Tools for parsimonious edge-colouring of graphs with maximum degree three

J.-L. Fouquet and J.-M. Vanherpe
L.I.F.O., Faculté des Sciences, B.P. 6759
Université d'Orléans, 45067 Orléans Cedex 2, FR

Abstract

The notion of a δ-minimum edge-colouring was introduced by J-L. Fouquet (in his french PhD Thesis [2]). Here we present some structural properties of δ-minimum edge-colourings, partially taken from the above thesis. The paper serves as an auxiliary tool for another paper submitted by the authors to Graphs and Combinatorics.

1 Introduction

Throughout this note, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [6] that these graphs can be edge-coloured with 4 colours. Let $\phi: E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G. It is often of interest to try to use one colour (say δ) as few as possible. In [1] we gave without proof (in French) results on δ-minimum edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [4] and [5]. The purpose of Section 2 is to give with their proofs those results as structural properties of δ-minimum edge-colourings.

An edge colouring of G using colours $\alpha, \beta, \gamma, \delta$ is said to be δ-*improper* provided that adjacent edges having the same colours (if any) are coloured with δ. It is clear that a proper edge colouring of G is a particular δ-improper edge colouring. For a proper or δ-improper edge colouring ϕ of G , it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges coloured with x by ϕ. For $x, y \in \{\alpha, \beta, \gamma, \delta\}$, $x \neq y$, $\phi(x, y)$ is the partial subgraph of G spanned by these two colours, that is $E_\phi(x) \cup E_\phi(y)$ (this subgraph being a union of paths and even cycles where the colours x and y alternate). Since any two δ-minimum edge-colourings of G have the same number of edges coloured δ we shall denote by $s(G)$ this number (the *colour number* as defined in[4]).

As usual, for any undirected graph G, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2 Structural properties of δ-minimum edge-colourings

The graph G considered in the following series of Lemmas will have maximum degree 3.

Lemma 1 *[2] Any 2-factor of G contains at least $s(G)$ disjoint odd cycles.*

Proof Assume that we can find a 2-factor of G with $k < s(G)$ odd cycles. Then let us colour the edges of this 2-factor with α and β, except one edge (coloured δ) on each odd cycle of our 2-factor and let us colour the remaining edges by γ. We get hence a new edge colouring ϕ with $E_\phi(\delta) < s(G)$, impossible. □

Lemma 2 *[2] Let ϕ be a δ-minimum edge-colouring of G. Any edge in $E_\phi(\delta)$ is incident to α, β and γ. Moreover each such edge has one end of degree 2 and the other of degree 3 or the two ends of degree 3.*

Proof Any edge of $E_\phi(\delta)$ is certainly adjacent to α, β and γ. Otherwise this edge could be coloured with the missing colour and we should obtain an edge colouring ϕ' with $|E_{\phi'}(\delta)| < |E_\phi(\delta)|$. □

Lemma 3 below was proven in [3], we give its proof for the sake of completeness.

Lemma 3 *[3] Let ϕ be a δ-improper colouring of G then there exists a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$. Moreover, if ϕ is improper, then ϕ' can be chosen so that $E_{\phi'}(\delta) \subsetneq E_\phi(\delta)$.*

Proof Let ϕ be a δ-improper edge colouring of G. If ϕ is a proper colouring, we are done. Hence, assume that uv and uw are coloured δ. If $d(u) = 2$ we can change the colour of uv to α, β or γ since v is incident to at most two colours in this set.

If $d(u) = 3$ assume that the third edge uz incident to u is also coloured δ, then we can change the colour of uv for the same reason as above.

If uz is coloured with α, β or γ, then v and w are incident to the two remaining colours of the set $\{\alpha, \beta, \gamma\}$ otherwise one of the edges uv, uw can be recoloured with the missing colour. W.l.o.g., consider that uz is coloured α then v and w are incident to β and γ. Since u has degree 1 in $\phi(\alpha, \beta)$ let P be the path of $\phi(\alpha, \beta)$ which ends on u. We can assume that v or w (say v) is not the other end vertex of P. Exchanging α and β along P does not change the colours incident to v. But now uz is coloured β and we can change the colour of uv to α.

In each case, we get hence a new δ-improper edge colouring ϕ_1 with $E_{\phi_1}(\delta) \subsetneq E_\phi(\delta)$. Repeating this process leads us to construct a proper edge colouring of G with $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$ as claimed. □

Corollary 4 *Any δ-minimum edge-colouring is proper.*

Lemma 5 *[2] Let ϕ be a δ-minimum edge-colouring of G. For any edge $e = uv \in E_\phi(\delta)$ with $d(v) \leq d(u)$ there is a colour $x \in \{\alpha, \beta, \gamma\}$ present at v and a colour $y \in \{\alpha, \beta, \gamma\} - \{x\}$ present at u such that one of connected components of $\phi(x, y)$ is a path of even length joining the two ends of e. Moreover, if $d(v) = 2$, then both colours of $\{\alpha, \beta, \gamma\} - \{x\}$ satisfy the above assertion.*

Proof Without loss of generality suppose that $x = \gamma$ is present at v and α, β are present at u (see Lemma 2). Then u is an endvertex of paths in both $\phi(\alpha, \gamma)$ and $\phi(\beta, \gamma)$, while there is $y \in \{\alpha, \beta\}$ such that v is an endvertex of a path in $\phi(x, y)$. Without loss of generality assume that both u and v are endvertices of paths in $\phi(\alpha, \gamma)$. If these paths are disjoint, we exchange the colours α and γ on the path with endvertex u and then recolour e with α; this yields a contradiction to the δ-minimality of ϕ. To conclude the proof note that if $d(v) = 2$, then v is an endvertex of paths in both $\phi(\alpha, \gamma)$ and $\phi(\beta, \gamma)$. □

An edge of $E_\phi(\delta)$ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$. From Lemma 5 it is clear that if $d(u) = 3$ and $d(v) = 2$ for an edge $e = uv \in E_\phi(\delta)$, the A_ϕ , B_ϕ and C_ϕ are not pairwise disjoint; indeed, if the colour γ is present at the vertex v, then $e \in A_\phi \cap B_\phi$.

When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with e. We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ.

Lemma 6 *[2] If G is a cubic graph then $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G)$ $(mod\ 2)$.*

Proof $\phi(\alpha, \beta)$ contains $2|A_\phi|+|B_\phi|+|C_\phi|$ vertices of degree 1 and must be even. Hence we get $|B_\phi| \equiv |C_\phi|$ $(mod\ 2)$. In the same way we get $|A_\phi| \equiv |B_\phi|$ $(mod\ 2)$ leading to $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G)$ $(mod\ 2)$. □

In the following lemma we consider an edge in A_ϕ, an analogous result holds true whenever we consider edges in B_ϕ or C_ϕ as well.

Lemma 7 *[2] Let ϕ be a δ-minimum edge-colouring of G and let e be an edge in A_ϕ then for any edge $e' \in C_{A_\phi}(e)$ there is a δ-minimum edge-colouring ϕ' such that $E_{\phi'}(\delta) = E_\phi(\delta) - \{e\} \cup \{e'\}$, $e' \in A_{\phi'}$ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. Moreover, each edge outside $C_{A_\phi}(e)$ but incident with this cycle is coloured γ, ϕ and ϕ' only differ on the edges of $C_{A_\phi}(e)$.*

Proof By exchanging colours δ and α and δ and β successively along the cycle $C_{A_\phi}(e)$, we are sure to obtain an edge colouring preserving the number of edges coloured δ. Since we have supposed that ϕ is δ-minimum, ϕ is proper by Corollary 4. At each step, the resulting edge colouring remains to be δ-minimum and hence proper. Hence, there is no edge coloured δ incident with $C_{A_\phi}(e)$, which means that every such edge is coloured with γ.

We can perform these exchanges until e' is coloured δ. In the δ-minimum edge-colouring ϕ' hence obtained, the two ends of e' are joined by a path of $\phi(\alpha, \beta)$. Which means that e' is in A_ϕ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. □

For each edge $e \in E_\phi(\delta)$ (where ϕ is a δ-minimum edge-colouring of G) we can associate one or two odd cycles following the fact that e is in one or two sets from among A_ϕ, B_ϕ, C_ϕ. Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_\phi(\delta)$.

Lemma 8 *[2] For each cycle $C \in \mathcal{C}$, there are no two consecutive vertices with degree two.*

Proof Otherwise, we exchange colours along C in order to put the colour δ on the corresponding edge and, by Lemma 2, this is impossible in a δ-minimum edge-colouring . □

Lemma 9 *[2] Let $e_1, e_2 \in E_\phi(\delta)$, such that $e_1 \neq e_2$ and let $C_1, C_2 \in \mathcal{C}$ be such that $C_1 \neq C_2$, $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ then C_1 and C_2 are (vertex) disjoint.*

Proof If e_1 and e_2 are contained in the same set A_ϕ, B_ϕ, or C_ϕ, we are done since their respective ends are joined by an alternating path of $\phi(x, y)$ for some two colours x and y in $\{\alpha, \beta, \gamma\}$.

Without loss of generality assume that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Assume moreover that there exists an edge e such that $e \in C_1 \cap C_2$. We have hence an edge $f \in C_1$ with exactly one end on C_2. We can exchange colours on C_1 in order to put the colour δ on f, which is impossible by Lemma 7. □

Lemma 10 *[2] Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G. Then v_1 is the only vertex in $N(u)$ of degree 2 and for any other edge $e_2 \in E_\phi(\delta)$, $\{e_1, e_2\}$ induces a $2K_2$.*

Proof We have seen in Lemma 2 that uv_1 has one end of degree 3 while the other has degree 2 or 3. Hence, we have $d(u) = 3$ and $d(v_1) = 2$. Let v_2 and v_3 the other neighbours of u. We can suppose without loss of generality that uv_2 is coloured α, uv_3 is coloured β and, finally v_1 is incident to an edge coloured γ, say v_1v_4.

Assume first that $d(v_2) = 2$. An alternating path of $\phi(\beta, \gamma)$ using the edge uv_3 ends with the vertex v_1, moreover v_2 is incident to an edge coloured γ since an alternating path of $\phi(\alpha, \gamma)$ using the edge uv_2 ends with v_1 (see Lemma 5), then, exchanging the colours along the component of $\phi(\beta, \gamma)$ containing v_2 allows us to colour uv_2 with γ and uv_1 with α. A new edge colouring ϕ' so obtained is such that $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, impossible.

Thus, $d(v_2) = 3$, and, by symmetry, $d(v_3) = 3$. We know that $e_1 \in B_\phi \cap C_\phi$ (see Lemma 5). By Lemma 7, since $e_1 \in C_{B_\phi}(e_1)$ the edges incident to v_3 in $C_{B_\phi}(e_1)$ are coloured with β and γ and the third edge incident to v_3 is coloured α. Similarly the vertex v_2 being on $C_{B_\phi}(e_1)$ is incident to colours α and γ and the third edge incident to v_2 is coloured β. Moreover the vertex v_4 being on both cycles $C_{B_\phi}(e_1)$ and $C_{C_\phi}(e_1)$ is incident to colours α, β, γ. Hence no edge coloured δ can be incident to v_2 nor v_3 nor v_4. It follows that for any edge e_2 in $E_\phi(\delta) - \{e_1\}$, the set $\{e_1, e_2\}$ induces a $2K_2$. □

Lemma 11 *[2] Let e_1 and e_2 be two edges of $E_\phi(\delta)$ $e_1 \neq e_2$. If e_1 and e_2 are contained in two distinct sets of A_ϕ, B_ϕ, C_ϕ then $\{e_1, e_2\}$ induces a $2K_2$, otherwise e_1, e_2 are joined by at most one edge.*

Proof By Lemma 10 one can suppose that all vertices incident with e_1, e_2 are of degree 3 so that there is exactly one of the sets A_ϕ, B_ϕ or C_ϕ containing e_1 (respectively e_2).

Assume in the first stage that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Since $C_{A_\phi}(e_1)$ and $C_{B_\phi}(e_2)$ are disjoint by Lemma 9, we know that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{A_\phi}(e_1)$ are coloured γ while those having exactly one end in $C_{B_\phi}(e)$ are coloured α. Hence there is no edge between e_1 and e_2 as claimed.

Assume in the second stage that $e_1 = u_1v_1, e_2 = u_2v_2 \in A_\phi$. Since $C_{A_\phi}(e_1)$ and $C_{B_\phi}(e_2)$ are disjoint by Lemma 9, we can consider that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{A_\phi}(e_1)$ or $C_{B_\phi}(e_2)$ are coloured γ. Assume that u_1u_2 and v_1v_2 are edges of G. We may suppose without loss of generality that u_1 and u_2 are incident to α while v_1 and v_2 are incident to β (if necessary, colours α and β can be exchanded on $C_{A_\phi}(e_1)$ and $C_{B_\phi}(e_2)$). We know that u_1u_2 and v_1v_2 are coloured γ. Let us colour e_1 and e_2 with γ and u_1u_2 with β and v_1v_2 with α. We get a new edge colouring ϕ' where $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 2$, contradiction since ϕ is a δ-minimum edge-colouring . □

Lemma 12 *[2] Let e_1, e_2 and e_3 be three distinct edges of $E_\phi(\delta)$ contained in the same set A_ϕ, B_ϕ or C_ϕ. Then $\{e_1, e_2, e_3\}$ induces a subgraph with at most four edges.*

Proof Without loss of generality assume that $e_1 = u_1v_1, e_2 = u_2v_2$ and $e_3 = u_3v_3 \in A_\phi$. From Lemma 11 we have just to suppose that (up to the names of vertices) $u_1u_3 \in E(G)$ and $v_1v_2 \in E(G)$. Possibly, by exchanging the colours α and β along the 3 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2 and e_3, we can suppose that u_1 and u_3 are incident to β while v_1 and v_2 are incident to α. Let ϕ' be obtained from ϕ when u_1u_3 is coloured with α, v_1v_2 with β and u_1v_1 with γ. It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a δ-minimum edge-colouring . □

Let us summarize most of the above results in a single Theorem.

Theorem 13 *Let G be a graph of maximum degree 3 and ϕ be a δ-minimum colouring of G. Then the following hold.*

1. *$E_\phi(\delta) = A_\phi \cup B_\phi \cup C_\phi$ where an edge e in A_ϕ (B_ϕ, C_ϕ respectively) belongs to a uniquely determined cycle $C_{A_\phi}(e)$ ($C_{B_\phi}(e)$, $C_{C_\phi}(e)$ respectively) with precisely one edge coloured δ and the other edges being alternately coloured α and β (β and γ, α and γ respectively).*

2. *Each edge having exactly one vertex in common with some edge in A_ϕ (B_ϕ, C_ϕ respectively) is coloured γ (α, β, respectively).*

3. *The multiset of colours of edges of $C_{A_\phi}(e)$ ($C_{B_\phi}(e)$, $C_{C_\phi}(e)$ respectively) can be permuted to obtain a (proper) δ-minimum edge-colouring of G in which the colour δ is moved from e to an arbitrarily prescribed edge.*

4. *No two consecutive vertices of $C_{A_\phi}(e)$ ($C_{B_\phi}(e)$, $C_{C_\phi}(e)$ respectively) have degree 2.*

5. *The cycles from* 1 *that correspond to distinct edges of* $E_\phi(\delta)$ *are vertex-disjoint.*

6. *If the edges* $e_1, e_2, e_3 \in E_\phi(\delta)$ *all belong to* A_ϕ *(*B_ϕ*,* C_ϕ *respectively), then the set* $\{e_1, e_2, e_3\}$ *induces in* G *a subgraph with at most* 4 *edges.*

References

[1] J.-L. Fouquet. Graphes cubiques d'indice chromatique quatre. *Annals of Discrete Mathematics*, 9:23–28, 1980.

[2] J.-L. Fouquet. *Contribution à l' étude des graphes cubiques et problèmes hamiltoniens dans les graphes orientés.* PhD thesis, Université Paris SUD, 1981.

[3] J.-L. Fouquet and J.-M. Vanherpe. On parcimonious edge-colouring of graphs with maximum degree three. Submitted to Graphs and Combinatorics.

[4] E. Steffen. Classifications and characterizations of snarks. *Discrete Mathematics*, 188:183–203, 1998.

[5] E. Steffen. Measurements of edge-uncolorability. *Discrete Mathematics*, 280:191–214, 2004.

[6] V.G. Vizing. On an estimate of the chromatic class of p-graphs. *Diskret. Analiz*, 3:25–30, 1964.