in \mathcal{W}_{2,3}^+$ the probability $S_2((z,y),.)$ doesn't depend on z, Theorem 7.1 easily follows from Proposition 8.5 when $k=2$.

For the general case one defines the random process $(Z^k(n), Y^k(n))_{n\geq 1}$ by letting

$$Z^k(n) = (X_1^k(n-\frac{1}{2}), \dots, X_{[\frac{k}{2}]}^k(n-\frac{1}{2})),$$
$$Y^k(n) = X^k(n),$$

for $n \geq 1$ and $Z^k(0) = Y^k(0) = 0$. Let us notice that Z^k is equal to X^k when k is even, whereas it is obtained from X^k by deleting its smallest component when k is odd. We consider the subset $\mathcal{W}_{k,k+1}^+$ of $\mathcal{W}_k^+ \times \mathcal{W}_{k+1}^+$ defined by

$$\mathcal{W}_{k,k+1}^+ = \{(z,y) \in \mathcal{W}_k^+ \times \mathcal{W}_{k+1}^+ : z \preceq y\},$$

and define a Markov kernel S_k on $\mathcal{W}_{k,k+1}^+$ by letting for every $(z,y),(z',y') \in \mathcal{W}_{k,k+1}^+$

$$S_k((z,y),(z',y')) = (1-q)^k \frac{s_k(y')}{s_k(y)} q^{\sum_{i=1}^r (y_i+y_i'-2z_i)} (1_{z_r>0} + \frac{1_{z_r=0}}{1+q}) 1_{z'\preceq y,y'} \tag{7}$$

when $k=2r$, and

$$S_k((z,y),(z',y')) = (1-q)^{k-1} \frac{s_k(y')}{s_k(y)} R(y_r, y_r') q^{\sum_{i=1}^{r-1} (y_i+y_i'-2z_i)} 1_{z'\preceq y,y'} \tag{8}$$

when $k=2r-1$. The fact that for $(z,y) \in \mathcal{W}_{k,k+1}^+$ the measure $S_k((z,y),.)$ is a probability measure is a consequence of Proposition 5.2 with $d=k+1$.

Notation. Since for $(z,y) \in \mathcal{W}_{k,k+1}^+$ the probability $S_k((z,y),.)$ doesn't depend on z, it is denoted by $S_k(y,.)$ when there is no ambiguity.

Even if the Markov kernel P_{k+1} is relevant for our purpose, we'll prove Theorem 7.1 showing that the Markov kernel S_k describes the evolution of the k^{th} row of the process $(X(t), t\geq 0)$ observed at the whole time. We'll prove it by induction on k.

Lemma 8.6. *If the random process*

$$(Z^{k-1}(n), Y^{k-1}(n))_{n\geq 1},$$

is a Markov process on $\mathcal{W}_{k-1,k}^+$ with transition kernel S_{k-1} then the random process

$$(Y^{k-1}(n), Z^k(n), Y^k(n))_{n\geq 1}$$

is a Markov process on the set

$$\{(x,(z,y)) \in \mathcal{W}_k^+ \times \mathcal{W}_{k,k+1}^+ : x \preceq y\}.$$

If we denote its transition kernel by Q_k then for $(u,z,y),(x,z',y') \in \mathcal{W}_k^+ \times \mathcal{W}_{k,k+1}^+$ such that $u \preceq y$ and $x \preceq y'$

$$Q_k((u,z,y),(x,z',y')) = \sum_{v\in\mathbb{N}^{r-1}} S_{k-1}(u,(v,x)) \overset{\rightarrow v_{r-1}}{R}(y_r \wedge v_{r-1}, y'_r)$$

$$\times \prod_{i=1}^{r-1} \overset{u_i \leftarrow}{P}(y_i \wedge v_{i-1}, z'_i) \prod_{i=1}^{r-1} \overset{\rightarrow v_{i-1}}{P}(z'_i \vee x_i, y'_i), \tag{9}$$

when $k = 2r-1$ and

$$Q_k((u,z,y),(x,z',y')) = \sum_{v\in\mathbb{N}^{r-1}} S_{k-1}(u,(v,x)) \overset{u_r \leftarrow}{P}(y_r \wedge v_{r-1}, z'_r)$$

$$\times \prod_{i=1}^{r-1} \overset{u_i \leftarrow}{P}(y_i \wedge v_{i-1}, z'_i) \prod_{i=1}^{r} \overset{\rightarrow v_{i-1}}{P}(z'_i \vee x_i, y'_i), \tag{10}$$

when $k = 2r$. In the odd and the even cases $v_0 = +\infty$ and the sum runs over $v = (v_1,\dots,v_{r-1}) \in \mathbb{N}^{r-1}$ such that $v_i \in \{y'_{i+1},\dots,x_i \wedge z'_i\}$, for $i \in \{1,\dots,r-1\}$

Proof. The dynamic of the model implies that the process

$$(Z^{k-1}(n), Y^{k-1}(n), Z^k(n), Y^k(n), n \geq 0)$$

is Markovian. Since for $(z,y) \in \mathcal{W}_{k-1,k}^+$ the transition probability $S_{k-1}((z,y),.)$ doesn't depend on z, the Markovianity of the process

$$(Y^{k-1}(n), Z^k(n), Y^k(n), n \geq 0)$$

follows. Identities (9) and (10) are deduced from the blocking and pushing interactions of the model. □

Proof of Theorem 7.1 for every integer k follows as in the case when $k = 2$ from an intertwining property. Let us define Markov Kernel L_k from $\mathcal{W}_{k,k+1}^+$ to $\mathcal{W}_k^+ \times \mathcal{W}_{k,k+1}^+$ by letting for $x \in \mathcal{W}_k^+$ and $(z,y),(z_0,y_0) \in \mathcal{W}_{k,k+1}^+$

$$L_k((z_0,y_0),(x,y,z)) = 1_{(z_0,y_0)=(z,y)} \frac{s_{k-1}(x)}{s_k(y)} 1_{x\preceq y}, \tag{11}$$

when k is odd and

$$L_k((z_0,y_0),(x,y,z)) = (1_{\{0\}}(x_{\frac{k}{2}}) + 2\,1_{\mathbb{N}^*}(x_{\frac{k}{2}}))1_{(z_0,y_0)=(z,y)} \frac{s_{k-1}(x)}{s_k(y)} 1_{x\preceq y}, \tag{12}$$

when k is even. The following proposition generalizes Lemma 8.4.

Proposition 8.7. *The Markov kernels S_k, L_k and Q_k defined as in the identities (7), (8) and (11), (12) and (9), (10) satsify the intertwining*

$$L_k Q_k = S_k L_k.$$

Proof. For $(z,y) \in \mathcal{W}_{k,k+1}^+$, $(x,z',y') \in \mathcal{W}_k^+ \times \mathcal{W}_{k,k+1}^+$ such that $x \preceq y'$,

$$L_k Q_k((z,y),(x,z',y')) = \sum_{u\in\mathcal{W}_k^+} L_k((z,y),(u,z,y)) Q_k((u,z,y),(x,z',y')).$$

We prove separately the even and the odd cases. When $k=2r$, the sum is equal to

$$\sum_{(u,v)\in\mathbb{N}^r\times\mathbb{N}^{r-1}} \frac{s_{k-1}(x)}{s_k(y)}(1_{\{0\}}(u_r)+2\,1_{\mathbb{N}^*}(u_r))(1-q)^{2r-2}R(u_r,x_r)q^{\sum_{i=1}^{r-1}(x_i+u_i-2v_i)} \times P(z_1'\vee x_1,y_1')\prod_{i=1}^r \overset{u_i\leftarrow}{P}(y_i\wedge v_{i-1},z_i')\prod_{i=2}^r \overset{\rightarrow v_{i-1}}{P}(z_i'\vee x_i,y_i').$$

where the sum runs over $(u,v)\in\mathbb{N}^r\times\mathbb{N}^{r-1}$ such that $u_r\in\{0,\dots,z_r'\}$, $v_i\in\{y_{i+1}',\dots,x_i\wedge z_i'\}$, $u_i\in\{v_i\vee y_{i+1},\dots,z_i'\}$, for $i\in\{1,\dots,r-1\}$. Thus the sum equals

$$\sum_{v\in\mathbb{N}^{r-1}}\frac{s_{k-1}(x)}{s_k(y)}(1-q)^{2r-2}q^{\sum_{i=1}^{r-1}x_i}P(z_1'\vee x_1,y_1')\prod_{i=2}^r q^{-2v_{i-1}}\overset{\rightarrow v_{i-1}}{P}(z_i'\vee x_i,y_i') \times\sum_{u\in\mathbb{N}^r}(1_{\{0\}}(u_r)+2\,1_{\mathbb{N}^*}(u_r))(1-q)^{2r-2}R(u_r,x_r)\prod_{i=1}^r q^{u_i}\overset{u_i\leftarrow}{P}(y_i\wedge v_{i-1},z_i').$$

For a fixed v the sum over u is equal to

$$\sum_{u_r=0}^{z_r'}(1_{\{0\}}(u_r)+2\,1_{\mathbb{N}^*}(u_r))R(u_r,x_r)\overset{u_r\leftarrow}{P}(y_r\wedge v_{r-1},z_r')\prod_{i=1}^{r-1}\sum_{u_i=v_i\vee y_{i+1}}^{z_i'}q^{u_i}\overset{u_i\leftarrow}{P}(y_i\wedge v_{i-1},z_i').$$

Since L_k and Q_k are Markov kernels it is sufficient to consider the case when $z_r>0$. In that case, identities (3) and (4) of Lemma 8.3 imply that the sum over u equals

$$(1_{\{0\}}(x_r)+2\,1_{\mathbb{N}^*}(x_r))q^{x_r\vee z_r'+y_r\wedge v_{r-1}-2z_r'}(1-q)\prod_{i=1}^{r-1}q^{y_i\wedge v_{i-1}-z_i'+v_i\vee y_{i+1}},$$

i.e.

$$(1_{\{0\}}(x_r)+2\,1_{\mathbb{N}^*}(x_r))q^{x_r\vee z_r'+y_r-2z_r'+\sum_{i=1}^{r-1}y_i+v_i-z_i'}(1-q).$$

Thus

$$L_{2r}Q_{2r}((z,y),(x,z',y'))$$

equals

$$\frac{s_{k-1}(x)}{s_k(y)}(1-q)^{2r-1}(1_{\{0\}}(x_r)+2\,1_{\mathbb{N}^*}(x_r))q^{x_r\vee z_r'+y_r-2z_r'+\sum_{i=1}^{r-1}y_i-z_i'}q^{\sum_{i=1}^{r-1}x_i} \times P(z_1'\vee x_1,y_1')\prod_{i=2}^r\sum_{v_{i-1}=y_i}^{x_{i-1}\vee z_{i-1}'}q^{-v_{i-1}}\overset{\rightarrow v_{i-1}}{P}(z_i'\vee x_i,y_i').$$

Identity (5) of Lemma 8.3 gives that

$$\prod_{i=2}^r\sum_{v_{i-1}=y_i}^{x_{i-1}\vee z_{i-1}'}q^{-v_{i-1}}\overset{\rightarrow v_{i-1}}{P}(z_i'\vee x_i,y_i')=\prod_{i=2}^r q^{y_i'-z_i'\vee x_i-x_{i-1}\wedge z_{i-1}'} = q^{y_r'-z_r'\vee x_r-x_1\wedge z_1'}q^{\sum_{i=2}^{r-1}y_i'-x_i-z_i'},$$

which implies

$$L_{2r}Q_{2r}((z,y),(x,z',y'))=\frac{s_{k-1}(x)}{s_k(y)}(1-q)^{2r}(1_{\{0\}}(x_r)+2\,1_{\mathbb{N}^*}(x_r))q^{\sum_{i=1}^r y_i+y_i'-2z_i'},$$

and achieves the proof for the even case. Similarly when $k = 2r-1$

$$L_{2r-1}Q_{2r-1}((z,y),(x,z',y')) = \sum_{u,v\in\mathbb{N}^{r-1}} \frac{s_{k-1}(x)}{s_k(y)} q^{\sum_{i=1}^{r-1} x_i - 2v_i} \overset{\rightarrow v_{r-1}}{R}(y_r \wedge v_{r-1}, y'_r)$$
$$\times \prod_{i=1}^{r-1} q^{u_i} \overset{u_i\leftarrow}{P}(y_i \wedge v_{i-1}, z'_i) \prod_{i=1}^{r-1} \overset{\rightarrow v_{i-1}}{P}(z'_i \vee x_i, y'_i),$$

where the sum runs over $(u,v) \in \mathbb{N}^{r-1} \times \mathbb{N}^{r-1}$ such that $v_i \in \{y'_{i+1}, \ldots, x_i \wedge z'_i\}$, $u_i \in \{v_i \vee y_{i+1}, \ldots, z'_i\}$, for $i \in \{1, \ldots, r-1\}$. We obtain the intertwining in a quite similar way as in the even case, using identities (4), (5) and (6) of Lemma 8.3. □

The following proposition generalizes Proposition 8.5.

Proposition 8.8. *The random process $(Z^k(n), Y^k(n))_{n\geq 1}$, is Markovian with transition kernel S_k defined in (7) and (8).*

Proof. The random process $(X^k(t), t \geq 0)$ is conditionally independent of the processes $(X^l(t), t \geq 0)$, for $l = 1, \ldots, k-2$, given the process $(X^{k-1}(t), t \geq 0)$. So the property can be proved by induction on k. Proposition 8.5 claims that Proposition 8.8 is true for $k = 2$. Suppose that the proposition is true for a fixed interger $k-1$ greater that 1. Lemma 8.6 implies that the process

$$(Y^{k-1}(n), Z^k(n), Y^k(n))_{n\geq 1}$$

is Markovian with transition kernel Q_k. The intertwining property of Proposition 8.7 implies, by using the Pitman and Rogers criterion given in [7], that the process

$$(Z^k(n), Y^k(n))_{n\geq 1}$$

is Markovian with probability S_k. □

The Markov kernels P_{k+1} and S_k satisfy

$$P_{k+1}(y,y') = \sum_{z'\in\mathcal{W}_k^+} S_k(y,(z',y')), \quad y,y' \in \mathcal{W}_{k+1}^+.$$

Thus Theorem 7.1 is an immediate corollary of Proposition 8.8.

8.2. **Proof of Corollary 7.2.** The proof of Corollary 7.2 rests on a similar argument as in section 2.7 of [2].

Lemma 8.9. *Let $T_1(q)$ and $T_2(q)$ be two (possibly infinite) lower and upper triangular matrices, whose matrix coefficients are polynomials in an indeterminate $q > 0$:*

$$\begin{cases} T_1(q) = A_0 + qA_1 + q^2A_2 + \ldots, \\ T_2(q) = B_0 + qB_1 + q^2B_2 + \ldots, \end{cases}$$

and assume that $A_0 = B_0 = I$. Then for $t \in \mathbb{R}_+$,

$$\lim_{q\to 0} (T_1(q)T_2(q))^{[t/q]} = \exp(t(A_1 + B_1)).$$

Proof. Because of the triangularity assumption, the lemma follows, as in the proof of Lemma 2.21 of [2], from the claim for finite size matrices, which is standard. □

Lemma 8.9 implies immediately the following proposition.

Proposition 8.10. *Letting $q = \frac{1}{N}$, the process $(X([Nt]), t \geq 0)$ converges in distribution towards the process $(Y(t), t \geq 0)$ as N goes to infinity.*

Proof. It follows from Lemma 8.9 by taking

$$\begin{cases} T_1(q)(x,y) = \mathbb{P}(X(n+\frac{1}{2}) = y | X(n) = x), \\ \\ T_2(q)(x,y) = \mathbb{P}(X(n+1) = y | X(n+\frac{1}{2}) = x), \end{cases}$$

for $x, y \in GT_k$. □

With the help of the identities of Proposition 5.2, Theorem 7.1 and Lemma 2.21 of [2] imply that the process

$$(X^k([Nt]), t \geq 0)$$

converges towards a Markov process with infinitesimal generator equal to A_k as N goes to infinity. The convergence stated in Proposition 8.10 achieves the proof of Corollary 7.2.

8.3. **Proof of Theorem 7.3.** Let $(x_N)_{N\geq 1}$ be a sequence of elements of $\mathcal{W}^+_{k+1}$ such that $\frac{x_N}{N}$ converges to $x \in \mathcal{C}_{k+1}$ as N goes to infinity and $(\nu_N)_{N \geq 1}$ be a sequence of probability measures on $\mathcal{W}^+_{k+1}$ defined by

$$\nu_N = \sum_{y \in \mathcal{W}^+_{k+1}} R_k(x_N, y) \delta_{\frac{1}{N}y}.$$

Propositions 5.2 and 6.2 imply that the measure ν_N converges to the measure p_{k+1} defined in Proposition 6.2 as N goes to infinity. Theorem 7.3 follows.

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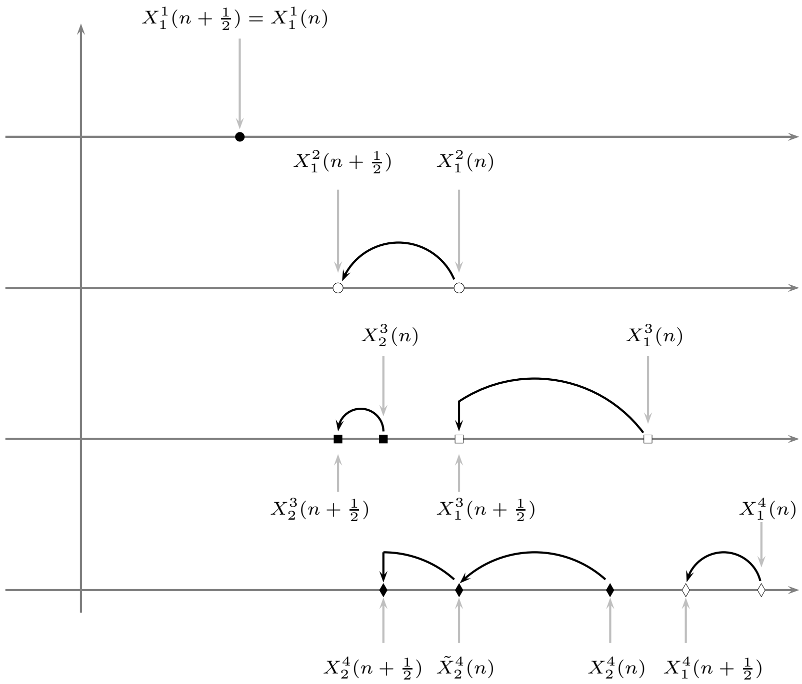
Interactions between times n and $n + \frac{1}{2}$



Interactions between times $n + \frac{1}{2}$ and $n + 1$

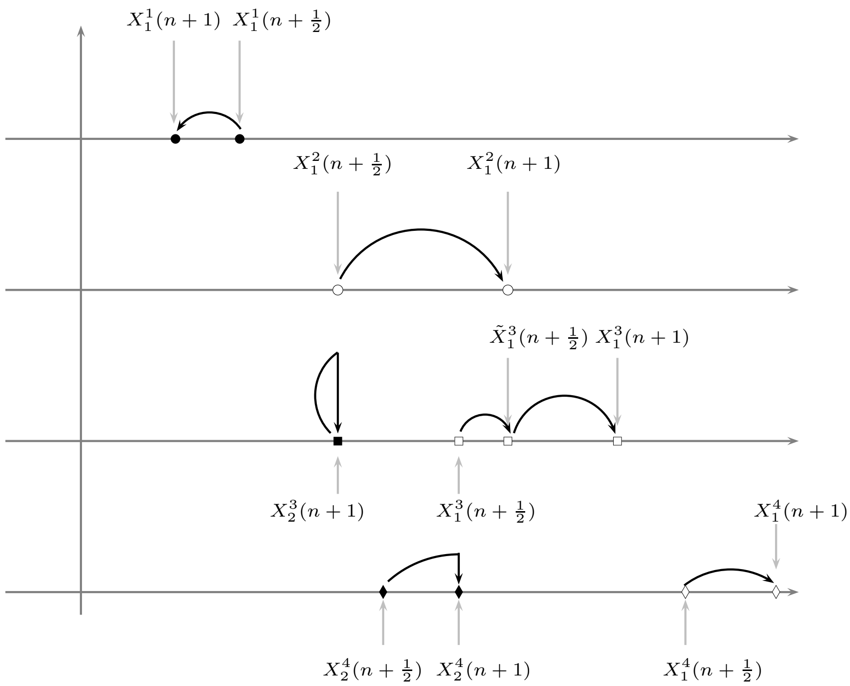


FIGURE 2. An example of blocking and pushing interactions between times n and $n+1$ for $k = 4$. Different kinds of dots represent different particles.