On parsimonious edge-colouring of graphs with maximum degree three

J.-L. Fouquet and J.-M. Vanherpe
L.I.F.O., Faculté des Sciences, B.P. 6759
Université d'Orléans, 45067 Orléans Cedex 2, FR

Abstract

In a graph G of maximum degree Δ let γ denote the largest fraction of edges that can be Δ edge-coloured. Albertson and Haas showed that $\gamma \geq \frac{13}{15}$ when G is cubic . We show here that this result can be extended to graphs with maximum degree 3 with the exception of a graph on 5 vertices. Moreover, there are exactly two graphs with maximum degree 3 (one being obviously the Petersen graph) for which $\gamma = \frac{13}{15}$. This extends a result given by Steffen. These results are obtained by using structural properties of the so called δ-minimum edge colourings for graphs with maximum degree 3.

Keywords : Cubic graph; Edge-colouring;
Mathematics Subject Classification (2010) : 05C15.

1 Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [15] that these graphs can be edge-coloured with 4 colours. Let $\phi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G. It is often of interest to try to use one colour (say δ) as few as possible. When it is optimal following this constraint, we shall say that such a parsimonious edge-colouring ϕ is $\delta - minimum$. In [3] we gave without proof (in French, see [6] for a translation) results on $\delta - minimum$ edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [13, 14]. Some other results which were not stated formally in [4] are needed here. The purpose of Section 2 is to give those results as structural properties of δ-minimum edge-colourings as well as others which will be useful in Section 3.

An edge colouring of G using colours $\alpha, \beta, \gamma, \delta$ is said to be δ-*improper* provided that adjacent edges having the same colours (if any) are coloured with δ. It is clear that a proper edge colouring (and hence a δ-minimum edge-colouring) of G is a particular δ-improper edge colouring. For a proper or δ-improper edge colouring ϕ of G, it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges coloured with x by ϕ. For $x, y \in \{\alpha, \beta, \gamma, \delta\}$,$x \neq y$, $\phi(x, y)$ is the partial subgraph of G spanned by these two colours, that is $E_\phi(x) \cup E_\phi(y)$ (this subgraph being a union of paths and even cycles where the colours x and y alternate). Since any two δ-minimum edge-colourings of G have the same number of edges coloured δ we shall denote by $s(G)$ this number (the *colour number* as defined by Steffen in [13]).

As usual, for any undirected graph G, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we suppose that $|V(G)| = n$ and $|E(G)| = m$. Moreover, $V_i(G)$ denotes the set of vertices of G of degree i, and when no confusion is possible we shall write V_i instead of $V_i(G)$. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2 On δ-minimum edge-colouring

The graph G considered in this section will have maximum degree 3.

Lemma 1 *Let ϕ be a $\delta-$improper colouring of G then there exists a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$*

Proof Let ϕ be a δ-improper edge colouring of G. If ϕ is a proper colouring, we are done. Hence, assume that uv and uw are coloured δ. If $d(u) = 2$ we can change the colour of uv to α, β or γ since v is incident to at most two colours in this set.

If $d(u) = 3$ assume that the third edge uz incident to u is also coloured δ, then we can change the colour of uv for the same reason as above.

If uz is coloured with α, β or γ, then v and w are incident to the two remaining colours of the set $\{\alpha, \beta, \gamma\}$ otherwise one of the edges uv, uw can be recoloured with the missing colour. W.l.o.g., consider that uz is coloured α then v and w are incident to β and γ. Since u has degree 1 in $\phi(\alpha, \beta)$ let P be the path of $\phi(\alpha, \beta)$ which ends on u. We can assume that v or w (say v) is not the other end vertex of P. Exchanging α and β along P does not change the colours incident to v. But now uz is coloured α and we can change the colour of uv with β.

In each case, we get hence a new δ-improper edge colouring ϕ_1 with $E_{\phi_1}(\delta) \subsetneq E_\phi(\delta)$. Repeating this process leads us to construct a proper edge colouring of G with $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$ as claimed. $\square$

Proposition 2 *Let $v_1, v_2, \ldots, v_k \in V(G)$ be such that $G - \{v_1, v_2, \ldots, v_k\}$ is 3-edge colourable. Then $s(G) \leq k$.*

Proof Let us consider a 3-edge colouring of $G - \{v_1, v_2, \ldots v_k\}$ with α, β and γ and let us colour the edges incident to $v_1, v_2, \ldots, v_k$ with δ. We get a δ-improper edge colouring ϕ of G. Lemma 1 gives a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$. Hence ϕ' has at most k edges coloured with δ and $s(G) \leq k$. $\square$

Proposition 2 above has been obtained by Steffen [13] for cubic graphs.

Lemma 3 *Let ϕ be a $\delta-$improper colouring of G then $|E_\phi(\delta)| \geq s(G)$.*

Proof Applying Lemma 1, let ϕ' be a proper edge colouring of G such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$. We clearly have $|E_\phi(\delta)| \geq |E_{\phi'}(\delta)| \geq s(G)$. $\square$

Theorem 4 *[6] Let G be a graph of maximum degree 3 and ϕ be a δ-minimum colouring of G. Then the following hold.*

1. *$E_\phi(\delta) = A_\phi \cup B_\phi \cup C_\phi$ where an edge e in A_ϕ (B_ϕ, C_ϕ respectively) belongs to a uniquely determined cycle $C_{A_\phi}(e)$ ($C_{B_\phi}(e)$, $C_{C_\phi}(e)$ respectively) with precisely one edge coloured δ and the other edges being alternately coloured α and β (β and γ, α and γ respectively).*
2. *Each edge having exactly one vertex in common with some edge in A_ϕ (B_ϕ, C_ϕ respectively) is coloured γ (α, β, respectively).*
3. *The multiset of colours of edges of $C_{A_\phi}(e)$ ($C_{B_\phi}(e)$, $C_{C_\phi}(e)$ respectively) can be permuted to obtain a (proper) δ-minimum edge-colouring of G in which the colour δ is moved from e to an arbitrarily prescribed edge.*
4. *No two consecutive vertices of $C_{A_\phi}(e)$ ($C_{B_\phi}(e)$, $C_{C_\phi}(e)$ respectively) have degree 2.*
5. *The cycles from 1 that correspond to distinct edges of $E_\phi(\delta)$ are vertex-disjoint.*
6. *If the edges $e_1, e_2, e_3 \in E_\phi(\delta)$ all belong to A_ϕ (B_ϕ, C_ϕ respectively), then the set $\{e_1, e_2, e_3\}$ induces in G a subgraph with at most 4 edges.*

Lemma 5 *[4] Let ϕ be a δ-minimum edge-colouring of G. For any edge $e = uv \in E_\phi(\delta)$ with $d(v) \leq d(u)$ there is a colour $x \in \{\alpha, \beta, \gamma\}$ present at v and a colour $y \in \{\alpha, \beta, \gamma\} - \{x\}$ present at u such that one of connected components of $\phi(x, y)$ is a path of even length joining the two ends of e. Moreover, if $d(v) = 2$, then both colours of $\{\alpha, \beta, \gamma\} - \{x\}$ satisfy the above assertion.*

An edge of $E_\phi(\delta)$ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$. From Lemma 5 it is clear that if $d(u) = 3$ and $d(v) = 2$ for an edge $e = uv \in E_\phi(\delta)$, the A_ϕ , B_ϕ and C_ϕ are not pairwise disjoint; indeed, if the colour γ is present at the vertex v, then $e \in A_\phi \cap B_\phi$.

When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with e. We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ.

For each edge $e \in E_\phi(\delta)$ (where ϕ is a δ-minimum edge-colouring of G) we can associate one or two odd cycles following the fact that e is in one or two sets among A_ϕ, B_ϕ or C_ϕ. Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_\phi(\delta)$.

By Theorem 4 any two cycles in $\mathcal{C}$ corresponding to edges in distinct sets A_ϕ, B_ϕ or C_ϕ are at distance at least 2. Assume that $C_1 = C_{A_\phi}(e_1)$ and $C_2 = C_{A_\phi}(e_2)$ for some edges e_1 and e_2 in A_ϕ. Can we say something about the subgraph of G whose vertex set is $V(C_1) \cup V(C_2)$? In general, we have no answer to this problem. However, when G is cubic and any vertex of G

lies on some cycle of $\mathcal{C}$ (we shall say that $\mathcal{C}$ is *spanning*), we have a property which will be useful later. Let us remark first that whenever $\mathcal{C}$ is spanning, we can consider that G is edge-coloured in such a way that the edges of the cycles of $\mathcal{C}$ are alternatively coloured with α and β (except one edge coloured δ) and the remaining perfect matching is coloured with γ. For this δ-minimum edge-colouring of G we have $B_\phi = \emptyset$ as well as $C_\phi = \emptyset$.

Lemma 6 *Assume that G is cubic and $\mathcal{C}$ is spanning. Let $e_1, e_2 \in A_\phi$ and let $C_1, C_2 \in \mathcal{C}$ such that $C_1 = C_{A_\phi}(e_1)$ and $C_2 = C_{A_\phi}(e_2)$. Then at least one of the following is true:*

(i) *C_1 and C_2 are at distance at least 2.*

(ii) *C_1 and C_2 are joined by at least 3 edges.*

(iii) *C_1 and C_2 have at least two chords each.*

Proof Since $e_1, e_2 \in A_\phi$ and $\mathcal{C}$ is spanning we have $B_\phi = C_\phi = \emptyset$. Let $C_1 = v_0v_1 \dots v_{2k_1}$ and $C_2 = w_0w_1 \dots w_{2k_2}$. Assume that C_1 and C_2 are joined by the edge v_0w_0. By Theorem 4, up to a re-colouring of the edges in C_1 and C_2, we can consider a δ-minimum edge-colouring ϕ such that $\phi(v_0v_1) = \phi(w_0w_1) = \delta$, $\phi(v_1v_2) = \phi(w_1w_2) = \beta$ and $\phi(v_0v_{2k_1}) = \phi(w_0w_{2k_2}) = \alpha$. Moreover each edge of G (in particular v_0w_0) incident with these cycles is coloured γ. We can change the colour of v_0w_0 in β. We obtain thus a new δ-minimum edge-colouring ϕ'. Performing that exchange of colours on v_0w_0 transforms the edges coloured δ v_0v_1 and w_0w_1 in two edges of $C_{\phi'}$ lying on odd cycles C'_1 and C'_2 respectively. We get hence a new set $\mathcal{C}' = \mathcal{C} - \{C_1, C_2\} \cup \{C'_1, C'_2\}$ of odd cycles associated to $\delta-$coloured edges in ϕ'.

From Theorem 4 C'_1 (C'_2 respectively) is at distance at least 2 from any cycle in $\mathcal{C} - \{C_1, C_2\}$. Hence $V(C'_1) \cup V(C'_2) \subseteq V(C_1) \cup V(C_2)$. It is an easy task to check now that (ii) or (iii) above must be verified. $\square$

Lemma 7 *[4] Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G. Then v_1 is the only vertex in $N(u)$ of degree 2 and for any other edge $e_2 \in E_\phi(\delta)$, $\{e_1, e_2\}$ induces a $2K_2$.*

3 Applications

3.1 On a result by Payan

In [10] Payan showed that it is always possible to edge-colour a graph of maximum degree 3 with three maximal matchings (with respect to the inclusion) and introduced henceforth a notion of *strong-edge colouring* where a strong edge-colouring means that one colour is a strong matching while the remaining colours are usual matchings. Payan conjectured that any $d-$regular graph has d pairwise disjoint maximal matchings and showed that this conjecture holds true for graphs with maximum degree 3.

The following result has been obtained first by Payan [10], but his technique does not exhibit explicitly the odd cycles associated to the edges of the strong matching and their properties.

Theorem 8 *Let G be a graph with maximum degree at most 3. Then G has a δ-minimum edge-colouring ϕ where $E_\phi(\delta)$ is a strong matching and, moreover, any edge in $E_\phi(\delta)$ has its two ends of degree 3 in G.*

Proof Let ϕ be a δ-minimum edge-colouring of G. From Theorem 4, any two edges of $E_\phi(\delta)$ belonging to distinct sets from among A_ϕ, B_ϕ and C_ϕ are at least at distance 2 and thus induce a strong matching. Hence, we have to find a δ-minimum edge-colouring where each set A_ϕ, B_ϕ or C_ϕ induces a strong matching (with the supplementary property that the end vertices of these edges have degree 3). That means that we can work on each set A_ϕ, B_ϕ and C_ϕ independently. Without loss of generality, we only consider A_ϕ here.

Assume that $A_\phi = \{e_1, e_2, \dots e_k\}$ and $A'_\phi = \{e_1, \dots e_i\}$ $(1 \leq i \leq k-1)$ is a strong matching and each edge of A'_ϕ has its two ends with degree 3 in G. Consider the edge e_{i+1} and let $C = C_{e_{i+1}}(\phi) = u_0, u_1 \dots u_{2p}$ be the odd cycle associated to this edge (Theorem 4).

Let us mark any vertex v of degree 3 on C with a $+$ whenever the edge of colour γ incident to this vertex has its other end which is a vertex incident to an edge of A'_ϕ and let us mark v with $-$ otherwise. By Theorem 4, no consecutive vertices on C have degree 2, that means that a vertex of degree 2 on C has its two neighbours of degree 3 and by Lemma 7 these two vertices are marked with a $-$. By Theorem 4 we cannot have two consecutive vertices marked with a $+$, otherwise we would have three edges of $E_\phi(\delta)$ inducing a subgraph with more than 4 edges, a contradiction. Hence, C must have two consecutive vertices of degree 3 marked with $-$ whatever is the number of vertices of degree 2 on C.

Let u_j and u_{j+1} be two vertices of C of degree 3 marked with $-$ (j being taken modulo $2p+1$). We can transform the edge colouring ϕ by exchanging colours on C uniquely, in such a way that the edge of colour δ of this cycle is u_ju_{j+1}. In the resulting edge colouring ϕ_1 we have $A_{\phi_1} = A_\phi - \{e_{i+1}\} \cup \{u_ju_{j+1}\}$ and $A'_{\phi_1} = A'_\phi \cup \{u_ju_{j+1}\}$ is a strong matching where each edge has its two ends of degree 3. Repeating this process we are left with a new δ-minimum colouring ϕ' where $A_{\phi'}$ is a strong matching. □

Corollary 9 *Let G be a graph with maximum degree 3 then there are $s(G)$ vertices of degree 3 pairwise non-adjacent $v_1 \dots v_{s(G)}$ such that $G - \{v_1 \dots v_{s(G)}\}$ is 3-colourable.*

Proof Pick a vertex on each edge coloured δ in a δ-minimum colouring ϕ of G where $E_\phi(\delta)$ is a strong matching (Theorem 8). We get a subset S of vertices satisfying our corollary. □

Steffen [13] obtained Corollary 9 for bridgeless cubic graphs.

3.2 Parsimonious edge colouring

Let $\chi'(G)$ be the classical chromatic index of G. For convenience let

$$c(G) = max\{|E(H)| : H \subseteq G \ , \chi'(H) = 3\}$$

$$\gamma(G) = \frac{c(G)}{|E(G)|}$$

Staton [12] (and independently Locke [9]) showed that whenever G is a cubic graph distinct from K_4 then G contains a bipartite subgraph (and hence a 3-edge colourable graph, by König's theorem [8]) with at least $\frac{7}{9}$ of the edges of G. Bondy and Locke [2] obtained $\frac{4}{5}$ when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever G is a cubic graph, we have $\gamma(G) \geq \frac{13}{15}$ while for graphs with maximum degree 3 they obtained $\gamma(G) \geq \frac{26}{31}$. Our purpose here is to show that $\frac{13}{15}$ is a lower bound for $\gamma(G)$ when G has maximum degree 3, with the exception of the graph G_5 depicted in Figure 1 below.

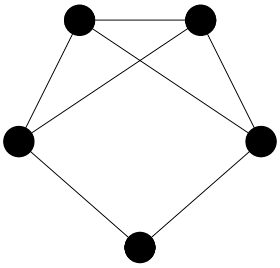


Figure 1: G_5

Lemma 10 *Let G be a graph with maximum degree 3 then $\gamma(G) = 1 - \frac{s(G)}{m}$.*

Proof Let ϕ be a δ-minimum edge-colouring of G. The restriction of ϕ to $E(G) - E_\phi(\delta)$ is a proper 3-edge-colouring, hence $c(G) \geq m - s(G)$ and $\gamma(G) \geq 1 - \frac{s(G)}{m}$.

If H is a subgraph of G with $\chi(H) = 3$, consider a proper 3-edge-colouring $\phi : E(H) \rightarrow \{\alpha, \beta, \gamma\}$ and let $\psi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be the continuation of ϕ with $\psi(e) = \delta$ for $e \in E(G) - E(H)$. By Lemma 1 there is a proper edge-colouring ψ' of G with $E_{\psi'}(\delta) \subseteq E_\psi(\delta)$ so that $|E(H)| = |E(G) - E_\psi(\delta)| \leq |E(G) - E_{\psi'}(\delta)| \leq m - s(G)$, $c(G) \leq m - s(G)$ and $\gamma(G) \leq 1 - \frac{s(G)}{m}$. $\square$

In [11], Rizzi shows that for triangle-free graphs of maximum degree 3, $\gamma(G) \geq 1 - \frac{2}{3g_o(G)}$ (where the *odd girth* of a graph G, denoted by $g_o(G)$, is the minimum length of an odd cycle in G).

Theorem 11 *Let G be a graph with maximum degree 3 then $\gamma(G) \geq 1 - \frac{2}{3g_o(G)}$.*

Proof Let ϕ be a δ-minimum edge-colouring of G and $E_\phi(\delta) = \{e_1, e_2 \ldots, e_{s(G)}\}$. $\mathcal{C}$ being the set of odd cycles associated to edges in $E_\phi(\delta)$, we write $\mathcal{C} = \{C_1, C_2 \ldots, C_{s(G)}\}$ and suppose that for $i = 1, 2 \ldots, s(G)$, e_i is an edge of C_i. We know by Theorem 4 that the cycles of $\mathcal{C}$ are vertex-disjoint.

Let us write $\mathcal{C} = \mathcal{C}_2 \cup \mathcal{C}_3$, where $\mathcal{C}_2$ denotes the set of odd cycles of $\mathcal{C}$ which have a vertex of degree 2, while $\mathcal{C}_3$ is for the set of cycles in $\mathcal{C}$ whose all vertices have degree 3. Let $k = |\mathcal{C}_2|$, obviously we have $0 \leq k \leq s(G)$ and $\mathcal{C}_2 \cap \mathcal{C}_3 = \emptyset$.

If $C_i \in \mathcal{C}_2$ we suppose without loss of generality that $C_i \in A_\phi$ and we have $|C_i| \geq g_o(G)$. Moreover, since any edge in C_i can be coloured δ (Theorem 4), we may assume that e_i has a vertex of degree 2. We can associate to e_i another odd cycle say $C_i^{'} \in B_\phi$ (Lemma 5) whose edges distinct from e_i form an even path of $\phi(\alpha, \gamma)$ using at least $\frac{g_o(G)-1}{2}$ edges, coloured γ, which are not edges of C_i.

When $|C_i| > g_o(G)$ or $|C_i'| > g_o(G)$ there are either at least $g_o(G)+2$ edges in C_i or at least $\frac{g_o(G)-1}{2}+1$ edges coloured γ in C_i'. If $|C_i| = |C_i'| = g_o(G)$ there is at least one edge coloured α in C_i' that is not an edge of C_i, otherwise all the edges coloured γ of C_i' would be chords of C_i, a contradiction since a such chord would form with vertices of C_i an odd cycle of length smaller than $g_o(G)$.

Hence, $C_i \cup C_i^{'}$ contains at least $g_o(G) + \frac{g_o(G)-1}{2} + 1 > \frac{3}{2} g_o(G)$ edges.

Consequently there are at least $\frac{3}{2} \times k \times g_o(G)$ edges in $\bigcup\limits_{C_i \in \mathcal{C}_2} (C_i \cup C_i^{'})$.

When $C_i \in \mathcal{C}_3$, C_i contains at least $g_o(G)$ edges, moreover, each vertex of C_i being of degree 3, there are at least $\frac{s(G)-k}{2} \times g_o(G)$ additionnal edges which are incident to a vertex of $\bigcup\limits_{C_i \in \mathcal{C}_3} C_i$.

Since $C_i \cap C_j = \emptyset$ and $C_i^{'} \cap C_j = \emptyset$ $(1 \leq i, j \leq s(G),\ i \neq j)$, we have

$$m \geq \frac{3}{2} g_o(G) \times k + (s(G)-k) \times g_o(G) + \frac{s(G)-k}{2} \times g_o(G) = \frac{3}{2} \times s(G) \times g_o(G).$$

Consequently $\gamma(G) = 1 - \frac{s(G)}{m} \geq 1 - \frac{2}{3 g_o(G)}$. □

As a matter of fact, $\gamma(G) > 1 - \frac{2}{3 g_o(G)}$ when the graph G contains vertices of degree 2. In a further work (see [5]) we refine the bound and prove that $\gamma(G) \geq 1 - \frac{2}{3 g_o(G)+2}$ when G is a graph of maximal degree 3 distinct from the Petersen graph.

Lemma 12 *[1] Let G be a graph with maximum degree 3. Assume that $v \in V(G)$ is such that $d(v) = 1$ then $\gamma(G) > \gamma(G - v)$.*

A *triangle* $T = \{a, b, c\}$ is said to be *reducible* whenever its neighbours are distinct. When T is a reducible triangle in G (G having maximum degree 3) we can obtain a new graph G' with maximum degree 3 by shrinking this triangle into a single vertex and joining this new vertex to the neighbours of T in G.

Lemma 13 *[1] Let G be a graph with maximum degree 3. Assume that $T = \{a, b, c\}$ is a reducible triangle and let G' be the graph obtained by reduction of this triangle. Then $\gamma(G) > \gamma(G')$.*

Theorem 14 *Let G be a graph with maximum degree 3. If $G \neq G_5$ then $\gamma(G) \geq 1 - \frac{\frac{2}{15}}{1+\frac{2}{3}\frac{|V_2|}{|V_3|}}$.*

Proof From Lemma 12 and Lemma 13 we can consider that G has only vertices of degree 2 or 3 and that G contains no reducible triangle.

Assume that we can associate a set P_e of at least 5 distinct vertices of V_3 for each edge $e \in E_\phi(\delta)$ in a δ-minimum edge-colouring ϕ of G. Assume moreover that

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P_{e'} = \emptyset \tag{1}$$

Then

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \geq 1 - \frac{\frac{|V_3|}{5}}{\frac{3}{2}|V_3| + |V_2|}$$

Hence

$$\gamma(G) \geq 1 - \frac{\frac{2}{15}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

It remains to see how to construct the sets P_e satisfying Property (1). Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_\phi(\delta)$. Let $e \in E_\phi(\delta)$, assume that e is contained in a cycle $C \in \mathcal{C}$ of length 3. By Theorem 4, the edges incident to that triangle have the same colour in $\{\alpha, \beta, \gamma\}$. This triangle is hence reducible, impossible. We can thus consider that each cycle of $\mathcal{C}$ has length at least 5. By Lemma 7, we know that whenever such a cycle contains vertices of V_2, their distance on this cycle is at least 3. Which means that every cycle $C \in \mathcal{C}$ contains at least 5 vertices in V_3 as soon as C has length at least 7 or C has length 5 but does not contain a vertex of V_2. For each edge $e \in E_\phi(\delta)$ contained in such a cycle we associate P_e as any set of 5 vertices of V_3 contained in the cycle.

There may exist edges in $E_\phi(\delta)$ contained in a 5-cycle of $\mathcal{C}$ having exactly one vertex in V_2. Let $C = a_1a_2a_3a_4a_5$ be such a cycle and assume that $a_1 \in V_2$. By Lemma 7, a_1 is the only vertex of degree 2 and by exchanging colours along this cycle, we can suppose that $a_1a_2 \in E_\phi(\delta)$. Since $a_1 \in V_2$, $e = a_1a_2$ is contained in a second cycle C' of $\mathcal{C}$ (see Remark **??**). If C' contains a vertex $x \in V_3$ distinct from a_2, a_3, a_4 and a_5 then we set $P_e = \{a_2, a_3, a_4, a_5, x\}$. Otherwise $C' = a_1a_2a_4a_3a_5$ and G is isomorphic to G_5, impossible.

The sets $\{P_e | \ e \in E_\phi(\delta)\}$ are pairwise disjoint since any two cycles of $\mathcal{C}$ associated to distinct edges in $E_\phi(\delta)$ are disjoint. Hence Property 1 holds and the proof is complete. □

Albertson and Haas [1] proved that $\gamma(G) \geq \frac{26}{31}$ when G is a graph with maximum degree 3 and Rizzi [11] obtained $\gamma(G) \geq \frac{6}{7}$. From Theorem 14 we get immediately for all graphs $G \neq G_5$ $\gamma(G) \geq \frac{13}{15}$, a better bound. Let us remark that we get also $\gamma(G) \geq \frac{13}{15}$ by Theorem 11 as soon as $g_o(G) \geq 5$.

Lemma 15 *Let G be a cubic graph which can be factored into $s(G)$ cycles of length 5 and has no reducible triangle. Then every 2-factor of G contains $s(G)$ cycles of length 5.*

Proof Since G has no reducible triangle, all cycles in a 2-factor have length at least 4. Let $\mathcal{C}$ be any 2-factor of G. Let us denote n_4 the number of cycles of length 4, n_5 the number of cycles of length 5 and n_{6+} the number of cycles on at least 6 vertices in $\mathcal{C}$. We have $5n_5 + 6n_{6+} \leq 5s(G) - 4n_4$.

If $n_4 + n_{6_+} = 0$, then $n_5 = s(G)$. If $n_4 + n_{6_+} > 0$, then the number of odd cycles in $\mathcal{C}$ is at most $n_5 + n_{6_+} \leq \frac{5s(G)-4n_4-n_{6_+}}{5} = \frac{5s(G)-(n_4+n_{6_+})-3n_4}{5} < s(G)$. A contradiction since a 2-factor of G contains at least $s(G)$ odd cycles. □

Corollary 16 *Let G be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. Then G is a cubic graph which can be factored into $s(G)$ cycles of length 5. Moreover every 2-factor of G has this property.*

Proof The optimum for $\gamma(G)$ in Theorem 14 is obtained whenever $s(G) = \frac{|V_3|}{5}$ and $|V_2| = 0$. That is, G is a cubic graph admitting a 2-factor of $s(G)$ cycles of length 5. Moreover by Lemma 13 G has no reducible triangle, the result comes from Lemma 15. $\square$

As pointed out by Albertson and Haas [1], the Petersen graph with $\gamma(G) = \frac{13}{15}$ supplies an extremal example for cubic graphs. Steffen [14] proved that the only cubic bridgeless graph with $\gamma(G) = \frac{13}{15}$ is the Petersen graph. In fact, we can extend this result to graphs with maximum degree 3 where bridges are allowed (excluding the graph G_5). Let P' be the cubic graph on 10 vertices obtained from two copies of G_5 (Figure 1) by joining by an edge the two vertices of degree 2.

Theorem 17 *Let G be a connected graph with maximum degree* 3 *such that $\gamma(G) = \frac{13}{15}$. Then G is isomorphic to the Petersen graph or to P'.*

Proof Let G be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$.

From Corollary 16, we can consider that G is cubic and G has a 2-factor of cycles of length 5. Let $\mathcal{C} = \{C_1 \dots C_{s(G)}\}$ be such a 2-factor ($\mathcal{C}$ is spanning). Let ϕ be a δ-minimum edge-colouring of G induced by this $2-$factor.

Without loss of generality consider two cycles in $\mathcal{C}$, namely C_1 and C_2, and let us denote $C_1 = v_1v_2v_3v_4v_5$ while $C_2 = u_1u_2u_3u_4u_5$ and assume that $v_1u_1 \in G$. From Lemma 6, C_1 and C_2 are joined by at least 3 edges or each of them has two chords. If $s(G) > 2$ there is a cycle $C_3 \in \mathcal{C}$. Without loss of generality, G being connected, we can suppose that C_3 is joined to C_1 by an edge. Applying one more time Lemma 6, C_1 and C_3 have two chords or are joined by at least 3 edges, contradiction with the constraints imposed by C_1 and C_2. Hence $s(G) = 2$ and G has 10 vertices and no 4-cycle, which leads to a graph isomorphic to P' or the Petersen graph as claimed. $\square$

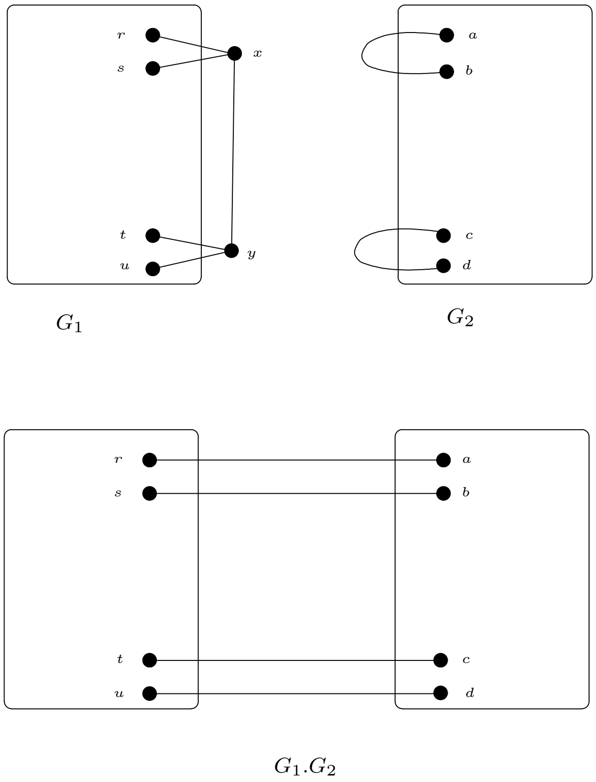
We can construct cubic graphs with chromatic index 4 (*snarks* in the litterature) which are cyclically 4- edge connected and having a 2-factor of C_5's.

Indeed, let G be a cubic cyclically 4-edge connected graph of order n and M be a perfect matching of G, $M = \{x_iy_i | i = 1 \dots \frac{n}{2}\}$. Let $P_1 \dots P_{\frac{n}{2}}$ be $\frac{n}{2}$ copies of the Petersen graph. For each P_i ($i = 1 \dots \frac{n}{2}$) we consider two edges at distance 1 apart e_i^1 and e_i^2. Let us observe that $P_i - \{e_i^1, e_i^2\}$ contains a 2-factor of two C_5's (C_i^1 and C_i^2).

We construct then a new cyclically 4-edge connected cubic graph H with chromatic index 4 by applying the well known operation dot-product (see Figure 2, see Isaacs [7] for a description and for a formal definition) on $\{e_i^1, e_i^2\}$ and the edge x_iy_i ($i = 1 \dots \frac{n}{2}$). We remark that the vertices of G vanish in the operation and the resulting graph H has a 2 factor of C_5, namely $\{C_1^1, C_1^2, \dots C_i^1, C_i^2, \dots C_{\frac{n}{2}}^1, C_{\frac{n}{2}}^2\}$.

We do not know an example of a cyclically 5-edge connected snark (except the Petersen graph) with a 2-factor of induced cycles of length 5.

Problem 18 Is there any cyclically 5-edge connected snark distinct from the Petersen graph with a 2-factor of C_5's ?



G_1

G_2

$G_1.G_2$

Figure 2: The dot product operation on graphs G_1, G_2.

As a first step towards the resolution of this Problem we propose the following Theorem. Recall that a *permutation graph* is a cubic graph having some 2-factor with precisely 2 odd cycles.

Theorem 19 *Let G be a cubic graph which can be factored into $s(G)$ induced odd cycles of length at least 5, then G is a permutation graph. Moreover, if G has girth 5 then G is the Petersen graph.*

Proof Let $\mathcal{F}$ be a 2-factor of $s(G)$ cycles of length at least 5 in G, every cycle of $\mathcal{F}$ being an induced odd cycle of G. We consider the δ-minimum edge-colouring ϕ such that the edges of all cycles of $\mathcal{F}$ are alternatively coloured α and β except for exactly one edge per cycle which is coloured with δ, all the remaining edges of G being coloured γ. By construction we have $B_\phi = C_\phi = \emptyset$ and $A_\phi = \mathcal{F}$.

Let xy be an edge connecting two distinct cycles of $\mathcal{F}$, say C_1 and C_2 ($x \in C_1$, $y \in C_2$). By Theorem 4, since any edge in C_1 or C_2 can be coloured δ, we may assume that there is an edge in C_1, say e_1, adjacent to x and coloured with δ, similarly there is on C_2 an edge e_2 adjacent to y and coloured with δ. Let z be the neighbour of y on C_2 such that $e_2 = yz$ and let t be the neighbour of z such that zt is coloured with γ. If $t \notin C_1$, there must be $C_3 \neq C_1$ such that $t \in C_3$, by Theorem 4 again there is an edge e_3 of C_3, adjacent to t and coloured with δ. But now $\{e_1, e_2, e_3\}$ induces a subgraph with at least 5 edges, a contradiction with Theorem 4.

It follows that $\mathcal{C}$ contains exactly two induced odd cycles and they are of equal length. Consequently G is a permutation graph. When these cycles have length 5, since G has girth 5 G is obviously the Petersen graph. □

When G is a cubic bridgeless planar graph, we know from the Four Colour Theorem that G is 3−edge colourable and hence $\gamma(G) = 1$. Albertson and Haas [1] gave $\gamma(G) \geq \frac{6}{7} - \frac{2}{35m}$ when G is a planar bridgeless graph with maximum degree 3. Our Theorem 14 improves this lower bound (allowing moreover bridges). On the other hand, they exhibit a family of planar graphs with maximum degree 3 (bridges are allowed) for which $\gamma(G) = \frac{8}{9} - \frac{2}{9n}$.

As Steffen in [14] we denote $g(\mathcal{F})$ the minimum length of an odd cycle in a 2-factor $\mathcal{F}$ and $g^+(G)$ the maximum of these numbers over all 2-factors. We suppose that $g^+(G)$ is defined, that is G has at least one 2-factor (when G is a cubic bridgeless graphs this condition is obviously fulfilled).

When G is cubic bridgeless, Steffen [14] showed that we have :

$$\gamma(G) \geq \quad \max\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\}$$

The difficult part being to show that $\gamma(G) \geq \frac{11}{12}$.

Theorem 20 *Let G be a graph with maximum degree 3. Then $\gamma(G) \geq 1 - \frac{2n}{(3n-|V_2|)g^+(G)}$.*

Proof By Lemma 12, we may assume $V_1 = \emptyset$. Hence, $m = \frac{1}{2}(2|V_2| + 3|V_3|)$, moreover $n = |V_2| + |V_3|$, henceforth $m = \frac{3n-|V_2|}{2}$. We have $\gamma(G) = 1 - \frac{s(G)}{m}$, obviously, $s(G) \leq \frac{n}{g^+(G)}$. The result follows. □

Theorem 21 *Let G be a graph with maximum degree 3 having at least one 2-factor. Assume that $|V_2| \leq \frac{n}{3}$ and $g^+(G) \geq 11$ then $\gamma(G) \geq \max\{1 - \frac{3}{4g^+(G)}, \frac{11}{12}\}$.*

Proof By assumption we have $V_1 = \emptyset$. From Theorem 20 we have just to prove that $\gamma(G) \geq \frac{11}{12}$. Following the proof of Theorem 14, we try to associate a set P_e of at least 8 distinct vertices of V_3 for each edge $e \in E_\phi(\delta)$ in a δ-minimum edge-colouring ϕ of G such that

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P_{e'} = \emptyset \tag{2}$$

Indeed, let $\mathcal{F}$ be a 2-factor of G where each odd cycle has length at least 11 and let $C_1, C_2 \dots C_{2k}$ be its set of odd cycles. We have, obviously $s(G) \leq 2k$. Let V'_3 and V'_2 be the sets of vertices of degree 3 and 2 respectively contained in these odd cycles . As soon as $|V'_3| \geq 8s(G)$ we have

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \geq 1 - \frac{\frac{|V'_3|}{8}}{\frac{3}{2}|V_3| + |V_2|} \tag{3}$$

which leads to

$$\gamma(G) \geq 1 - \frac{\frac{2|V'_3|}{24|V_3|}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

Since $|V_3| \geq |V'_3|$, we have

$$\gamma(G) \geq 1 - \frac{\frac{2}{24}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

and

$$\gamma(G) \geq \frac{11}{12}$$

as claimed.

It remains the case where $|V'_3| < 8s(G)$. Since each odd cycle has at least 11 vertices we have $|V'_2| > 11 \times 2k - |V'_3| > 3s(G)$.

$$\gamma(G) = \frac{m - s(G)}{m} \geq \frac{m - \frac{|V'_2|}{3}}{m}$$

We have

$$\frac{m - \frac{|V'_2|}{3}}{m} \geq \frac{11}{12}$$

when

$$m \geq 4|V'_2| \tag{4}$$

Since $|V_2| \leq \frac{n}{3}$ we have $|V_3| \geq \frac{2n}{3}$ and

$$m = 3\frac{|V_3|}{2} + |V_2| = 3\frac{n - |V_2|}{2} + |V_2| = 3\frac{n}{2} - \frac{|V_2|}{2} \geq 4\frac{n}{3} \geq 4|V'_2| \tag{5}$$

and the result holds. $\square$

Acknowledgment : the authors are grateful to the anonymous referees whose observations led to a number of improvments of this paper.

References

[1] M.O. Albertson and R. Haas. Parsimonious edge colouring. *Discrete Mathematics*, 148:1–7, 1996.

[2] J.A. Bondy and S. Locke. Largest bipartite subgraphs in triangle free graphs with maximum degree three. *J. Graph Theory*, 10:477–504, 1986.

[3] J.-L. Fouquet. Graphes cubiques d'indice chromatique quatre. *Annals of Discrete Mathematics*, 9:23–28, 1980.

[4] J.-L. Fouquet. *Contribution à l' étude des graphes cubiques et problèmes hamiltoniens dans les graphes orientés.* PhD thesis, Université Paris SUD, 1981.

[5] J.-L. Fouquet and J.-M. Vanherpe. A new bound for parsimonious edge-colouring of graphs with maximum degree three. http://hal.archives-ouvertes.fr/hal-00516702/PDF/Parcimonious_15_17_Version_27_Feb_2011–.pdf, 2011.

[6] J.-L. Fouquet and J.-M. Vanherpe. Tools for parsimonious edge-colouring of graphs with maximum degree three, January 2012. http://hal.archives-ouvertes.fr/hal-00502201/PDF/ToolsForParcimoniousColouring_Revision4_HAL.pdf.

[7] R. Isaacs. Infinite families of non-trivial trivalent graphs which are not Tait colorable. *Am. Math. Monthly*, 82:221–239, 1975.

[8] D. König. Über Graphen und ihre Anwendung auf Determinantentheorie un Mengenlehre. *Math.Ann*, 77:453–465, 1916.

[9] S.C. Locke. Maximum k-colourable subgraphs. *Journal of Graph Theory*, 6:123–132, 1982.

[10] C. Payan. Sur quelques problèmes de couverture et de couplage en combinatoire. Thèse d'état, 1977.

[11] R. Rizzi. Approximating the maximum 3-edge-colorable subgraph problem. *Discrete Mathematics*, 309(12):4166–4170, 2009.

[12] W. Staton. Edge deletions and the chromatic number. *Ars Combin*, 10:103–106, 1980.

[13] E. Steffen. Classifications and characterizations of snarks. *Discrete Mathematics*, 188:183–203, 1998.

[14] E. Steffen. Measurements of edge-uncolorability. *Discrete Mathematics*, 280:191–214, 2004.

[15] V.G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz*, 3:25–30, 1964.