

# ON THE HYPERBOLICITY OF SURFACES OF GENERAL TYPE WITH SMALL $c_1^2$

XAVIER ROULLEAU, ERWAN ROUSSEAU

ABSTRACT. Surfaces of general type with positive second Segre number  $s_2 := c_1^2 - c_2 > 0$  are known by results of Bogomolov to be quasi-hyperbolic i.e. with finitely many rational and elliptic curves. These results were extended by McQuillan in his proof of the Green-Griffiths conjecture for entire curves on such surfaces. In this work, we study hyperbolic properties of minimal surfaces of general type with minimal  $c_1^2$ , known as Horikawa surfaces. In principle these surfaces should be the most difficult case for the above conjecture as illustrate the quintic surfaces in  $\mathbb{P}^3$ . Using orbifold techniques, we exhibit infinitely many irreducible components of the moduli of Horikawa surfaces whose very generic member has no rational curves or even is algebraically hyperbolic. Moreover, we construct explicit examples of algebraically hyperbolic and (quasi-)hyperbolic orbifold Horikawa surfaces.

## 1. INTRODUCTION

Our motivations are the following conjectures of Green-Griffiths and Lang:

**Conjecture 1** (Lang). *Let  $X$  be a variety of general type. Then a proper Zariski closed subset  $Z$  of  $X$  contains all its subvarieties not of general type. In particular,  $X$  has only a finite number of codimension-one subvarieties not of general type.*

Even in the case of surfaces, this conjecture is still open. It has attracted a lot of attention because of its important conjectural links with arithmetic: according to the Bombieri-Lang conjecture, the rational points on a variety of general type defined over a number field should not be Zariski dense.

A surface satisfying Conjecture 1, i.e. with finitely many rational and elliptic curves, is said *algebraically quasi-hyperbolic*. A projective variety  $X \subset \mathbb{P}^n$  is called *algebraically hyperbolic* if there exists a positive real number  $\varepsilon$  such that

$$2g(C) - 2 \geq \varepsilon \deg C$$

for each reduced irreducible curve  $C \subset X$ , where  $g(C)$  and  $\deg C$  are the geometric genus and the degree of the curve  $C \hookrightarrow X$  respectively.

The analytic version of Conjecture 1 is:

**Conjecture 2** (Green-Griffiths,-Lang). *Let  $X$  be a variety of general type. Then a proper Zariski closed subset of  $X$  contains all images of entire curves  $f : \mathbb{C} \rightarrow X$ .*

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A projective variety satisfying Conjecture 2 is said *quasi-hyperbolic* and a variety that contains no non-constant entire curve is said *hyperbolic*.

Let us recall that minimal surfaces of general type  $X$  are classified according to their Chern numbers. These Chern numbers  $c_1^2, c_2$  satisfy:

$$\frac{1}{5}(c_2 - 36) \leq c_1^2 \leq 3c_2.$$

If  $c_1^2 = 3c_2$  then, by Yau [34],  $X$  is a quotient of the unit ball. Therefore  $X$  is hyperbolic.

A striking overhang toward Conjecture 1 is its proof by Bogomolov for surfaces whose Segre number  $s_2 = c_1^2 - c_2$  is positive. The key point in Bogomolov's proof is that for surfaces with positive Segre number, the sufficiently high symmetric differentials have global sections.

An extension of Bogomolov's result to its analytic version was obtained by Lu and Yau [20] proving conjecture 2 for surfaces with  $c_1^2 > 2c_2$ .

Then McQuillan [21] proved Conjecture 2 for surfaces with positive second Segre number.

For surfaces with  $s_2 \leq 0$  there is no such a good result. Demailly and El Goul [11] proved Conjecture 2 for some surfaces with  $13c_1^2 > 9c_2$ .

In the extreme case, a surface that reaches the equality  $c_2 = 5c_1^2 + 36$  if  $c_1^2$  is even and  $c_2 = 5c_1^2 + 30$  otherwise is called a Horikawa surface. Thus the Horikawa surfaces are the one for which the Segre number  $s_2 = c_1^2 - c_2$  is the most negative as possible in the geography of (minimal) surfaces of general type.

As the previous list of results suggests, one is naturally lead to believe that it is for these surfaces that the above conjectures will be the most difficult to establish.

Even for hypersurfaces in  $\mathbb{P}^3$ , (which have  $s_2 < 0$ ), we are far from complete results. The case of quintics ( $c_1^2 = 5, c_2 = 55$ ) is particularly difficult to treat and this may be explained by the fact that they are Horikawa surfaces, thus with minimal second Segre number. Indeed, a conjecture by Kobayashi predicts that generic quintics should be hyperbolic. We know by Geng Xu [33] that a very generic surface of degree  $d \geq 5$  in  $\mathbb{P}^3$  contains no curve of geometric genus 0 or 1. But we still do not know that they are hyperbolic, even algebraically hyperbolic.

Moreover, we have very few examples of hyperbolic surfaces of low degree. A striking fact is that no example of hyperbolic surface is known for  $d = 5$ .

Here we investigate the hyperbolic properties of Horikawa surfaces using their very interesting geometric properties. Indeed, we know that most Horikawa surfaces realizing the equality  $c_2 = 5c_1^2 + 36$  appear as ramified coverings. Therefore it is very natural to associate to such surface an orbifold. Our philosophy is to use this orbifold to study the geometry of curves in Horikawa surfaces.

We will see that "orbifold" techniques as systematically introduced by Campana ([5], [6], see also [26]) are useful in this context.

Let  $N \geq 0$  be an integer and let  $\mathbb{F}_N$  be the  $N^{\text{th}}$  Hirzebruch surface. The surface  $\mathbb{F}_N$  has a natural fibration  $\mathbb{F}_N \rightarrow \mathbb{P}^1$ . We denote by  $F$  a fiber, and by  $T$  the section of this fibration such that  $T^2 = N$ . Any divisor  $D$  on  $\mathbb{F}_N$  is numerically equivalent to  $aT + bF$  for  $a, b$  integers. We denote by  $(a, b)$  the equivalence class of  $D$ . The holomorphic Euler characteristic of a surface is  $\chi := \frac{c_1^2 + c_2}{12}$ . Horikawa obtained the following classification: an Horikawa surface with even  $c_1^2$  is either

- (1) a double covering of  $\mathbb{P}^2$  branched along an octic ( $\chi = 4$ ),
- (2) a double covering of  $\mathbb{P}^2$  branched along a curve of degree 10 ( $\chi = 7$ ),
- (3) a double covering of  $\mathbb{F}_N$  branched along a curve of type  $(6, 2a)$ , ( $2a \geq -N$ ) ( $\chi = 3N + 2a - 1$ ).

Here the branch curve has at most *ADE* singularities. We show:

- Theorem 3.** (1) *Let  $X$  be a very generic Horikawa surface of type (1). Then  $X$  has no rational curves.*
- (2) *Let  $X$  be a very generic Horikawa surface of type (2). Then  $X$  is algebraically hyperbolic, in particular  $X$  has no rational or elliptic curves.*
- (3) *Let  $X$  be a very generic Horikawa surface of type (3) with  $a = 3$  and  $\chi = 3N + 5$ . Then  $X$  has no rational curves.*

This result is obtained as a consequence of more general results on the algebraic hyperbolicity of branched covers which may be of independent interest.

One of the most natural examples of surfaces with  $s_2 < 0$  are hypersurfaces in  $\mathbb{P}^3$ . In [4], Bogomolov and De Oliveira studied the hyperbolicity of hypersurfaces of degree  $d > 5$  with sufficiently many nodes. Translated into the language of orbifold, their key observation is that for such a surface  $X$ , the natural structure of orbifold  $\mathcal{X}$  has positive orbifold Segre number  $s_2(\mathcal{X}) > 0$ . Here we investigate hyperbolic properties of Hirzebruch quintic surfaces, degree 5 covers of  $\mathbb{P}^2$  branched over 5 lines in general position. In this study, we develop the theory of jet differentials in the orbifold setting and, as an application, prove

**Theorem 4.** *Let  $X$  be a Hirzebruch quintic surface. Then any entire orbifold curve  $f : \mathbb{C} \rightarrow X$  satisfies a differential equation of order 2.*

Using Nevanlinna theory and constructions of Persson [25], we exhibit some explicit examples of (quasi) hyperbolic Horikawa orbifold surfaces

**Theorem 5.** *For  $\chi$  equal to 4 and  $2k - 1$  (for any integer  $k > 2$ ), there exists (quasi) hyperbolic orbifold Horikawa surfaces whose minimal resolutions have Euler characteristic  $\chi$ .*

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## 2. ORBIFOLD SET-UP

**2.1. Orbifolds.** As in [13], we define orbifolds as a particular type of log pairs. The data  $(X, \Delta)$  is a log pair if  $X$  is a normal algebraic variety (or a normal complex space) and  $\Delta = \sum_i d_i D_i$  is an effective  $\mathbb{Q}$ -divisor where the  $D_i$  are distinct, irreducible divisors and  $d_i \in \mathbb{Q}$ .

For orbifolds, we need to consider only pairs  $(X, \Delta)$  such that  $\Delta$  has the form

$$\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i,$$

where the  $D_i$  are prime divisors and  $m_i \in \mathbb{N}$ .

**Definition 6.** An orbifold chart on  $X$  compatible with  $\Delta$  is a Galois covering  $\varphi : U \rightarrow \phi(U) \subset X$  such that

- (1)  $U$  is a domain in  $\mathbb{C}^n$  and  $\varphi(U)$  is open in  $X$ ,
- (2) the branch locus of  $\varphi$  is  $[\Delta] \cap \varphi(U)$ ,
- (3) for any  $x \in U'' := U \setminus \varphi^{-1}(X_{\text{sing}} \cup \Delta_{\text{sing}})$  such that  $\varphi(x) \in D_i$ , the ramification order of  $\varphi$  at  $x$  verifies  $\text{ord}_\varphi(x) = m_i$ .

**Definition 7.** An orbifold  $\mathcal{X}$  is a log pair  $(X, \Delta)$  such that  $X$  is covered by orbifold charts compatible with  $\Delta$ .

*Remark 8.* (1) In the language of stacks, we have a smooth Deligne-Mumford stack  $\pi : \mathcal{X} \rightarrow X$ , with coarse moduli space  $X$ .

- (2) More generally, Campana introduces *geometric orbifolds* in [5] as pairs of this type but which are not necessarily locally uniformizable.

**Example 9.** Let  $X$  be a complex manifold and  $\Delta = \sum_i (1 - \frac{1}{m_i}) D_i$  with a support  $[\Delta]$  which is a normal crossing divisor, i.e. for any point  $x \in X$  there is a holomorphic coordinate system  $(V, z_1, \dots, z_n)$  such that  $\Delta$  has equation

$$z_1^{(1-\frac{1}{m_1})} \dots z_n^{(1-\frac{1}{m_n})} = 0.$$

Then  $(X, \Delta)$  is an orbifold. Indeed, fix a coordinate system as above. Set

$$\varphi : U \rightarrow V, \phi(x_1, \dots, x_n) = (x_1^{m_1}, \dots, x_n^{m_n}).$$

Then  $(U, \phi)$  is an orbifold chart on  $X$  compatible with  $\Delta$ .

We have more examples of orbifolds looking, in the case of surfaces, at singularities that naturally appear in the logarithmic Mori program.

**Definition 10.** Let  $(X, \Delta)$ ,  $\Delta = \sum_i (1 - \frac{1}{m_i}) C_i$ , be a pair where  $X$  is a normal surface and  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. Let  $\pi : \tilde{X} \rightarrow X$  be a resolution of the singularities of  $(X, \Delta)$ , so that the exceptional divisors,  $E_i$  and the components of  $\tilde{\Delta}$ , the strict transform of  $\Delta$ , have normal crossings and

$$K_{\tilde{X}} + \tilde{\Delta} + \sum_i E_i = \pi^*(K_X + \Delta) + \sum_i a_i E_i.$$

We say that  $(X, \Delta)$  is *klt* (Kawamata log terminal) if  $m_i < \infty$  and  $a_i > 0$  for every exceptional curve  $E_i$ .

Thanks to a result of [23] (see also the appendix of [7]), we have

**Example 11.** Let  $(X, \Delta)$ ,  $\Delta = \sum_i (1 - \frac{1}{m_i}) C_i$ , be a klt pair with  $X$  a surface, then  $(X, \Delta)$  is an orbifold.

**2.2. Chern classes.** Let  $\pi : \mathcal{X} \rightarrow (X, \Delta)$  be a two dimensional orbifold. Let  $S$  be the singular points of  $X$  and of the divisor  $[\Delta]$ ,  $\Delta = \sum (1 - \frac{1}{m_i}) D_i$ . The orbifold canonical line bundle is:  $K_{\mathcal{X}} = \pi^*(K_X + \Delta)$ , therefore:

$$c_1(\mathcal{X}) = -\pi^*(K_X + \sum (1 - \frac{1}{m_i}) D_i).$$

To each point  $p$  of  $S$ , there is a well defined integer  $\beta(p)$ , the order of the isotropy group, such that by the orbifold Gauss-Bonnet formula [30]:

$$c_2(\mathcal{X}) = e(X) - \sum (1 - \frac{1}{m_i})e(D_i \setminus S) - \sum_{p \in S} (1 - \frac{1}{\beta(p)}).$$

Let  $p \in S$  be a smooth point of  $X$ .

**Lemma 12.** *Suppose that  $p$  is an ADE singularity of  $[\Delta]$ . We have :*

Type	$\beta(p)$
$A_1$	$m_i m_j$
$A_{2n}$	$\frac{2}{2n+1} (\frac{1}{m_i} + \frac{1}{2n+1} - \frac{1}{2})^{-2}$
$A_{2n-1}$	$\frac{4}{n} (\frac{1}{m_i} + \frac{1}{m_j} + \frac{1}{n} - 1)^{-2}, n \geq 2$
$D_{2n+2}$	$\frac{4}{n} (\frac{1}{m_i} + \frac{1}{m_j} + \frac{1}{nm_k} - 1)^{-2}$
$D_{2n+3}$	$2(2n+1)m_i^2, n \geq 1$
$E_7$	96

where the branches are as follows:

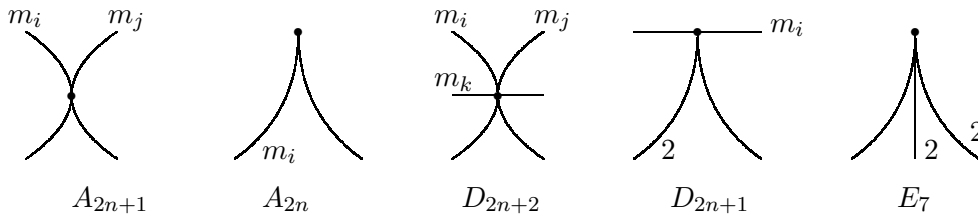


FIGURE 2.1. ADE singularities and conditions on the multiplicities

*Proof.* See [32] Table 2.3, [31], [7] and [18]. □

Let  $C_1, \dots$  be disjoint reduced divisors on  $X$  whose irreducible components are  $(-2)$ -curves. There exist a surface  $X'$  and map  $X \rightarrow X'$  such that the  $C_i$ 's are contracted onto ADE-singularities and that is an isomorphism outside. Let  $a_n$  (resp  $d_n, e_n$ ) be the number of  $A_n$  (resp.  $D_n, E_n$ ) singularities on  $X'$ .

**Proposition 13.** *The surface  $X'$  has a natural structure of orbifold  $\mathcal{X}$  and its Chern numbers are  $c_1^2(\mathcal{X}) = c_1^2(X)$  and:*

$$c_2(\mathcal{X}) = c_2(X) - \sum (n+1)(a_n + d_n + e_n) + \sum \frac{a_n}{n+1} + \frac{d_n}{4(n-2)} + \frac{e_6}{24} + \frac{e_7}{48} + \frac{e_8}{120}.$$

The denominators  $4(n-2), 24, 48, 120$  are the order of the binary dihedral  $BD_{4(n-2)}$ , the binary tetrahedral, the binary octahedral, and the binary icosahedral group respectively.

**2.3. Orbifold Riemann-Roch.** Let  $L$  be an orbifold line bundle on the orbifold  $\mathcal{X}$  of dimension  $n$ . From Kawazaki's orbifold Riemann-Roch theorem [17] or Toën's for Deligne-Mumford stacks [30], we obtain the asymptotic:

$$\chi(\mathcal{X}, L^k) = \frac{c_1(L)^n}{n!} k^n + O(k^{n-1}),$$

using orbifold Chern classes.

It should be remarked that we use here the fact that we work only with effective orbifolds: the stabilizers are generically trivial therefore the correction terms of the orbifold Riemann-Roch do not affect the leading term.

We will apply this result to the case of orbifold surfaces  $\mathcal{X}$  of general type with big orbifold cotangent bundle  $\Omega_{\mathcal{X}}$ . Indeed, let  $T_{\mathcal{X}}$  be the orbifold tangent bundle. Then  $\mathbb{P}(T_{\mathcal{X}})$  is naturally an orbifold and we can apply the previous Riemann-Roch formula to the tautological line bundle  $\mathcal{O}_{\mathbb{P}(T_{\mathcal{X}})}(1)$ . As a corollary, we obtain using orbifold Serre duality: if  $c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0$  then

$$H^0(S^m \Omega_{\mathcal{X}}) \geq c.m^3$$

for a positive constant  $c$ .

**2.4. Orbifold hyperbolicity.** In the context of manifolds, complex hyperbolicity is concerned with the geometry of rational, elliptic curves and more generally entire curves i.e. holomorphic maps from  $\mathbb{C}$ . For orbifold hyperbolicity, algebraic curves are replaced by orbifold Riemann surfaces  $\mathcal{C}$  i.e. the data of a Riemann surface  $C$  of genus  $g \geq 0$  and  $r$  points  $p_1, \dots, p_r \in X$  marked by orders of stabilizer groups  $m_1, \dots, m_r \geq 2$  corresponding to the  $\mathbb{Q}$ -divisor  $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) p_i$ .

**Definition 14.** An orbifold Riemann surface  $\mathcal{C}$  is *rational* (resp. *elliptic*) if  $\deg(K_{\mathcal{C}}) < 0$  i.e.  $2g - 2 + \sum_i \left(1 - \frac{1}{m_i}\right) < 0$  (resp.  $\deg(K_{\mathcal{C}}) = 0$ ).

**Example 15.** A case-by-case check gives that  $(C, \Delta)$  is elliptic if it is isomorphic to one of the following orbifold curve:

- $(E, \emptyset)$  where  $E$  is an elliptic curve
- $(\mathbb{P}^1, \left(1 - \frac{1}{m_1}\right) \{0\} + \left(1 - \frac{1}{m_2}\right) \{1\} + \left(1 - \frac{1}{m_3}\right) \{\infty\})$  where  $(m_1, m_2, m_3)$  is either  $(2, 3, 6), (2, 4, 4), (3, 3, 3)$
- $(\mathbb{P}^1, \left(1 - \frac{1}{2}\right) \{0\} + \left(1 - \frac{1}{2}\right) \{1\} + \left(1 - \frac{1}{2}\right) \{p_3\} + \left(1 - \frac{1}{2}\right) \{\infty\})$  with  $p_3 \in \mathbb{C} \setminus \{0, 1\}$ .

Recall that an *orbifold map* between orbifolds  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  is a map between the underlying spaces  $X_1$  and  $X_2$  which lifts to an equivariant map in orbifold charts.

**Definition 16.** An orbifold map  $p : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  is an *orbifold covering* if every  $x \in X_2$  is in some  $U \subset X_2$  such that for every component  $V$  of  $p^{-1}(U)$  the corresponding orbifold chart  $\phi_1 : \tilde{V} \rightarrow V$  verifies that  $p \circ \phi_1 : \tilde{V} \rightarrow U$  is an orbifold chart for  $U$ .

**Definition 17.** Let  $\mathcal{X}$  be an orbifold. An orbifold rational (resp. elliptic) curve in  $\mathcal{X}$  is the image of an orbifold map  $f : \mathcal{C} \rightarrow \mathcal{X}$  where  $\mathcal{C}$  is rational (resp. elliptic).

A hyperbolic orbifold should not have rational and elliptic curves and more generally:

**Definition 18.** An orbifold  $\mathcal{X}$  is *(Brody-)hyperbolic* if there is no non-constant orbifold map  $f : \mathbb{C} \rightarrow \mathcal{X}$ . An orbifold  $\mathcal{X}$  is *quasi-hyperbolic* if the Zariski closure of the union of the images of orbifold maps  $f : \mathbb{C} \rightarrow \mathcal{X}$  is a proper sub-variety.

**Example 19.** From the uniformization theorem available in the orbifold setting, we have that an orbifold curve  $\mathcal{C}$  is hyperbolic if and only if  $\deg(K_{\mathcal{C}}) > 0$ .

In the algebraic setting, we can generalize the notion of algebraic hyperbolicity to orbifolds:

**Definition 20.** A compact orbifold  $\mathcal{X}$ , with  $\omega$  an orbifold hermitian metric, is *algebraically hyperbolic* if there exists  $\epsilon > 0$  such that for any orbifold morphism  $f : \mathcal{C} \rightarrow \mathcal{X}$  with  $\mathcal{C}$  compact,

$$\deg(K_{\mathcal{C}}) \geq \epsilon \int_{\mathcal{C}} f^* \omega.$$

We can generalize Green-Griffiths-Lang conjecture in the orbifold context

**Conjecture 21** (Orbifold Green-Griffiths-Lang conjecture). *Let  $\mathcal{X}$  be an orbifold of general type. Then there exists a proper sub-orbifold  $\mathcal{Z} \subset \mathcal{X}$  which contains the image of all holomorphic maps  $f : \mathbb{C} \rightarrow \mathcal{X}$ .*

Even in the setting of manifolds, this conjecture is widely open. An important result was obtained by McQuillan [21] with the confirmation of the conjecture for surfaces of general type with  $c_1^2 - c_2 > 0$ . The positivity of the second Segre number ensures by a Riemann-Roch computation as already explained above that  $H^0(S^m \Omega_X) > cm^3$ . McQuillan has extended his result to 2-dimensional Deligne-Mumford stacks with projective moduli [22] (see also [26] and [27]).

Since orbifolds are examples of DM stacks, a consequence of his result that we shall use is

**Theorem 22.** *Let  $\mathcal{X}$  be an orbifold surface of general type with positive orbifold second Segre number*

$$c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0.$$

*Then there exists a proper sub-orbifold  $\mathcal{Z} \subset \mathcal{X}$  which contains the image of all orbifold maps  $f : \mathbb{C} \rightarrow \mathcal{X}$ , in other terms: the orbifold  $\mathcal{X}$  is quasi-hyperbolic.*

Let  $\mathcal{C} \rightarrow \mathcal{X}$  be an orbifold rational or elliptic curve. The subjacent space to  $\mathcal{C}$  is  $\mathbb{P}^1$  or a smooth elliptic curve, thus such a curve gives rise to an entire orbifold curve  $\mathbb{C} \rightarrow \mathcal{X}$ . Therefore a quasi-hyperbolic orbifold is algebraically quasi-hyperbolic in the sense that it has only finitely many rational and elliptic orbifold curves.

**Example 23.** Following [4] (see also [26]), we see that nodal surfaces in  $\mathbb{P}^3$  of degree  $d$  with  $l$  nodes provide examples of orbifolds with positive orbifold second Segre number if

$$l > \frac{8}{3} \left( d^2 - \frac{5}{2}d \right).$$

It should be remarked that this numerical condition can be satisfied only for  $d \geq 6$ . Moreover, in [4], the authors claim that a subspace  $H^0(\mathcal{X}, S^m \Omega_{\mathcal{X}})$  of symmetric orbifold differentials can be extended to the resolutions of such surfaces but it seems to us that there is a gap in the argument of [4] Lemma 2.2.

**2.5. Orbifold jet differentials.** Since Bloch [2], jet differentials have turned out to be a useful tool to study hyperbolic properties of complex manifolds. We would like to use these technics in the orbifold setting extending the construction of Demailly [10] for manifolds to this situation.

Start with an orbifold  $\mathcal{X}$  and an orbifold subbundle  $\mathcal{V} \subset T_{\mathcal{X}}$ . Then we consider the orbifold  $\tilde{\mathcal{X}} := \mathbb{P}(\mathcal{V})$  with its natural orbifold line bundle

$$\mathcal{O}_{\tilde{\mathcal{X}}}(-1).$$

We have an orbifold map  $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ . We define

$$\tilde{\mathcal{V}} := \pi_*^{-1} \mathcal{O}_{\tilde{\mathcal{X}}}(-1) \subset T_{\tilde{\mathcal{X}}},$$

here  $\pi_*$  is the differential map between  $T_{\tilde{\mathcal{X}}}$  and  $\pi^* T_{\mathcal{X}}$  (see [12]).

Therefore, as in the case of manifolds, we have an inductive process which gives the orbifold Demailly-Semple jet bundles:

$$(\mathcal{X}_0, \mathcal{V}_0) = (\mathcal{X}, \mathcal{V}), (\mathcal{X}_k, \mathcal{V}_k) = (\tilde{\mathcal{X}}_{k-1}, \tilde{\mathcal{V}}_{k-1}).$$

Taking  $\mathcal{V}_0 := T_{\mathcal{X}_0}$ , one can define jet differentials  $E_{k,m}$  as the direct image sheaf

$$E_{k,m} := (\pi_{0,k})_* \mathcal{O}_{\tilde{\mathcal{X}}_k}(m),$$

where  $\pi_{0,k} := \mathcal{X}_k \rightarrow \mathcal{X}$  is the natural projection map.

One can use the orbifold Riemann-Roch to compute the Euler characteristic. In the case of orbifold surfaces, one obtains

$$\chi(\mathcal{X}, E_{k,m}) = m^{k+2}(\alpha_k c_1^2 - \beta_k c_2) + O(m^{k+1}),$$

where  $c_1$  and  $c_2$  denote the orbifold Chern classes. Using the semi-stability of  $T_{\mathcal{X}}$  with respect to  $K_{\mathcal{X}}$ , which is also true for orbifolds ([18] and [29]), one can control the higher cohomology as in [14]. In particular, one obtains for an orbifold surface of general type

$$(2.1) \quad h^0(\mathcal{X}, E_{2,m}) \geq \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

In the case of smooth hypersurfaces of degree  $d$  of  $\mathbb{P}^3$ , one obtains that

$$h^0(X, E_{2,m}) \geq C.m^4$$

for  $d \geq 15$  (and  $C > 0$ ). In general, the bigger  $c_1^2/c_2$  is, the easier it is to obtain global jet differentials. This is another illustration that Horikawa surfaces are the most difficult cases to deal with. Therefore, for smooth surfaces in  $\mathbb{P}^3$  of degree  $d = 5$ , one has to take  $k$  much larger. In contrast, we will provide below an example of an orbifold surface of  $\mathbb{P}^3$  of degree 5 where  $h^0(\mathcal{X}, E_{2,m}) \geq C.m^4$  (for  $C > 0$ ).

As in the case of manifolds, one obtains as a consequence of Ahlfors-Schwarz lemma:

**Theorem 24.** *Let  $\pi : \mathcal{X} \rightarrow X$  be a projective orbifold and an ample line bundle  $A$  on  $X$  such that  $H^0(\mathcal{X}_k, \mathcal{O}_{\tilde{\mathcal{X}}_k}(m) \otimes \pi_{0,k}^* A^{-1}) \simeq H^0(\mathcal{X}, E_{k,m} \otimes A^{-1})$  has a non-zero section  $P$ . Then every orbifold entire curve  $f : \mathbb{C} \rightarrow \mathcal{X}$  must satisfy the algebraic differential equation  $P(f) = 0$ .*

The generalization of jet differentials to orbifolds can be applied to classical problems which are not yet solved by Nevanlinna theory. Among these problems, there is the following one.

**Conjecture 25.** *Let  $H_1, \dots, H_q$  be hypersurfaces of degrees  $d_i$  of  $\mathbb{P}^n$  in general position. Assume that  $f : \mathbb{C} \rightarrow \mathbb{P}^n$  is a holomorphic curve which ramifies over  $H_i$  with multiplicity divisible by  $m_i$  and that*

$$\sum_i \left(1 - \frac{1}{m_i}\right) d_i > n + 1,$$

*then  $f$  is algebraically degenerate i.e. its image is contained in a proper algebraic hypersurface.*

The idea is that if  $f : \mathbb{C} \rightarrow \mathbb{P}^n$  is as above, then it defines an orbifold entire curve  $f : \mathbb{C} \rightarrow (\mathbb{P}^n, \Delta)$  where  $\Delta := \sum_i \left(1 - \frac{1}{m_i}\right) H_i$ . The numerical condition above is equivalent to

$$K_{\mathbb{P}^n} + \Delta > 0,$$

and Conjecture 25 is a particular case of Conjecture 21.

**2.6. Horikawa orbifolds.** Let us recall that for minimal surfaces of general type, we have the inequalities  $5c_1^2 + 36 \geq c_2$  if  $c_1^2$  is even,  $5c_1^2 + 30 \geq c_2$  if  $c_1^2$  is odd. Surfaces realizing the equalities are called Horikawa surfaces. Horikawa gave a classification of these surfaces [16]:

**Theorem 26.** *Let  $Z$  be a Horikawa surface with  $c_1^2$  even, then there is a birational map  $Z \rightarrow X'$  where  $X'$  is a surface branched along a curve with ADE singularities, more precisely  $X'$  is either*

- (1) *a double covering of  $\mathbb{P}^2$  branched along an octic ( $\chi = 4$ ),*
- (2) *a double covering of  $\mathbb{P}^2$  branched along a curve of degree 10 ( $\chi = 7$ ),*
- (3) *a double covering of  $\mathbb{F}_N$  branched along a curve of type  $(6, 2a)$ , ( $2a \geq -N$ ) ( $\chi = 3N + 2a - 1$ ).*

In the sequel, we will say that a Horikawa surface is of type  $(i)$  according to the above classification.

It is therefore natural to associate orbifolds to Horikawa surfaces.

**Definition 27.** We say that an orbifold  $\mathcal{X} = (X, \Delta)$  is Horikawa if there is a Horikawa surface  $Z$  with a birational map  $Z \rightarrow X'$  where  $X' \rightarrow X$  is a branched covering with ramification divisor  $\Delta$ .

**Example 28.** From Horikawa's theorem  $(\mathbb{P}^2, (1 - \frac{1}{2})C)$  where  $C$  is a curve of degree 8 or 10 with at most ADE singularities is a Horikawa orbifold.

Hirzebruch considered Horikawa surfaces with  $c_1^2 = 5$  that are (resolutions of) 5-fold covers of  $\mathbb{P}^2$  branched along five lines. More generally, examples of Horikawa orbifolds are provided by  $(\mathbb{P}^2, (1 - \frac{1}{5})C)$  where  $C$  is a curve of degree 5 with at most nodes.

### 3. ALGEBRAIC HYPERBOLICITY

**3.1. Horikawa surfaces with even first Chern number.** Having in mind Lang's conjecture claiming that there are only a finite number of curves of genus 0 or 1 on surfaces of general type, the aim of this section is to provide a lower bound on the genus of a curve on some explicit surfaces.

To study curves in Horikawa surfaces, one strategy is to study orbifold curves in Horikawa orbifolds. Using this philosophy, we obtain

- Theorem 29.** (1) *Let  $X$  be a very generic Horikawa surface of type (1). Then  $X$  has no rational curves.*
- (2) *Let  $X$  be a very generic Horikawa surface of type (2). Then  $X$  is algebraically hyperbolic, in particular  $X$  has no rational or elliptic curves.*
- (3) *Let  $X$  be a very generic Horikawa surface of type (3) with  $a = 3$  and  $\chi = 3N + 5$ . Then  $X$  has no rational curves.*

This theorem will be a consequence of more general results which we establish now.

**3.2. Covers of the plane.** Let  $D = \cup D_k \subset \mathbb{P}^2$ , where  $D_k$  is a very general curve of degree  $d_k$ . For  $n > 1$  dividing  $d = \sum d_k$ , let us consider  $p : X \rightarrow \mathbb{P}^2$  the  $n$ -cyclic covering branched along  $D$  (for the construction of cyclic cover see [1]).

The singularities of  $\Delta$  are the intersection points of the  $D_k$ 's. For such a singularity, there exist local coordinates  $z_1, z_2$  such that  $\Delta$  has equation  $z_1 z_2 = 0$ ; the singularity on the cover is therefore  $\{t^n = z_1 z_2\}$  it is a  $A_{n-1}$ . The map  $p$  is an orbifold covering between  $X$  and  $(\mathbb{P}^2, \Delta)$  where  $\Delta = (1 - \frac{1}{n}) \sum D_i$ .

**Theorem 30.** *Let  $f : C \rightarrow X$  be an orbifold compact curve in  $X$  not contained in the branch locus  $D$ . Then*

$$\deg(K_C) \geq (d - \frac{d}{n} - 4) \deg C,$$

where  $\deg C$  is the degree of  $C$  computed as  $\deg p(f(C))$ .

For  $d > 4$  and  $(d, n) \neq (5, 5), (6, 2), (6, 3), (8, 2)$ , the surface  $X$  is algebraically hyperbolic. For  $(d, n) = (5, 5), (6, 3)$  or  $(8, 2)$ ,  $X$  has no rational curves.

*Remark 31.* The Chern numbers of the desingularisation of  $X$  are:

$$\begin{aligned} c_1^2 &= n(-3 + (1 - \frac{1}{n})d)^2 \\ c_2 &= 3n + (n - 1)(d^2 - 3d). \end{aligned}$$

For  $(d, n) = (6, 2)$ , the surface  $X$  is  $K3$ : such a surface contains an infinite number of elliptic curves [14]. The three other cases  $(d, n) = (5, 5), (6, 3)$  or  $(8, 2)$  are Horikawa surfaces. The case  $(5, 5)$  is a quintic surface. For  $(d, n) = (6, 3)$ , we have  $c_1^2 = 3$  and  $5c_1^2 + 30 = c_2 = 45$ . The case  $(d, n) = (8, 2)$  is a Horikawa surface of type (1).

First, let us prove a lemma about the branch locus:

**Lemma 32.** *Consider the orbifold structure  $\Delta_i$  induced on  $D_i$  by  $(1 - \frac{1}{n}) \sum_{j \neq i} D_j$ . If  $d \geq 5$  then  $(D_i, \Delta_i)$  is hyperbolic.*

*Proof.* We can suppose that  $D_i$  is a rational or an elliptic curve. By the genericity hypothesis on  $D$ ,  $D_i$  is of degree 1, 2 or 3. If  $\deg D \geq 5$ , then  $D_i(D - D_i)$  is  $\geq 4$ , and it is even  $> 4$  if  $\deg D > 5$ , therefore  $(1 - \frac{1}{n}) D_i(D - D_i) > 2$  since for  $d = 5$  we must have  $n = 5$ . Thus  $(D_i, \Delta_i)$  is hyperbolic.  $\square$

For the proof of Theorem 30, we use the main Theorem of Xi Chen in [8] (see also [24]):

**Theorem 33.** *For all reduced curve  $C \subset \mathbb{P}^2$ , we have:*

$$(3.1) \quad 2g(C) - 2 + i(C, D) \geq (d - 4) \deg C,$$

where  $d = \sum d_k$  and  $i(C, D)$  is the number of distinct points of  $\nu^*D$  if  $\nu : C' \rightarrow C$  is the normalization.

*Proof.* (Of Theorem 30). Let  $f : C \rightarrow X$  be an orbifold map. We have the following commuting diagram:

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ h \downarrow & & \downarrow p \\ (C'_1, \Delta') & \xrightarrow{g} & (\mathbb{P}^2, \Delta) \end{array}$$

where  $C_1 = p(f(C))$ ,  $C'_1$  is the desingularisation of  $C_1$ , and  $g$  is an orbifold morphism. Let

$$\begin{aligned} g^*(D_j) &= \sum_{i=1}^{i(C_1, D)} t_{i,j} p_i, \\ g^*(D) &= \sum_{i=1}^{i(C_1, D)} t_i p_i. \end{aligned}$$

Now, let  $\tilde{\Delta} = \sum_{i=1}^{i(C_1, D)} \left(1 - \frac{1}{\tilde{m}_i}\right) p_i$  be the minimal orbifold structure on  $C'_1$  such that  $g : (C'_1, \tilde{\Delta}) \rightarrow (\mathbb{P}^2, \Delta)$  is an orbifold morphism. The conditions for  $g$  to be an orbifold morphism are  $n | \tilde{m}_i t_{i,j}$  for all  $j$ . Therefore the minimal orbifold structure is given by

$$\tilde{m}_i = \text{lcm}_j \left( \frac{n}{\gcd(n, t_{i,j})} \right).$$

As  $t_i = \sum_j t_{i,j}$ , we have  $\gcd(n, t_{i,j}) \leq t_i$  for all  $i, j$ , thus:

$$\frac{n}{t_i} \leq \frac{n}{\gcd(n, t_{i,j})} \leq \text{lcm}_j \left( \frac{n}{\gcd(n, t_{i,j})} \right) = \tilde{m}_i.$$

So, we have:

$$\sum_{i=1}^{i(C_1, D)} \left(1 - \frac{1}{\tilde{m}_i}\right) \geq \sum_{i=1}^{i(C_1, D)} \left(1 - \frac{t_i}{n}\right) = i(C_1, D) - \frac{d}{n} \deg C_1.$$

Now, inequality 3.1 of Chen gives:

$$2g(C'_1) - 2 + i(C_1, D) \geq (d - 4) \deg C_1,$$

therefore:

$$2g(C'_1) - 2 + \sum_{i=1}^{i(C, D)} \left(1 - \frac{1}{\tilde{m}_i}\right) \geq \left(d - \frac{d}{n} - 4\right) \deg C_1.$$

To conclude, we observe that

$$\deg(K_C) \geq 2g(C'_1) - 2 + \sum_{i=1}^{i(C, D)} \left(1 - \frac{1}{\tilde{m}_i}\right)$$

because  $C$  is a uniformisation of  $C'_1$ .

For  $d > 4$ , by Lemma 32, the curves over the branch locus are hyperbolic. We then check that for  $d > 4$ , we have  $d - \frac{d}{n} - 4 > 0$  if and only if  $(d, n) \neq (5, 5), (6, 2), (6, 3), (8, 2)$ . Moreover, for  $(d, n) = (5, 5), (6, 3), (8, 2)$ , we have  $d - \frac{d}{n} - 4 = 0$ .

□

This implies (1) and (2) of Theorem 29.

When  $n = d$ , the surface  $X$  is the degree  $d$  hypersurface in  $\mathbb{P}^3$  defined by:

$$X = \{x_4^d - G(x_1, x_2, x_3) = 0\},$$

where  $G$  is a homogeneous polynomial in  $x_1, x_2, x_3$  of degree  $d$  defining the curve  $C \subset \mathbb{P}^2$ . We can apply the result to smooth  $X$ :

**Corollary 34.** *If  $C$  is very generic and  $d \geq 6$ , then the algebraic surface  $X$  is algebraically hyperbolic. If  $C$  is very generic and  $d = 5$ , then the algebraic surface  $X$  contains no curves of geometric genus 0.*

This result should be compared to results of Clemens, Ein, Pacienza, Voisin establishing the algebraic hyperbolicity of very general hypersurfaces of  $\mathbb{P}^n$  of large degree (see [12] for a survey).

Moreover let us mention that, even if here we are interested in surfaces, since inequality 3.1 of Chen generalizes to higher dimension (see [8] and [24]), the proof above immediately generalizes to

**Theorem 35.** *If  $D \subset \mathbb{P}^n$  is a very generic hypersurface of degree  $d$  and  $d \geq 2n + 2$ , then the degree  $d$  branched cover  $X \subset \mathbb{P}^{n+1}$  ramified over  $D$  is algebraically hyperbolic.*

**3.3. Covers of Hirzebruch surfaces  $\mathbb{F}_N$ , Algebraically hyperbolic Horikawa surfaces.** For  $N \geq 0$ , let  $\mathbb{F}_N := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(N))$  be the Hirzebruch surface and let  $T$  and  $F$  the divisors on  $\mathbb{F}_N$  generating  $\text{Pic}(\mathbb{F}_N)$  with  $T^2 = N$ ,  $T.F = 1$  and  $F^2 = 0$ . Let  $D = \cup D_k \subset \mathbb{F}_N$  where each  $D_k$  is a very general member of a base point free complete linear series. And let  $D \sim aT + bF$  with  $a(N - 1) + b \geq 0$ .

**Theorem 36.** (Chen, [9] Corollary 1.12). *We have:*

$$(3.2) \quad 2g(C) - 2 + i(C, D) \geq \min(a - 3, b - 2) \deg C$$

for all reduced irreducible curves  $C \subset \mathbb{F}_N$  with  $C \not\subset D$  where  $\deg C = (T + F).C$ .

As before, for  $n$  dividing  $a$  and  $b$ , we consider  $p : X \rightarrow \mathbb{F}_N$  a  $n$ -cyclic covering branched along  $D$ . Then we have

**Theorem 37.** *Let  $f : C \rightarrow X$  be an orbifold compact curve in  $X$  not contained in the branch locus. Let  $c, d$  be the integers such that  $p(f(C)) = C_1 \sim cM + dF$ , then:*

$$\deg(K_C) \geq \min(a - 3, b - 2)(c(N + 1) + d) - \frac{1}{n}(bc + ad + acN).$$

*Proof.* We follow the notations and proof of theorem 30 :  $C'_1$  is the desingularisation of  $C_1$ . First we remark that

$$\deg C_1 = c(1 + N) + d, \quad C_1.D = bc + ad + acN.$$

Then, using the inequality 3.2, we obtain:

$$2g(C'_1) - 2 + \sum_{i=1}^{i(C, D)} \left(1 - \frac{1}{\tilde{m}_i}\right) \geq \min(a - 3, b - 2) (\deg C_1) - \frac{C_1.D}{n}.$$

□

To wish to apply Theorem 37 to the Horikawa surfaces (for which  $a = 6, n = 2$ ). To obtain a non-trivial result, we need  $b \geq 3$  if  $N \geq 1$  and  $b \geq 6$  if  $N = 0$  (recall that  $a(N - 1) + b \geq 0$ ). Recall moreover that  $b$  must be divisible by 2. The case  $b = 4$  gives :

$$\text{deg}(K_C) \geq c(N + 1) + d - \frac{1}{2}(bc + 6d + 6cN) = c(-1 - 3N) - 2d,$$

and therefore is not interesting. If  $b \geq 6$ , we have:

$$\text{deg}(K_C) \geq 3(c(N + 1) + d) - \frac{1}{2}(bc + 6d + 6cN) = c(3 - \frac{b}{2}),$$

thus we obtain a result ( $\text{deg}(K_C) \geq 0$ ) only for  $b = 6$ .

Let us suppose that  $D \sim 6T + 6F$  on  $\mathbb{F}_N$  ( $N \geq 0$ ) is a very general member of a base point free complete linear series. The double cover of  $\mathbb{F}_N$  ramified over  $D$  is a Horikawa surface  $X$  with  $\chi = 3N + 5$ . That implies (3) of theorem 29:

**Corollary 38.** *The Horikawa surface  $X$  contains no rational curve.*

We would like to emphasize here the fact that this result is valid for a very generic surface in some irreducible component of the moduli space of Horikawa surfaces.

**3.4. The example of Hirzebruch Quintic surfaces.** Our starting point was the fact that we do not know quasi-hyperbolic quintics. An interesting construction of quintics is due to Hirzebruch: consider 5 lines  $L_1, \dots, L_5$  in general position on  $\mathbb{P}^2$  and then the degree 5 ramified cover of  $\mathbb{P}^2$  branched over the  $L_i$ 's which is a quintic surface  $X$  in  $\mathbb{P}^3$ . If  $\ell_i = \ell_i(x_1, x_2, x_3)$  is an equation of  $L_i$ , an equation of  $X$  is :

$$X = \{x_4^5 - \ell_1 \ell_2 \ell_3 \ell_4 \ell_5 = 0\}.$$

This surface  $X$  is an orbifold which has a singularity  $A_4$  over over each of the 10 nodes of  $L_1 + \dots + L_5$ . This surface was constructed to exhibit a quintic that has Picard number at least 41.

A corollary of Theorem 30 and proposition 32 is

**Corollary 39.**  *$X$  has no orbifold rational curves.*

As  $X$  is an orbifold of general type, a natural question is

**Problem 40.** Show that  $X$ , a Hirzebruch quintic surface, has only finitely many (orbifold) elliptic curves.

A first result toward this is

**Theorem 41.** *If  $X$  is a Hirzebruch quintic, then the image in  $\mathbb{P}^2$  of any orbifold elliptic curve  $f : C \rightarrow X$  is not rational. More precisely, it is mapped in  $\mathbb{P}^2$  to a singular genus one curve which intersects every line  $L_i$  with multiplicity divisible by 5.*

*Proof.* As above, we have the following diagram:

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ h \downarrow & & \downarrow p \\ (C'_1, \Delta') & \xrightarrow{g} & (\mathbb{P}^2, \Delta) \end{array}$$

where  $C_1 = p(f(C))$ ,  $C'_1$  is the desingularisation of  $C_1$ , and  $g$  is an orbifold morphism. Therefore  $(C'_1, \Delta' = \sum (1 - \frac{1}{m_i}) p_i)$  is an orbifold elliptic curve i.e.  $K_{C'_1} + \Delta' = 0$ . A case-by-case check gives that  $(C'_1, \Delta')$  is isomorphic to one of the following orbifold curve:

- $(E, \emptyset)$  where  $E$  is an elliptic curve
- $(\mathbb{P}^1, (1 - \frac{1}{m_1}) \{0\} + (1 - \frac{1}{m_2}) \{1\} + (1 - \frac{1}{m_3}) \{\infty\})$  where  $(m_1, m_2, m_3)$  is either  $(2, 3, 6), (2, 4, 4), (3, 3, 3)$
- $(\mathbb{P}^1, (1 - \frac{1}{2}) \{0\} + (1 - \frac{1}{2}) \{1\} + (1 - \frac{1}{2}) \{p_3\} + (1 - \frac{1}{2}) \{\infty\})$  with  $p_3 \in \mathbb{C} \setminus \{0, 1\}$ .

Let

$$\begin{aligned} g^*(L_j) &= \sum_{i=1}^{i(C_1, L)} t_{i,j} p_i. \\ g^*(L) &= \sum_{i=1}^{i(C_1, L)} t_i p_i. \end{aligned}$$

The conditions for  $g$  to be an orbifold morphism are  $5|m_i \cdot t_{i,j}$  for all  $j$ . As  $\gcd(5, m_i) = 1$  by the preceding case-by-case analysis, we conclude that  $5|t_{i,j}$ . Therefore  $g : C'_1 \rightarrow (\mathbb{P}^2, \Delta)$  is an orbifold morphism. From theorem 30, we deduce that this excludes the case  $C'_1 = \mathbb{P}^1$ . Therefore in the above list, only the first case one can occur. So  $C_1$  is a genus one curve. The condition  $5|t_{i,j}$  implies that  $\deg(C_1) = \sum_j t_{ij}$  is divisible by 5 and  $C_1$  cannot be a smooth elliptic curve. □

Therefore we have reduced the problem of counting orbifold elliptic curves in Hirzebruch quintics to counting genus one curves in  $\mathbb{P}^2$  which intersects every line  $L_i$  with multiplicity divisible by 5.

*Remark 42.* The following computation shows that it is natural to believe that there are finitely many of these curves.

$V^{d,\delta}$  the Severi variety of plane curves of degree  $d$  and  $\delta$  nodes has dimension  $3d + g - 1$ . Let  $C \in V^{d,\delta}$  be a curve which intersects  $L_i$  in  $\sum_j \beta_j$  points with multiplicity  $5j$ . The total number of conditions is

$$D := \sum_{i=1}^5 \sum_{j=1}^{l_i} \beta_j (5j - 1),$$

with  $\sum_j 5j\beta_j = d$ . Then  $D = \sum_i (d - \sum_j \beta_j)$ , et  $\sum \beta_j \leq \frac{d}{5}$ . Therefore  $D \geq \sum_i d(1 - \frac{1}{5}) = 4d$ . Thus for  $g \leq 1$  the number of conditions is greater than the number of parameters.

#### 4. ANALYTIC HYPERBOLICITY

**4.1. Nevanlinna theory.** Nevanlinna theory can be used to study entire curves in ramified covers. We will briefly recall in this context the truncated defect relation of Cartan following the notations of [19] to which we refer for details.

Let  $f : \mathbb{C} \rightarrow \mathbb{P}^n$  be a linearly-non degenerate entire curve and  $D_1, \dots, D_q$  be  $q$  hyperplanes in general position. Let us denote as usual  $N^{[n]}(f, r, D_i)$  and  $T(f, r, D_i)$ , the truncated counting function and the characteristic function.

The defect is defined by

$$\delta^{[n]}(f, D_i) := \liminf_{r \rightarrow \infty} \left( 1 - \frac{N^{[n]}(f, r, D_i)}{T(f, r, D_i)} \right).$$

Then we have the truncated defect relation of Cartan:

**Theorem 43.**  $\sum \delta^{[n]}(f, D_i) \leq n + 1.$

We can apply this to ramified covers of the plane. Let  $X \subset \mathbb{P}^3$  be the degree  $d$  cover of  $\mathbb{P}^2$  ramified over  $d$  lines  $D_i$ .

**Theorem 44.** *If  $d \geq 6$  then  $X$  is an hyperbolic orbifold.*

*Proof.* Composing with the projection, we obtain an orbifold map  $g : \mathbb{C} \rightarrow (\mathbb{P}^2, \Delta)$ . If  $g$  is linearly degenerate, from theorem 30, it would have to lie in the branch locus and would be constant by proposition 32. So we suppose, it is linearly non-degenerate. By the First Main Theorem of Nevanlinna theory, we have

$$\frac{d}{2} N^{[2]}(f, r, D_i) \leq N(f, r, D_i) \leq T(f, r, D_i).$$

Therefore

$$\delta^{[2]}(f, D_i) \geq 1 - \frac{2}{d},$$

which implies

$$\sum \delta^{[2]}(f, D_i) \geq d - 2,$$

contradicting Cartan's relation if  $d \geq 6$ . □

Unfortunately we see in the previous proof that we cannot say anything for  $d = 5$ , the case of Hirzebruch quintics. An interesting question is to prove that we have again algebraic degeneracy of entire curves in this case. Note that, according to Conjecture 25, this should be true since  $K_{\mathbb{P}^2} + \Delta = \mathcal{O}(1)$ . Surprisingly, it seems that this problem is still open.

**Problem 45.** Prove that any entire curve  $f : \mathbb{C} \rightarrow \mathbb{P}^2$  intersecting 5 lines in general position with multiplicity at least 5 is algebraically degenerate.

Nevertheless, we can use Nevanlinna theory to construct some hyperbolic orbifold Horikawa surfaces.

**Theorem 46.** *Let  $a \geq 3$  be an integer. Then there exists a hyperbolic orbifold Horikawa surface whose minimal resolution has  $\chi = 2a - 1$ .*

*Proof.* Let  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ . Consider the divisor

$$\Delta = \left( 1 - \frac{1}{2} \right) \sum_{i=1}^6 G_i + \left( 1 - \frac{1}{2} \right) \sum_{i=1}^{2a} H_i,$$

where the  $G_i$  are fibers of the first projection  $p_1$ , and the  $H_i$  are fibers of the second projection  $p_2$ . Then  $(\mathbb{F}_0, \Delta)$  is an Horikawa orbifold. The corresponding Horikawa surface has  $\chi = 2a - 1$ .

Let  $f : \mathbb{C} \rightarrow (\mathbb{F}_0, \Delta)$  be an orbifold entire curve. Then  $p_1 \circ f : \mathbb{C} \rightarrow \mathbb{P}^1$  is an entire curve with multiplicity at least 2 over 6 points. The truncated defect relation (i.e. Nevanlinna Second Main Theorem in this case) implies that  $p_1 \circ f$  is

constant. Therefore  $f$  has its image contained in a fiber of  $p_2$  and has multiplicity at least 2 over  $2a$  points of this fiber. This implies again that  $f$  is constant since  $2a \geq 6$ .  $\square$

We can apply this kind of construction to numerical quintics. A quintic surface has  $c_1^2 = 5$  and  $c_2 = 55$ . Horikawa [15] proved that a surface with  $c_1^2 = 5$  and  $c_2 = 55$ , called a numerical quintic, is either a quintic or a double cover.

Let us consider surfaces of the second type with their attached natural geometric orbifolds  $(X, \Delta)$ .

**Theorem 47.** *There exists a numerical quintic whose geometric orbifold is quasi-hyperbolic.*

*Proof.* Let  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ . Consider the divisor

$$\Delta = \left(1 - \frac{1}{2}\right) \sum_{i=1}^5 G_i + \left(1 - \frac{1}{2}\right) (H_1 + H_2 + C + D_1 + D_2),$$

where the  $G_i$  are fibers of the first projection  $p_1$ , the  $H_i$  are fibers of the second projection  $p_2$ ,  $C$  is a curve of type  $(2, 1)$ , and the  $D_i$  are curves of type  $(1, 1)$  such that:

- (1)  $G_1, H_1, C$  and  $D_1$  meet in a point  $a$ .
- (2)  $G_1, H_2, C$  and  $D_2$  meet in a point  $b$ .

Then  $[\Delta]$  is a curve of type  $(6, 8)$  with two quadruple points lying on a single fiber of  $\mathbb{F}_0$ .

$(\mathbb{F}_0, \Delta)$  is a numerical quintic orbifold.

Let  $f : \mathbb{C} \rightarrow (\mathbb{F}_0, \Delta)$  be an orbifold entire curve. Then  $p_1 \circ f : \mathbb{C} \rightarrow \mathbb{P}^1$  is an entire curve with multiplicity at least 2 over 5 points. The truncated defect relation (i.e. Nevanlinna Second Main Theorem in this case) implies that  $p_1 \circ f$  is constant. Therefore  $f$  has its image contained in a fiber of  $p_2$ . For a generic fiber i.e. not  $G_1$ ,  $f$  has multiplicity at least 2 over at least 5 points of this fiber. This implies again that  $f$  is constant. Therefore the image of  $f$  is contained in  $G_1$ .  $\square$

One can remark that Cartan's theorem unfortunately says nothing in the case of Horikawa surfaces which are degree 2 covers of the plane  $\mathbb{P}^2$ .

**4.2. Jet differentials.** As we have just seen, applications of Nevanlinna theory are quite limited. This is one motivation to develop the tool of orbifold jet differentials.

Let us illustrate it in the case of Hirzebruch quintics. First one could think of using jets of order 1 to apply theorem 22.

As already mentioned, the surface  $X$  has a natural structure of orbifold  $\mathcal{X} = (X, 0)$ .

**Proposition 48.** *The Chern numbers of  $\mathcal{X}$  are:*

$$\begin{aligned} c_1^2(\mathcal{X}) &= 5 \\ c_2(\mathcal{X}) &= 7, \end{aligned}$$

*Proof.* Let  $X' \rightarrow X$  be the resolution of singularities of  $X$ , its Chern numbers are equals to the Chern numbers of a smooth quintic (Brieksorn resolution Theorem)

i.e.  $K_{X'}^2 = 5$ ,  $e(X') = 55$ . We have  $K_{\mathcal{X}}^2 = K_X^2 = 5$  and  $c_2(\mathcal{X}) = e(X') - 10 \cdot (5 - \frac{1}{5}) = 7$ . □

Therefore the orbifold second Segre number is negative and we cannot apply theorem 22.

Turning to jet differentials of order 2, one sees that

$$13c_1^2(\mathcal{X}) - 9c_2(\mathcal{X}) = 2 > 0.$$

Therefore by inequality (2.1), we have

**Corollary 49.** *Let  $X$  be an Hirzebruch quintic surface. We have:*

$$h^0(X, E_{2,m} \otimes L^{-1}) \geq 2 \cdot \frac{m^4}{648} + O(m^3),$$

for  $L$  an ample line bundle on  $X$ . As a consequence, any entire orbifold curve  $f : \mathbb{C} \rightarrow X$  satisfies a differential equation of order 2.

There are many known quintic surfaces with quotient singularities (see e.g. [28]), but as far as we know, the Hirzebruch surface is the only one such that the associated orbifold satisfies  $13c_1^2(\mathcal{X}) - 9c_2(\mathcal{X}) > 0$ .

This result can also be seen as a first step toward problem 45.

## 5. PERSSON-HORIKAWA SURFACES

Let us recall some notations. For  $N$  a positive integer, the  $N^{\text{th}}$  Hirzebruch surface is  $\mathbb{F}_N = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(N))$ . We denote by  $F$  denote a fiber of the  $\mathbb{P}^1$ -bundle  $\mathbb{F}_N \rightarrow \mathbb{P}^1$  and by  $T$  a section s.t.  $T^2 = N$ . We denote by  $(a, b)$  a divisor equivalent to  $aT + bF$  in the Néron-Severi group.

Recall that a Horikawa surface is the double cover of the plane branched along an octic (case  $\chi = 4$ ), or a curve of degree 10 (case  $\chi = 7$ ) or a double cover of  $\mathbb{F}_N$  branched over a curve  $D$  of type  $(6, 2a)$ , with  $2a \geq -N$ . In this section we will exhibit quasi-hyperbolic Horikawa orbifolds of each type.

**5.1. Horikawa surfaces with  $\chi = 2k - 1$ .** Let us prove the following Proposition:

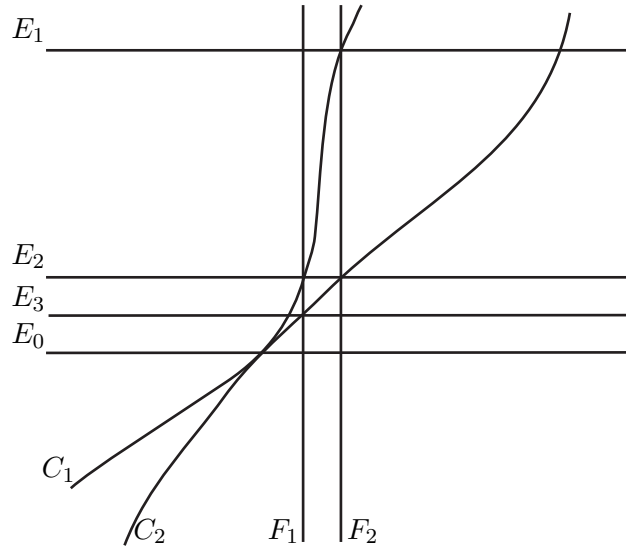
**Proposition 50.** *For any integer  $k > 2$ , there exists a quasi-hyperbolic Horikawa orbifold  $\mathcal{X}$  whose minimal resolution has Euler characteristic equals to  $\chi = 2k - 1$ .*

The divisor  $-2(F + T)$  is the canonical divisor on Hirzebruch surface  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ . Following Persson's construction in [25] Lemma 4.5, there exists on  $\mathbb{F}_0$  a  $\mathbb{Q}$ -divisor:

$$\Delta = (1 - \frac{1}{k})(F_1 + F_2) + (1 - \frac{1}{2})(C_1 + C_2 + E_0 + E_1 + E_2 + E_3)$$

such that the orbifold  $(\mathbb{F}_0, \Delta)$  is uniformizable for all  $k \geq 2$  and the desingularisation of the uniformization is a Horikawa surface with Chern invariants  $\chi = 2k - 1$  and  $c_1^2 = 2\chi - 6$ . The curves  $C_1, C_2$  are  $(1, 1)$ -curves meeting with multiplicity 2 into one point. The curves  $F_1, F_2$  are fibers of the first projection  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , the curves  $E_i$  are sections of this fibration. The singularities of  $[\Delta]$  are  $6A_1 + 4D_4 + D_6$ .

Two singularities  $A_1$  are on branches with multiplicities  $m_i$  equal to 2 (thus  $\beta(p) = 4$ ),  $4A_1$  are on branches with multiplicities 2 and  $k$  (with  $\beta(p) = 2k$ ), the



branches of the  $4D_4$  have the same multiplicities :  $2, 2, k$  (giving  $\beta(p) = 4k^2$ ) and the branches of the  $D_6$  point have multiplicities 2, thus  $\beta(p) = 32$  by Lemma 12.

Let us compute the orbifold Chern numbers of the associated orbifold  $\mathcal{X} : K + \Delta$  is numerically equivalent to:  $(1, 1 - \frac{2}{k})$  thus:

$$c_1(\mathcal{X})^2 = 2(1 - \frac{2}{k}).$$

For the second Chern number of  $\mathcal{X}$ :

$$c_2(\mathcal{X}) = 4 - 2(1 - \frac{1}{k})(2 - 4) - \frac{1}{2}(3(2 - 3) + (2 - 2) + 2(2 - 4)) \\ - (2(1 - \frac{1}{4}) + 4(1 - \frac{1}{2k}) + 4(1 - \frac{1}{4k^2}) + (1 - \frac{1}{32}))$$

and

$$c_2(\mathcal{X}) = \frac{33}{32} - \frac{2}{k} + \frac{1}{k^2}.$$

Therefore :

$$c_1(\mathcal{X})^2 - c_2(\mathcal{X}) = \frac{31}{32} - \frac{2}{k} - \frac{1}{k^2},$$

and the orbifold  $\mathcal{X}$  is quasi-hyperbolic for  $k > 2$ .

*Remark 51.* Let  $X \rightarrow \mathbb{F}_0$  be the  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$ -cover of  $F_0$  branched over  $\Delta$ . The desingularisation of  $X$  is a Horikawa surface. The surface  $X$  has only  $ADE$  singularities and has a natural structure of orbifold  $\mathcal{X}'$ . The map  $\mathcal{X}' \rightarrow \mathcal{X}$  is an orbifold covering, in particular  $c_1(\mathcal{X}')^2 = 2kc_1(\mathcal{X})^2$  and  $c_2(\mathcal{X}') = 2kc_2(\mathcal{X})$ .

5.1.1. *Generalization : a family of quasi-hyperbolic Horikawa orbifold with  $\chi = 2k - 1$ ,  $k \geq 4$ .* Let us consider the configuration of curves in figure 5.1:

The two  $(1, 1)$ -curves  $C_1, C_2$  are in general position. The singularities of  $\Delta$  are  $7A_1 + 5D_4$ . One  $D_4$  with multiplicities  $(2, 2, 2)$ ,  $4D_4$  with multiplicities  $(2, 2, k)$ ,  $4A_1$  with  $(2, k)$  and  $3A_1$  with  $(2, 2)$ .

Let  $\mathcal{X}$  be the associated orbifold. Its Chern numbers are:  $c_1(\mathcal{X})^2 = 2(1 - \frac{2}{k})$  and

$$c_2(\mathcal{X}) = 4 - 2(1 - \frac{1}{k})(2 - 4) - \frac{1}{2}(3(2 - 3) + (2 - 2) + 2(2 - 5)) \\ - (3(1 - \frac{1}{4}) + 4(1 - \frac{1}{2k}) + 4(1 - \frac{1}{4k^2}) + (1 - \frac{1}{16}))$$

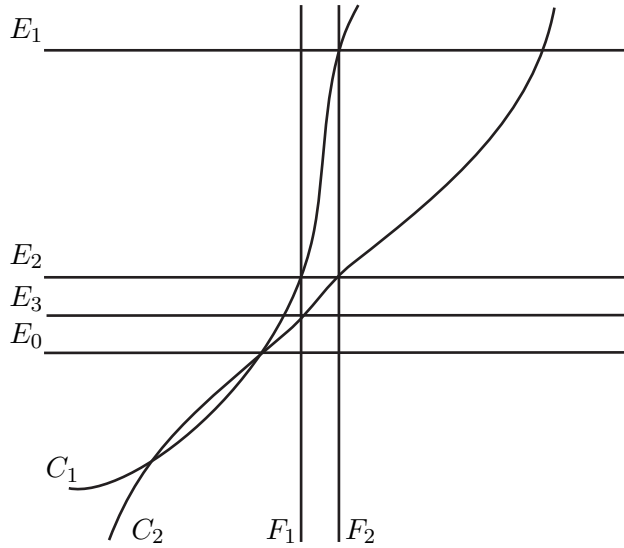


FIGURE 5.1. The divisor  $\Delta$  on  $\mathbb{F}_0$ .

thus  $c_2(\mathcal{X}) = \frac{21}{16} - \frac{2}{k} + \frac{2}{k^2}$  and

$$c_1(\mathcal{X})^2 - c_2(\mathcal{X}) = \frac{11}{16} - \frac{2}{k} - \frac{2}{k^2}.$$

It is positive if  $k \geq 4$ .

**5.2. Horikawa surfaces with  $\chi = 4k - 1$ .** Let us give another construction of quasi-hyperbolic Horikawa surfaces:

**Proposition 52.** *For every integer  $k > 1$ , there exists a quasi-hyperbolic orbifold Horikawa surface  $\mathcal{X}$  with*

$$\begin{aligned} c_1^2(\mathcal{X}) &= 4\left(1 - \frac{1}{k}\right) \\ c_2(\mathcal{X}) &= \frac{17}{12} - \frac{2}{k} - \frac{1}{k^2}. \end{aligned}$$

*The desingularisation of the  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$ -cover of the subadjacent space to  $\mathcal{X}$  is a Horikawa surface with  $\chi = 4k - 1$ .*

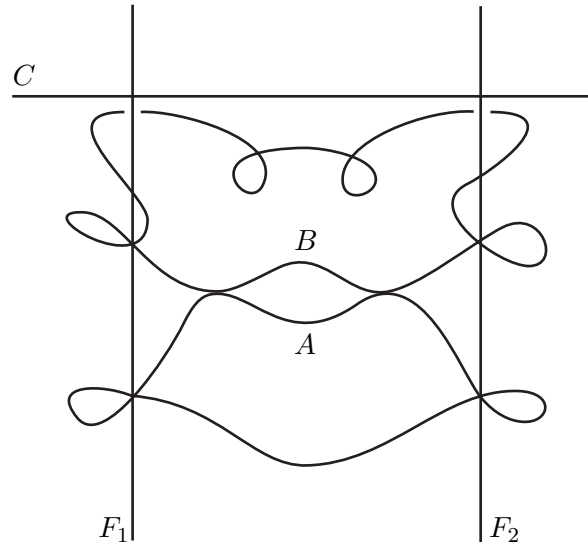
*Proof.* The canonical divisor on  $\mathbb{F}_2$  is  $K_{\mathbb{F}_2} = -2T$ . Let  $k > 1$  be an integer. Persson ([25] proposition 4.7) constructed on  $\mathbb{F}_2$  a divisor :

$$\Delta = \left(1 - \frac{1}{k}\right)(F_1 + F_2) + \left(1 - \frac{1}{2}\right)(A + B + C)$$

whose irreducible components  $F_i, A, B, C$  have the following configuration :  $F_1$  and  $F_2$  are fibers,  $C$  has type  $(1, -2)$ ,  $A$  has type  $(2, 0)$ ,  $B$  has type  $(3, 0)$ .

We have  $e(A) = e(B) = 0$ ,  $CA = CB = 0$  and  $F_iA = 2$ ,  $F_iB = 3$ . The common points to  $A$  and  $B$  are  $2A_{11}$  singularities of  $[\Delta]$ .

The singularities on  $\Delta$  are  $2A_1 + 2A_{11}$  with multiplicities  $(2, 2)$ ,  $4A_1$  with multiplicities  $(2, k)$ , and  $4D_4$  with multiplicities  $(2, 2, k)$  (see figure 5.2).

FIGURE 5.2. The divisor  $\Delta$  on  $\mathbb{F}_0$ .

Let us compute the orbifold Chern numbers of  $\mathcal{X} = (\mathbb{F}_2, \Delta)$ :

$$c_1^2(\mathcal{X}) = (K_{\mathbb{F}_2} + \Delta)^2 = \left( \left(1 - \frac{1}{k}\right)(0, 2) + \left(1 - \frac{1}{2}\right)(6, -2) + (-2, 0) \right)^2 = 4\left(1 - \frac{1}{k}\right),$$

and

$$c_2(\mathcal{X}) = 4 - 2\left(1 - \frac{1}{k}\right)(2 - 4) - \frac{1}{2}((0 - 4) + (0 - 8) + 0) - (2\left(1 - \frac{1}{4}\right) + 2\left(1 - \frac{1}{24}\right) + 4\left(1 - \frac{1}{2k}\right) + 4\left(1 - \frac{1}{4k^2}\right)).$$

Therefore :  $c_2(\mathcal{X}) = \frac{31}{12} - \frac{2}{k} + \frac{1}{k^2}$  and

$$c_1(\mathcal{X})^2 - c_2(\mathcal{X}) = \frac{17}{12} - \frac{2}{k} - \frac{1}{k^2}$$

is positive for  $k > 1$ . □

### 5.3. Examples of quasi-hyperbolic orbifold double octic.

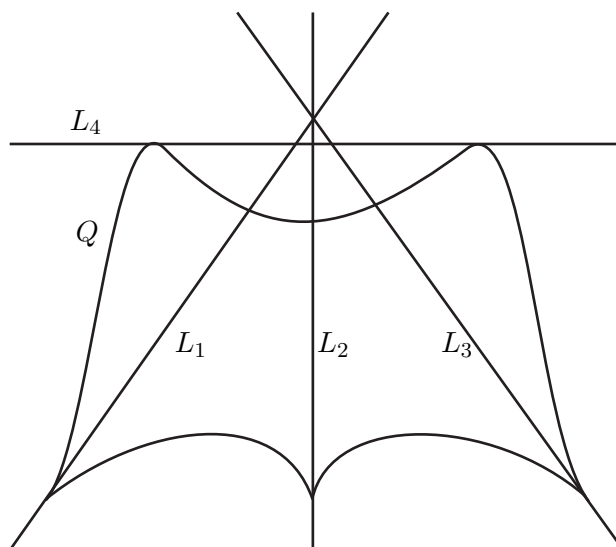
5.3.1. *First construction* :  $\frac{c_1^2}{c_2} = 3 - \frac{1}{11}$ . The Horikawa surfaces with  $\chi = 4$  are double cover of the plane ramified over an octic with at most *ADE* singularities.

**Proposition 53.** *There exists a quasi-hyperbolic orbifold Horikawa surface whose minimal resolution has  $\chi = 4$ .*

Let  $\mathcal{X}$  be the orbifold whose subadjacent variety is  $\mathbb{P}^2$  and with  $\Delta = \frac{1}{2}(Q + L_1 + L_2 + L_3 + L_4)$  where  $Q$  is the Steiner quartic (the unique quartic curve with 3 cusps),  $L_1, L_2, L_3$  are the tangents to the 3 cusps and  $L_4$  is the unique bitangent of  $Q$ .

The singularities of the curve  $[\Delta]$  are  $6A_1 + 2A_3 + D_4 + 3E_7$ . We have:

$$c_1^2(\mathcal{X}) = (-3 + (1 - \frac{1}{2})8)^2 = 1.$$



Moreover:

$$c_2(\mathcal{X}) = 3 - \frac{1}{2}(e(Q \setminus S) + \sum_i e(L_i \setminus S)) - \sum_{p \in S} 1 - \frac{1}{\beta(p)},$$

where  $S$  is the set of singularities of  $[\Delta]$ . Therefore:

$$c_2(\mathcal{X}) = 3 - \frac{1}{2}(2 - 8 + 3(2 - 4) + (2 - 5)) - 12 + \frac{6}{4} + \frac{2}{8} + \frac{1}{16} + \frac{3}{96} = \frac{11}{32},$$

and  $c_1^2(\mathcal{X}) - c_2(\mathcal{X}) = \frac{21}{32} > 0$ . The desingularisation of the degree 2 cover ramified over  $[\Delta]$  is a Horikawa surface with  $\chi = 4$ .

5.3.2. *Second construction: a pencil of quasi-hyperbolic Horikawa surfaces.*

**Proposition 54.** *There exists a pencil of quasi-hyperbolic orbifold Horikawa surfaces whose minimal resolutions have  $\chi = 4$ .*

Let us consider the curve in Proposition 53, and replace the line  $L_3$  by a generic line  $L'_3$  going through the intersection point of  $L_1$  and  $L_2$ .

That gives a degree 8 curve  $\Delta$  with singularities

$$9A_1 + 2A_3 + D_4 + 2E_7$$

For which  $c_2(\mathcal{X}) = \frac{3}{4}$  and  $c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0$ .

5.4. **A family of quasi-hyperbolic double covers branched over a degree 10 curve.** Let us prove:

**Proposition 55.** *There exists a 4 dimensional family of quasi-hyperbolic Horikawa orbifolds whose minimal resolutions have  $\chi = 7$ .*

*Proof.* Let  $C$  be the degree 10 curve that is the union of the degree 8 curve in Proposition 53 and two lines in general position. It has singularities:

$$23A_1 + 2A_3 + d_4 + 3E_7.$$

The orbifold Chern classes of  $\mathcal{X} = (\mathbb{P}^2, C)$  are :

$$c_1^2(\mathcal{X}) = \left(-3 + \frac{10}{2}\right)^2 = 4$$

and

$$c_2(\mathcal{X}) = 3 - \frac{1}{2}(2 - 16 + 3(2 - 6) + (2 - 7) + 2(2 - 9)) - 29 + \frac{23}{4} + \frac{2}{8} + \frac{1}{16} + \frac{3}{96} = \frac{83}{32},$$

thus :  $c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0$ . The moduli of 2 lines in  $\mathbb{P}^2$  is 4 dimensional.  $\square$

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Xavier Roulleau,  
roulleau@math.ist.utl.pt  
Départemento de Mathematica,  
Instituto Superior Técnico,  
Avenida Rovisco Pais  
1049-001 Lisboa  
Portugal

Erwan Rousseau,  
erwan.rousseau@cmi.univ-mrs.fr  
Laboratoire d’Analyse, Topologie, Probabilités  
Université d’Aix-Marseille et CNRS  
39, rue Frédéric Joliot-Curie  
13453 Marseille Cedex 13  
France