DISTRIBUTION OF POINTS AND HARDY TYPE INEQUALITIES IN SPACES OF HOMOGENEOUS TYPE

E. ROUTIN

Abstract. In the setting of spaces of homogeneous type, we study some Hardy type inequalities, which notably appeared in the proofs of local $T(b)$ theorems as in [AR]. We give some sufficient conditions ensuring their validity, related to the geometry and distribution of points in the homogeneous space. We study the relationships between these conditions and give some examples and counterexamples in the complex plane.

Contents

1. Introduction

The goal of this paper is to study, in the setting of a space of homogeneous type, what we call Hardy type inequalities. They notably appear in the proofs of local $T(b)$ theorems as in [H], [AR], where they play a crucial role to estimate some of the matrix coefficients involved in the arguments. The prototype of a Hardy type inequality is the following in the Euclidean space of dimension 1:

$$\left|\int_I\int_J \frac{f(y)g(x)}{x-y}\mathrm{d}y\mathrm{d}x\right| \leq C\left(\int_I |f(y)|^{\nu}\mathrm{d}y\right)^{\frac{1}{\nu}}\left(\int_J |g(x)|^{\nu'}\mathrm{d}x\right)^{\frac{1}{\nu'}},$$

2010 *Mathematics Subject Classification.* 42B20.

Key words and phrases. Space of homogeneous type, geometry and distribution of points, Hardy type inequalities, Layer decay properties, Monotone geodesic property.

where I, J are adjacent intervals, $\operatorname{supp} f \subset I$, $\operatorname{supp} g \subset J$, and $1 < \nu < \infty$, $\nu' = \frac{\nu}{\nu-1}$. The integral here is absolutely convergent, and this is an immediate consequence of the boundedness of the Hardy operator $H(f)(x) = \frac{1}{x}\int_0^x f(t)\mathrm{d}t$ (hence our terminology). The above 1−dimensional inequality easily extends to the Euclidean space in any dimension: for every $1 < \nu < +\infty$, there exists $C < +\infty$ such that for every disjoint dyadic cubes Q, Q' in $\mathbb{R}^n$, every function $f \in L^\nu(Q)$ supported on Q, $g \in L^{\nu'}(Q')$ supported on Q', the following integral is absolutely convergent and we have

$$\left|\int_Q \int_{Q'} \frac{f(y)g(x)}{|x-y|^n}\mathrm{d}x\mathrm{d}y\right| \leq C\|f\|_{L^\nu(Q)}\|g\|_{L^{\nu'}(Q')}.$$

This estimate, well known by the specialists, follows from the 1−dimensional one by expressing the fact that the singularity in the integral is supported along one direction, transverse to an hyperplane separating the disjoint cubes Q, Q'.

A similar result in the setting of a space of homogeneous type is to be expected, that is

$$\left|\int_I \int_J \frac{f(y)g(x)}{\mathrm{Vol}(B(x, \mathrm{dist}(x,y)))} d\mu(x)d\mu(y)\right| \leq C\|f\|_{L^\nu(I,d\mu)}\|g\|_{L^{\nu'}(J,d\mu)},$$

where $I \cap J = \varnothing$, and $1 < \nu < +\infty$, $1/\nu + 1/\nu' = 1$. Expectedly, it turns out that it holds without any restriction if I, J are Christ's dyadic cubes (in the sense of [C], see Section 2), even if the previous argument cannot be valid as the dyadic cubes in such a space do not follow any geometry. It seemed not to have been noticed in the literature before our work with P. Auscher [AR]. It relies in particular on the small layers for dyadic cubes. However, if I is a ball B and J, say, $2B\backslash B$, then it is not clear in general. It clearly depends on how B and $2B\backslash B$ see each other through their boundary. In [AR], we came up with some small boundary hypothesis on the space of homogeneous type (called the relative layer decay property) ensuring that the inequality was satisfied. We also showed that this property held in all doubling complete Riemannian manifolds, geodesic spaces and more generally in any monotone geodesic space of homogeneous type. The latter notion arose in geometric measure theory from the work of R. Tessera [T], and was recently proved by Lin, Nakai and Yang [LNY] to be equivalent to a chain ball notion introduced by S. Buckley [B2].

We continue this study in the present paper, investigating further these different conditions and the relationships they entertain. It appears that they are all connected to the way points are distributed in the homogeneous space. We produce some interesting examples and counterexamples in the complex plane (Theorem 2.5, see Section 2). A natural question that also arises is the following: if the Hardy type inequality for balls is satisfied for a fixed couple of exponents, is it satisfied for every couple of exponents? We show the answer is positive if the homogeneous space satisfies some additional hypothesis (Proposition 2.6).

The paper is organized as follows. We give some basic definitions, recall the results already obtained in [AR] and present our results in Section 2. We recall the proof of Proposition 2.3 in Section 3 and give the proof of Proposition 2.6 in Section 4. We then devote Section 5 to the layer decay and annular decay properties that appeared in [AR]. We study some geometric properties ensuring that the latter are satisfied in Section 6. Finally, we present in Sections 7 and 8 some examples and counterexamples in the complex plane, inspired from a curve conceived by R. Tessera in [T].

This work is part of a doctorate dissertation that was conducted at Université Paris-Sud under the supervision of P. Auscher. The author would like to warmly thank him for his kind support. I also thank T. Hytönen for his insightful comments and discussions related to this work.

2. Definitions and main results

Throughout this paper, we will work in the setting of a space of homogeneous type, that is a triplet (X,ρ,μ) where X is a set equipped with a metric ρ and a non-negative Borel measure μ for which there exists a constant $C_D<+\infty$ such that all the associated balls $B(x,r)=\{y\in X;\rho(x,y)<r\}$ satisfy the doubling property

$$0<\mu(B(x,2r))\leq C_D\mu(B(x,r))<\infty$$

for every $x\in X$ and $r>0$. We suppose that $\mu(X)\in]0,+\infty]$, and we allow the presence of atoms in X, that is points $x\in X$ such that $\mu(\{x\})\neq 0$.

Remark 2.1. One usually only assumes that ρ is a quasi-distance on X in the definition of a space of homogeneous type in the sense of Coifman and Weiss [CW]. For the sake of simplicity, we limit ourselves to the metric setting. However, our work can easily be carried out to the quasi-metric setting, though one then has to assume the balls to be Borel sets to give a sense to the objects we will define in the following. Note that this is not necessarily the case as the quasi-distance, in contrast to a distance, may not be Hölder-regular, and quasi-metric balls might not be open nor even Borel sets with respect to the topology defined by the quasi-distance. Other kind of assumptions and arguments appeared in the literature, see for example [AH] for a discussion on the subject.

We will use the notation $A\lesssim B$ (resp. $A\approx B$) to denote the estimate $A\leq CB$ (resp. $(1/C)B\leq A\leq CB$) for some absolute constant C which may vary from line to line. Denote by $\operatorname{supp} f$ the support of a function f defined on X, $\operatorname{diam} E$ the diameter of a subset $E\subset X$, $\bar{E}$ the topological closure of a set $E\subset X$, $\operatorname{Card} I$ the cardinal of a finite set I, $|E|$ the Lebesgue measure of a set $E\subset\mathbb{R}^n$, and $\rho(E,F)=\inf_{x\in E,y\in F}\rho(x,y)$ the distance between two subsets $E,F\subset X$.

For $1\leq p\leq\infty$, let $p'=\frac{p}{p-1}$ be the dual exponent of p. The space of p-integrable complex valued functions on X with respect to μ is denoted by $L^p(X)$, the norm of a function $f\in L^p(X)$ by $\|f\|_p$, the duality bracket given by $\langle f,g\rangle=\int_X fgd\mu$ (we do mean the bilinear form), and the mean of a function f on a set E denoted by $[f]_E=\mu(E)^{-1}\int_E fd\mu$.

Finally, for any $x,y\in X$, we set

$$\lambda(x,y)=\mu(B(x,\rho(x,y))).$$

It is easy to see that, because of the doubling property, $\lambda(x,y)$ is comparable to $\lambda(y,x)$, uniformly in $x,y\in X$.

The following result, due to M. Christ (see [C]), states the existence of sets analogous to the dyadic cubes of $\mathbb{R}^n$ in a space of homogeneous type.

Lemma 2.2. *There exist a collection of open subsets $\{Q_\alpha^j\subset X:j\in\mathbb{Z},\alpha\in I_j\}$, where I_j denotes some (possibly finite) index set depending on j, and constants $0<\delta<1$, $a_0>0$, $\eta>0$, and $C_1,C_2<+\infty$ such that*

1. *For all $j\in\mathbb{Z}$, $\mu(\{X\backslash\bigcup_{\alpha\in I_j}Q_\alpha^j\})=0$.*
2. *If $j<j'$, then either $Q_\beta^{j'}\subset Q_\alpha^j$, or $Q_\beta^{j'}\cap Q_\alpha^j=\varnothing$.*
3. *For each (j,α) and each $j'<j$ there is a unique β such that $Q_\alpha^j\subset Q_\beta^{j'}$.*
4. *For each (j,α), we have $\operatorname{diam}(Q_\alpha^j)\leq C_1\delta^j$.*
5. *Each Q_α^j contains some ball $B(z_\alpha^j,a_0\delta^j)$. We say that z_α^j is the center of the cube Q_α^j.*
6. *Small boundary condition:*

$$\mu\left(\left\{x\in Q_\alpha^j:\rho(x,X\backslash Q_\alpha^j)\leq t\delta^j\right\}\right)\leq C_2t^\eta\mu(Q_\alpha^j)\quad\forall j,\alpha,\quad\forall t>0. \tag{2.1}$$

We point out that this result had already been proved by G. David in the case of Ahlfors-regular Euclidean sets (see [D1]). The construction of these open sets by M. Christ is quite abstract, and, as such, these sets do not follow any geometry. It means that they can be rather "ugly" sets in practice, in spite of their nice properties. We will call those open sets dyadic cubes of the space of homogeneous type X. For a cube $Q = Q^j_\alpha$, j is called the generation of Q, and we set $l(Q) = \delta^j$. By (4) and (5), $l(Q)$ is comparable to the diameter of Q, and we call it, in analogy with $\mathbb{R}^n$, the length of Q. Whenever $Q^{j+1}_\alpha \subset Q^j_\beta$, we will say that Q^{j+1}_α is a child of Q^j_β, and Q^j_β the parent of Q^{j+1}_α. It is easy to check that each dyadic cube has a number of children uniformly bounded. A neighbor of Q is any dyadic cube Q' of the same generation with $\rho(Q, Q') < l(Q)$. The notation $\widehat{Q}$ will denote the union of Q and all its neighbors. It is clear that Q and $\widehat{Q}$ have comparable measures. It is also easy to check that a cube Q has a number of neighbors that is uniformly bounded.

Our first result is a Hardy type inequality in the dyadic setting, analogous to the Euclidean one, valid in general spaces of homogeneous type.

Proposition 2.3. *Let (X, ρ, μ) be a space of homogeneous type. Let Q, Q' be two disjoint dyadic cubes in X. Let $1 < \nu < +\infty$, with dual exponent ν'. There exists $C < +\infty$ such that for every function f supported on Q, $f \in L^\nu(Q)$, and every function g supported on Q', $g \in L^{\nu'}(Q')$, we have*

$$\int_{Q'} \int_Q \frac{|f(y)g(x)|}{\lambda(x,y)} \mathrm{d}\mu(x)\mathrm{d}\mu(y) \leq C\|f\|_\nu \|g\|_{\nu'}. \tag{2.2}$$

The constant C only depends on C_D and ν.

This is Lemma 2.4 of [AR], where it has already been proved. However, for the sake of a better understanding of what follows, we will recall the proof in Section 3, which includes some minor corrections and adjustments to the one present in [AR]. Let us remark that this result is crucial to the estimation of some of the matrix coefficients appearing in the argument to prove the local $T(b)$ theorem central to that paper.

With the motivation to state, for practical reasons, a local $T(b)$ theorem with hypotheses on balls rather than cubes in [AR], we also wanted to know if we could get a similar result outside of the dyadic setting. Let us precise what we mean with the following definition.

Definition 2.4. Hardy property.
Let (X, ρ, μ) be a space of homogeneous type. We say that X has the Hardy property (HP) if for every $1 < \nu < +\infty$, with dual exponent ν', there exists $C < +\infty$ such that for every ball B in X, with $2B$ denoting the concentric ball with double radius, and all functions f supported on B, $f \in L^\nu(B)$, g supported on $2B \backslash B$, $g \in L^{\nu'}(2B\backslash B)$, we have

$$\int_B \int_{2B\backslash B} \frac{|f(y)g(x)|}{\lambda(x,y)} \mathrm{d}\mu(y)\mathrm{d}\mu(x) \leq C\|f\|_\nu \|g\|_{\nu'}. \tag{H}$$

Again, this is Definition 3.4 of [AR]. One could have equivalently replaced $2B$ by cB for fixed $c > 1$ in (H). This property is not always valid, as we will show in Section 7. The difficulty owes to the fact that balls obviously do not satisfy in general the nice properties satisfied by the dyadic cubes, and particularly the property of small boundary (2.1). We thus looked for conditions on the way points are distributed in the space of homogeneous type ensuring that the Hardy property would be satisfied. Our main result is the following:

Theorem 2.5. *Let (X, ρ, μ) be a space of homogeneous type. We have the following diagram of implications in X:*

$$
\begin{tikzcd}
 & & \text{(HB)} \arrow[d, Rightarrow, shift left] \arrow[dr, Rightarrow, shift right] & \\
\text{(M)} \arrow[r, Rightarrow, shift left] & \text{(RAD)} \arrow[l, Rightarrow, "/" marking, shift left] \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow] & \text{(RLD)} \arrow[l, Rightarrow, "/" marking, shift left] \arrow[r, Rightarrow] \arrow[u, Rightarrow, "/" marking, shift left] \arrow[d, Rightarrow, shift left] \arrow[dl, Rightarrow, "/" marking] & \text{(HP)} \arrow[ul, Rightarrow, "/" marking, shift right] \\
 & \text{(AD)} \arrow[r, Rightarrow, shift left] & \text{(LD)} \arrow[l, Rightarrow, "/" marking, shift left] \arrow[u, Rightarrow, "/" marking, shift left] &
\end{tikzcd}
$$

A few comments are in order.

(1) Theorem 2.5 sums up our study of sufficient conditions for the Hardy property (HP), and of the relationships between these conditions. This work was initiated in [AR], where some of the implications in the above diagram have already been proved. Every property except (HP) is related to the geometry and the distribution of points in the homogeneous space. (M) is a purely metric property, while the others are geometric conditions both metric and measure related.

(2) The properties (LD), (RLD), (AD) and (RAD), already introduced in [AR], will be recalled in Section 5, but here we slightly modify (RLD) and (RAD), which does not affect the statements already proved in [AR]. Next, the properties (HB) and (M) will be defined and studied in Section 6. Section 7 will be devoted to the presentation of a counterexample for some of the false implications in Theorem 2.5, while we will give in Section 8 examples of spaces not satisfying (HP) and complete the proof of the theorem.

(3) Observe that, conversely, we do not know necessary conditions for the Hardy property (HP). In particular, does (HP) imply that for every ball B of the homogeneous space, $\mu(\overline{B}\backslash B) = 0$? We think that the answer should be positive, but we have been unable to prove it. Similarly, does (HP) imply (RLD) ? We think this has to be false, but we have not come up with a counterexample yet.

Our last result deals with a natural question regarding these Hardy type inequalities, inspired from the Calderón-Zygmund theory. The question is the following: in a general space of homogeneous type, is it possible to deduce the Hardy property (HP) from the Hardy type inequality (H) for a particular couple (p, p')? We have proved that the answer is positive, provided some additional hypothesis is assumed on the homogeneous space, as shown by the following result.

Proposition 2.6. *Let (X, ρ, μ) be a space of homogeneous type. Assume that for every ball $B \subset X$, $\mu(\bar{B}\backslash B) = 0$. Then, if X satisfies the Hardy type inequality* (H) *for one particular couple of exponents (p, p'), X has the Hardy property* (HP).

We defer the proof to Section 4. Remark that the assumption made on the homogeneous space is in particular satisfied if X satisfies the layer decay property (LD), see Section 5.

3. Proof of Proposition 2.3

Proof. [1] Let I be the integral in (2.2). For a locally integrable function f, we will denote by $M_\mu(f)$ the centered maximal function

$$M_\mu(f)(x) = \sup_{\tau>0} \frac{1}{\mu(B(x,\tau))} \int_{B(x,\tau)} |f(y)| \mathrm{d}\mu(y).$$

Recall that by the Hardy-Littlewood maximal theorem, M_μ is of strong type (p,p) for any $1 < p < +\infty$, and of weak type $(1,1)$. We refer for example to [S] for further details. We prove that for all $1 < r, s < \infty$, we have

$$I \lesssim \left\langle (M_\mu(|f|^r))^{1/r}, |g| \right\rangle + \left\langle |f|, (M_\mu(|g|^s))^{1/s} \right\rangle, \tag{3.1}$$

and the Hardy-Littlewood maximal theorem then gives the desired result, choosing $1 < r < \nu$ and $1 < s < \nu'$. Remark that this inequality was not known even in the Euclidean setting. Without loss of generality, we can assume $f, g \geq 0$. We have

$$I = \int_{\substack{x\in Q' \\ \rho(x,Q)>0}} g(x) \int_{\substack{y\in Q \\ \rho(y,Q')\leq\rho(x,Q)}} \frac{f(y)}{\lambda(x,y)} \mathrm{d}\mu(y)\mathrm{d}\mu(x) + \int_{\substack{y\in Q \\ \rho(y,Q')>0}} f(y) \int_{\substack{x\in Q' \\ \rho(y,Q')>\rho(x,Q)}} \frac{g(x)}{\lambda(x,y)} \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

Indeed, $\mu\left(\{x \in Q' \mid \rho(x,Q) = 0\}\right) = \mu\left(\{y \in Q \mid \rho(y,Q') = 0\}\right) = 0$, because of (2.1): we have for example

$$\{x \in Q' \mid \rho(x,Q) = 0\} \subset \bigcap_{j\in\mathbb{N}} \{x \in Q' \mid \rho(x, Q'^c) \leq 2^{-j}\delta^k\},$$

and all those sets have their measure bounded by $\mu(Q')2^{-j\eta} \underset{j\to\infty}{\longrightarrow} 0$. By symmetry, it is enough to estimate the first integral for example, which we call I_1. For $x \in Q'$, let $E_x = \{y \in Q \mid \rho(y, Q^c) \leq \rho(x,Q)\}$. For a fixed x in Q', we prove that for all $1 < r < \infty$,

$$I_1(x) = \int_{y\in E_x} \frac{f(y)}{\lambda(x,y)} \mathrm{d}\mu(y) \lesssim M_\mu(f^r)(x)^{1/r},$$

and (3.1) will follow. Consider the dyadic subcubes Q_α of Q, which are maximal for the relation $l(Q_\alpha) \leq \rho(Q_\alpha, x)$. Call them $Q_\alpha^l(x)$, where l denotes their generation. They realize what we call a Whitney partition of the cube Q, as for μ-almost every $y \in Q$, there exists a sufficiently small cube Q_y containing y such that $l(Q_y) \leq \rho(x,Q) \leq \rho(x,Q_y)$, and Q_y is then included in one of those maximal $Q_\alpha^l(x)$. The case where there is a unique cube $Q_\alpha^l(x) = Q$ means that $l(Q) \leq \rho(x,Q)$. Let $n \in \mathbb{N}$ be such that $2^n l(Q) \leq \rho(x,Q) < 2^{n+1} l(Q)$. Then $\lambda(x,y) \geq \mu(B(x, 2^n l(Q)))$ while $Q \subset B(x, (C_1+2^{n+1})l(Q))$ for the dimensional constant C_1 of Lemma 2.2. Hence, by the doubling property,

$$I_1(x) \leq \frac{1}{\mu(B(x,2^n l(Q)))} \int_{B(x,(C_1+2^{n+1})l(Q))} |f(y)| \mathrm{d}\mu(y) \lesssim M_\mu f(x).$$

The second case is when Q is not a maximal cube. By maximality, if $Q_\beta^{l-1} \subset Q$ is the unique parent of a $Q_\alpha^l(x)$, then we have $\rho(x, Q_\beta^{l-1}) < l(Q_\beta^{l-1})$, and thus $\rho(x, Q_\alpha^l(x)) \lesssim \rho(x, Q_\beta^{l-1}) + l(Q_\beta^{l-1}) \lesssim l(Q_\beta^{l-1}) = \delta^{-1} l(Q_\alpha^l(x))$. Note that this implies that for every l such that there exists a maximal cube $Q_\alpha^l(x)$, we have $\rho(x,Q) \leq \rho(x, Q_\alpha^l(x)) \lesssim \delta^l$ with implicit constant independent of x and l. For a fixed l as above, let

$$C^l(x) = \bigcup_{\alpha : Q_\alpha^l(x) \cap E_x \neq \varnothing} Q_\alpha^l(x).$$

[1] We thank J.M. Martell for the suggestion of using Whitney coverings that led to an improvement of our earlier argument.

Observe that if $y \in Q^l_\alpha(x) \cap E_x$, then $\rho(y, (Q^l_\alpha(x))^c) \le \rho(y, Q^c) \le \rho(x, Q)$. Thus, $\mu(Q^l_\alpha(x) \cap E_x) \lesssim \left(\frac{\rho(x,Q)}{\delta^l}\right)^\eta \mu(Q^l_\alpha(x))$ by the small boundary property (2.1) of Christ's dyadic cubes. Summing over the cubes in $C^l(x)$, we have

$$\mu(C^l(x)) \lesssim \left(\frac{\rho(x,Q)}{\delta^l}\right)^\eta \mu(C^l(x)) \lesssim \left(\frac{\rho(x,Q)}{\delta^l}\right)^\eta \mu(B(x, \delta^l)),$$

the last inequality being a consequence of doubling and the fact that each of the cubes $Q^l_\alpha(x)$ in $C^l(x)$ is of length δ^l and at a distance comparable to δ^l from x, so that $C^l(x) \subset B(x, C\delta^l)$ for some C independent of l and x. Now, write

$$\begin{aligned} I_1(x) &\lesssim \sum_l \int_{C^l(x)\cap E_x} \frac{|f(y)|}{\mu(B(x,\rho(x,y)))} \mathrm{d}\mu(y) \\ &\lesssim \sum_l \frac{1}{\mu(B(x,\delta^l))} \int_{B(x,C\delta^l)} |f| 1_{C^l(x)\cap E_x} \mathrm{d}\mu \\ &\lesssim \sum_l \left(\frac{1}{\mu(B(x,\delta^l))} \int_{B(x,C\delta^l)} |f|^r \mathrm{d}\mu\right)^{1/r} \left(\frac{\mu(C^l(x)\cap E_x)}{\mu(B(x,\delta^l))}\right)^{\frac{1}{r'}} \\ &\lesssim M_\mu(|f|^r)(x)^{1/r} \sum_l \left(\frac{\rho(x,Q)}{\delta^l}\right)^{\frac{\eta}{r'}}, \end{aligned}$$

where the last inequality is obtained by applying the Hölder inequality with $r > 1$. Now, observe that every integer l intervening in the above sum is such that $\rho(x, Q) \lesssim \delta^l$. Summing over such integers provides us with a geometric sum, uniformly bounded with respect to x. Hence,

$$I_1(x) \lesssim M_\mu(|f|^r)(x)^{1/r}.$$

This concludes our proof. □

4. Proof of Proposition 2.6

Proof. Fix a ball $B = B(z_B, r) \subset X$. We assume that (H) holds for the couple (p, p'). By Fubini's theorem, this implies that one can define $T : L^p(B) \to L^p(2B \backslash B)$ by the absolutely convergent integral

$$(Tf)(y) = \int_B \frac{f(x)}{\lambda(x,y)} \mathrm{d}\mu(x) \quad \text{for almost every} \quad y \in 2B\backslash B.$$

We will proceed in three steps.

(1) The first step in the argument consists in regularizing the kernel $\lambda(x, y)^{-1}$: we show that we can freely assume it satisfies a Lipschitz regularity estimate[2]. To do this, let φ be a function of the real variable such that $\varphi \in C^1([0, +\infty[)$, $\varphi \ge 0$, $\operatorname{supp} \varphi \subset [1, 4]$ and $\int_0^{+\infty} \varphi(t) \frac{\mathrm{d}t}{t} = 1$. For $r \ge 0$, set $\lambda(y, r) = \mu(B(y, r))$. Set

$$\widetilde{\lambda}(x,y) = \int_0^{+\infty} \lambda(y,r)\, \varphi\left(\frac{\rho(x,y)}{r}\right) \frac{\mathrm{d}r}{r}.$$

[2]We thank T. Hytönen for this idea which nicely improved our earlier result.

First, observe that $\widetilde{\lambda}(x,y)$ is comparable to $\lambda(x,y)$, uniformly in $x,y\in X$. Indeed, by an easy change of variable, write

$$\widetilde{\lambda}(x,y)=\int_0^{+\infty}\lambda\left(y,\frac{\rho(x,y)}{u}\right)\varphi(u)\frac{\mathrm{d}u}{u}.$$

Because of the support conditions and size estimate of φ, it comes

$$\lambda(y,\frac{1}{4}\rho(x,y))\leq\widetilde{\lambda}(x,y)\leq\lambda(y,\rho(x,y)).$$

By the doubling property, it follows that, uniformly in $x,y\in X$, we have

$$\widetilde{\lambda}(x,y)\approx\lambda(x,y). \tag{4.1}$$

Then we say that $\widetilde{\lambda}^{-1}$ satisfies a Lipschitz regularity estimate in the first variable. Indeed, fix $x,x',y\in X$ such that $\rho(x,x')\leq\rho(x,y)/2$ and $\rho(x,y)>0$. Because of the support conditions and regularity of φ, we have

$$\begin{aligned}
|\widetilde{\lambda}(x,y)-\widetilde{\lambda}(x',y)| &\leq \int_0^{+\infty}\lambda(y,r)\left|\varphi\left(\frac{\rho(x,y)}{r}\right)-\varphi\left(\frac{\rho(x',y)}{r}\right)\right|\frac{\mathrm{d}r}{r}\\
&\lesssim \int_{r\in[\frac14\rho(x,y),\rho(x,y)]\cup[\frac14\rho(x',y),\rho(x',y)]}\lambda(y,r)\frac{|\rho(x,y)-\rho(x',y)|}{r}\frac{\mathrm{d}r}{r}\\
&\lesssim \rho(x,x')\lambda(x,y)\int_{\frac14\rho(x,y)\leq r\leq\rho(x,y)}\frac{\mathrm{d}r}{r^2}+\rho(x,x')\lambda(x',y)\int_{\frac14\rho(x',y)\leq r\leq\rho(x',y)}\frac{\mathrm{d}r}{r^2}\\
&\lesssim \rho(x,x')\left(\frac{\lambda(x,y)}{\rho(x,y)}+\frac{\lambda(x',y)}{\rho(x',y)}\right).
\end{aligned}$$

But since $\rho(x,x')\leq\rho(x,y)/2$, we have $\rho(x,y)\approx\rho(x',y)$ and $\lambda(x,y)\approx\lambda(x',y)$, uniformly in x,x',y. It follows that

$$|\widetilde{\lambda}(x,y)^{-1}-\widetilde{\lambda}(x',y)^{-1}|=\frac{|\widetilde{\lambda}(x,y)-\widetilde{\lambda}(x',y)|}{\widetilde{\lambda}(x,y)\widetilde{\lambda}(x',y)}\lesssim\frac{\rho(x,x')}{\rho(x,y)}\frac{1}{\widetilde{\lambda}(x,y)}, \tag{4.2}$$

because $\widetilde{\lambda}(x,y)\approx\widetilde{\lambda}(x',y)\approx\lambda(x,y)$.

But because of (4.1), one can define an operator $\widetilde{T}:L^p(B)\to L^p(2B\backslash B)$ by the absolutely convergent integral

$$(\widetilde{T}f)(y)=\int_B\frac{f(x)}{\widetilde{\lambda}(x,y)}\mathrm{d}\mu(x)\quad\text{for almost every}\quad y\in 2B\backslash B,$$

and, for every $1<\nu<+\infty$, the boundedness of $T:L^\nu(B)\to L^\nu(2B\backslash B)$ is equivalent to the boundedness of $\widetilde{T}:L^\nu(B)\to L^\nu(2B\backslash B)$. Obviously, by symmetry, we can apply the same argument with respect to the second variable. It shows that we can freely assume the kernel $\lambda(x,y)^{-1}$ to satisfy the Lipschitz regularity estimate (4.2) in both variables, which we will do in the following, forgetting this operator $\widetilde{T}$. Observe however that, under this assumption, we can no longer use the fact that $\lambda(x,y)=\mu(B(x,\rho(x,y)))$, only that these two quantities are comparable.

(2) The second step in the argument now consists in applying a standard Calderón-Zygmund decomposition. We show that T is of weak type $(1,1)$: we prove that for all $f\in L^1(B)$, with $\operatorname{supp} f\subset B$

$$\mu(\{x\in 2B\backslash B\mid |Tf(x)|>\alpha\})\lesssim\frac{1}{\alpha}\|f\|_{L^1(B)}.$$

The idea is to write a Calderón-Zygmund decomposition of f on X. However, we have to be a bit careful, because if we write the standard decomposition $f = g + \sum_{i\in I} b_i$ directly on the ball B, g will not be supported inside B. To avoid this problem, let us use a Whitney partition of the ball B, as in Section 3: consider the dyadic cubes $Q \subset B$ which are maximal for the relation $l(Q) \leq \rho(Q, B^c)$. Call them Q_j, $j \in J$. They are mutually disjoint and they realize a partition of the ball B but for a set of measure zero. Now, for $f \in L^1(B)$, with $\operatorname{supp} f \subset B$, we have $f = \sum_j f 1_{Q_j}$ $\mu\, a.e.$ Set $f_j = f 1_{Q_j}$, and for every fixed j, write a Calderón-Zygmund decomposition of f_j on Q_j: $f_j = g_j + \sum_{i\in I} b_{i,j}$ with

- $g_j = 1_{\Omega_{j,\alpha}^c} f_j$ where $\Omega_{j,\alpha} = \{x \in Q_j \mid |f_j^*(x)| > \alpha\}$, f_j^* denoting the dyadic maximal function of f_j on Q_j. Thus $\operatorname{supp} g_j \subset \overline{Q_j}$, $g_j \in L^\infty$, $\|g_j\|_\infty \leq \alpha$, and $\|g_j\|_p \leq (\|g_j\|_\infty^{p-1}\|g_j\|_1)^{1/p} \leq \alpha^{1/p'}\|f_j\|_1^{1/p}$.
- $\operatorname{supp} b_{i,j} \subset \overline{Q_{i,j}}$ where the sets $Q_{i,j}$ are dyadic subcubes of Q_j of center $z_{Q_{i,j}}$, realizing in turn a Whitney partition of the open set $\Omega_{j,\alpha}$: $\mu(\Omega_{j,\alpha} \setminus \cup_i Q_{i,j}) = 0$, $Q_{i,j}$ are mutually disjoint, and there exists a dimensional constant $C > C_1$ such that for every i, $B(z_{Q_{i,j}}, Cl(Q_{i,j})) \cap \Omega_{j,\alpha}^c \neq \varnothing$. In addition, we have $[|b_{i,j}|]_{Q_{i,j}} \lesssim \alpha$ and $[b_{i,j}]_{Q_{i,j}} = 0$.
- $\|f_j\|_1 = \sum_i \|b_{i,j}\|_1 + \|g_j\|_1$.

Now, set $g = \sum_j g_j$. Observe that we have $\operatorname{supp} g \subset B$, and $f = g + \sum_{i,j} b_{i,j}$. Applying (H) for the couple (p, p'), and the disjointness of the dyadic cubes Q_j, we have

$$\begin{aligned}
\mu(\{x \in 2B\backslash B \mid |Tg(x)| > \alpha/2\}) &\lesssim \frac{1}{\alpha^p}\int_{2B\backslash B} |Tg|^p \mathrm{d}\mu \lesssim \frac{1}{\alpha^p}\int_B |g|^p \mathrm{d}\mu \\
&= \frac{1}{\alpha^p}\sum_j \int_{Q_j} |g_j|^p \mathrm{d}\mu = \frac{1}{\alpha^p}\sum_j \|g_j\|_p^p \\
&\leq \frac{1}{\alpha}\sum_j \|f_j\|_1 = \frac{1}{\alpha}\|f\|_{L^1(B)}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mu(\{x \in 2B\backslash B \mid |T(\sum_{i,j} b_{i,j})(x)| > \alpha/2\}) &\leq \mu(\bigcup_{i,j}\widehat{Q_{i,j}}) + \mu\Big((\bigcup_{i,j}\widehat{Q_{i,j}})^c \cap \{x \in X \,|\sum_{i,j}|Tb_{i,j}(x)| > \alpha/2\}\Big) \\
&\lesssim \sum_{i,j}\mu(Q_{i,j}) + \mu(\{x \in X \mid \sum_{i,j}|Tb_{i,j}(x)| 1_{\widehat{Q_{i,j}}^c} > \alpha/2\}) \\
&\lesssim \sum_{i,j}\mu(Q_{i,j}) + \sum_{i,j}\frac{1}{\alpha}\int_{\widehat{Q_{i,j}}^c} |Tb_{i,j}| \mathrm{d}\mu.
\end{aligned}$$

Since $b_{i,j}$ is of mean 0 on $Q_{i,j}$, and because the kernel $\lambda(x,y)^{-1}$ satisfies the Hölder standard estimate, we can apply the standard Calderón-Zygmund estimates to obtain

$$\begin{aligned}
\int_{\widehat{Q_{i,j}}^c} |Tb_{i,j}| \mathrm{d}\mu &\lesssim \int_{\widehat{Q_{i,j}}^c}\int_{Q_{i,j}} |b_{i,j}(x)| \left|\lambda(z_{Q_{i,j}}, y)^{-1} - \lambda(x,y)^{-1}\right| \mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&\lesssim \|b_{i,j}\|_1 \int_{\widehat{Q_{i,j}}^c} \frac{1}{\lambda(z_{Q_{i,j}}, y)}\left(\frac{l(Q_{i,j})}{\rho(z_{Q_{i,j}}, y)}\right) \mathrm{d}\mu(y) \\
&\lesssim \|b_{i,j}\|_1 \sum_{k\geq 0}\int_{2^k l(Q_{i,j}) \leq \rho(y, z_{Q_{i,j}}) < 2^{k+1} l(Q_{i,j})} \frac{2^{-k}}{\mu(B(z_{Q_{i,j}}, 2^k l(Q_{i,j})))} \mathrm{d}\mu(y) \\
&\lesssim \|b_{i,j}\|_1 \sum_{k\geq 0} 2^{-k} \lesssim \alpha\mu(Q_{i,j}).
\end{aligned}$$

Finally, we have

$$\mu(\{x\in 2B\backslash B \mid |T(\sum_{i,j} b_{i,j})(x)|>\alpha/2\}) \lesssim \sum_{i,j}\mu(Q_{i,j}) \lesssim \sum_j \mu(\Omega_{j,\alpha}) \lesssim \sum_j \frac{1}{\alpha}\|f_j\|_1 = \frac{1}{\alpha}\|f\|_{L^1(B)}.$$

Thus, we get as expected

$$\mu(\{x\in 2B\backslash B \mid |Tf(x)|>\alpha\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(B)}.$$

By interpolation, it follows that we have $Tf\in L^q(2B\backslash B)$ for every $1<q\le p$ and $f\in L^q(B)$, with $\operatorname{supp} f\subset B$.

(3) The last part of our argument consists in applying some duality argument to conclude. (H) for the couple (p,p') implies that the adjoint operator T^* is bounded from $L^{p'}(2B\backslash B)$ to $L^{p'}(B)$. We show again that T^* is of weak type $(1,1)$: we prove that for all $f\in L^1(2B\backslash B)$, with $\operatorname{supp} f\subset 2B\backslash B$,

$$\mu(\{x\in B \mid |T^*f(x)|>\alpha\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(2B\backslash B)}.$$

The idea is to use as before a Whitney partition of the open set $\overline{B}^c$: consider the dyadic cubes $Q_j\subset \overline{B}^c$ that are maximal for the relation $l(Q)\le \rho(Q,B)$. They partition $\overline{B}^c$ but for a set of measure 0. Of course, there are some of these Q_j that intersect $(2B)^c$. To overcome this problem, let us keep only the cubes $Q_j, j\in J$, intersecting the set $cB\backslash B$, with $1<c<\frac{C_1+2}{C_1+1}$. These cubes satisfy, for every $j\in J$, $l(Q_j)\le \rho(Q_j,B)\le (c-1)r$. Because of property (4) of Christ's dyadic cubes (Lemma 2.2), it implies that for every $x\in Q_j$, $\rho(x,z_B)\le cr+\operatorname{diam} Q_j\le cr+C_1(c-1)r<2r$, so that $Q_j\subset 2B\backslash B$. Now, for $f\in L^1(2B\backslash B)$, with $\operatorname{supp} f\subset 2B\backslash B$, write

$$f=\sum_{j\in J} f1_{Q_j} + f1_{(2B\backslash B)\backslash \cup_{j\in J}Q_j} = \sum_{j\in J} f_j+\tau \quad \mu\, a.e.$$

Remark that this is where we use the assumption $\mu(\bar{B}\backslash B)=0$, and it is the only time that we use it in our proof. Apply the same argument as in step (2) to f_j, for every $j\in J$, to get

$$\mu(\{x\in B \mid |T^*(\sum_{j\in J} f_j)(x)|>\alpha/2\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(2B\backslash B)}.$$

For the remaining term τ, observe that since $\Big((2B\backslash B)\backslash \cup_{j\in J} Q_j\Big)\subset (cB)^c$, $\operatorname{supp}\tau\subset (cB)^c$, and we trivially have $T^*\tau\in L^1(B)$. Indeed, by the doubling property, we have

$$\int_B |T^*\tau|\mathrm{d}\mu \le \int_B\int_{(cB)^c} \frac{|\tau(y)|}{\lambda(x,y)}\mathrm{d}\mu(y)\mathrm{d}\mu(x) \le \|\tau\|_1 \frac{\mu(B(x,r))}{\mu(B(x,cr))} \lesssim \|f\|_{L^1(2B\backslash B)}.$$

Hence,

$$\mu(\{x\in B \mid |T^*\tau(x)|>\alpha\}) \lesssim \frac{1}{\alpha}\|f\|_{L^1(2B\backslash B)},$$

and T^* is of weak type $(1,1)$. By interpolation, it follows that $T^*f\in L^q(B)$ for every $1<q\le p'$ and $f\in L^q(2B\backslash B)$, with $\operatorname{supp} f\subset 2B\backslash B$. By duality, the Hardy property (HP) is satisfied on X. □

Remark 4.1. Observe that the first step in our argument does not directly extend to the quasi-metric setting. However, R. Macias and S. Segovia [MS] proved that if ρ is a quasi-distance on X, there exists another quasi-distance ρ' equivalent to ρ (in the sense that $\rho(x,y)\eqsim \rho'(x,y)$ for every $x,y\in X$) which is Hölder regular. See also [PS] for an elegant proof of this result. To adapt our proof, one only has to define the new kernel $\widetilde{\lambda}$ of the first step of the argument with this quasi-metric ρ' instead of ρ. The estimate (4.2) is then replaced by a Hölder regularity estimate, where the $\rho'(x,x'), \rho'(x,y)$ that appear can be replaced by $\rho(x,x'), \rho(x,y)$ because $\rho \approx \rho'$. For the next steps in the argument, one again assumes the kernel to satisfy the Hölder regularity estimate obtained and works as in the metric setting with the quasi-metric ρ, forgetting ρ'. Note that the existence of Christ's dyadic cubes given by Lemma 2.2 remains valid in the quasi-metric setting (see [C]), allowing us to do the same kind of Whitney coverings.

5. Sufficient conditions for the Hardy property

5.1. **Layer decay properties** (LD)**,** (RLD)**.** It is not clear when (H) is true in a space of homogeneous type for a given ball B. In fact, it is false in general, as will be illustrated by some counterexamples in the following sections. It obviously depends on how the sets B and B^c see each other in X. By analogy with Christ's dyadic cubes, natural objects are the outer and inner layers $\{x \in B | \rho(x, B^c) \leq \varepsilon\}$ and $\{y \in B^c | \rho(y, B) \leq \varepsilon\}$. We shall assume they tend to zero in measure as $\varepsilon \to 0$, and in a scale invariant way, as expressed by the following definition.

Definition 5.1. Layer decay and relative layer decay properties.
Let (X, ρ, μ) be a space of homogeneous type. For a ball B in X, set $B_\varepsilon = \{x \in B | \rho(x, B^c) \leq \varepsilon\} \cup \{y \in B^c | \rho(y, B) \leq \varepsilon\}$ the union of the inner and outer layers.

- We say that X has the **layer decay property** if there exist constants $\eta > 0$, $C < +\infty$ such that for every ball $B = B(z,r)$ in X and every $\varepsilon > 0$, we have

$$\mu(B_\varepsilon) \leq C \left(\frac{\varepsilon}{r}\right)^\eta \mu(B(z,r)). \tag{LD}$$

- We say that X has the **relative layer decay property** if there exist constants $\eta > 0$, $C < +\infty$ such that for every ball $B = B(z,r)$ in X, every ball $B(w,R)$ with $R \leq 2r$, and every $\varepsilon > 0$, we have

$$\mu\left(B_\varepsilon \cap B(w,R)\right) \leq C \left(\frac{\varepsilon}{R}\right)^\eta \mu(B(w,R)). \tag{RLD}$$

This is Definition 9.1 of [AR], with a minor modification to the relative layer decay property where we have substituted the condition $R \leq 2r$ to $z \notin B(w,R)$. We think this is a better definition, though equivalent, because this property is only relevant for small R (else it says nothing), and when R is small enough compared to r, if $B_\varepsilon \cap B(w,R) \neq \varnothing$ then necessarily $z \notin B(w,R)$.

The layer decay property already appeared in [B1] (with only $\mu\left(\{x \in B | \rho(x, B^c) \leq \varepsilon\}\right)$ in the left hand side of (LD)). These properties express the fact that the points are distributed in such a way in the homogeneous space that they never concentrate too much in the inner and outer layers of balls. One can note that while (LD) is a global condition, a sort of averaging property over the whole layer, (RLD) is a rather local condition. Remark also that (RLD) implies (LD) (see [AR]). Because of the opposed local and global nature of (RLD) and (LD) though, it is sensible to think that these properties should not be equivalent. We will prove it in Section 8.2.

It turns out that the relative layer decay property constitutes a sufficient condition for the Hardy property (HP):

Proposition 5.2. *Let (X, ρ, μ) be a space of homogeneous type, and suppose that X has the relative layer decay property* (RLD)*. Then the Hardy property* (HP) *is satisfied on X.*

We refer to [AR] (Proposition 9.2) for the proof of this result. Let us only recall here that the argument is actually quite similar to the proof of Proposition 2.3, with adaptations owing to the fact one cannot use exact coverings with balls as for dyadic cubes. The modification we made in the definition of the relative layer decay property does not change the argument.

Remarks 5.3. (1) Obviously, if X satisfies the layer decay property (LD), then we have $\mu(\bar{B}\backslash B) = 0$ for every ball $B \subset X$. Thus, with Proposition 5.2 and Proposition 2.6, we have proved that if X satisfies (RLD), then X satisfies (HP), and if X satisfies only (LD), then X satisfies (HP) provided it satisfies the Hardy inequality (H) for one particular couple of exponents (p, p').

(2) Note that if f and g are taken in $L^{\nu_1}(X)$ and $L^{\nu_2}(X)$ with $1/\nu_1 + 1/\nu_2 < 1$ (which is stronger than $g \in L^{\nu_1'}$), then if I denotes the integral in (H), the normalised inequality

$$\frac{1}{\mu(B)} I \lesssim \|f\|_{L^{\nu_1}(2B\backslash B, \frac{\mathrm{d}\mu}{\mu(B)})} \|g\|_{L^{\nu_2}(B, \frac{\mathrm{d}\mu}{\mu(B)})}$$

is true in any space of homogeneous type only satisfying (LD). The proof is in this case much easier and we refer to [AR] for the detail.

5.2. **Annular decay properties** (AD)**,** (RAD)**.** Interestingly, another very similar condition already appeared in the literature (see for example [MS], [DJS], [B2], [T]). It is the notion of annular decay that we recall now.

Definition 5.4. Annular decay and relative annular decay properties.
Let (X, ρ, μ) be a space of homogeneous type. For $z \in X$ and $r > 0, 0 < s < r$, set $C_{r,r-s}(z) = B(z,r)\backslash B(z, r-s)$.

- We say that X has the **annular decay property** if there exist constants $\eta > 0$ and $C < +\infty$ such that for every $z \in X, r > 0, 0 < s < r$, we have

$$\text{(AD)} \qquad \mu(C_{r,r-s}(z)) \leq C \left(\frac{s}{r}\right)^{\eta} \mu(B(z,r)).$$

- We say that X has the **relative annular decay property** if there exist constants $\eta > 0$ and $C < +\infty$ such that for every $z \in X$, $r > 0, 0 < s < r$, and every ball $B(w, R)$ with $R \leq 2r$, we have

$$\text{(RAD)} \qquad \mu(C_{r,r-s}(z) \cap B(w,R)) \leq C \left(\frac{s}{R}\right)^{\eta} \mu(B(w,R)).$$

Once again, this is Definition 9.4 of [AR], with the same minor modification in the definition of the relative annular decay property than before. Note that this condition (AD) was made an assumption in [DJS] for the first proof of the global Tb theorem in a space of homogeneous type. Again, (AD) is a global property while (RAD) is a local one. Similarly as for layer decay properties, we have that (RAD) implies (AD). Observe that for a ball $B = B(x, r)$, with $x \in X$, $r > 0$, we have, if $\varepsilon > 0$,

$$B_\varepsilon = \{y \in C_{r,r-\varepsilon}(x) \mid \rho(y, B^c) \leq \varepsilon\} \cup \{y \in C_{r+2\varepsilon,r}(x) \mid \rho(y, B) \leq \varepsilon\}.$$

It follows that $B_\varepsilon \subset C_{r+2\varepsilon,r-\varepsilon}(x)$ and thus (RAD) (respectively (AD)) implies (RLD) (respectively (LD)). In particular, (RAD) is a sufficient condition for the Hardy property (HP) because of Proposition 5.2.

6. Geometric properties ensuring the relative layer decay property

6.1. **Monotone geodesic property** (M)**.** In [B2], Buckley introduces the notion of chain ball spaces and proves that under that condition, a doubling metric measure space satisfies (AD). Colding and Minicozzi II already had proved that this property was satisfied by doubling complete riemannian manifolds in [CM]. Tessera introduced a notion of monotone geodesic property in [T], and proved that this property also implies (AD) (called there the Følner property for balls) in a doubling metric measure space. Lin, Nakai and Yang recently showed in [LNY] that chain ball and a slightly stronger scale invariant version of the monotone geodesic are equivalent. It is the latter that will interest us.

Definition 6.1. Let (X,ρ) be a metric space. We say that X has the **monotone geodesic property** (M) if there exists a constant $0 < C < +\infty$ such that for all $u > 0$ and all $x, y \in X$ with $\rho(x,y) \geq u$, there exists a point $z \in X$ such that

$$\rho(z,y) \leq Cu \quad \text{and} \quad \rho(z,x) \leq \rho(y,x) - u. \tag{M}$$

Remark that C must satisfy $C \geq 1$. Remark also that iterating this property, one gets that for every $x, y \in X$ with $\rho(x,y) \geq u$, there exists a sequence of points $y_0 = y, y_1, ..., y_m = x$ such that for every $i \in \{0, ..., m-1\}$

$$\rho(y_{i+1}, y_i) \leq Cu \quad \text{and} \quad \rho(y_{i+1}, x) \leq \rho(y_i, x) - u.$$

Observe that this is a purely metric property. It is obviously satisfied by complete doubling Riemannian manifolds. It is also satisfied by any geodesic space or length space (see [BBI] for a definition). It appears that (M) not only yields the annular decay property, but also, as was proved in [AR], the stronger relative annular decay property.

Proposition 6.2. *Let (X,ρ,μ) be a space of homogeneous type, and suppose that X has the monotone geodesic property* (M)*. Then X has the relative annular decay property* (RAD).

We refer again to [AR] (Proposition 9.6) for the proof of this result (our modification on (RAD) has no impact). The argument essentially adapts the one in [T] with more care on localization.

Remark 6.3. Observe that conversely, neither (HP) nor (RAD) imply (M). Let us give two examples to illustrate this. First consider the space formed by the real line from which an arbitrary interval has been withdrawn, equipped with the Euclidean distance and Lebesgue measure. This space obviously does not have the monotone geodesic property, as, to put it roughly, there is a hole in it. On the other hand, this space clearly satisfies the Hardy property, as a consequence of (HP) on the real line, as well as (RAD). The second example is a connected one: consider the space made of the three edges of an arbitrary triangle in the plane, again equipped with the induced Euclidean distance and Lebesgue measure. This space has the Hardy property, once again as a straightforward consequence of the fact that the unit circle has it and easy change of variables. It easily follows from the fact that one of the angles must be less than $\pi/2$ that it does not have the monotone geodesic property: one of the pairs (x,y) with x a vertex and y its orthogonal projection on the opposite side cannot meet condition (M). In passing, it proves that this property is not stable under bi-Lipschitz mappings (see also [T]).

6.2. **Homogeneous balls property** (HB)**.** Studying in detail the proof of Proposition 2.3, one realizes that if B and $X\backslash B$ are themselves spaces of homogeneous types with uniform constants, then there will be no difficulty to prove that (HP) is satisfied, as one can then adapt the proof using Christ's dyadic cubes both on B and $X\backslash B$. This motivates the following definition.

Definition 6.4. Let (X,ρ,μ) be a space of homogeneous type. Let B be a ball in X, and suppose that B and B^c are themselves spaces of homogeneous type with doubling constant C_B, *i.e.* for all $x\in B$, $y\in B^c$, and all $r>0$, we have

$$\mu(B(x,2r)\cap B)\leq C_B\ \ \mu(B(x,r)\cap B),\qquad \mu(B(y,2r)\cap B^c)\leq C_B\ \ \mu(B(y,r)\cap B^c).$$

We say that X has the **homogeneous balls property** (HB) if this is satisfied by all the balls in X and if

$$\sup_{B\subset X} C_B<+\infty.$$

Proposition 6.5. *Let (X,ρ,μ) be a space of homogeneous type.*

(1) *If X has the homogeneous balls property* (HB)*, then X has the relative layer decay property* (RLD).
(2) *If X has the homogeneous balls property* (HB)*, then X has the Hardy property* (HP).

Proof. (1) If X has the homogeneous balls property, then every ball in X as well as its complement in X can be partitioned into dyadic cubes, with uniform constants (see [C]). But these cubes themselves have the layer decay property, and it is easy to see that this property transposes to the balls. Let $x\in X$, $r>0$, $B=B(x,r)$, $w\in X$, $0<R\leq 2r$, and fix $\varepsilon>0$. Let us estimate for example the measure of the inner layer $C_\varepsilon=\{x\in B:\rho(x,B^c)\leq\varepsilon\}$. We want to prove that there exists $\eta'>0$ such that $\mu(C_\varepsilon\cap B(w,R)))\lesssim\left(\frac{\varepsilon}{R}\right)^{\eta'}\mu(B(w,R))$. As B constitutes by itself a space of homogeneous type with uniform doubling constant, there exists at every scale a partitioning of B into Christ's dyadic cubes (with uniform constants for these cubes). The idea is to pave B by dyadic cubes of a well chosen generation. Let $N,m,l\in\mathbb{Z}$ be such that $\delta^{N+1}<r\leq\delta^N$, $\delta^{N+m+1}<\varepsilon\leq\delta^{N+m}$, and $\delta^{N+l+1}<R\leq\delta^{N+l}$. We can assume that $\varepsilon\leq R$, since otherwise the result is trivial, which means $m\geq l$. We look at the cubes of generation $j=N+[\frac{m+l}{2}]\geq N$ because $R\leq 2r$ (otherwise there would be no such cubes): for all $y\in C_\varepsilon$, there exists a unique β such that $y\in Q^j_\beta$, and $\rho(y,(Q^j_\beta)^c)\leq\rho(y,B^c)\leq\varepsilon\leq\delta^{m+N}$. By the small boundary property (2.1) of Christ's dyadic cubes, one gets that for all β

$$\mu(C_\varepsilon\cap Q^j_\beta)\lesssim\left(\frac{\delta^{n+m}}{\delta^j}\right)^\eta\mu(Q^j_\beta)\lesssim\left(\frac{\varepsilon}{R}\right)^{\eta/2}\mu(Q^j_\beta).$$

On the other hand, if $C_\varepsilon\cap Q^j_\beta\cap B(w,R)\neq\varnothing$ and if $y\in C_\varepsilon\cap Q^j_\beta\cap B(w,R)$, $x\in Q^j_\beta$, then, using the fact that since $m\geq l$, $\delta^j\leq\delta^{n+l}\lesssim R$, we have

$$\rho(x,w)\leq\rho(x,y)+\rho(y,w)\leq C_1\delta^j+R\leq CR$$

for a dimensional constant $C>0$, and then $Q^j_\beta\subset B(w,CR)$. Finally, we get

$$\begin{aligned}\mu(C_\varepsilon\cap B(w,R))&=\sum_{\beta:C_\varepsilon\cap Q^j_\beta\cap B(w,R)\neq\varnothing}\mu(C_\varepsilon\cap Q^j_\beta\cap B(w,R))\\&\lesssim\sum_{\beta:C_\varepsilon\cap Q^j_\beta\cap B(w,R)\neq\varnothing}\left(\frac{\varepsilon}{R}\right)^{\eta/2}\mu(Q^j_\beta)\\&\lesssim\left(\frac{\varepsilon}{R}\right)^{\eta/2}\mu(B(w,CR))\lesssim\left(\frac{\varepsilon}{R}\right)^{\eta/2}\mu(B(w,R)),\end{aligned}$$

where the last line is obtained using the disjointness of the cubes Q^j_β and then the doubling property. We can do the same for the outer layer as B^c also constitutes a space of homogeneous type. It proves (RLD).

(2) It is a direct consequence of (1) and Proposition 5.2. But it can also be proved directly slightly adapting the proof of Proposition 2.3. Indeed, observe that the homogeneous balls property allows to use exact coverings of B and $X\backslash B$ by Christ's dyadic cubes as above, and then everything works out as in the dyadic setting, taking care of those "large" cubes which are not contained in $2B\backslash B$ in a simple manner (see Section 4). □

It is easy to see that a space of homogeneous type does not satisfy the homogeneous balls property (HB) in general. Let us give a counterexample.

Example 6.6. Consider the real line, from which one has withdrawn the interval $I_\varepsilon =]1-\varepsilon, 1-\varepsilon^2[$, with ε a fixed small constant. Consider the ball in this set of center $1/2$ and of radius $1/2$. It is easy to see that it has, as a space of homogeneous type, a doubling constant of at least $1/\varepsilon$: inside this ball, consider the ball of center $1-\varepsilon^2$ and radius $\varepsilon-\varepsilon^2$, and its concentric double. Now, set

$$I_{n,\varepsilon} =]n - \varepsilon/2^{n-1}, n - (\varepsilon/2^{n-1})^2[,$$

and consider the space $X = \mathbb{R}\backslash \cup_{n\geq 1} I_{n,\varepsilon}$, equipped with the Euclidean distance and the Lebesgue measure. It is clear that X does not satisfy the homogeneous balls property since the doubling constants explode. Neither does X satisfy the monotone geodesic property (M) as, to put it roughly, it has holes in it. However, observe that X satisfies both (RLD) and (HP). As a matter of fact, proving (HP) on X is exactly the same as proving it for the real line, and this is trivial (it is the same for (RLD)). In particular, this example shows that neither the monotone geodesic property nor the homogeneous balls property are necessary conditions for (HP).

Remark 6.7. We have given two sufficient conditions for (RLD), one which is purely metric, the monotone geodesic property, while the other is rather a measure property. Let us examine how these two properties are connected. It is clear that (HB) does not imply (M), as is shown by the trivial example of the real line from which an arbitrary interval has been withdrawn. Now, if we suppose that X satisfies (M), we have a partial result regarding the homogeneous balls property. As a matter of fact, let $B = B(x, r_0)$, and $y \in B$. We prove that for all $r > 0$,

$$\mu(B(y, 2r) \cap B) \lesssim \mu(B(y, r) \cap B).$$

First, if $r \geq 2r_0$, then $B \subset B(y, r) \subset B(y, 2r)$ and $B(y, 2r) \cap B = B(y, r) \cap B = B$. Then, if $r \leq 2r_0$ and $\rho(x, y) \leq r_0/2$, we have $B(y, r/4) \subset (B \cap B(y, r))$ and by the doubling property, $\mu(B(y, 2r)) \lesssim \mu(B(y, r/4)) \lesssim \mu(B(y, r) \cap B)$. It only remains to study the case $r \leq 2r_0$ and $r_0/2 < \rho(x, y) < r_0$. Let $\alpha < \min(\frac{2}{1+C}, \frac{1}{2})$, and $\beta < \alpha/2$. We have $\rho(x, y) \geq r_0/2 \geq r\alpha/2$ and by (M), there exists a point z in X such that

$$\rho(y, z) \leq Cr\frac{\alpha}{2} \quad \text{and} \quad \rho(x, z) \leq \rho(y, x) - r\frac{\alpha}{2}.$$

Consider the ball $B(z, \beta r)$. If $w \in B(z, \beta r)$, then

$$\rho(w, x) \leq \rho(w, z) + \rho(z, x) \leq \beta r + \rho(y, x) - r\frac{\alpha}{2} \leq r_0.$$

Thus $B(z, \beta r) \subset B$. Furthermore,

$$\rho(w, y) \leq \rho(w, z) + \rho(z, y) \leq \beta r + Cr\frac{\alpha}{2} < r,$$

since $\alpha < \frac{2}{1+C}$. Thus $B(z,\beta r) \subset B(y,r)$. Finally,

$$\mu(B(y,2r)\cap B) \lesssim \mu(B(y,r)) \lesssim \mu(B(z,2r)) \lesssim \mu(B(z,\beta r)) \lesssim \mu(B(y,r)\cap B).$$

This means that every ball in X is itself a space of homogeneous type, with uniform constant. However, we cannot obtain the same result for the complement of balls, and (HB) cannot be inferred.

With the results of these first sections, we have already proved a part of Theorem 2.5: we have proved the following

$$\begin{tikzcd}
& & \text{(HB)} \arrow[d, Rightarrow, shift left] \arrow[dr, Rightarrow, shift left] & \\
\text{(M)} \arrow[r, Rightarrow, shift left] & \text{(RAD)} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{(RLD)} \arrow[u, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{(HP)} \arrow[ul, Rightarrow, shift left, "/" marking] \\
& \text{(AD)} \arrow[r, Rightarrow] & \text{(LD)} &
\end{tikzcd}$$

In particular, all positive implications are proved and we look now for the negative ones.

7. An example in the complex plane: the curve of Tessera

To further understand these properties of the homogeneous space, we will consider in this section a curve introduced by Tessera in [T]. This curve is given by a stairway-like construction in the complex plane, starting from 0, and containing for every $k \in \mathbb{N}$ a half-circle of center 0 and radius 2^k. More precisely, consider in the complex plane the parametric curve $\gamma(t)$ defined for $t \geq 0$, and constructed as follows with $|\gamma'(t)| = 1$ for every $t \geq 0$:

- $\gamma(0) = 0$.
- $\{\gamma(t) \mid 0 \leq t \leq t_1\}$ is the segment $[0,1]$.
- $\{\gamma(t) \mid t_1 \leq t \leq t_2\}$ is the half-circle of center 0 and radius 1 in the half-plan $\{\Im z \geq 0\}$.
- By induction, for $k \geq 1$, $\{\gamma(t) \mid t_{2k} \leq t \leq t_{2k+1}\}$ is the segment $[2^{k-1}, 2^k]$, and $\{\gamma(t) \mid t_{2k+1} \leq t \leq t_{2k+2}\}$ is the half-circle of center 0 and radius 2^k in the half-plan $\{\Im z \geq 0\}$ if k is even, in the half-plan $\{\Im z \leq 0\}$ if k is odd.

Set $t_0 = 0$. An easy computation shows that we have for all $k \geq 1$,

$$t_{2k} = 1 + \pi + (1+2\pi)(2^{k-1} - 1) = -\pi + (1+2\pi)2^{k-1},$$
$$t_{2k+1} = 1 + \pi + (1+2\pi)(2^{k-1} - 1) + 2^{k-1} = -\pi + (2+2\pi)2^{k-1}.$$

Set $X_T = \{\gamma(t) \mid t \geq 0\}$, equipped with the Euclidean distance d in $\mathbb{C}$ and the Hausdorff length Λ. See Figure 2 for a representation of X_T. For $x, y \in X_T$, denote by (x, y) the arc in X_T between x and y. For $z \in X_T$, and $r > 0$, we denote by $B(z,r)$ the open ball of center z and radius r in X_T : $B(z,r) = \{x \in X_T \mid |z - x| < r\}$. We recall that a bounded set $E \subset \mathbb{C}$ is said to be Ahlfors-David regular (of dimension 1) when there exists a constant $0 < C < +\infty$ such that for every $z \in \mathbb{C}$ and $0 < r \leq 1$, $\frac{1}{C} r \leq \Lambda(E \cap B(x,r)) \leq Cr$ (see [A], [D2]).

Proposition 7.1. (1) $(X_T, \mathrm{d}, \Lambda)$ *is an Ahlfors-David set of dimension* 1*, and thus* $(X_T, \mathrm{d}, \Lambda)$ *can be seen as a space of homogeneous type.*

(2) X_T *does not satisfy the annular decay property* (AD) *(nor the relative annular decay property* (RAD)*).*

(3) X_T *does not satisfy the homogeneous balls property* (HB), *nor the monotone geodesic property* (M).
(4) X_T *satisfies the relative layer decay property* (RLD).
(5) X_T *satisfies the Hardy property* (HP).

Proof. (1) Let us first make a preliminary observation: as $|\gamma'(t)| = 1$ for every $t \geq 0$, we have $\Lambda((\gamma(\alpha), \gamma(\beta))) = |\alpha - \beta|$. But there exists a dimensional constant $1 < C_{AD} < +\infty$ such that for all $\alpha, \beta > 0$

$$\frac{1}{C_{AD}} \mathrm{d}(\gamma(\alpha), \gamma(\beta)) \leq \Lambda((\gamma(\alpha), \gamma(\beta))) \leq C_{AD} \, \mathrm{d}(\gamma(\alpha), \gamma(\beta)). \tag{7.1}$$

The left inequality is trivial. For the right inequality, let $n, m \in \mathbb{N}$ such that $t_n \leq \alpha < t_{n+1}$, $t_m \leq \beta < t_{m+1}$. If $n = m$, then $\gamma(\alpha)$ and $\gamma(\beta)$ are either on the same half-circle, or on the same segment, and the result is clear. Assume that $|n - m| = 1$, then one of these two points is on a segment, and the other on a connected half-circle. Suppose for example that $t_{2k} \leq \alpha < t_{2k+1}$ and $t_{2k+1} \leq \beta < t_{2k+2}$, so that $\gamma(\alpha)$ is on the segment $[2^{k-1}, 2^k]$ and $\gamma(\beta)$ on a half-circle of center 0 and radius 2^k. Let $\omega = \gamma(t_{2k+1}) = 2^k$. Set $a = \mathrm{d}(\gamma(\alpha), \omega)$, $c = \mathrm{d}(\gamma(\beta), \omega)$ and $b = \mathrm{d}(\gamma(\alpha), \gamma(\beta))$. Applying elementary triangle geometry (see Figure 1), and the fact that $a \leq b$, $c \leq a + b \leq 2b$, write

$$\Lambda((\gamma(\alpha), \gamma(\beta)))^2 \lesssim (a + c)^2 \lesssim b^2 = \mathrm{d}(\gamma(\alpha), \gamma(\beta))^2.$$

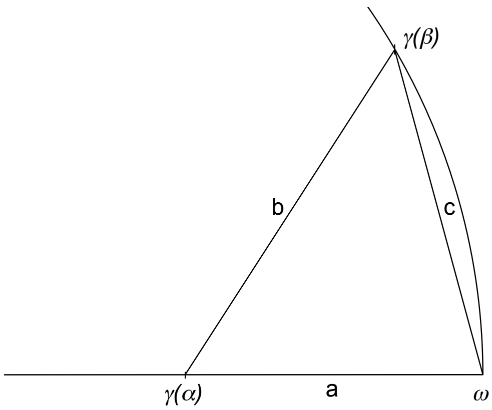

FIGURE 1. X_T is an Ahlfors-David set of dimension 1.

Finally, if $|n - m| \geq 2$, assume that for example $n \geq m$, then $\mathrm{d}(\gamma(\alpha), \gamma(\beta)) \geq |\gamma(\alpha)| - |\gamma(\beta)| \geq 2^{\frac{n}{2}-1} - 2^{\frac{m}{2}}$. On the other hand, $\Lambda((\gamma(\alpha), \gamma(\beta))) = \alpha - \beta \leq t_{n+1} - t_m \lesssim 2^{n/2} - 2^{m/2}$. The result follows. Now, let us check that $(X_T, \mathrm{d}, \Lambda)$ is Ahlfors-David. Let $x \in X_T$, $r > 0$, and set $B = B(x, r)$. As X_T is connected and not bounded, the connected component of x in B, denoted by $C(x)$, is at distance zero from the complement of B. Thus, there exists $y \in C(x)$ such that $\mathrm{d}(x, y) \geq r/2$. But then, by (7.1), we have $\Lambda(B) \geq \Lambda((x, y)) \geq C_{AD}^{-1} \mathrm{d}(x, y) \geq C_{AD}^{-1} (r/2)$. Moreover, set $t_0 = \inf\{t \geq 0 \mid \gamma(t) \in B\} \geq 0$ and $t_1 = \sup\{t \geq 0 \mid \gamma(t) \in B\} < +\infty$ because X_T is unbounded. Then $\Lambda(B) \leq \Lambda((\gamma(t_0), \gamma(t_1))) \leq C_{AD} \mathrm{d}(\gamma(t_0), \gamma(t_1)) \leq C_{AD} \, 2r$. This proves that $(X_T, \mathrm{d}, \Lambda)$ is an Ahlfors-David set of dimension 1.

(2) As a consequence of (1), observe that for any $k \geq 1$, $\Lambda(B(0,2^k)) \approx 2^k$. However, for every $\varepsilon > 0$, we have

$$\Lambda(B(0,2^k+\varepsilon)\backslash B(0,2^k)) \geq \pi 2^k.$$

If X_T had the annular decay property, the measure of this set would be going to 0 with ε. We thus have a contradiction. Hence, X_T cannot satisfy (AD), nor (RAD).

(3) It is obvious that X_T does not satisfy the monotone geodesic property (pick any two points on different half circles in X_T). For the homogeneous balls property, fix $\varepsilon > 0$ and consider the ball in X_T of center $x = (0,4)$ and radius $r = 3+\varepsilon$. Let $y = (0,1)$ and $\rho = 3$. The set $B(x,r) \cap B(y,\rho)$ has only one connected component, containing y, and its length is comparable to ε. On the other hand, the set $B(x,r) \cap B(y,2\rho)$ has two connected components, one of which containing x and of length comparable to 1. Thus, the doubling constant of the ball $B(x,r)$ seen as a space of homogeneous type exceeds C/ε. It follows that X_T cannot satisfy (HB), as the doubling constants of the balls cannot be uniform.

(4) Let $0 < \alpha < 1$ and let $c_1 > 0$ be such that $\forall t < c_1, \quad t^{1-\alpha}|\ln t| \leq 1$. We first prove that X_T satisfies the layer decay property (LD). Fix $z \in X_T, r > 0, 0 < \varepsilon \leq r$, set $B = B(z,r)$. Set as before $B_\varepsilon = \{x \in B | \mathrm{d}(x,B^c) \leq \varepsilon\} \cup \{y \in B^c | \mathrm{d}(y,B) \leq \varepsilon\}$ the union of the inner and outer layers. Let $\theta = \varepsilon/r$. We show that there exists a dimensional constant $C < +\infty$ such that

$$\Lambda(B_\varepsilon) \leq C\theta^\alpha \Lambda(B). \tag{7.2}$$

Observe that if $c_1 \leq \theta \leq 1$, the result is trivial:

$$\Lambda(B_\varepsilon) \leq \Lambda(B) \leq \left(\frac{1}{c_1}\right)^\alpha \theta^\alpha \Lambda(B).$$

So assume now that $\theta < c_1$. Observe that the points in B_ε are elements of X_T at distance less or equal to ε from $\overline{B}\backslash B$, where $\overline{B}$ is the adherence of B in X_T: $\gamma(t) \in \overline{B}\backslash B$ if $\gamma(t) \in B^c$ and either for every $s > 0$ small enough $\gamma(t+s) \in B$ or for every $s > 0$ small enough $\gamma(t-s) \in B$. If $r \leq 1$, remark that B and B^c are connected sets, so that there are less than two points in $\overline{B}\backslash B$. But since X_T is Ahlfors-David, we get $\Lambda(B_\varepsilon) \lesssim 2\varepsilon$, and also $\Lambda(B) \approx 1$. (7.2) follows.
We suppose now $r > 1$. Assume first that $0 \notin B$, that is $|z| > r$. Denote by C_i, respectively C'_j, $0 \leq i \leq p$, $0 \leq j \leq p+1$, the different connected components of B, respectively B^c, starting from the one closest to the origin. Because of (7.1), it is easy to see that each C_i, C'_j will roughly contribute to ε towards $\Lambda(B_\varepsilon)$. More precisely, set $B^i_\varepsilon = B_\varepsilon \cap (C_i \cup C'_i)$ for $0 \leq i \leq p$, and $B^{p+1}_\varepsilon = B_\varepsilon \cap C'_{p+1}$. Then, it follows from (7.1) that we have for every $0 \leq i \leq p+1$,

$$\Lambda(B^i_\varepsilon) \leq 4C_{AD}\,\varepsilon \lesssim \varepsilon. \tag{7.3}$$

Now, the idea is to estimate the number of components C_i, and to take care of the fact that some of them can contribute to $\Lambda(B_\varepsilon)$ for less than ε, as their length can be less than that if r and ε are large enough. It is easy to see that p can be roughly bounded by $\ln r$. Indeed, let $k_0 \geq 0$ be such that $2^{k_0} \leq \mathrm{dist}(C_0,0) < 2^{k_0+1}$ (remember that we have assumed $|z| > r$), and observe (see Figure 2) that we have, for $0 \leq i \leq p-1$, $\mathrm{d}(C_i,C_{i+1}) \geq 2^{k_0+2i} \geq 2^{2i}$. Consequently, we must have

$$\sum_{i=0}^{p-1} 2^{2i} \leq 2r \Rightarrow 4^p \lesssim r \Rightarrow p \lesssim \ln r.$$

On the other hand, observe that for every $0 \leq i \leq p$, $(C_i \cup C'_i) \subset B(0,2^{k_0+2i+2})$. Since X_T is Ahlfors-David, it follows that

$$\Lambda(C_i \cup C'_i) \lesssim 2^{k_0+2i+2} \lesssim 4^{k_0+i}. \tag{7.4}$$

Applying (7.3), we get $\Lambda(B_\varepsilon^i) \lesssim \min(\varepsilon, 4^{k_0+i})$. Finally, we obtain

$$\Lambda(B_\varepsilon) = \sum_{i=0}^{p+1} \Lambda(B_\varepsilon^i) \lesssim \sum_{i\geq 0 : 4^{k_0+i} \leq \varepsilon} 4^{k_0+i} + \varepsilon \operatorname{Card}\{0 \leq i \leq p+1 \mid \varepsilon < 4^{k_0+i}\}.$$

But $\operatorname{Card}\{0 \leq i \leq p+1 \mid \varepsilon < 4^{k_0+i}\} \leq \operatorname{Card}\{0 \leq i \leq p+1 \mid \varepsilon < 4^i\}$. Remark that if $4^i > \varepsilon$, then $i \geq \frac{\ln \varepsilon}{\ln 4}$. The cardinal intervening in the second term is thus bounded by $C(\ln r - \ln \varepsilon)$ where C is an absolute constant. Consequently, we have

$$\Lambda(B_\varepsilon) \lesssim \varepsilon + \varepsilon \ln\left(\frac{r}{\varepsilon}\right) \lesssim \varepsilon(1 - \ln \theta) \lesssim \theta^\alpha r,$$

because $\varepsilon(1-\ln\theta)\theta^{-\alpha} r^{-1} = \theta^{1-\alpha} - \theta^{1-\alpha}\ln\theta \leq 2$ as $\theta < c_1$. But since X_T is Ahlfors-David, we have $\Lambda(B) \approx r$ and (7.2) follows.
It remains to consider the case when $0 \in B$. But the same argument still works, only the notations have to be slightly modified because, this time, the origin belongs to the connected component C_0 of B. Thus, there is in this case the same number of components C_i and C_i', and it is the distance $\mathrm{d}(C_0', 0)$ that plays a role in the argument instead of $\mathrm{d}(C_0, 0)$. Apart from this, the argument is mostly unchanged, so we do not elaborate on it here.

It remains to prove (RLD). Let $B = B(z, r)$ be a ball in X_T as before. Let $0 < R \leq 2r$, $w \in X_T$. We prove that

$$\Lambda(B_\varepsilon \cap B(w,R)) \lesssim \left(\frac{\varepsilon}{R}\right)^\alpha \Lambda(B(w,R)), \tag{7.5}$$

Once again, when $\varepsilon/R \geq c_1$, the result is trivial, so we can assume that $\varepsilon/R < c_1$. Now, observe that we can apply exactly the same argument as above. The only difference is that instead of estimating the total number p of connected components C_i of B, we now have to estimate the number of these connected components that intersect $B(w,R)$. But by the same argument as before, this number is bounded by $(\ln R)$ as soon as $R > 1$ (and the result is trivial when $R \leq 1$). Going through with the argument, this provides the bound

$$\Lambda(B_\varepsilon \cap B(w,R)) \lesssim \left(\frac{\varepsilon}{R}\right)^\alpha R.$$

But since X_T is Ahlfors-David, we have $\Lambda(B(w,R)) \approx R$ and (7.5) follows.

(5) Applying Proposition 5.2, (5) is a direct consequence of (4), but to better understand this example, we will give a direct proof here. Fix $z \in X_T$, $r > 0$, $1 < \nu < +\infty$, set $B = B(z,r)$, and let $f \in L^\nu(B)$, f supported on B, $g \in L^{\nu'}(2B\backslash B)$, g supported on $2B\backslash B$. Remark that because of Proposition 2.6, we could limit ourselves to the case when $\nu = \nu' = 2$, but we will keep on working with undefined exponents to show that they do not play any part in our argument and that the latter does not rely on any specific geometry brought by L^2 integrability. Assume as before that for example $0 \notin B$, the argument is unchanged when $0 \in B$, only the notations have to be adapted. We adopt the same notations as in (4): denote by C_i, respectively C_j', $0 \leq i \leq p$, $0 \leq j \leq p+1$, the different connected components of B, respectively B^c, starting from the one closest to the origin. Set $I_{2i} = \{t \geq 0 \mid \gamma(t) \in C_{i-1}\}$ for $1 \leq i \leq p+1$, $I_{2j+1} = \{t \geq 0 \mid \gamma(t) \in C_j' \cap (2B\backslash B)\}$ for $0 \leq j \leq p+1$. We want to estimate the following quantity

$$H(f,g) = \int_B \int_{2B\backslash B} \frac{f(x)g(y)}{\Lambda(B(x, \mathrm{d}(x,y)))} \mathrm{d}\Lambda(x)\mathrm{d}\Lambda(y).$$

As X_T is Ahlfors-David, and applying (7.1), we have

$$H(f,g) \approx \int_B \int_{2B\backslash B} \frac{f(x)g(y)}{\mathrm{d}(x,y)} \mathrm{d}\Lambda(x)\mathrm{d}\Lambda(y) \approx \sum_{i=0}^{p} \sum_{j=-1}^{p} \int_{I_{2i}} \int_{I_{2j+1}} \frac{f(\gamma(t))g(\gamma(s))}{|s-t|} \mathrm{d}t\mathrm{d}s, \tag{7.6}$$

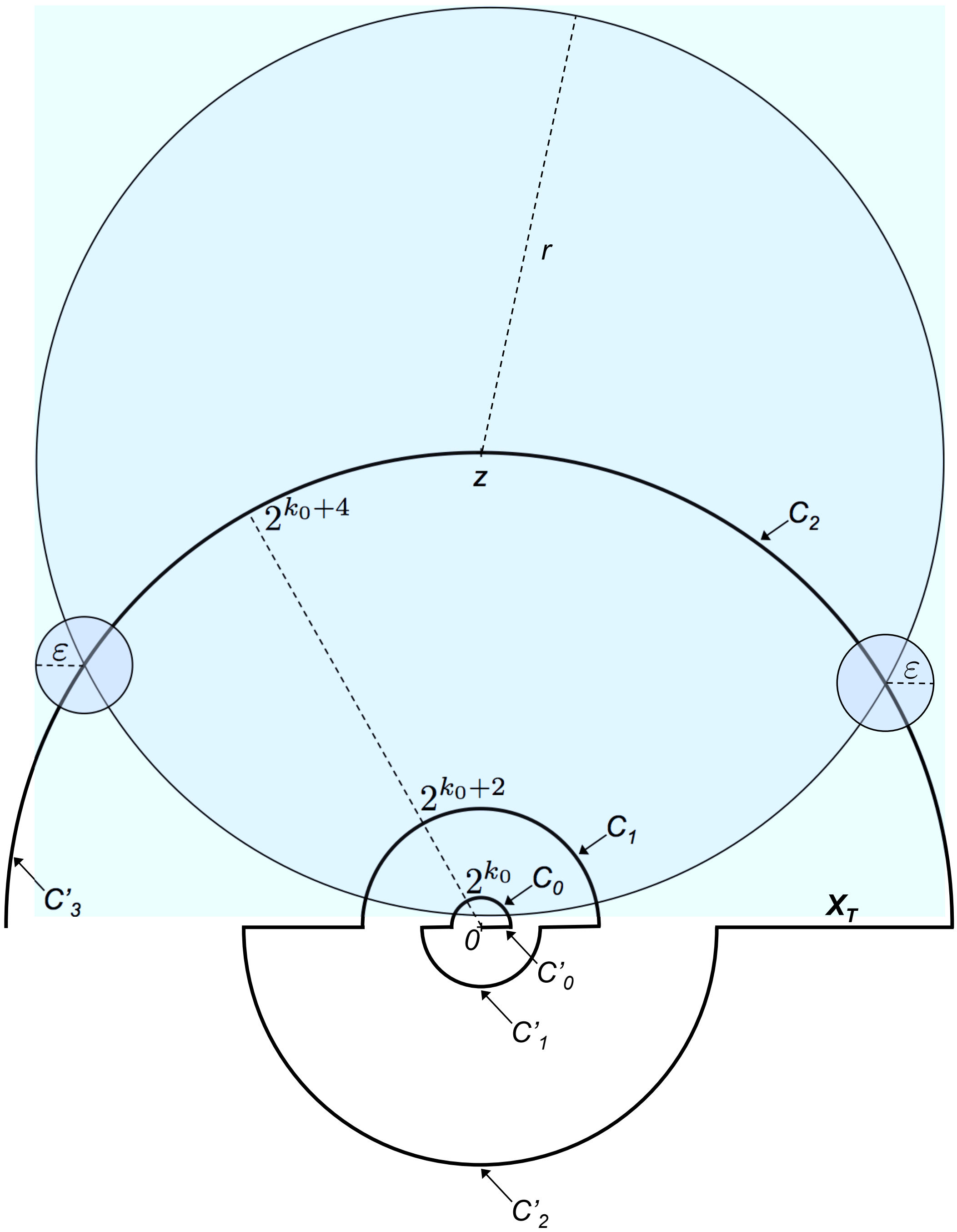

FIGURE 2. Layer decay property (LD) in the space X_T.

because $|\gamma'(t)| = 1$ for every $t \geq 0$. Set $\widetilde{f} = f \circ \gamma$, $\widetilde{g} = g \circ \gamma$, $\widetilde{f} \in L^\nu$, supported on $\bigcup_{i=1}^{p+1} I_{2i}$, $\widetilde{g} \in L^{\nu'}$, supported on $\bigcup_{j=0}^{p+1} I_{2j+1}$, with $\|\widetilde{f}\|_\nu = \|f\|_\nu$, $\|\widetilde{g}\|_{\nu'} = \|g\|_{\nu'}$. Let again $k_0 \geq 0$ be such that $2^{k_0} \leq \mathrm{dist}(C_0, 0) < 2^{k_0+1}$ (remember that we have assumed $|z| > r$). Because of the fact that $|\gamma'(t)| = 1$ for every $t \geq 0$, observe that we have, by (7.4), for every $1 \leq i \leq p+1$, $0 \leq j \leq p$,

$$|I_{2i}| = \Lambda(C_{i-1}) \leq \Lambda(C_{i-1} \cup C'_{i-1}) \lesssim 2^{k_0} 4^i, \qquad |I_{2j+1}| \leq \Lambda(C'_j) \leq \Lambda(C_j \cup C'_j) \lesssim 2^{k_0} 4^j.$$

For $j=p+1$, remark that as for $0\le i\le p$, $C_i\subset B(0,2^{k_0+2i+2})$, we have $B\subset B(0,2^{k_0+2p+2})$. It follows that $2B\subset B(0,3\times 2^{k_0+2p+2})$. As a matter of fact, if $\mathrm{d}(z,x)<2r$, then $|x|\le \mathrm{d}(x,z)+|z|<2r+|z|<3|z|<3\times 2^{k_0+2p+2}$, because we have assumed $|z|>r$. Thus, since X_T is Ahlfors-David, it follows that

$$|I_{2p+3}|\le \Lambda(2B\backslash B)\le \Lambda(B(0,3\times 2^{k_0+2p+2}))\lesssim 2^{k_0}4^p.$$

Finally, it is easy to see that if $j\notin\{i-1,i\}$, we have $\mathrm{dist}(I_{2i},I_{2j+1})\gtrsim 2^{k_0}|4^i-4^j|$. Now, set

$$f_i=\left(\int_{I_{2i}}|\widetilde{f}(t)|^\nu \mathrm{d}t\right)^{1/\nu}\in \ell^\nu(\{1,...,p+1\}),\quad \text{with}\quad \|(f_i)_i\|_{\ell^\nu}=\|\widetilde{f}\|_\nu=\|f\|_\nu,$$

$$g_j=\left(\int_{I_{2j+1}}|\widetilde{g}(s)|^{\nu'} \mathrm{d}s\right)^{1/\nu'}\in \ell^{\nu'}(\{0,...,p+1\}),\quad \text{with}\quad \|(g_j)_j\|_{\ell^{\nu'}}=\|\widetilde{g}\|_{\nu'}=\|g\|_{\nu'}.$$

Split the sum in (7.6) for the neighboring I_k and the ones that are far from one another: we have

$$\begin{aligned}H(f,g)&\lesssim\sum_{i=1}^{p+1}\sum_{j=i-1}^{i}\int_{I_{2i}}\int_{I_{2j+1}}\frac{|\widetilde{f}(t)||\widetilde{g}(s)|}{|s-t|}\mathrm{d}t\mathrm{d}s+\sum_{i=1}^{p+1}\sum_{j\notin\{i-1,i\}}\int_{I_{2i}}\int_{I_{2j+1}}\frac{|\widetilde{f}(t)||\widetilde{g}(s)|}{|s-t|}\mathrm{d}t\mathrm{d}s\\&=H_1(f,g)+H_2(f,g).\end{aligned}$$

For H_1, apply (H) on $\mathbb{R}$ and then the Cauchy-Schwarz inequality to get

$$\begin{aligned}H_1(f,g)&\lesssim\sum_{i=1}^{p+1}\sum_{j\in\{i-1,i\}}f_ig_j\lesssim\left(\sum_{i=1}^{p+1}\sum_{j\in\{i-1,i\}}f_i^\nu\right)^{1/\nu}\left(\sum_{j=0}^{p+1}\sum_{\substack{1\le i\le p+1\\ i\in\{j+1,j\}}}g_j^{\nu'}\right)^{1/\nu'}\\&\lesssim\|f\|_\nu\|g\|_{\nu'}.\end{aligned}$$

To estimate H_2, write

$$\begin{aligned}H_2(f,g)&\lesssim\sum_{i=1}^{p+1}\sum_{j\notin\{i-1,i\}}\frac{1}{\mathrm{dist}(I_{2i},I_{2j+1})}f_i|I_{2i}|^{\frac{1}{\nu'}}g_j|I_{2j+1}|^{\frac{1}{\nu}}\\&\lesssim\sum_{i=1}^{p+1}\sum_{j\notin\{i-1,i\}}\frac{4^{\frac{i}{\nu'}}4^{\frac{j}{\nu}}}{|4^i-4^j|}f_ig_j\end{aligned}$$

By symmetry, we will be done if we can bound for example the sum for $j>i$. But if $j>i$ observe that $|4^i-4^j|=4^i(4^{j-i}-1)\gtrsim 4^j$. Applying the Cauchy-Schwarz inequality, write then

$$\begin{aligned}\sum_{1\le i\le p+1,\,j>i}\frac{4^{\frac{i}{\nu'}}4^{\frac{j}{\nu}}}{|4^i-4^j|}f_ig_j&\lesssim\sum_{1\le i\le p+1,\,j>i}f_ig_j4^{-\frac{j-i}{\nu'}}\\&\lesssim\left(\sum_{1\le i\le p+1,\,j>i}f_i^\nu 4^{-\frac{j-i}{\nu'}}\right)^{\frac{1}{\nu}}\left(\sum_{1\le i\le p+1,\,j>i}g_j^{\nu'}4^{-\frac{j-i}{\nu'}}\right)^{\frac{1}{\nu'}}\\&\lesssim\left(\sum_{1\le i\le p+1}f_i^\nu\sum_{j>i}4^{-\frac{j-i}{\nu'}}\right)^{\frac{1}{\nu}}\left(\sum_{2\le j\le p+1}g_j^{\nu'}4^{-\frac{j}{\nu'}}\sum_{1\le i<j}4^{\frac{i}{\nu'}}\right)^{\frac{1}{\nu'}}\\&\lesssim\left(\sum_{1\le i\le p+1}f_i^\nu\right)^{\frac{1}{\nu}}\left(\sum_{2\le j\le p+1}g_j^{\nu'}\right)^{\frac{1}{\nu'}}\lesssim\|f\|_\nu\|g\|_{\nu'}.\end{aligned}$$

Finally, one gets $H(f,g)\lesssim\|f\|_\nu\|g\|_{\nu'}$, and thus X_T satisfies the Hardy property (HP).

□

Consequently, the space X_T satisfies (RLD) and (HP), but neither (HB), (M) nor (RAD). As we have already pointed out before, there is a tangible difference between the definitions of layer decay and annular decay properties. Thus, it is not surprising to find out that these properties are not equivalent. We now have a counterexample for most of the false implications in Theorem 2.5. It only remains to build a counterexample to prove that (LD) $\not\Rightarrow$ (RLD) to complete the proof.

Remark 7.2. Slightly modifying the construction, basically just by truncating the space X_T, one can get a much simpler example of a space satisfying (RLD) but not (RAD). Indeed, let $X'_T = \{\gamma(t) \mid 0 \leq t \leq t_8\} \cup [8, +\infty[$ where $[8, +\infty[$ denotes the half-line on the real axis. Then the argument in (3) obviously still holds and X'_T does not satisfy (RAD). On the other hand, it is immediate to see that for any given ball B of X'_T, $\mathrm{Card}(\overline{B}\backslash B) \leq 6$, and (RLD) follows easily.

8. Counterexamples and end of the proof of Theorem 2.5

We now present some variations of the space X_T in order to provide a space where the Hardy property cannot be satisfied. It was originally inspired from the curve of Tessera, but is actually in the end only marginally connected to it. Still in the complex plane, consider the space formed by the union of the segment $[0, 1]$ on the real axis, the half circle of center 0 and radius 1 in the half-plan $\Im z \geq 0$, and the half-line $]-\infty, -1]$ on the real axis. Parameter this curve so that it is traveled at constant speed. This is someway a truncation of the space X_T. Now introduce a small perturbation ϵ of the half-circle: for $t \in [0, \pi]$, set $\rho(t) = 1 + \epsilon(t)$, where ϵ is a rapidly oscillating function in the neighborhood of the origin, and set

$$X_\epsilon = \{\gamma(t) \mid t \geq -1\}, \quad \text{with} \quad \gamma(t) = \begin{cases} (t+1, 0) & \text{for} \quad -1 \leq t \leq 0, \\ (\rho(t)\cos t, \rho(t)\sin t) & \text{for} \quad 0 < t < \pi, \\ (\pi - t, 0) & \text{for} \quad t \geq \pi. \end{cases}$$

Choose an oscillating function ϵ, ensuring that X_ϵ keeps finite arclength: $\epsilon(t) = a(t)\sin(b(t))$ for functions a, b appropriately chosen.

8.1. **Exponential oscillation.** Let

$$\epsilon_1(t) = e^{-\frac{1}{t^2}} \sin(\pi e^{\frac{1}{t} - \frac{1}{\pi}}),$$

and set $X_1 = X_{\epsilon_1}$. See Figure 3 for a representation of the space X_1.

Remark that choosing $\frac{1}{t^\alpha}$ instead of $\frac{1}{t}$ would not change anything in the following. Observe that with this choice of ϵ, γ is C^1 and $|\gamma'|$ is uniformly bounded below and above, which obviously makes of $(X_1, \mathrm{d}, \Lambda)$ an Ahlfors-David space. In the following of this section, $B(z, r)$ will always denote the ball in $\mathbb{R}^2$ of center z and radius r. For a ball B in $\mathbb{R}^2$ centered at a point of $X_1 \subset \mathbb{R}^2$, we will denote by B^{X_1} the corresponding ball in the space X_1. Let $B^0 = B(0, 1)$.

Proposition 8.1. (1) X_1 *does not satisfy the layer decay property* (LD).
(2) X_1 *does not satisfy the Hardy property* (HP).

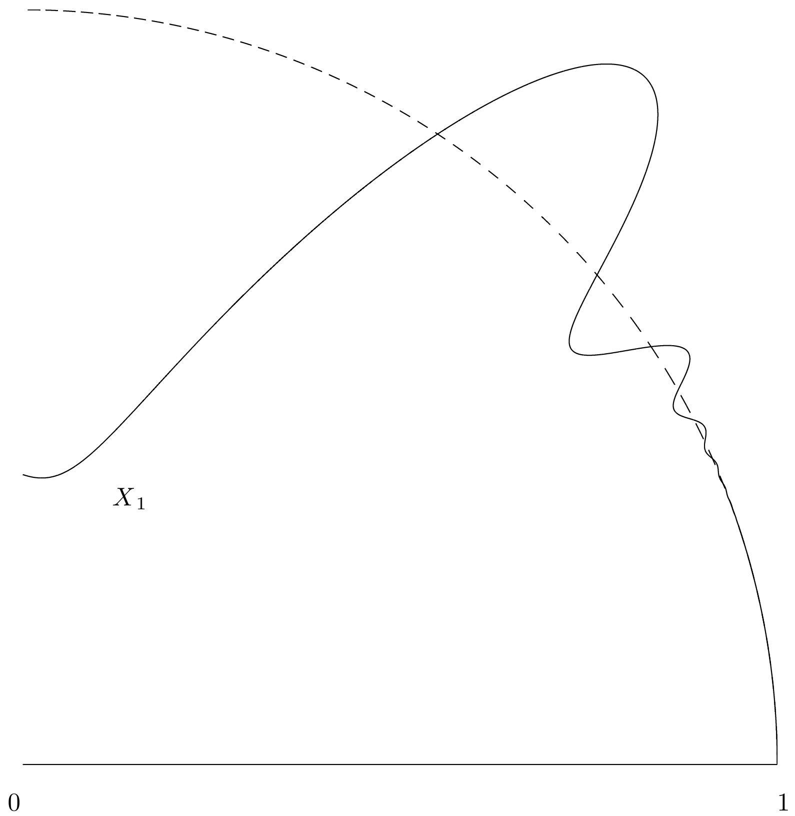

FIGURE 3. A representation of the space X_1.

Proof. We prove that neither of these properties are satisfied for the ball B^{0,X_1}.

(1) Let $\varepsilon > 0$. Remark that if $\gamma(t) \in \overline{B^{0,X_1}} \backslash B^{0,X_1}$, then $\epsilon(t) = 0$. Denote by t_k these points with $(t_k)_{k\in\mathbb{N}}$ a sequence decreasing to zero, and $t_0 = \pi$. Then we have

$$B_\varepsilon^{0,X_1} = \{\gamma(t) \mid \exists k \in \mathbb{N} \;\; \mathrm{d}(\gamma(t), \gamma(t_k)) < \varepsilon\}.$$

Because of the uniform boundedness of $|\gamma'|$ above and below, observe that we have

$$\Lambda(B_\varepsilon^{0,X_1}) \approx |\{t \in [0,\pi] \mid \exists k \;\; |t_k - t| < \varepsilon\}|,$$

On the other hand, for $k \geq 1$, we have

$$\pi e^{-\frac{1}{\pi}} e^{\frac{1}{t_k}} = k\pi \Rightarrow t_k = \frac{1}{\ln k + 1/\pi} \Rightarrow t_k - t_{k+1} \lesssim \frac{1}{k}.$$

Thus, if $k \geq \frac{1}{\varepsilon}$, and $t \leq t_k \lesssim \frac{1}{-\ln\varepsilon}$, then $t_k - t_{k+1} \lesssim \varepsilon$. It implies that $\gamma(t)$ stays inside B_ε^{0,X_1} for all the $t \leq \frac{C}{-\ln\varepsilon}$ for some constant $C < +\infty$. We obtain

$$\Lambda(B_\varepsilon^{0,X_1}) \gtrsim \left|\left\{t \in [0,\pi] \mid \exists k \geq \frac{1}{\varepsilon} \;\; |t_k - t| < \varepsilon\right\}\right| \geq \left|\left[0, \frac{C}{-\ln\varepsilon}\right]\right| \gtrsim \frac{1}{-\ln\varepsilon}.$$

It follows that there cannot be any upper bound of the form ε^η for $\Lambda(B_\varepsilon^{0,X_1})$, and (LD) cannot be satisfied.

(2) Let $1 < \nu < +\infty$. Let $f \in L^\nu(B^{0,X_1})$, f supported on B^{0,X_1}, $g \in L^{\nu'}(2B^{0,X_1} \backslash B^{0,X_1})$, g supported on $2B^{0,X_1} \backslash B^{0,X_1}$. Denote by $I_{2k}, k \geq 0$ the connected sets of t for which $\gamma(t) \in B^{0,X_1}$,

$I_{2k} =]t_{2k}, t_{2k+1}[$, and $I_{2p+1}, p \geq 0$ the connected sets of t for which $\gamma(t) \in 2B^{0,X_1} \backslash B^{0,X_1}$, $I_{2p+1} = [t_{2p+1}, t_{2p+2}]$. By the Ahlfors-David property of X_1, we have

$$|H(f,g)| = \left| \int_{B^{0,X_1}} \int_{2B^{0,X_1} \backslash B^{0,X_1}} \frac{f(x)g(y)}{\lambda(x,y)} \mathrm{d}\Lambda(y)\mathrm{d}\Lambda(x) \right| \approx \left| \sum_{k,p \geq 0} \int_{t \in I_{2k}} \int_{s \in I_{2p+1}} \frac{\widetilde{f}(t)\widetilde{g}(s)}{|t-s|} \mathrm{d}s\mathrm{d}t \right|,$$

where $\widetilde{f} = f \circ \gamma$, $\widetilde{g} = g \circ \gamma$, $\widetilde{f} \in L^{\nu}$, supported on $\cup_{k \geq 0} I_{2k}$, $\widetilde{g} \in L^{\nu'}$, supported on $\cup_{p \geq 0} I_{2p+1}$, and $\|\widetilde{f}\|_{\nu} \approx \|f\|_{\nu}$, $\|\widetilde{g}\|_{\nu'} \approx \|g\|_{\nu'}$, because $|\gamma'| \approx 1$. Now, assume that $\widetilde{f}, \widetilde{g}$ are constant and positive on each I_{2k}, I_{2p+1}. Set

$$f_k = \left(\int_{I_{2k}} |\widetilde{f}(t)|^{\nu} \mathrm{d}t \right)^{1/\nu} \in \ell^{\nu}(\mathbb{N}), \quad \text{with} \quad \|(f_k)_k\|_{\ell^{\nu}} = \|\widetilde{f}\|_{\nu},$$

$$g_p = \left(\int_{I_{2p+1}} |\widetilde{g}(s)|^{\nu'} \mathrm{d}s \right)^{1/\nu'} \in \ell^{\nu'}(\mathbb{N}), \quad \text{with} \quad \|(g_p)_p\|_{\ell^{\nu'}} = \|\widetilde{g}\|_{\nu'}.$$

Remark that if $p, k \geq 1$ with $p \notin \{k, k-1\}$, and $t \in I_{2k}$, $s \in I_{2p+1}$, then

$$|s-t| \lesssim \left| \frac{1}{\ln p} - \frac{1}{\ln k} \right| \lesssim \frac{|\ln(p/k)|}{\ln p \ln k}.$$

We thus have

$$\begin{aligned} |H(f,g)| &\gtrsim \sum_{\substack{k,p \geq 1 \\ p \notin \{k,k-1\}}} \frac{\ln p \ln k}{|\ln(p/k)|} \int_{t \in I_{2k}} |\widetilde{f}(t)| \mathrm{d}t \int_{s \in I_{2p+1}} |\widetilde{g}(s)| \mathrm{d}s \\ &= \sum_{\substack{k,p \geq 1 \\ p \notin \{k,k-1\}}} f_k g_p \frac{\ln p \ln k}{|\ln(p/k)|} |I_{2k}|^{\frac{1}{\nu'}} |I_{2p+1}|^{\frac{1}{\nu}}. \end{aligned}$$

But since

$$|I_l| = |t_{l+1} - t_l| \underset{l \to +\infty}{\sim} \frac{1}{l(\ln l)^2},$$

there exists $N \in \mathbb{N}^*$ such that if $l \geq N$, then $|I_l| \geq \frac{1}{2l(\ln l)^2}$. It follows that

$$|H(f,g)| \gtrsim \sum_{\substack{k,p \geq N \\ p \notin \{k,k-1\}}} f_k g_p \frac{(\ln p)^{1-\frac{2}{\nu}} (\ln k)^{1-\frac{2}{\nu'}}}{k^{\frac{1}{\nu'}} p^{\frac{1}{\nu}} |\ln(p/k)|}.$$

It is easy to see that this is an unbounded operator. Fix $0 < \eta < 1/2$, and set for example for $k, p \geq N$

$$f_k = \frac{1}{k^{\frac{1}{\nu}} (\ln k)^{\frac{1}{\nu}+\eta}}, \quad g_p = \frac{1}{p^{\frac{1}{\nu'}} (\ln p)^{\frac{1}{\nu'}+\eta}}.$$

Then

$$\begin{aligned} \sum_{p \geq k+1} g_p \frac{1}{p^{\frac{1}{\nu}}} \frac{(\ln p)^{1-\frac{2}{\nu}}}{|\ln(p/k)|} &= \sum_{p \geq k+1} \frac{1}{p (\ln p)^{\frac{1}{\nu}+\eta} \ln(p/k)} \\ &\geq \int_{k+1}^{+\infty} \frac{\mathrm{d}t}{t (\ln t)^{\frac{1}{\nu}+\eta} \ln(t/k)} = \int_{1+\frac{1}{k}}^{+\infty} \frac{\mathrm{d}u}{u \ln u \ln(ku)^{\frac{1}{\nu}+\eta}} \\ &\geq \frac{1}{\ln(3k)^{\frac{1}{\nu}+\eta}} \int_{1+\frac{1}{k}}^{3} \frac{\mathrm{d}u}{u \ln u} \\ &\gtrsim \frac{|\ln(\ln(1+1/k))|}{(\ln k)^{\frac{1}{\nu}+\eta}} \underset{k \to +\infty}{\sim} (\ln k)^{\frac{1}{\nu'}-\eta}. \end{aligned}$$

It follows that

$$|H(f,g)| \gtrsim \sum_{k\geq N} f_k \frac{(\ln k)^{1-\frac{2}{\nu'}}}{k^{\frac{1}{\nu'}}} (\ln k)^{\frac{1}{\nu'}-\eta} = \sum_{k\geq N} \frac{1}{k(\ln k)^{2\eta}} = +\infty.$$

Thus (H) cannot be satisfied for any $1 < \nu < +\infty$. □

Remark 8.2. Observe that although the layer decay property is not satisfied here, we still have $\Lambda(B_\varepsilon^{0,X_1}) \underset{\varepsilon\to 0}{\to} 0$. Indeed, $\bigcap_{n\geq 1} B_{1/n}^{0,X_1} = \overline{B^{0,X_1}} \backslash B^{0,X_1}$, but this set is of measure 0 as it is countable. Thus, we have

$$\Lambda(\overline{B^{0,X_1}} \backslash B^{0,X_1}) = 0 = \lim_{n\to+\infty} \Lambda(B_{1/n}^{0,X_1}).$$

Observe furthermore that it is always the case in this kind of example with a continuous function ϵ, because $\overline{B^{0,X_\epsilon}} \backslash B^{0,X_\epsilon}$ must necessarily be a countable set, hence of measure zero. Indeed, $\overline{B^{0,X_\epsilon}} \backslash B^{0,X_\epsilon}$ is the set of points $\gamma(t_0)$ for which $\epsilon(t_0) = 0$ and for every $\eta > 0$, there exists $0 < \varepsilon < \eta$ such that $\epsilon(t_0 \pm \varepsilon) < 0$. By continuity of ϵ, we deduce from this that for every point $\gamma(t_0) \in \overline{B^{0,X_\epsilon}} \backslash B^{0,X_\epsilon}$, there exists $q_0 \in \mathbb{Q}$ with $\epsilon(q_0) < 0$ and $|t_0 - q_0|$ as small as one wants. We can thus construct an injection from $\overline{B^{0,X_\epsilon}} \backslash B^{0,X_\epsilon}$ to $\mathbb{Q}$ and it follows that $\overline{B^{0,X_\epsilon}} \backslash B^{0,X_\epsilon}$ is countable, hence of measure zero.

8.2. **Polynomial oscillation.** This time, let $b(t) = \frac{\pi^2}{t}$: choose

$$\epsilon_2(t) = A_0 \left(\frac{t}{\pi}\right)^3 \sin\left(\frac{\pi^2}{t}\right),$$

with A_0 a sufficiently small constant to be specified later, and construct a space X_2 as before. Again, $(X_2, \mathrm{d}, \Lambda)$ is an Ahlfors-David space (and thus a space of homogeneous type).

Proposition 8.3. (1) X_2 *does not satisfy the Hardy property* (HP).

(2) X_2 *does satisfy the layer decay inequality* (LD), *but not the relative layer decay inequality* (RLD).

Proof. (1) We prove again that (H) is not satisfied for the unit ball B^{0,X_2}. Let us use the same notations as before. For functions $f \in L^2(B^{0,X_2})$, f supported inside B^{0,X_2}, and $g \in L^2(2B^{0,X_2} \backslash B^{0,X_2})$, g supported inside $2B^{0,X_2} \backslash B^{0,X_2}$, we have

$$|H(f,g)| \approx \left| \sum_{k,p\geq 0} \int_{t\in I_{2k}} \int_{s\in I_{2p+1}} \frac{\widetilde{f}(t)\widetilde{g}(s)}{|t-s|} \mathrm{d}s\mathrm{d}t \right|.$$

This time, we have for $k \geq 1$, $t_k = \frac{\pi}{k}$, and thus for $l \geq 1$, we have $|I_l| = \frac{\pi}{l(l+1)} \gtrsim \frac{1}{l^2}$. Besides, if $p, k \geq 1$ with $p \notin \{k, k-1\}$, and $t \in I_{2k}$, $s \in I_{2p+1}$, then

$$|s-t|^{-1} \gtrsim \left|\frac{1}{p} - \frac{1}{k}\right|^{-1} \gtrsim \frac{pk}{|p-k|}.$$

Assume again that $\widetilde{f}, \widetilde{g}$ are constant and positive on each I_{2k}, I_{2p+1}, and set

$$f_k = \left(\int_{I_{2k}} |\widetilde{f}(t)|^2 \mathrm{d}t\right)^{1/2} \in \ell^2(\mathbb{N}), \quad \text{with} \quad \|(f_k)_k\|_{\ell^2} = \|\widetilde{f}\|_2,$$

$$g_p = \left(\int_{I_{2p+1}} |\widetilde{g}(s)|^2 \mathrm{d}s \right)^{1/2} \in \ell^2(\mathbb{N}), \quad \text{with} \quad \|(g_p)_p\|_{\ell^2} = \|\widetilde{g}\|_2.$$

Then, we have

$$|H(f,g)| \gtrsim \sum_{\substack{k,p\geq 1 \\ p\notin\{k,k-1\}}} \frac{f_k g_p}{|p-k|}.$$

But it is well known that this operator is unbounded on ℓ^2. It follows that (H) cannot be satisfied for $\nu = 2$. Thus, X_2 does not satify (HP), nor (RLD) because of Proposition 5.2.

(2) Let $\varepsilon > 0$. We are going to prove (LD) for all the balls $B^{X_2}(z,r)$ centered at a point $z \in X_2$ of radius $r > 0$. We classify these balls in three categories, each of which will be taken care of differently: first there are the balls $B(z,r)$ of radius $r \geq 1/2$, then there are the balls $B(z,r)$ of radius $0 < r < 1/2$ tangential to the ball B^0 at the point of affix 1, and finally the balls $B(z,r)$ of radius $0 < r < 1/2$ non tangential to the ball B^0 at the point of affix 1. We begin by taking care of the first category. We first show that X_2 satisfies (LD) for the unit ball B^{0,X_2}, for some exponent $\eta < 1$. Indeed, remark that if $k \geq \varepsilon^{-1/2}$, then $|I_k| = |t_k - t_{k+1}| \lesssim \frac{1}{k^2} \lesssim \varepsilon$, and it implies that $\gamma(t)$ stays inside B_ε^{0,X_2} for all the $t \leq C\varepsilon^{1/2}$ for some uniform constant $0 < C < +\infty$. Besides, observe that we have

$$\mathrm{Card}\left\{ k \in \mathbb{N} \mid k \lesssim \varepsilon^{-1/2} \right\} \lesssim \varepsilon^{-1/2},$$

and that for each one of the corresponding t_k, there is a contribution of at most ε to $\Lambda(B_\varepsilon^{0,X_2})$. Thus, we have

$$\Lambda(B_\varepsilon^{0,X_2}) \lesssim |[0, C\varepsilon^{1/2}]| + \varepsilon \times \mathrm{Card}\left\{ k \in \mathbb{N} \mid k \lesssim \varepsilon^{-1/2} \right\} \lesssim \varepsilon^{1/2} + \varepsilon \times \varepsilon^{-1/2} \lesssim \varepsilon^{1/2}.$$

Since X_2 is Ahlfors-David, we have $\Lambda(B^{0,X_2}) \approx 1$, and (LD) follows.

Now, observe that this extends to all the balls $B = B(z,r)$ of radius $r \geq 1/2$. As a matter of fact, remark that we necessarily have $\Lambda(B_\varepsilon^{X_2}) \leq \Lambda(B_\varepsilon^{0,X_2}) \lesssim \varepsilon^{1/2}$. Indeed, there are at most two elements in $\overline{B^{X_2}} \backslash B^{X_2}$ outside of $\{\gamma(t) \mid 0 \leq t \leq \pi\}$. And it is also easy to see that $\overline{B^{X_2}} \backslash B^{X_2} \cap \{\gamma(t) \mid 0 \leq t \leq \pi\}$ can be injected inside $\overline{B^{0,X_2}} \backslash B^{0,X_2}$. Thus the preceding argument still applies. It follows that

$$\Lambda(B_\varepsilon^{X_2}) \lesssim \left(\frac{\varepsilon}{r}\right)^{1/2} r^{1/2} \lesssim \left(\frac{\varepsilon}{r}\right)^{1/2} r,$$

because $r \geq 1/2$. But since X_2 is Ahlfors-David, we have $\Lambda(B^{X_2}) \approx r$ and (LD) follows.

Now, we consider the balls $B = B(z,r)$ with $z = 1-r$ and $0 < r < 1/2$, tangential to B^0 at the point of affix 1. Let $\mathcal{C}$ denote in $\mathbb{R}^2$ the circle of center z and radius r. Switching to polar coordinates, for $t > 0$ sufficiently small ($t \leq t_{\max} = \arctan \frac{r}{1-r}$), denote by $M(t) = (t, v(t))$ the point of the circle $\mathcal{C}$ farthest from the origin, and let $u(t) = 1 - v(t)$ (see Figure 4). Then $v(t)$ satisfies the following equation

$$v(t)^2 - 2v(t)(1-r)\cos t + (1-r)^2 = r^2,$$

so that we have

$$u(t)^2 + 2u(t)[(1-r)\cos t - 1] + 2(1-r)[1-\cos t] = 0,$$

hence

$$u(t) = 1 - (1-r)\cos t - [(1-r)^2\cos^2 t + 1 - 2(1-r)]^{1/2}.$$

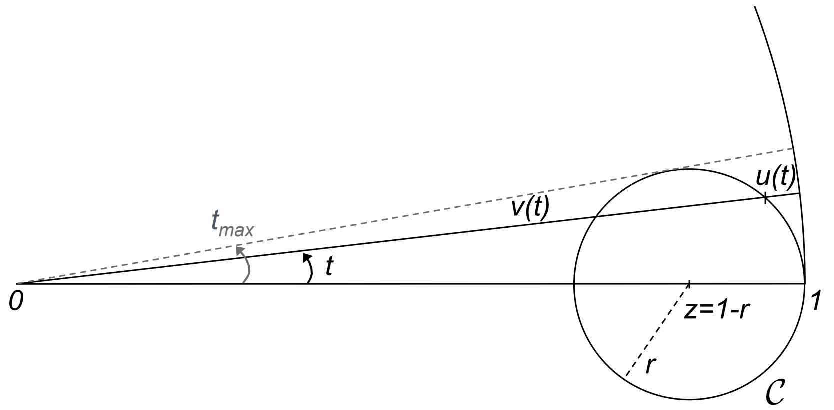

FIGURE 4. Existence of a separation between $\mathcal{C}$ and the unit circle.

There exists $t_0 > 0$ such that for every $0 < t < t_0$, $\cos t \le 1 - \frac{t^2}{4}$ and $t^4 \le t^2$. Then, for every $0 < t < t_0$ and $0 < r < 1/2$, we have

$$\begin{aligned}
u(t) &\ge 1 - (1-r)(1 - \frac{t^2}{4}) - \left[(1-r)^2(1 - \frac{t^2}{4})^2 + 2r - 1\right]^{1/2} \\
&\ge r + \frac{1-r}{4}t^2 - \left[(1-r)^2(1 - \frac{t^2}{2} + \frac{t^4}{16}) + 2r - 1\right]^{1/2} \\
&\ge r + \frac{1-r}{4}t^2 - r\left[1 - \frac{(1-r)^2}{2r^2}t^2 + \frac{(1-r)^2}{16r^2}t^4\right]^{1/2} \\
&\ge r + \frac{1-r}{4}t^2 - r\left[1 - \frac{7(1-r)^2}{16r^2}t^2\right]^{1/2} \ge r + \frac{1-r}{4}t^2 - r\left[1 - \frac{1}{2}\frac{7(1-r)^2}{16r^2}t^2\right] \\
&\ge \frac{7 - 6r - r^2}{32r}t^2 \ge \frac{3}{16}t^2.
\end{aligned}$$

Furthermore, since $u(t)$ clearly is an increasing function, if $t_0 \le t \le t_{\max}$, we have $u(t) \ge \frac{3}{16}t_0^2$. Now, as $|\epsilon(t)| \le A_0(t/\pi)^3$, it is clear that if $A_0 < \frac{3}{16}t_0^2$, then $\gamma(t)$ stays outside of B for every $0 < t \le \pi$. Thus, B^{X_2} reduces to the open segment $]1-2r, 1[$ which is connected. It implies that $\mathrm{Card}(\overline{B^{X_2}} \backslash B^{X_2}) \le 2$, and consequently, we have

$$\Lambda(B_\varepsilon^{X_2}) \lesssim 2\varepsilon \lesssim \left(\frac{\varepsilon}{r}\right) \Lambda(B^{X_2}),$$

as, once again, by the Ahlfors-David property of X_2 we have $\Lambda(B^{X_2}) \simeq r$.

It remains only to consider the balls $B = B(z, r)$ of radius $0 < r < 1/2$ non tangential to the ball B^0 at the point of affix 1. But it is immediate to see that the number of connected components of such a ball B^{X_2} is at most 2: B^{X_2} is a connected set in most cases, but there can be two connected components if $z = \gamma(\tau)$ with $0 < \tau \le \pi$ is small and r is small enough (then $1 \notin B^{X_2}$ but $\gamma(t) \in B^{X_2}$ for some $-1 < t < 0$). Thus, as for the balls of the previous category, we have $\mathrm{Card}(\overline{B^{X_2}} \backslash B^{X_2}) \le 4$, and

$$\Lambda(B_\varepsilon^{X_2}) \lesssim 4\varepsilon \lesssim \left(\frac{\varepsilon}{r}\right) \Lambda(B^{X_2}).$$

Putting all this together, we have proven that X_2 satisfies (LD) for $\eta = 1/2$. $\square$

The space X_2 is thus a counterexample to the implication (LD) $\Rightarrow$ (RLD). The proof of Theorem 2.5 is now complete.

Remarks 8.4.
- Choosing $b(t) = \frac{\pi^{1+\alpha}}{t^\alpha}$ with $\alpha > 0$, and $a(t)$ accordingly, would give a space very similar to X_2, satisfying the same properties. Actually, the choice of a does not really matter as long as it ensures that γ keeps finite arclength.
- One could pick similar examples for other types of decreasing functions b. This range of examples shows that the Hardy property, as well as the relative and non relative layer decay properties, are very unstable, as it suffices to apply a slight perturbation to the initial space, where they were satisfied, to lose them.

References

[A] L. Ahlfors. Zur Theorie der Uberlagerungsflächen. *Acta Math.*, 65:157–194, 1935.

[AH] P. Auscher and T. Hytönen. Orthonormal bases of regular wavelets in spaces of homogeneous type. *Preprint, arXiv:1110.5766v2*, 2011.

[AR] P. Auscher and E. Routin. Local Tb theorems and Hardy inequalities. *To appear in J. Geom. Anal., Preprint: arXiv:1011.1747v2*, 2011.

[B1] S. M. Buckley. Inequalities of John-Nirenberg type in doubling spaces. *J. Anal. Math.*, 79:215–240, 1999.

[B2] S. M. Buckley. Is the maximal function of a Lipschitz function continuous? *Ann. Acad. Sci. Fenn. Math.*, 24:519–528, 1999.

[BBI] D. Burago, Y. Burago, and S. Ivanov. *A Course in Metric Geometry*, volume 33 of *Graduate studies in mathematics*. American Mathematical Society, 2001.

[C] M. Christ. A T(b) theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.*, 60/61:601–628, 1990.

[CM] T. H. Colding and W. P. Minicozzi II. Liouville theorems for harmonic sections and applications. *Comm. Pure Appl. Math.*, 51:113–138, 1998.

[CW] R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes*, volume 242 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1971.

[D1] G. David. Morceaux de graphes lipschitziens et intégrales singulières sur une surface. *Rev. Mat. Iberoamericana*, 4(1):73–114, 1988.

[D2] G. David. *Wavelets and singular integrals on curves and surfaces*, volume 1465 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1991.

[DJS] G. David, J. Journé, and S. Semmes. Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation. *Rev. Mat. Iberoamericana*, 1:1–56, 1985.

[H] S. Hofmann. A proof of the local Tb theorem for standard Calderón-Zygmund operators. Unpublished, arXiv:0705.0840v1, 2007.

[LNY] H. Lin, E. Nakai, and D. Yang. Boundedness of lusin-area and g_λ^* functions on localized bmo spaces over doubling metric measure spaces. *Bull. Sci. Math. (to appear), arXiv:0903.4587v2*, 2010.

[MS] R.A. Macías and C. Segovia. Lipschitz functions on spaces of homogeneous type. *Adv. in Math.*, 33(3):257–270, 1979.

[PS] M. Paluszyński and K. Stempak. On quasi-metric and metric spaces. *Proc. Amer. Math. Soc.*, 137(12):4307–4312, 2009.

[S] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Number 43 in Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993.

[T] R. Tessera. Volume of spheres in doubling metric measured spaces and in groups of polynomial growth. *Bull. Soc. Math. France*, 135:47–64, 2007.

Eddy Routin, Université Paris-Sud, laboratoire de Mathématiques, UMR 8628, Orsay F-91405; CNRS, Orsay, F-91405; France

E-mail address: eddy.routin@gmail.com