

# Global existence of strong solution for shallow water system with large initial data on the irrotational part

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## Abstract

We show existence of global strong solutions with large initial data on the irrotational part for the shallow-water system in dimension  $N \geq 2$ . We introduce a new notion of *quasi-solutions* when the initial velocity is assumed to be irrotational, these last one exhibit regularizing effects both on the velocity and in a very surprising way also on the density (indeed the density is a priori governed by an hyperbolic equation). We would like to point out that this smoothing effect is purely non linear and is absolutely crucial in order to deal with the pressure term as it provides new damping effects in high frequencies. In particular our result gives a first kind of answer to the problem of the existence of global weak solution for the shallow-water system. We conclude by giving new point wise decay estimates on the solution which improves the previous works [18, 19].

## 1 Introduction

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla P(\rho) = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \quad (1.1)$$

Here  $u = u(t, x) \in \mathbb{R}^N$  stands for the velocity field and  $\rho = \rho(t, x) \in \mathbb{R}^+$  is the density. The pressure  $P$  is a suitable smooth function of  $\rho$ . We denote by  $\lambda$  and  $\mu$  the two viscosity coefficients of the fluid, which are assumed to satisfy  $\mu > 0$  and  $\lambda + 2\mu > 0$ . Such a condition ensures ellipticity for the momentum equation and is satisfied in the physical cases where  $\lambda + \frac{2\mu}{N} > 0$ . In the sequel we shall only consider the shallow-water system which corresponds to:

$$\mu(\rho) = \mu\rho \quad \text{with } \mu > 0 \quad \text{and} \quad \lambda(\rho) = 0.$$

We supplement the problem with initial condition  $(\rho_0, u_0)$ . Throughout the paper, we assume that the space variable  $x \in \mathbb{R}^N$  or to the periodic box  $\mathcal{T}_a^N$  with period  $a_i$ , in the  $i$ -th direction. We restrict ourselves to the case  $N \geq 2$ .

In this paper we are interested in proving the announced result in [14]. Before giving

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our main result, let us recall some important results concerning the existence of strong solutions for the compressible Navier Stokes system (1.1). The existence and uniqueness of local classical solutions for (1.1) with smooth initial data such that the density  $\rho_0$  is bounded and bounded away from zero has been stated by Nash in [26]. Let us emphasize that no stability condition was required there. On the other hand, for small smooth perturbations of a stable equilibrium with constant positive density, global well-posedness has been proved in [23]. More precisely Matsumura and Nishida in [23] obtained the existence of global strong solutions for three-dimensional polytropic ideal fluids and no outer force with initial data chosen small in the following spaces  $(\rho_0 - 1, u_0) \in H^3 \times H^3$ . Refined functional analysis has been used during the last decades, ranging from Sobolev, Besov, Lorentz and Triebel spaces to describe the regularity and long time behavior of solutions to the compressible model [30], [20], [4], [9].

Guided in our approach by numerous works dedicated to the incompressible Navier-Stokes equation (see e.g [25]):

$$(NS) \quad \begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = 0, \\ \operatorname{div} v = 0, \end{cases}$$

we aim at solving (1.1) in the case where the data  $(\rho_0, u_0)$  have *critical* regularity. By critical, we mean that we want to solve the system (1.1) in functional spaces with norm invariant by the changes of scales which leave (1.1) invariant. In the case of barotropic fluids, it is easy to see that the transformations:

$$(\rho(t, x), u(t, x)) \longrightarrow (\rho(l^2 t, lx), lu(l^2 t, lx)), \quad l \in \mathbb{R}, \quad (1.2)$$

have that property, provided that the pressure term has been changed accordingly. One of the main difficulty of compressible fluid mechanics is to deal with the vacuum, indeed in this case the momentum equation loses its parabolicity. That is why in the sequel we will work around stable equilibrium in order to control the vacuum.

**Definition 1.1** *In the sequel we shall note:  $q = \rho - 1$ .*

The use of critical functional frameworks led to several new well-posedness results for compressible fluids (see [11, 9, 10, 6, 3]). In addition to have a norm invariant by (1.2), appropriate functional spaces for solving (1.1) must provide a control on the  $L^\infty$  norm of the density (in order to avoid vacuum and loss of parabolicity but also for dealing with the non linear term as the pressure). Additionally we also must have some Lipschitz control on the velocity in order to control the density via the transport equation on the density. For that reason, we restricted our study to the case where the initial data  $(\rho_0, u_0)$  are in homogeneous Besov spaces such that:

$$q_0 \in B_{p,1}^{\frac{N}{q}} \quad \text{and} \quad u_0 \in B_{p_1,1}^{\frac{N}{p_1}-1}$$

with  $(p, p_1) \in [1, +\infty[$  suitably chosen.

Recall that R. Danchin in [6] shows for the first time a result of existence of global strong solution close with **small** initial data in critical space for the scaling of the system. More precisely the initial data are chosen as follows  $(q_0, u_0) \in (B_{2,1}^{\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-1}) \times B_{2,1}^{\frac{N}{2}-1}$ . The

main difficulty consists in obtaining estimates on the linearized system given that the velocity and the density are coupled via the pressure. The crucial points is the obtention of damping effect on the density in order to control the pressure term. This last result has been generalized to the case of Besov space constructed on  $L^p$  space by F. Charve and R. Danchin in [3] and the author in [10] by using two different methods. In [3] the authors in order to obtain estimates on the linearized system associated to (1.1) including the convection terms combines a parilinearization method developed by T. Hmidi and an accurate study of the linear system without the convection term in order to get estimates in Besov space. In [10] we extend the results of Charve and Danchin [3] to the case where the Lebesgue index of Besov spaces are not the same for the density and the velocity and by reaching general index  $p$  for the density. Indeed in [3] the authors need the following restriction  $p < 2N$  where  $p$  is the Lebesgue index for the density. To do that, as in [9] we introduce a new notion of *effective velocity* in high frequencies which allows us to cancel out the coupling between the velocity and the pressure. This effective velocity enables us to get as in R. Danchin in [6] a  $L^1$  decay on  $q$  in the high frequency regime. In low frequencies, the first order terms predominate, so that (1.1) has to be treated by means of hyperbolic energy methods (roughly  $q$  and the potential part of the velocity verify a wave equation). This implies that we can treat the low regime only in spaces constructed on  $L^2$  as it is classical that hyperbolic systems are ill-posed in general  $L^p$  spaces. In particular the system has to be handled differently in low and high frequencies.

Recall that in [17], Hoff stated the existence of global weak solutions with small initial data including discontinuous initial data (namely  $q_0 \in L^2 \cap L^\infty$  and is small in  $L^2$  and  $u_0$  is small in  $L^4$  if  $N = 2$  and small in  $L^8$  if  $N = 3$ ). One of the major interest of the results of Hoff is to exhibit some smoothing effects on the incompressible part of the velocity  $u$  and on the so-called *effective viscous flux*  $F = (2\mu + \lambda)\operatorname{div}u - P(\rho) + P(\bar{\rho})$ . This also plays a crucial role in the proof of Lions for the existence of global weak solution (see [22]). However if the results of Hoff are critical in the sense of the scaling for the density, it is not the case for the initial velocity. We would like to mention that the results of [10] makes the link in term of scaling between the results obtained in [3] and [17].

However the existence of global strong solution with large initial data remains open even in dimension  $N = 2$  except for some very specific viscosity coefficients (see [31]). Indeed in a remarkable work Vaigant and Kazhikhov prove the existence of global strong solution when the viscosity coefficients are chosen such that  $\mu(\rho) = \mu$  and  $\lambda(\rho) = \rho^\beta$  with  $\beta > 3$ . To do this, they use very clever energy inequalities by taking profit of the structure of effective flux that introduce the choice of the viscosity coefficient. We would like to mention that the condition  $\beta > 3$  is crucial in order to get  $L^\infty$  estimates on the density. In [10, 3, 17] all these works need to assume a smallness condition on the initial data in order to obtain global solution.

In this paper we would give a first kind of answer to this problem for a large family of initial data, more precisely we are going to prove the existence of global strong solution for initial data with large irrotational part. To do this we shall work around a irrotational *quasi-solution* of the system (1.1) (we also refer to [13] for a such type of solution in the context of Korteweg system). It is not clear as for the Euler system how to write the Saint-Venant system under a purely irrotational form (it means with a solution such that  $u = \nabla\theta$ ) because the strong coupling between the velocity and the density, however we can check that  $(\rho^1, -\mu\nabla\ln\rho^1)$  is a particular irrotational solution of the system (1.1)



simplify the notation, and we will write  $\tilde{B}_{p_1, p_2, 1}^{s_1, s_2}$  for  $\tilde{B}_{(p_1, 1), (p_2, 1)}^{s_1, s_2}$ . For more details on the definition of these spaces we refer to the definition 2.6.

One can now state our main result.

**Theorem 1.1** *Let  $1 \leq p < \max(4, N)$  such that  $\sup(\frac{1}{q}, \frac{1}{p}) \leq 1 + \inf(\frac{1}{p}, \frac{1}{q})$ ,  $\sup(\frac{1}{q_1}, \frac{1}{2}) \leq 1 + \inf(\frac{1}{q_1}, \frac{1}{2})$  and  $\rho_0 = \rho_0^1 e^{h_0^2}$  and  $u_0 = -\mu \nabla \ln \rho_0^1 + u_0^2$ . Furthermore we assume that  $\rho_0^1 \geq c > 0$ ,  $q^1 \in B_{\infty, 1}^0 \cap \tilde{B}_{q^1, q, \infty}^{\frac{N}{2}-1, \frac{N}{q}} \cap \tilde{B}_{2, p, 1}^{\frac{N}{2}-2, \frac{N}{p}-2}$ ,  $h^2 \in \tilde{B}_{2, p, 1}^{\frac{N}{2}-1, \frac{N}{p}}$  and  $u_2 \in \tilde{B}_{2, p, 1}^{\frac{N}{2}-1, \frac{N}{p}-1}$ . Then it exists  $\epsilon$  such that if:*

$$\|q_0^1\|_{\tilde{B}_{2, p, 1}^{\frac{N}{2}-2, \frac{N}{p}-2}} + \|h_0^2\|_{\tilde{B}_{2, p, 1}^{\frac{N}{2}-1, \frac{N}{p}}} + \|u_0^2\|_{\tilde{B}_{2, p, 1}^{\frac{N}{2}-1, \frac{N}{p}-1}} \leq \epsilon,$$

then if  $\frac{1}{2} + \frac{1}{p} > \frac{1}{N}$ ,  $\frac{1}{2} + \frac{1}{q_1} > \frac{1}{N}$ ,  $\frac{1}{p} + \frac{1}{q} > \frac{1}{N}$  there exists a global solution  $(\rho, u)$  of the system (1.1) written under the following form:  $\rho = \rho^1 e^{h^2}$  and  $u = -\mu \nabla \ln \rho^1 + u^2$  with:

$$\begin{cases} \partial_t \rho^1 - \mu \Delta \rho^1 = 0, \\ \rho_{t=0}^1 = \rho_0^1. \end{cases} \quad (1.5)$$

and such that:

$$h^2 \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2, p, 1}^{\frac{N}{2}-1, \frac{N}{p}}) \cap L^1(\mathbb{R}^+, \tilde{B}_{2, p, 1}^{\frac{N}{2}+1, \frac{N}{p}})$$

$$\text{and } u^2 \in \tilde{C}(\mathbb{R}^+; \tilde{B}_{2, p, 1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap L^1(\mathbb{R}^+, \tilde{B}_{2, p, 1}^{\frac{N}{2}+1, \frac{N}{p}+1}).$$

We refer to the definition for the definition of the hybrid Besov spaces.

**Remark 1** *We would like to emphasize on the fact that the density consists in the product of a regular function  $\rho^1$  and of a small perturbation of the equilibrium  $e^{h^2}$ . This point is very surprising in the sense that the density is governed by a hyperbolic equation which means that a priori we do not wait for any regularizing effects on the density. It seems that there is a singular behavior around the quasi-solution  $(\rho^1, -\mu \nabla \ln \rho^1)$  and we note that this effect is strictly non-linear. Indeed it depends on the convection term  $u \cdot \nabla u$ . Furthermore this regularizing effect is crucial in order to deal with the term  $u^2 \cdot \nabla \ln \rho^1$ , indeed we lose one derivate on  $\nabla \ln \rho^1$ , it is then crucial to obtain regularizing effects in order to treat this term.*

**Remark 2** *Up our knowledge, it is the first result of global strong solution for compressible Navier-Stokes equations with large initial data in dimension  $N \geq 2$  in critical Besov spaces for the scaling of the equations. In particular it improves strongly the results of [3, 10]. Indeed the only condition of smallness on  $\rho$  is on the following Besov space  $\tilde{B}_{2, p, 1}^{\frac{N}{2}-2, \frac{N}{p}-2}$  which is largely subcritical, it is the same for the velocity for the Besov space  $\tilde{B}_{2, p, 1}^{\frac{N}{2}-3, \frac{N}{p}-3}$ . We observe that the density is not necessary small in norm  $L^\infty$  as it is the case in [10, 3].*

*In particular we emphasize on the fact that the we can choose initial data with large initial data in the energy space. Indeed we recall that the energy space is such that  $\rho_0 \in L_1^2(\mathbb{R}^N)$*

(we refer to [22] for the definition of the Orlicz space) and  $\sqrt{\rho_0}u_0 \in L^2(\mathbb{R}^N)$ . In our case in dimension  $N = 3$  if  $p = q = 2$  we need in high frequencies an assumption of smallness on  $q^1$  in  $B_{2,1}^{-\frac{1}{2}}$ . And in high frequencies  $L^2$  is embedded in  $B_{2,1}^{-\frac{1}{2}}$ . We can then choose initial density with large norm. It is exactly the same for the initial velocity as the smallness condition in high frequencies correspond to the norm  $B_{2,1}^{-\frac{3}{2}}$ .

In particular it gives a first kind of answer to the problem of the existence of global weak solution for the shallow-water system. Indeed in [24], Mellet and Vasseur prove the stability of the global weak solution for the shallow-water system. However it seems extremely difficult to construct a sequence  $(\rho_n, u_n)_{n \in \mathbb{N}}$  of regular approximate global solution of the system (1.1) which verifies uniformly in  $n$  the different entropies that use Mellet and Vasseur for proving their result. The theorem 1.1 allows us to obtain the existence of global weak solution for a family of large initial data in the energy space (it means some initial density such that  $\rho^0$  is small in high frequencies in  $B_{2,1}^{-\frac{1}{2}}$  but not necessary in  $L^2$ ). To do this it suffices to regularize the energy initial data to construct approximate global solution  $(\rho_n, u_n)_{n \in \mathbb{N}}$  of the system (1.1) and to pass to the limit by the compactness arguments developed in [24].

**Remark 3** By using the same idea than in [10], it would be possible to cancel out the coupling between  $h^2$  and  $u^2$ , and in particular obtaining the existence of global strong solution with  $h_0^2 \in B_{p,1}^{\frac{N}{p}}$  with  $p$  close from the infinity if we assume additionally that  $\rho^0$  is in  $H^1(\mathbb{R}^N)$  with is conform with the energy space used in [24].

Let also mention that it would be possible to improve the regularity hypothesis in low frequencies, indeed by studying only the variable  $u = u^1 + u^2$  in high frequencies and by working directly with  $u$  in low frequencies following the same idea than in [10].

**Remark 4** We would like to mention that this result is strongly related to the structure of the viscosity coefficient as we are able to construct quasi-solutions. Indeed with constant viscosity coefficient it seems not clear how to construct quasi-solution.

**Remark 5** We would like to point out that our initial density is continuous as we assume that  $q_0^1$  is in  $B_{\infty,1}^0$ . In [15], we are able to prove the existence of global strong solution with discontinuous initial density for the Korteweg system, this is due to the fact that for the Korteweg system we have also regularizing effects on  $h^2$ . Here the term  $\nabla h^2 \cdot Du^1$  imposed a control  $L^\infty$  on  $Du^1$  (it means a Lipschitz control). To have this, by using the proposition 2.5 we see that the condition  $q_0^1 \in B_{\infty,1}^0$  is optimal.

**Remark 6** We could weaken the condition on  $(h^2, u^2)$  by following also the idea of [4].

We are now going a result on the time asymptotic behavior of our solution  $(\rho, u)$ .

**Corollary 1** Let  $(\rho, u)$  the global strong solution constructed in the theorem 1.1 with the following additional condition on the initial data,  $q_0^1$  belongs to  $L^1$ . We set:

$$\epsilon = \|h^2\|_{\tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}})} + \|u^2\|_{\tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})}.$$

Then the global solution  $(\rho, u)$  satisfies the following decay estimates for:

$$\|(\rho - 1)(t, \cdot)\|_{L^\infty} \leq C \left( \frac{\|q_0^1\|_{L^1}}{(1+t)^{\frac{N}{2}}} (1 + \epsilon) + \epsilon \right), \quad (1.6)$$

and

$$\|u(t, \cdot)\|_{B_{\infty, \infty}^{-1}} \leq C \left( \frac{\|q_0^1\|_{L^1}}{(1+t)^{\frac{N}{2} + \frac{1}{2}}} + \epsilon \right), \quad (1.7)$$

where  $C$  depends on the viscosity  $\mu$  and the initial data  $h_0^0, u_0^2$ .

**Remark 7** This result improves the works of [18, 19] as we are working with critical initial data without assuming that the regularity of  $\rho_0$  is the same than  $u_0$ . Indeed in [18, 19] consider the compressible Navier-Stokes equations as a regularizing system of the compressible Euler system.

We refer also to the thesis of Rodrigues (see [28]) for more details on the time asymptotic decay of the global strong solution with small initial data.

We shall come back on this problem in a forthcoming paper.

We are now going to consider the viscous shallow water model with friction. This model is also called by the french community the Saint-Venant equations and is generally used in oceanography. Indeed it allows to model vertically averaged flows in terms of the horizontal mean velocity field  $u$  and the depth variation  $\rho$ . In the rotating framework, the model is described by the following system::

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu \rho D(u)) + \frac{\nabla \rho}{Fr^2} + r \rho u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \quad (1.8)$$

$Fr > 0$  denotes the Froude number. The turbulent regime ( $r \geq 0$ ) is obtained from the friction condition on the bottom, see [27]. We shall work in the sequel with  $r = \frac{1}{Fr^2}$ .

In this case we are able to exhibit explicit solution  $(\rho^1, -\mu \nabla \ln \rho^1)$  which verifies (1.5), by using similar idea than in theorem 1.1 we shall obtain the following result.

**Theorem 1.2** Let  $1 \leq p < \max(4, N)$  such that  $\sup(\frac{1}{q}, \frac{1}{p}) \leq 1 + \inf(\frac{1}{p}, \frac{1}{q})$ ,  $\sup(\frac{1}{q_1}, \frac{1}{2}) \leq 1 + \inf(\frac{1}{q_1}, \frac{1}{2})$  and  $\rho_0 = \rho_0^1 e^{h_0^2}$  and  $u_0 = -\mu \nabla \ln \rho_0^1 + u_0^2$ . Furthermore we assume that  $\rho_0^1 \geq c > 0$ ,  $q^1 \in B_{\infty, 1}^0 \cap \tilde{B}_{q_1, q, \infty}^{\frac{N}{q_1} - 1, \frac{N}{q}} \cap \tilde{B}_{2, p, 1}^{\frac{N}{2} - 2, \frac{N}{p} - 2}$ ,  $h^2 \in \tilde{B}_{2, p, 1}^{\frac{N}{2} - 1, \frac{N}{p}}$  and  $u_2 \in \tilde{B}_{2, p, 1}^{\frac{N}{2} - 1, \frac{N}{p} - 1}$ . Then it exists  $\epsilon$  such that if:

$$\|h_0^2\|_{\tilde{B}_{2, p, 1}^{\frac{N}{2} - 1, \frac{N}{p}}} + \|u_0^2\|_{\tilde{B}_{2, p, 1}^{\frac{N}{2} - 1, \frac{N}{p} - 1}} \leq \epsilon,$$

then if  $\frac{1}{2} + \frac{1}{p} > \frac{1}{N}$  there exists a global solution  $(\rho, u)$  of the system (1.1) written under the following form:  $\rho = \rho^1 e^{h^2}$  and  $u = -\mu \nabla \ln \rho^1 + u^2$  with:

$$\begin{cases} \partial_t \rho^1 - \mu \Delta \rho^1 = 0, \\ \rho_{t=0}^1 = \rho_0^1. \end{cases}$$

and such that:

$$h^2 \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \cap L^1(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}}) \quad \text{and} \quad u^2 \in \tilde{C}(\mathbb{R}^+; \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap L^1(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1}).$$

We refer to the definition for the definition of the hybrid Besov spaces.

**Remark 8** Compared with the theorem 1.1, we do not need any assumption of smallness on the density  $\rho_0^1$ , it is completely a result of global strong solution for large initial data when  $N \geq 2$ . It is the first result up our knowledge of global strong solution with large initial data for a compressible system.

Our paper is structured as follows. In section 2, we give a few notation and briefly introduce the basic Fourier analysis techniques needed to prove our result. In section ??, we prove estimates on a linear system with convection terms. In section 3 we prove the theorems 1.1, the corollary 1 and the theorem 1.2. Some technical continuity results for the paraproduct in hybrid Besov spaces have been postponed in appendix.

## 2 Littlewood-Paley theory and Besov spaces

Throughout the paper,  $C$  stands for a constant whose exact meaning depends on the context. The notation  $A \lesssim B$  means that  $A \leq CB$ . For all Banach space  $X$ , we denote by  $C([0, T], X)$  the set of continuous functions on  $[0, T]$  with values in  $X$ . For  $p \in [1, +\infty]$ , the notation  $L^p(0, T, X)$  or  $L_T^p(X)$  stands for the set of measurable functions on  $(0, T)$  with values in  $X$  such that  $t \rightarrow \|f(t)\|_X$  belongs to  $L^p(0, T)$ . Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We can use for instance any  $\varphi \in C^\infty(\mathbb{R}^N)$ , supported in  $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  such that:

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1 \quad \text{if} \quad \xi \neq 0.$$

Denoting  $h = \mathcal{F}^{-1}\varphi$ , we then define the dyadic blocks by:

$$\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x - y) dy \quad \text{and} \quad S_l u = \sum_{k \leq l-1} \Delta_k u.$$

Formally, one can write that:

$$u = \sum_{k \in \mathbb{Z}} \Delta_k u.$$

This decomposition is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in  $\mathcal{S}'(\mathbb{R}^N)$  for two reasons:

1. The right hand-side does not necessarily converge in  $\mathcal{S}'(\mathbb{R}^N)$ .
2. Even if it does, the equality is not always true in  $\mathcal{S}'(\mathbb{R}^N)$  (consider the case of the polynomials).

## 2.1 Homogeneous Besov spaces and first properties

**Definition 2.3** For  $s \in \mathbb{R}$ ,  $p \in [1, +\infty]$ ,  $q \in [1, +\infty]$ , and  $u \in \mathcal{S}'(\mathbb{R}^N)$  we set:

$$\|u\|_{B_{p,q}^s} = \left( \sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^p})^q \right)^{\frac{1}{q}}.$$

The Besov space  $B_{p,q}^s$  is the set of temperate distribution  $u$  such that  $\|u\|_{B_{p,q}^s} < +\infty$ .

**Remark 9** The above definition is a natural generalization of the nonhomogeneous Sobolev and Hölder spaces: one can show that  $B_{\infty,\infty}^s$  is the nonhomogeneous Hölder space  $C^s$  and that  $B_{2,2}^s$  is the nonhomogeneous space  $H^s$ .

**Proposition 2.1** The following properties holds:

1. there exists a constant universal  $C$  such that:  

$$C^{-1} \|u\|_{B_{p,r}^s} \leq \|\nabla u\|_{B_{p,r}^{s-1}} \leq C \|u\|_{B_{p,r}^s}.$$
2. If  $p_1 < p_2$  and  $r_1 \leq r_2$  then  $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-N(1/p_1-1/p_2)}$ .
3.  $B_{p,r_1}^{s'} \hookrightarrow B_{p,r}^s$  if  $s' > s$  or if  $s = s'$  and  $r_1 \leq r$ .

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [2, 1]).

**Proposition 2.2** We have the following laws of product:

- For all  $s \in \mathbb{R}$ ,  $(p, r) \in [1, +\infty]^2$  we have:

$$\|uv\|_{B_{p,r}^s} \leq C(\|u\|_{L^\infty} \|v\|_{B_{p,r}^s} + \|v\|_{L^\infty} \|u\|_{B_{p,r}^s}). \quad (2.9)$$

- Let  $(p, p_1, p_2, r, \lambda_1, \lambda_2) \in [1, +\infty]^2$  such that:  $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ ,  $p_1 \leq \lambda_2$ ,  $p_2 \leq \lambda_1$ ,  $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$  and  $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$ . We have then the following inequalities:  
 if  $s_1 + s_2 + N \inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$ ,  $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$  and  $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$  then:

$$\|uv\|_{B_{p,r}^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}} \lesssim \|u\|_{B_{p_1,r}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}, \quad (2.10)$$

when  $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$  (resp  $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ ) we replace  $\|u\|_{B_{p_1,r}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}$  (resp  $\|v\|_{B_{p_2,\infty}^{s_2}}$ ) by  $\|u\|_{B_{p_1,1}^{s_1}} \|v\|_{B_{p_2,r}^{s_2}}$  (resp  $\|v\|_{B_{p_2,\infty}^{s_2} \cap L^\infty}$ ), if  $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$  and  $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$  we take  $r = 1$ .

If  $s_1 + s_2 = 0$ ,  $s_1 \in (\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}]$  and  $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$  then:

$$\|uv\|_{B_{p,\infty}^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}} \lesssim \|u\|_{B_{p_1,1}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}. \quad (2.11)$$

If  $|s| < \frac{N}{q}$  for  $p \geq 2$  and  $-\frac{N}{p'} < s < \frac{N}{q}$  else, we have:

$$\|uv\|_{B_{p,r}^s} \leq C \|u\|_{B_{p,r}^s} \|v\|_{B_{p,\infty}^{\frac{N}{q}} \cap L^\infty}. \quad (2.12)$$

**Remark 10** In the sequel  $p$  will be either  $p_1$  or  $p_2$  and in this case  $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$  if  $p_1 \leq p_2$ , resp  $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$  if  $p_2 \leq p_1$ .

**Corollary 1** Let  $r \in [1, +\infty]$ ,  $1 \leq p \leq p_1 \leq +\infty$  and  $s$  such that:

- $s \in (-\frac{N}{p_1}, \frac{N}{p_1})$  if  $\frac{1}{p} + \frac{1}{p_1} \leq 1$ ,
- $s \in (-\frac{N}{p_1} + N(\frac{1}{p} + \frac{1}{p_1} - 1), \frac{N}{p_1})$  if  $\frac{1}{p} + \frac{1}{p_1} > 1$ ,

then we have if  $u \in B_{p,r}^s$  and  $v \in B_{p_1,\infty}^{\frac{N}{p_1}} \cap L^\infty$ :

$$\|uv\|_{B_{p,r}^s} \leq C \|u\|_{B_{p,r}^s} \|v\|_{B_{p_1,\infty}^{\frac{N}{p_1}} \cap L^\infty}.$$

The study of non stationary PDE's requires space of type  $L^\rho(0, T, X)$  for appropriate Banach spaces  $X$ . In our case, we expect  $X$  to be a Besov space, so that it is natural to localize the equation through Littlewood-Paley decomposition. But, in doing so, we obtain bounds in spaces which are not type  $L^\rho(0, T, X)$  (except if  $r = p$ ). We are now going to define some useful spaces in which we will work, which are a refinement of the spaces  $L_T^\rho(B_{p,r}^s)$ .

**Definition 2.4** Let  $\rho \in [1, +\infty]$ ,  $T \in [1, +\infty]$  and  $s_1 \in \mathbb{R}$ . We set:

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} = \left( \sum_{l \in \mathbb{Z}} 2^{lr s_1} \|\Delta_l u(t)\|_{L^\rho(L^p)}^r \right)^{\frac{1}{r}}.$$

We then define the space  $\tilde{L}_T^\rho(B_{p,r}^{s_1})$  as the set of temperate distribution  $u$  over  $(0, T) \times \mathbb{R}^N$  such that  $\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} < +\infty$ .

We set  $\tilde{C}_T(\tilde{B}_{p,r}^{s_1}) = \tilde{L}_T^\infty(\tilde{B}_{p,r}^{s_1}) \cap \mathcal{C}([0, T], B_{p,r}^{s_1})$ . Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} \leq \|u\|_{L_T^\rho(B_{p,r}^{s_1})} \text{ if } r \geq \rho, \quad \|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} \geq \|u\|_{L_T^\rho(B_{p,r}^{s_1})} \text{ if } r \leq \rho.$$

**Remark 11** It is easy to generalize proposition 2.2, to  $\tilde{L}_T^\rho(B_{p,r}^{s_1})$  spaces. The indices  $s_1$ ,  $p$ ,  $r$  behave just as in the stationary case whereas the time exponent  $\rho$  behaves according to Hölder inequality.

In the sequel we will need of composition lemma in  $\tilde{L}_T^\rho(B_{p,r}^s)$  spaces.

**Lemma 1** Let  $s > 0$ ,  $(p, r) \in [1, +\infty]$  and  $u \in \tilde{L}_T^\rho(B_{p,r}^s) \cap L_T^\infty(L^\infty)$ .

1. Let  $F \in W_{loc}^{[s]+2, \infty}(\mathbb{R}^N)$  such that  $F(0) = 0$ . Then  $F(u) \in \tilde{L}_T^\rho(B_{p,r}^s)$ . More precisely there exists a function  $C$  depending only on  $s$ ,  $p$ ,  $r$ ,  $N$  and  $F$  such that:

$$\|F(u)\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq C(\|u\|_{L_T^\infty(L^\infty)}) \|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)}.$$

2. Let  $F \in W_{loc}^{[s]+3,\infty}(\mathbb{R}^N)$  such that  $F(0) = 0$ . Then  $F(u) - F'(0)u \in \tilde{L}_T^\rho(B_{p,r}^s)$ . More precisely there exists a function  $C$  depending only on  $s, p, r, N$  and  $F$  such that:

$$\|F(u) - F'(0)u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq C(\|u\|_{L_T^\infty(L^\infty)})\|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)}^2.$$

Now we give some result on the behavior of the Besov spaces via some pseudodifferential operator (see [1]).

**Definition 2.5** Let  $m \in \mathbb{R}$ . A smooth function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is said to be a  $\mathcal{S}^m$  multiplier if for all multi-index  $\alpha$ , there exists a constant  $C_\alpha$  such that:

$$\forall \xi \in \mathbb{R}^N, \quad |\partial^\alpha f(\xi)| \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}.$$

**Proposition 2.3** Let  $m \in \mathbb{R}$  and  $f$  be a  $\mathcal{S}^m$  multiplier. Then for all  $s \in \mathbb{R}$  and  $1 \leq p, r \leq +\infty$  the operator  $f(D)$  is continuous from  $B_{p,r}^s$  to  $B_{p,r}^{s-m}$ .

Let us now give some estimates for the heat equation:

**Proposition 2.4** Let  $s \in \mathbb{R}$ ,  $(p, r) \in [1, +\infty]^2$  and  $1 \leq \rho_2 \leq \rho_1 \leq +\infty$ . Assume that  $u_0 \in B_{p,r}^s$  and  $f \in \tilde{L}_T^{\rho_2}(B_{p,r}^{s-2+\rho_2})$ . Let  $u$  be a solution of:

$$\begin{cases} \partial_t u - \mu \Delta u = f \\ u_{t=0} = u_0. \end{cases}$$

Then there exists  $C > 0$  depending only on  $N, \mu, \rho_1$  and  $\rho_2$  such that:

$$\|u\|_{\tilde{L}_T^{\rho_1}(\tilde{B}_{p,r}^{s+2/\rho_1})} \leq C(\|u_0\|_{B_{p,r}^s} + \mu^{\frac{1}{\rho_2}-1} \|f\|_{\tilde{L}_T^{\rho_2}(B_{p,r}^{s-2+\rho_2})}).$$

If in addition  $r$  is finite then  $u$  belongs to  $C([0, T], B_{p,r}^s)$ .

**Proposition 2.5** Let  $s$  be a positive real number and  $(p, r) \in [1, +\infty]^2$ . A constant  $C$  exists which satisfies:

$$C^{-1} \|u\|_{B_{p,r}^{-2s}} \leq \| \|t^s e^{t\Delta} u\|_{L^p} \|_{L^r(\mathbb{R}^+, \frac{dt}{t})} \leq C \|u\|_{B_{p,r}^{-2s}}.$$

## 2.2 Hybrid Besov spaces

The homogeneous Besov spaces fail to have nice inclusion properties: owing to the low frequencies, the embedding  $B_{p,1}^s \hookrightarrow B_{p,1}^t$  does not hold for  $s > t$ . Still, the functions of  $B_{p,1}^s$  are locally more regular than those of  $B_{p,1}^t$ : for any  $\phi \in C_0^\infty$  and  $u \in B_{p,1}^s$ , the function  $\phi u \in B_{p,1}^t$ . This motivates the definition of Hybrid Besov spaces introduced by R. Danchin (see [1, 10]) where the growth conditions satisfied by the dyadic blocks and the coefficient of integrability are not the same for low and high frequencies. Hybrid Besov spaces have been used in [10] to prove global well-posedness for compressible Navier-Stokes equation. We generalize here a little bit the definition by allowing for different Lebesgue norms in low and high frequencies.

**Definition 2.6** Let  $l_0 \in \mathbb{N}$ ,  $s, t, \in \mathbb{R}$ ,  $(r, r_1) \in [1, +\infty]^2$  and  $(p, q) \in [1, +\infty]$ . We set:

$$\|u\|_{\tilde{B}_{p,q,1}^{s,t}} = \sum_{l \leq l_0} 2^{ls} \|\Delta_l u\|_{L^p} + \sum_{l > l_0} 2^{lt} \|\Delta_l u\|_{L^q},$$

and:

$$\|u\|_{\tilde{B}_{(p,r),(q,r_1)}^{s,t}} = \left( \sum_{l \leq l_0} (2^{ls} \|\Delta_l u\|_{L^p})^r \right)^{\frac{1}{r}} + \left( \sum_{l > l_0} (2^{lt} \|\Delta_l u\|_{L^q})^{r_1} \right)^{\frac{1}{r_1}}.$$

**Remark 12** It will be important in the sequel to chose  $l_0$  big enough.

**Notation 1** We will often use the following notation:

$$u_{BF} = \sum_{l \leq l_0} \Delta_l u \quad \text{and} \quad u_{HF} = \sum_{l > l_0} \Delta_l u.$$

**Remark 13** We have the following properties:

- We have  $\tilde{B}_{p,p,1}^{s,s} = B_{p,1}^s$ .
- If  $s_1 \geq s_3$  and  $s_2 \geq s_4$  then  $\tilde{B}_{p,q,1}^{s_3,s_2} \hookrightarrow \tilde{B}_{p,q,1}^{s_1,s_4}$ .

We shall also make use of hybrid Besov-spaces. The basic idea of paradifferential calculus is that any product of two distributions  $u$  and  $v$  can be formally decomposed into:

$$uv = T_u v + T_v u + R(u, v) = T_u v + T'_v u$$

where the paraproduct operator is defined by  $T_u v = \sum_q S_{q-1} u \Delta_q v$ , the remainder operator  $R$ , by  $R(u, v) = \sum_q \Delta_q u (\Delta_{q-1} v + \Delta_q v + \Delta_{q+1} v)$  and  $T'_v u = T_v u + R(u, v)$ .

We recall here an important proposition on the paraproduct for hybrid Besov spaces (see [10]).

**Proposition 2.6** Let  $p_1, p_2, p_3, p_4 \in [1, +\infty]$ ,  $(s_1, s_2, s_3, s_4) \in \mathbb{R}^4$  and  $(p, q) \in [1, +\infty]^2$ , we have then the following inequalities:

- If  $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda} \leq 1$ ,  $\frac{1}{q} \leq \frac{1}{p_4} + \frac{1}{\lambda'} \leq 1$  with  $(\lambda, \lambda') \in [1, +\infty]^2$  and  $p_1 \leq \lambda'$ ,  $p_1 \leq \lambda$ ,  $p_3 \leq \lambda'$  then:

$$\|T_u v\|_{\tilde{B}_{p,q,r}^{s_1+s_2+\frac{N}{q}-\frac{N}{p_1}-\frac{N}{p_2}, s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4}}} \lesssim \|u\|_{\tilde{B}_{p_1,p_3,1}^{s_1,s_3}} \|v\|_{\tilde{B}_{p_2,p_4,r}^{s_2,s_4}}, \quad (2.13)$$

if  $s_1 + \frac{N}{\lambda} \leq \frac{N}{p_1}$ ,  $s_1 + \frac{N}{\lambda} \leq \frac{N}{p_1}$  and  $s_3 + \frac{N}{\lambda} \leq \frac{N}{p_3}$ .

- If  $\frac{1}{q} \leq \frac{1}{p_3} + \frac{1}{p_4}$  and  $s_3 + s_4 + N \inf(0, 1 - \frac{1}{p_3} - \frac{1}{p_4}) > 0$  then

$$\sum_{l \geq 4} 2^{l(s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4})} \|\Delta_l R(u, v)\|_{L^q} \lesssim \|u\|_{\tilde{B}_{p_1,p_3,1}^{s_1,s_3}} \|v\|_{\tilde{B}_{p_2,p_4,r}^{s_2,s_4}}. \quad (2.14)$$



We are going to generalize this proposition to the case when  $(q^1, u^1)$  is different from  $(0, 0)$ . More precisely we have the following proposition:

**Proposition 3.8** *Let  $p \leq \max(4, N)$ ,  $q$  such that  $\frac{N}{p} - 1 \leq \frac{N}{q}$  and  $(F, G) \in \tilde{L}^1(B_{2,p,1}^{s-1,s}) \times \tilde{L}^1(B_{2,p,1}^{s-1})$  and  $(h^2, u^2)$  a solution of (3.18), let  $s = \frac{N}{p}$  and  $s' = \frac{N}{2}$ . The following estimate holds:*

$$\begin{aligned} & \| (h^2, u^2) \|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1})} + \| (h^2, u^2) \|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{s'+1,s} \times \tilde{B}_{2,p,1}^{s'+1,s+1})} \\ & \leq C e^{V(T)} \left( \| (h_0^2, u_0^2) \|_{\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1}} + \| (F, G) \|_{\tilde{L}^1(\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1})} \right). \end{aligned}$$

with  $V(T) = \int_0^T (\|q^1(s)\|_{B_{q_1,q,\infty}^{\frac{N}{q_1}-\frac{1}{2},\frac{N}{q}+\frac{1}{2}}}^4 + \|q^1(s)\|_{B_{q_1,q,\infty}^{\frac{N}{q_1}+1,\frac{N}{q}+2}} + \|\nabla u^1(s)\|_{L^\infty \cap B_{q_1,q,\infty}^{\frac{N}{q_1}-1,\frac{N}{q}}} + \|\nabla v(s)\|_{L^\infty}) ds$ .

**Proof:** By using the proposition 3.7 we obtain that:

$$\begin{aligned} & \| (h^2, u^2) \|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1})} + \| (h^2, u^2) \|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{s'+1,s} \times \tilde{B}_{2,p,1}^{s'+1,s+1})} \\ & \leq C e^{V(T)} \left( \| (h_0^2, u_0^2) \|_{\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1}} + \| (F_1, G_1) \|_{\tilde{L}^1(\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1})} \right). \end{aligned} \quad (3.19)$$

with:

$$\begin{aligned} F_1 &= F - u^2 \cdot \nabla \ln \rho^1, \\ G_1 &= G - u_2 \cdot \nabla u^1 + \mu \nabla \ln \rho^1 \cdot Du^2 + \mu \nabla h^2 \cdot Du^1. \end{aligned}$$

Therefore, it is only a matter of proving appropriate estimates for  $F_1, G_1$  by using properties of continuity on the paraproduct (we refer in particular to the appendix of [10] when the Besov spaces are hybrid).

We start with the first term  $u^2 \cdot \nabla \ln \rho^1$ , we have then by proposition 5.9, interpolation (with  $s + \frac{1}{2} = \frac{1}{4}(s-1) + \frac{3}{4}(s+1)$ ), composition estimates and Hölder inequality:

$$\begin{aligned} & \int_0^T \|u^2 \cdot \nabla \ln \rho^1(s)\|_{\tilde{B}_{2,p,1}^{s'-1,s}} ds \leq C \int_0^T (\|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'-1,s-1}} \|\nabla \ln \rho^1(s)\|_{\tilde{B}_{q_1,q,\infty}^{\frac{N}{q_1},\frac{N}{q}+1}} \\ & \quad + \|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'+\frac{1}{2},s+\frac{1}{2}}} \|\nabla \ln \rho^1(s)\|_{\tilde{B}_{q_1,q,\infty}^{\frac{N}{q_1}-\frac{3}{2},\frac{N}{q}-\frac{1}{2}}}) ds, \\ & \leq \int_0^T (\|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'-1,s-1}} \|q^1(s)\|_{B_{q_1,q,\infty}^{\frac{N}{q_1}+1,\frac{N}{q}+2}} + \|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'+\frac{1}{2},s+\frac{1}{2}}} \|q^1(s)\|_{B_{q_1,q,\infty}^{\frac{N}{q_1}-\frac{1}{2},\frac{N}{q}+\frac{1}{2}}}) ds, \\ & \leq \int_0^T (\|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'-1,s-1}} \|q^1(s)\|_{B_{q_1,q,\infty}^{\frac{N}{q_1}+1,\frac{N}{q}+2}} \\ & \quad + \|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'-1,s-1}}^{\frac{1}{4}} \|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'+1,s+1}}^{\frac{3}{4}} \|q^1(s)\|_{B_{q_1,q,\infty}^{\frac{N}{q_1}-\frac{1}{2},\frac{N}{q}+\frac{1}{2}}}) ds, \\ & \leq \int_0^T (\|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'-1,s-1}} \|q^1(s)\|_{B_{q_1,q,\infty}^{\frac{N}{q_1}+1,\frac{N}{q}+2}} \\ & \quad + \frac{1}{4\epsilon} \|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'-1,s-1}} \|q^1(s)\|_{B_{q_1,q,\infty}^{\frac{N}{q_1}-\frac{1}{2},\frac{N}{q}+\frac{1}{2}}}^4 + \frac{3\epsilon^{\frac{1}{3}}}{4} \|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'+1,s+1}}) ds, \end{aligned} \quad (3.20)$$

We are going to deal with the terms of  $G_1$ , let us start with the term  $u^2 \cdot \nabla u^1$ , we have as  $-\frac{N}{q} < s-1 < \frac{N}{q}$  and as  $\nabla u^1 \in L_T^1(L^\infty)$ :

$$\int_0^T \|u^2 \cdot \nabla u^1(s)\|_{\tilde{B}_{2,p,\infty}^{s'-1, s-1}} ds \leq \int_0^T \|u_2\|_{\tilde{B}_{2,p,1}^{s'-1, s-1}} \|\nabla u^1(s)\|_{\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}-1, \frac{N}{q}} \cap L^\infty} ds. \quad (3.21)$$

We now have by Hölder inequality:

$$\begin{aligned} \int_0^T \|\nabla \ln \rho^1 \cdot Du^2(s)\|_{\tilde{B}_{2,p,1}^{s'-1, s-1}} ds &\leq \int_0^T \|\nabla \ln q^1\|_{\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}-\frac{3}{2}, \frac{N}{q}-\frac{1}{2}}} \|Du^2(s)\|_{\tilde{B}_{2,p,1}^{s'-\frac{1}{2}, s-\frac{1}{2}}} \\ &\quad + \|\nabla \ln q^1\|_{\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}-\frac{1}{2}, \frac{N}{q}+\frac{1}{2}}} \|Du^2(s)\|_{\tilde{B}_{2,p,1}^{s'-\frac{3}{2}, s-\frac{3}{2}}} ds \\ &\leq \int_0^T \left( \frac{1}{\epsilon} \|\nabla q^1\|_{\tilde{B}_{q_1, q, 1}^{\frac{N}{q}-1, \frac{N}{q}}}^2 \|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'-1, s-1}} + \epsilon \|u^2(s)\|_{\tilde{B}_{2,p,1}^{s'+1, s+1}} \right) ds \end{aligned} \quad (3.22)$$

It remains to deal with  $\nabla h^2 \cdot Du^1$  and by using the fact that  $\frac{N}{q}$ :

$$\int_0^T \|\nabla \ln h^2 \cdot Du^1\|_{\tilde{B}_{2,p,1}^{s'-1, s-1}} \leq \int_0^T \|h^2\|_{\tilde{B}_{2,p,1}^{s'-1, s}} \|Du^1(s)\|_{\tilde{B}_{q_1, q, 1}^{\frac{N}{q}-1, \frac{N}{q}}} ds, \quad (3.23)$$

By combining the estimates (3.19), (3.20), (3.21), (3.22), (3.23) and Gronwall lemma, we achieve the proposition.

## 4 The proof of theorem 1.1

### 4.1 Proof of the existence

We recall here that  $(\rho^1, u^1) = (\rho^1, -\mu \nabla \ln \rho^1)$  is a quasi-solution of the system (1.1) with:

$$\begin{aligned} \partial_t \rho^1 - \mu \Delta \rho^1 &= 0, \\ \rho_{t=0}^1 &= \rho_0^1. \end{aligned} \quad (4.24)$$

It means that  $(\rho^1, u^1)$  is an irrotational solution of the approximate system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu \rho D(u)) = 0, \\ (\rho, u)|_{t=0} = (\rho_0^1, u_0^1). \end{cases} \quad (4.25)$$

As we assume that  $q_0^1$  is in  $\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}-1, \frac{N}{q}} \cap B_{\infty, 1}^0 \cap \tilde{B}_{2,p,1}^{\frac{N}{2}-2, \frac{N}{p}-2}$ , by using proposition ?? and the fact that  $q^1$  verifies an heat equation, we show that for any  $T > 0$ :

$$\begin{aligned} \|q^1\|_{\tilde{L}^\infty(\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}-1, \frac{N}{q}} \cap B_{\infty, 1}^0)} + \|q^1\|_{\tilde{L}^1(\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}+1, \frac{N}{q}+2} \cap B_{\infty, 1}^2)} &\leq C \|q_0^1\|_{\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}-1, \frac{N}{q}} \cap B_{\infty, 1}^0}, \\ \|u^1\|_{\tilde{L}^\infty(\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}-2, \frac{N}{q}-1} \cap B_{\infty, 1}^{-1})} + \|u^1\|_{\tilde{L}^\infty(\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}, \frac{N}{q}+1} \cap B_{\infty, 1}^1)} &\leq C \|u_0^1\|_{\tilde{B}_{q_1, q, \infty}^{\frac{N}{q}-1, \frac{N}{q}} \cap B_{\infty, 1}^0}, \end{aligned} \quad (4.26)$$

and:

$$\|q^1\|_{\tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-2, \frac{N}{p}-2})} + \|q^1\|_{\tilde{L}^1(\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}})} \leq C \|q_0^1\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-2, \frac{N}{p}-2}}, \quad (4.27)$$

By the maximum principle, we also obtain that for all  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$ :

$$0 \leq c \leq \rho^1(t, x) \leq \|\rho_0^1\|_{L^\infty} \quad (4.28)$$

Our goal now consists in solving the system (1.4) in order to obtain solution of the system (1.1) under the form  $\rho = \rho^1 \epsilon^{h^2}$ ,  $u = u^1 + u^2$  with  $\rho \geq c' > 0$ . To do this, we use a standard scheme:

1. We smooth out the data and get a sequence of local solutions  $(h_n^2, u_n^2)_{n \in \mathbb{N}}$  on  $[0, T_n]$  to (1.4) by using the result of [9].
2. We prove uniform estimates on  $(h_n^2, u_n^2)$  on  $[0, T_n]$  by using the proposition 3.8 and we deduce that  $T_n = +\infty$ .
3. We use compactness to prove that the sequence  $(h_n^2, u_n^2)$  converges, up to extraction, to a solution of (1.4).

### Construction of approximate solutions

We smooth out the data as follows:

$$(h_0^2)_n = S_n h_0^2 \quad \text{and} \quad (u_0^2)_n = S_n u_0^2 \quad .$$

Note that we have:

$$\forall l \in \mathbb{Z}, \quad \|\Delta_l (h_0^2)_n\|_{L^p} \leq \|\Delta_l h_0^2\|_{L^p} \quad \text{and} \quad \|(h_0^2)_n\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}} \leq \|h_0^2\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}},$$

and similar properties for  $(u_0^2)_n$ , a fact which will be used repeatedly during the next steps. Now, according [9], one can solve (1.4) with the smooth data  $((q_0^2)_n, (u_0^2)_n)$ . We get a solution  $(h_n^2, u_n^2)$  on a non trivial time interval  $[0, T_n]$  such that:

$$h_n^2 \in \tilde{C}([0, T_n], B_{2,1}^N \cap B_{2,1}^{\frac{N}{2}-1}) \quad u_n^2 \in \tilde{C}([0, T_n], B_{2,1}^{\frac{N}{2}-1}) \cap \tilde{L}^1([0, T_n], B_{2,1}^{\frac{N}{2}+1}). \quad (4.29)$$

### Uniform bounds

In the sequel we set:

$$\ln \rho_n = \ln \rho^1 + h_n^2 \quad \text{and} \quad u_n = u^1 + u_n^2.$$

We recall that  $(h_n^2, u_n^2)$  satisfies the following system:

$$\begin{cases} \partial_t h_n^2 + u_n \cdot \nabla h_n^2 + \operatorname{div} u_n^2 = u_n^2 \cdot \nabla \ln \rho^1, \\ \partial_t u_n^2 + u_n \cdot \nabla u_n^2 - \mu \Delta u_n^2 + a \nabla h_n^2 = -u_n^2 \cdot \nabla u^1 + \mu \nabla \ln \rho^1 \cdot D u_n^2 \\ \quad - a \nabla \ln \rho^1 + \mu \nabla h_n^2 \cdot D u^1 + \mu \nabla h_n^2 \cdot D u_n^2, \\ (h_n^2, u_n^2)_{/t=0} = ((h_0^2)_n, (u_0^2)_n). \end{cases} \quad (4.30)$$

In this part, we aim at getting uniform estimates on  $(h_n^2, u_n^2)$  in the following space  $F_T$  with the norm  $\|\cdot\|_{F_T}$ :

$$F_T = (\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}})) \times (\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1})).$$

$$\|(h_n^2, u_n^2)\|_{F_T} = \|h_n^2\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}})} + \|u_n^2\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap \tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1})}.$$

We can observe that  $(h_n^2, u_n^2)$  verifies exactly the system (3.18) with  $v = u^n$  and:

$$\begin{aligned} F^n &= 0, \\ G^n &= -a \nabla \ln \rho^1 + \mu \nabla h_n^2 \cdot Du^1 + \mu \nabla h_n^2 \cdot Du_n^2. \end{aligned}$$

By using proposition 3.8, we have the following estimate on  $(h_n^2, u_n^2)$ :

$$\|(h_n^2, u_n^2)\|_{F_T} \leq C e^{V_1^n(T)} (\|(h_0^2)_n\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}} + \|(u_0^2)_n\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}} + \|G^n\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})}).$$

Therefore, it is only a matter of proving appropriate estimates for  $G_1^n$  by using proposition 5.9. We begin by estimating  $h_n^2 \cdot Du_n^2$  and we obtain:

$$\|\nabla h_n^2 \cdot Du_n^2\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \leq \|\nabla h_n^2\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \|Du_n^2\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}})}. \quad (4.31)$$

Similarly, we obtain as  $\frac{N}{2} - 1 \leq \frac{N}{q_1}$  and  $\frac{N}{p} - 1 \leq \frac{N}{q}$ :

$$\|\nabla h_n^2 \cdot Du^1\|_{\tilde{L}_T^1(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \leq \|\nabla h_n^2\|_{\tilde{L}_T^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \|Du^1\|_{\tilde{L}_T^1(\tilde{B}_{2,q,1}^{\frac{N}{q_1}, \frac{N}{q}})}. \quad (4.32)$$

And finally by using (4.27), we have:

$$\|q^1\|_{\tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-2, \frac{N}{p}-2})} + \|q^1\|_{\tilde{L}^1(\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}})} \leq C \|q_0^1\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-2, \frac{N}{p}-2}}, \quad (4.33)$$

We would like to point out the fact that  $\|q_0^1\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-2, \frac{N}{p}-2}}$  small is mandatory in order to deal with the bootstrap argument. Here we use strongly the regularizing effect on  $q^1$ . From a standard bootstrap argument, it is now easy to conclude that there exists a positive constant  $c$  such that if the data has been chosen so small as to satisfy:

$$\|q_0^1\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-2, \frac{N}{p}-2}} + \|h_0^2\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}} + \|u_0^2\|_{\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}} \leq \epsilon,$$

then  $T_n = +\infty$ . Furthermore it exists  $C > 0$  such that for all  $t \in \mathbb{R}$ :

$$\|(q^n, u^n)\|_{F_t} \leq C, \quad \forall t \in \mathbb{R}.$$

## Compactness arguments

Let us first focus on the convergence of  $(h_n^2)_{n \in \mathbb{N}}$ . We claim that, up to extraction,  $(h_n^2)_{n \in \mathbb{N}}$  converges in the distributional sense to some function  $h$  such that:

$$h \in \tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \cap \tilde{L}^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}}). \quad (4.34)$$

The proof is based on Ascoli's theorem and compact embedding for Besov spaces. As similar arguments have been employed in [1], we only give the outlines of the proof. We may write that:

$$\partial_t h_n^2 + u_n \cdot \nabla h_n^2 + \operatorname{div} u_n^2 = u_n^2 \cdot \nabla \ln \rho^1, .$$

Since  $(u_n^2)_{n \in \mathbb{N}}$  is uniformly bounded in  $\tilde{L}^2(\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}})$  and  $\nabla \ln \rho^1 \in \tilde{L}^\infty(\tilde{B}_{2,q,1}^{\frac{N}{q_1}-2, \frac{N}{q}-1})$ , we have  $u_n^2 \cdot \nabla \ln \rho^1$  which is bounded in  $\tilde{L}^2(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})$ . Similarly  $u^n \cdot \nabla h_n^2$  is bounded in  $\tilde{L}^2(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})$ . Finally we have proved that  $\partial_t h_n^2$  is bounded in  $\tilde{L}^2(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})$ , it means that  $(h_n^2)_{n \in \mathbb{N}}$  seen as a sequence of  $\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}$  valued functions is locally equicontinuous in  $\mathbb{R}^+$ . In addition  $(h_n^2)_{n \in \mathbb{N}}$  is bounded in  $C(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})$ . As the embedding  $\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1} \cap \tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}}$  is locally compact (see [1], Chap2), one can thus conclude by means of Ascoli's theorem and Cantor diagonal extraction process that there exists some distribution  $h$  such that up to an omitted extraction  $(\psi h_n^2)_{n \in \mathbb{N}}$  converges to  $\psi h$  in  $C(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})$  for all smooth  $\psi$  with compact support in  $\mathbb{R}^+ \times \mathbb{R}^N$ . Then by using the so-called Fatou property for the Besov spaces, one can conclude that (4.34) is satisfied. (the reader may consult [1], Chap 10 too). By proceeding similarly, we can prove that up to extraction,  $(u_n^2)_{n \in \mathbb{N}}$  converges in the distributional sense to some function  $u^2$  such that:

$$u^2 \in \tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1}) \cap \tilde{L}^1(\tilde{B}_{2,p,1}^{\frac{N}{2}+1, \frac{N}{p}+1}). \quad (4.35)$$

In order to complete the proof of the existence part of theorem 1.1, it is only a matter of checking the continuity properties with respect to time, namely that:

$$h^2 \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}) \quad \text{and} \quad u^2 \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}}).$$

As regards  $h^2$ , it suffices to notice that, according to (4.34), (4.35) and to the product laws in the Besov spaces, we have:

$$\partial_t h^2 + u \cdot \nabla h^2 + \operatorname{div} u^2 = u^2 \cdot \nabla \ln \rho^1 \in \tilde{L}^1(\tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}}).$$

As  $h_0^2 \in \tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}}$ , classical results for the transport equation (see [1], Chap 3) ensure that  $h^2 \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}, \frac{N}{p}})$ . And as previously, we have shown that  $h^2 \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})$ , it means clearly that  $q \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}})$ .

For getting the continuity result for  $u^2$ , one follows the same ideas.

## The proof of the uniqueness

In the case  $\frac{2}{N} \leq \frac{1}{p} + \frac{1}{p_1}$ , the uniqueness has been established in [1].

### 4.2 Proof of the theorems 1.2

We follow here exactly the lines of the proof of the theorem 1.1 except that we take profit that  $(\rho^1, -\mu \nabla \ln \rho^1)$  is an exact solution of our system (1.8). Indeed we have no coupling by the pressure term in the momentum equation on  $u^2$ , it explains why we do not assume any smallness condition on  $q_0^1$ .

### 4.3 Proof of the corollary 1

We remind the reader that if  $K_\mu$  is the standard heat kernel, that is the solution of the following system:

$$\begin{cases} \partial_t K_\mu - \mu \Delta K_\mu = 0, \\ K_\mu(0, \cdot) = \delta_0 \end{cases}$$

then:

$$\|D^\alpha K_\mu(t, \cdot)\|_{L^p(\mathbb{R}^N)} \leq C(\alpha, \mu) t^{-r_{\alpha,p}} \quad (4.36)$$

with  $r_{\alpha,p} = \frac{N}{2}(1 - \frac{1}{p}) + \frac{\alpha}{2}$ . In the sequel we shall note:

$$M_1 = \|h^2\|_{\tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}})} \quad \text{and} \quad M^2 = \|u^2\|_{\tilde{L}^\infty(\tilde{B}_{2,p,1}^{\frac{N}{2}-1, \frac{N}{p}-1})} \quad \text{such that} \quad \max(M_1, M_2) \leq \epsilon,$$

with  $\epsilon$  small enough if we choose the initial data on  $(h_0^2, u_0^2)$  small enough. As  $q^1$  verifies an heat equation, we have that  $q^1 = K_\mu * q_0^1$ . We obtain then by using (4.36) that:

$$\begin{aligned} \|(\rho - 1)(t, \cdot)\|_{L^\infty} &\leq \|q^1(e^{h^2} - 1)(t, \cdot)\|_{L^\infty} + \|q^1(t, \cdot)\|_{L^\infty} + \|(e^{h^2} - 1)(t, \cdot)\|_{L^\infty}, \\ &\leq C \left( \frac{\|q_0^1\|_{L^1}}{(1+t)^{\frac{N}{2}}} (1 + \epsilon) + \epsilon \right). \end{aligned} \quad (4.37)$$

Similarly we have:

$$\begin{aligned} \|u(t, \cdot)\|_{B_{\infty,\infty}^{-1}} &\leq \|u^1(t, \cdot)\|_{B_{\infty,\infty}^{-1}} + \|u^2(t, \cdot)\|_{B_{\infty,\infty}^{-1}}, \\ &\leq C \left( \frac{\|q_0^1\|_{L^1}}{(1+t)^{\frac{N}{2} + \frac{1}{2}}} + \epsilon \right) \end{aligned} \quad (4.38)$$

It achieves the proof of the corollary 1.

## 5 Appendix

This section is devoted to the proof of paraproduct estimates which have been used in section 2 and 3. They are based on paradifferential calculus, a tool introduced by J.-M. Bony in [2]. The basic idea of paradifferential calculus is that any product of two distributions  $u$  and  $v$  can be formally decomposed into:

$$uv = T_u v + T_v u + R(u, v) = T_u v + T'_v u$$

where the paraproduct operator is defined by  $T_u v = \sum_q S_{q-1} u \Delta_q v$ , the remainder operator  $R$ , by  $R(u, v) = \sum_q \Delta_q u (\Delta_{q-1} v + \Delta_q v + \Delta_{q+1} v)$  and  $T'_v u = T_v u + R(u, v)$ .

**Proposition 5.9** *Let  $p_1, p_2, p_3, p_4 \in [1, +\infty]$ ,  $(s_1, s_2, s_3, s_4) \in \mathbb{R}^4$  and  $(p, q) \in [1, +\infty]^2$ , we have then the following inequalities:*

- If  $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda} \leq 1$ ,  $\frac{1}{q} \leq \frac{1}{p_4} + \frac{1}{\lambda'} \leq 1$  with  $(\lambda, \lambda') \in [1, +\infty]^2$  and  $p_1 \leq \lambda'$ ,  $p_1 \leq \lambda$ ,  $p_3 \leq \lambda'$  then:

$$\|T_u v\|_{\tilde{B}_{p,q,r}^{s_1+s_2+\frac{N}{q}-\frac{N}{p_1}-\frac{N}{p_2}, s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4}}} \lesssim \|u\|_{\tilde{B}_{p_1,p_3,1}^{s_1,s_3}} \|v\|_{\tilde{B}_{p_2,p_4,r}^{s_2,s_4}}, \quad (5.39)$$

if  $s_1 + \frac{N}{\lambda} \leq \frac{N}{p_1}$ ,  $s_1 + \frac{N}{\lambda} \leq \frac{N}{p_1}$  and  $s_3 + \frac{N}{\lambda} \leq \frac{N}{p_3}$ .

- If  $\frac{1}{q} \leq \frac{1}{p_3} + \frac{1}{p_4}$  and  $s_3 + s_4 + N \inf(0, 1 - \frac{1}{p_3} - \frac{1}{p_4}) > 0$  then

$$\sum_{l \geq 4} 2^{l(s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4})} \|\Delta_l R(u, v)\|_{L^q} \lesssim \|u\|_{\tilde{B}_{p_1,p_3,1}^{s_1,s_3}} \|v\|_{\tilde{B}_{p_2,p_4,r}^{s_2,s_4}}. \quad (5.40)$$

- If  $\frac{1}{p} \leq \frac{1}{p_3} + \frac{1}{p_4} \leq 1$ ,  $\frac{1}{p} \leq \frac{1}{p_3} + \frac{1}{p_2} \leq 1$ ,  $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_4} \leq 1$ ,  $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1$  and  $s_3 + s_4 > 0$ ,  $s_3 + s_2 > 0$ ,  $s_4 + s_1 > 0$ ,  $s_1 + s_2 > 0$  then

$$\sum_{l \leq 4} 2^{l(s_1+s_2+\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2})} \|\Delta_l R(u, v)\|_{L^p} \lesssim \|u\|_{\tilde{B}_{p_1,p_3,1}^{s_1,\frac{N}{p_3}-\frac{N}{p_1}+s_1}} \|v\|_{\tilde{B}_{p_2,p_4,r}^{s_2,\frac{N}{p_4}-\frac{N}{p_2}+s_2}}. \quad (5.41)$$

with  $s_3 = \frac{N}{p_3} - \frac{N}{p_1} + s_1$  and  $s_4 = \frac{N}{p_4} - \frac{N}{p_2} + s_2$ .

- If  $u \in L^\infty$ , we also have:

$$\|T_u v\|_{\tilde{B}_{p,q,r}^{s_1,s_2}} \lesssim \|u\|_{L^\infty} \|v\|_{\tilde{B}_{p,q,r}^{s_1,s_2}}, \quad (5.42)$$

and if  $\min(s_1, s_2) > 0$  then:

$$\|R(u, v)\|_{\tilde{B}_{p,q,r}^{s_1,s_2}} \lesssim \|u\|_{L^\infty} \|v\|_{\tilde{B}_{p,q,r}^{s_1,s_2}}. \quad (5.43)$$

**Proof:** Let us prove (5.39). According to the decomposition of J.-M. Bony [2], we have:

$$wv = T_u v + T_v u + R(u, v),$$

so for all  $l > 0$ :

$$\Delta_l T_u v = \sum_{|l-l'|\leq 3} \Delta_l (S_{l'-1} u \Delta_{l'} v),$$

For  $\alpha, \beta \in \mathbb{R}$ , let us define the following characteristic function on  $\mathbb{Z}$

$$\begin{aligned} \varphi^{\alpha,\beta} &= \alpha \quad \text{if } r \leq 0, \\ \varphi^{\alpha,\beta} &= \beta \quad \text{if } r \geq 1. \end{aligned}$$

if  $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda} \leq 1$  and  $\frac{1}{q} \leq \frac{1}{p_4} + \frac{1}{\lambda'} \leq 1$  then

$$\|\Delta_l T_u v\|_{L^{\varphi^{p,q}}(l)} \lesssim 2^{lN} \varphi^{\frac{1}{p_2} + \frac{1}{\lambda} - \frac{1}{p} - \frac{1}{p_4} + \frac{1}{\lambda'} - \frac{1}{q}}(l) \sum_{|l-l'|\leq 3} \|S_{l'-1} u\|_{L^{\varphi^{\lambda,\lambda'}}(l')} \|\Delta_{l'} v\|_{L^{\varphi^{p_2,p_4}}(l')}.$$

We have by Bernstein inequalities and as  $p_1 \leq \lambda'$ ,  $p_3 \leq \lambda'$ ,  $p_1 \leq \lambda$  and  $s_1 + \frac{N}{\lambda} \leq \frac{N}{p_1}$ ,  $s_1 + \frac{N}{\lambda'} \leq \frac{N}{p_1}$ ,  $s_3 + \frac{N}{\lambda'} \leq \frac{N}{p_3}$ :

$$\begin{aligned} \|S_{l'-1} u\|_{L^{\varphi^{\lambda,\lambda'}}(l')} &\lesssim \sum_{k \leq l'-2} 2^{k(\frac{N}{p_1}, \frac{N}{p_3}(k) - \varphi^{\frac{N}{\lambda}, \frac{N}{\lambda'}}(l'))} \|\Delta_k u\|_{L^{\varphi^{p_1,p_3}}(k)}, \\ &\lesssim \sum_{k \leq l'-2} 2^{k(\frac{N}{p_1} - s_1, \frac{N}{p_3} - s_3)(k) - \varphi^{\frac{N}{\lambda}, \frac{N}{\lambda'}}(l')} 2^{k\varphi^{s_1,s_3}}(k) \|\Delta_k u\|_{L^{\varphi^{p_1,p_3}}(k)}, \\ &\lesssim 2^{l'(\frac{N}{p_1} - s_1, \frac{N}{p_3} - s_3)(l') - \varphi^{\frac{N}{\lambda}, \frac{N}{\lambda'}}(l')} \|u\|_{\tilde{B}_{p_1,p_3,1}^{s_1,s_3}}. \end{aligned}$$

Since  $\|\Delta_{l'} v\|_{L^{\varphi^{p_2,p_4}}(l')} = c_{l'} 2^{-l'(\varphi^{s_2,s_4}(l'))} \|v\|_{\tilde{B}_{p_2,p_4,1}^{s_2,s_4}}$  with  $\sum_{l' \in \mathbb{Z}} c_{l'} \leq 1$  we finally gather as  $l > 0$ :

$$\|\Delta_l T_u v\|_{L^q} \lesssim c_l 2^{l\varphi^{\frac{N}{p_1} + \frac{N}{p_2} - \frac{N}{p} - s_1 - s_2, \frac{N}{p_2} + \frac{N}{p_4} - \frac{N}{q} - s_3 - s_4}}(l) \|u\|_{\tilde{B}_{p_1,p_3,1}^{s_1,s_3}} \|v\|_{\tilde{B}_{p_2,p_4,1}^{s_2,s_4}}.$$

And we obtain (5.39).

Straightforward modification give (5.42). In this case as  $\|S_{k-1} u\|_{L^\infty} \leq \|u\|_{L^\infty}$  we have:

$$\|\Delta_l T_u v\|_{L^{\varphi^{p,q}}(l)} \lesssim \sum_{|l-l'|\leq 3} \|u\|_{L^\infty} \|\Delta_{l'} v\|_{L^{\varphi^{p_2,p_4}}(l')}.$$

Next we have:

$$2^{l\varphi^{p_2,p_4}}(l) \|\Delta_l T_u v\|_{L^{\varphi^{p,q}}(l)} \lesssim \|u\|_{L^\infty} \sum_{|l-l'|\leq 3} 2^{l\varphi^{p_2,p_4}}(l) - l' \varphi^{p_2,p_4}(l') 2^{\varphi^{p_2,p_4}}(l') \|\Delta_{l'} v\|_{L^{\varphi^{p_2,p_4}}(l')}.$$

We conclude by convolution.

To prove (5.41), we write:

$$\Delta_l R(u, v) = \sum_{k \geq l-2} \Delta_l (\Delta_k u \tilde{\Delta}_k v).$$

We consider now the case  $l > 3$ . By Bernstein and Hölder inequalities we obtain when  $\frac{1}{q} \leq \frac{1}{p_3} + \frac{1}{p_4} \leq 1$ :

$$\|\Delta_l R(u, v)\|_{L^q} \lesssim 2^{Nl(\frac{1}{p_3} + \frac{1}{p_4} - \frac{1}{q})} \sum_{k \geq l-2} \|\Delta_k u\|_{L^{p_3}} \|\tilde{\Delta}_k v\|_{L^{p_4}}.$$

Next we have:

$$\begin{aligned} 2^{l(s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4})} \|\Delta_l R(u, v)\|_{L^q} &\lesssim \sum_{k \geq l-2} 2^{(l-k)(s_3+s_4)} 2^{ks_3} \|\Delta_k u\|_{L^{p_3}} 2^{ks_4} \|\tilde{\Delta}_k v\|_{L^{p_4}}, \\ &\lesssim (c_k) * (d_{k'}), \end{aligned}$$

with  $c_k = 1_{[-\infty, 2]}(k) 2^{k(s_3+s_4)}$  and  $d_{k'} = 2^{k's_3} \|\Delta_{k'} u\|_{L^{p_3}} 2^{k's_4} \|\tilde{\Delta}_{k'} v\|_{L^{p_4}}$ . We conclude by Young inequality as  $s_3 + s_4 > 0$ .

We have to treat now the case when  $l < 0$ . We have then as  $\frac{1}{p} \leq \frac{1}{p_3} + \frac{1}{p_4} \leq 1$  and  $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ :

$$\begin{aligned} \|\Delta_l R(u, v)\|_{L^p} &\lesssim 2^{Nl(\frac{1}{p_3}+\frac{1}{p_4}-\frac{1}{p})} \sum_{k \geq 2} \|\Delta_k u\|_{L^{p_3}} \|\tilde{\Delta}_k v\|_{L^{p_4}} \\ &+ \sum_{0 \leq k \leq 1, |k-k'| \leq 1} \|\Delta_k u \Delta_{k'} v\|_{L^p} + 2^{Nl(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})} \sum_{l-2 \leq k \leq -1} \|\Delta_k u\|_{L^{p_1}} \|\tilde{\Delta}_k v\|_{L^{p_2}}. \end{aligned}$$

And by convolution on the middle frequencies:

$$\begin{aligned} \|\Delta_l R(u, v)\|_{L^p} &\lesssim 2^{Nl(\frac{1}{p_3}+\frac{1}{p_4}-\frac{1}{p})} \sum_{k \geq 2} \|\Delta_k u\|_{L^{p_3}} \|\tilde{\Delta}_k v\|_{L^{p_4}} + (2^{l(\frac{N}{p_3}+\frac{N}{p_2}-\frac{N}{p}-s_3-s_2)})_{c_l} \\ &+ 2^{l(\frac{N}{p_1}+\frac{N}{p_4}-\frac{N}{p}-s_1-s_4)} c_l + 2^{Nl(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})} \sum_{l-2 \leq k \leq -1} \|\Delta_k u\|_{L^{p_1}} \|\tilde{\Delta}_k v\|_{L^{p_2}}, \end{aligned}$$

with  $c_l \in l^1(\mathbb{Z})$ . Next by convolution we obtain:

$$\begin{aligned} \|\Delta_l R(u, v)\|_{L^p} &\lesssim c_l (2^{l(\frac{N}{p_3}+\frac{N}{p_4}-\frac{N}{p}-s_3-s_4)} + 2^{l(\frac{N}{p_3}+\frac{N}{p_2}-\frac{N}{p}-s_3-s_2)} + 2^{l(\frac{N}{p_1}+\frac{N}{p_4}-\frac{N}{p}-s_1-s_4)} \\ &+ 2^{l(\frac{N}{p_1}+\frac{N}{p_2}-\frac{N}{p}-s_1-s_2)}) \|u\|_{\tilde{B}_{p_1, p_3, 1}^{s_1, s_3}} \|v\|_{\tilde{B}_{p_2, p_4, r}^{s_2, s_4}}. \end{aligned}$$

And we can conclude.

We want prove now the inequality (5.42). We have then:

$$2^{l\varphi^{s_1, s_2}(l)} \|\Delta_l R(u, v)\|_{L^p} \lesssim \sum_{k \geq l-2} 2^{(l-k)\varphi^{s_1, s_2}(l)} 2^{k\varphi^{s_1, s_2}(l)} \|\Delta_k u\|_{L^\infty} \|\tilde{\Delta}_k v\|_{L^{\varphi^{p, q}(k)}},$$

And we conclude by Young inequality.  $\square$

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