

On the Hausdorff dimension of Julia sets of some real polynomials

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Résumé

We show that the supremum for c real of the Hausdorff dimension of the Julia set of the polynomial $z \mapsto z^d + c$ (d is an even natural number) is greater than $2d/(d+1)$.

1 Introduction and statement of the result

The Julia set of a non-linear polynomial $P : \mathbf{C} \rightarrow \mathbf{C}$ is the set of points having no neighborhood on which the family of iterates (P^n) is normal. It is a compact non-empty set which is, except for very special polynomials P , a fractal set. For $c \in \mathbf{C}$ we denote by f_c the polynomial

$$f_c(z) = z^d + c,$$

where $d \geq 2$ is an even integer number, which we fix. Denote by J_c the Julia set of f_c .

In the quadratic case $d = 2$, the values of c for which the Hausdorff dimension (HD) of J_c is big has attracted a lot of attention. It has been known since the pioneering work of Douady-Hubbard [3] that the Hausdorff dimension of the Julia set is less than 2 for every hyperbolic polynomial. Thus $HD(J_c) < 2$ outside ∂M and the (hypothetic) non-hyperbolic components of (the interior) M . We recall that M stands for the Mandelbrot set, that is the compact subset of $c \in \mathbf{C}$ such that J_c is connected. Shishikura [9] was the first to find quadratic Julia sets with Hausdorff dimension 2. He indeed proved that this property holds on a dense \mathcal{G}_δ subset of ∂M or even on a dense \mathcal{G}_δ subset on the boundary of every hyperbolic component of M .

More recently, Buff and Chéritat [12] have found quadratic Julia sets with positive Lebesgue measure (see also <http://annals.math.princeton.edu/articles/3682>).

Both Shishikura's and Buff-Chéritat's results are based on the phenomenon of parabolic implosion which has been discovered and studied by Douady-Hubbard [3]. It should be pointed out that Buff-Chéritat's result is very involved and that we will make no use of it. There is no doubt that if they exist, values of $c \in \mathbf{R}$ such that the Julia set J_c has positive measure must be as hard to find as Buff-Chéritat's ones. The aim of present note is much more modest. Its starting point is the second author re-readind of Shishikura's result [10] : it states that if one implodes a polynomial with a parabolic cycle having q petals, then the dimension of its Julia set automatically gets bigger than $2q/(q+1)$. Shishikura's result follows from this by a Baire argument.

Very little is known about Hausdorff dimension of J_c for real c . In particular, it is not known if, for a given degree d ,

$$\sup\{HD(J_c), c \in \mathbf{R}\} = \mathbf{2}.$$

Possible candidates for high dimension are of course (just look at them!) infinitely renormalizable polynomials but the analysis seems to be very delicate and at least no result concerning dimension 2 has been proven so far (for results in the opposite direction, see [1] though). It is for example unknown if the Julia set of the Feigenbaum polynomial has dimension 2 or not (see [4]- [5] for the Julia set of the Feigenbaum universal map though). The only known result about this set is a very general result of Zdunik [14] : it has dimension bigger than 1.

If one tries to use the same ideas as in [9] for real polynomials f_c , one immediately faces the problem that if f_c , for c real, has a parabolic cycle that may be imploded along the real axis then the number of petals is 1 and this does not imply more than Zdunik's general result. The only trick of this paper is to make use of a virtual doubling of petals when the critical point is mapped to a parabolic point (by Lavaurs map). It was inspired by Douady et al paper [13] and implies the following theorem, which is the main result of this work.

Theorem 1 *Let $f_c(z) = z^d + c$, d even. Let N be the set of parameters $c \in \mathbf{R}$, such that f_c has a parabolic cycle of period at least 2 and multiplier 1. Then there exists an open set Y of \mathbf{R} whose closure contains N such that J_c is connected and*

$$HD(J_c) > \frac{2d}{d+1},$$

for every $c \in Y$.

Comment 1 *In fact, we prove a stronger statement : hyperbolic dimension [9] of J_c is bigger than $2d/(d+1)$.*

Comment 2 *By [11] ([6], [8] for $d = 2$), the set of real c such that f_c is hyperbolic is dense in \mathbf{R} . In particular, hyperbolic parameters c are dense in Y .*

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2 Proof of the theorem

We fix an even integer $d \geq 2$ and consider the family $f_c(x) = x^d + c$, for real c . Then the Julia set J_c is connected if and only if $c \in [a, b]$, where $a < 0$ is such that $f_a^2(0)$ is a fixed point, and $b > 0$ is such that f_b has a fixed point with multiplier 1. It is sufficient to prove that, given $c_0 \in (a, b)$ such that f_{c_0} has a neutral cycle of period $k > 1$ with multiplier 1, that is, with one petal, there is an open set Z accumulating at c_0 for which $HD(J_c) > 2d/(d+1)$ for $c \in Z$.

We begin with three pictures. The first two ones illustrate how to choose the parameter (for $d = 2$) and how the corresponding Julia set looks like. The third one has been kindly drawn for us by the referee and shows the case $d = 4$.

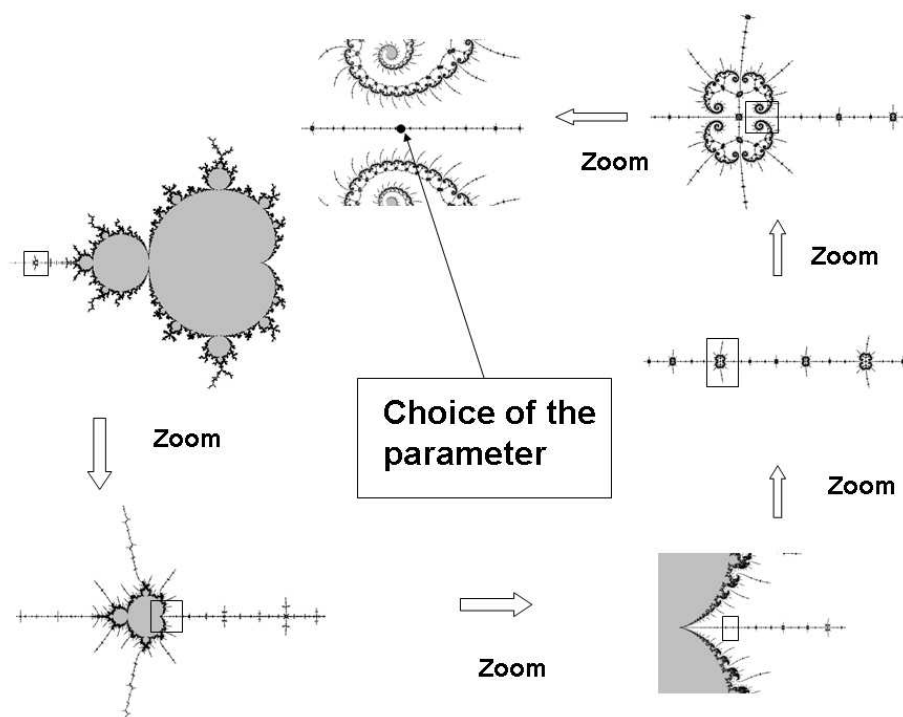


FIGURE 1 – Choice of the parameter

Since $c_0 \in (a, b)$, the set J_c is connected for c in a small neighborhood U of c_0 . It is known, that f_c has an attracting periodic orbit of period k for $c \in U$ on the left side of c_0 . Since $k > 1$, the corresponding filled-in Julia set K_{c_0} is such that

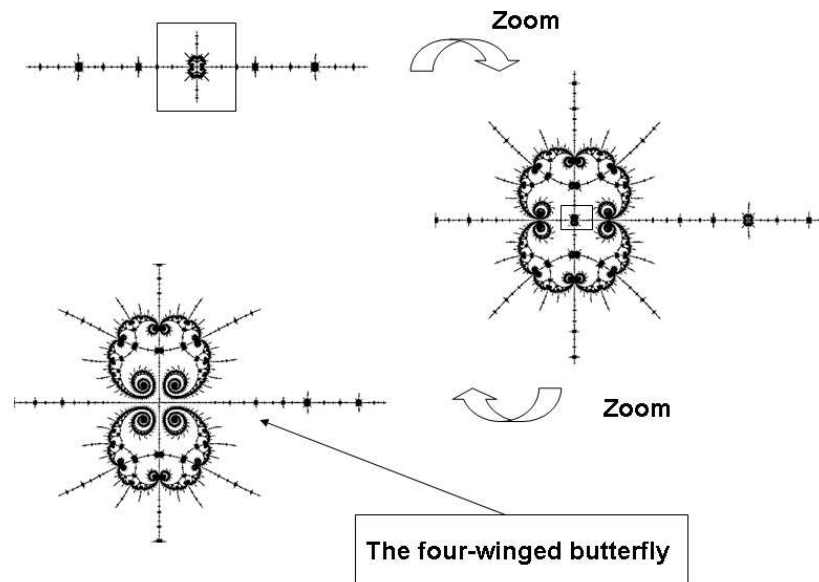


FIGURE 2 – The corresponding Julia set

its interior has a component Δ containing 0, and infinitely many preimages of Δ . The boundary $\partial\Delta$ contains a parabolic point α of period k . In particular, there is a sequence of preimages of Δ , which intersect the real line and accumulate at α .

For definitivity, one can assume that $\alpha > 0$. Then $F = f_{c_0}^k$ has the following local form :

$$F(z) = z + a(z - \alpha)^2 + b(z - \alpha)^3 + \dots, \quad (1)$$

where $a > 0$. This implies that a parabolic implosion phenomenon occurs as $\epsilon \rightarrow 0$, $\epsilon > 0$, for the maps $f_{c_0+\epsilon}$.

At this point we digress somewhat and describe briefly the theory of parabolic implosion.

Let $\delta > 0$ be very small and D_{\pm} be the disks centered at $\alpha \pm \delta$ with radius δ . The map F sends D_- into itself while $D_+ \subset F(D_+)$. This defines, after identification of z with $F(z)$ at the boundary, two cylinders $U_- = D_- \setminus F(D_-)$ and $U_+ = F(D_+) \setminus D_+$. The fact that U_{\pm} are actual cylinders is best seen in the approximate Fatou coordinate $I : z \mapsto -1/(a(z - \alpha))$ which sends α to ∞ and

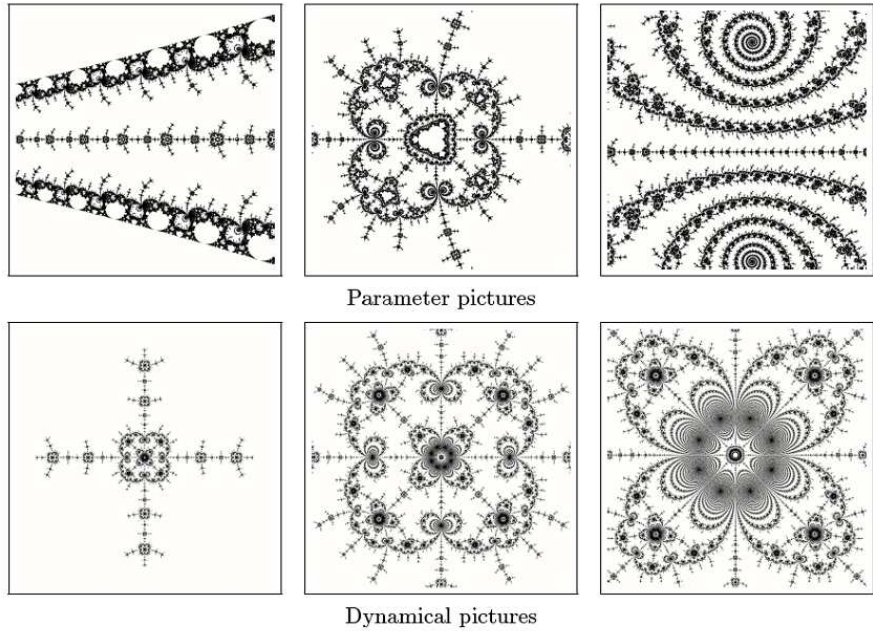


FIGURE 3 – The case $d=4$

conjugates F to a map which is asymptotically the translation by 1 at ∞ :

$$F_\infty(w) = I \circ F \circ I^{-1}(w) = w + 1 + \frac{A}{w} + O\left(\frac{1}{|w|^2}\right), \tag{2}$$

where $A = 1 - b/a^2$. The real number A is a conformal invariant. In the case of real polynomial F which has a parabolic fixed point α with multiplier 1 and with a single critical point in its immediate basin Δ , it is known [9] that $A > 0$.

By Riemann mapping theorem these two cylinders may be uniformized by "straight" cylinders. In other words there exists φ_\pm mapping the cylinders U_\pm to vertical strips V_\pm of width 1 conjugating F to the translation by 1. For further use we notice that, by symmetry, c_0 being real, we may assume that $\varphi_\pm(\bar{z}) = \overline{\varphi_\pm(z)}$. We also notice that these maps are unique up to post-composition by a real translation. These two maps are called respectively repelling (+) and attracting (-) Fatou coordinates. We normalize them as follows. For every $\kappa > 0$ and $R > 0$, consider two sectors $\Sigma_-(\kappa, R) = \{w : \text{Re}(w) > R - \kappa|\text{Im}(w)|\}$, $\Sigma_+(\kappa, R) = \{w : \text{Re}(w) < -R + \kappa|\text{Im}(w)|\}$. Then for any $\kappa > 0$ there is a big enough $R(\kappa)$, such

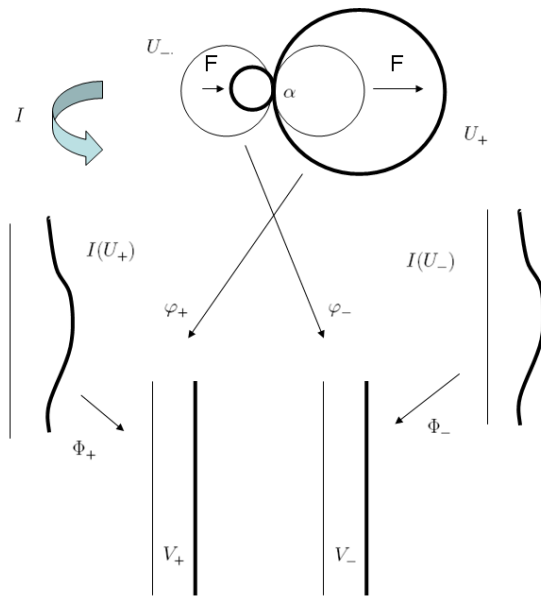


FIGURE 4 – Fatou coordinates

that, if we introduce two sectors $\Sigma_{\pm}(\kappa) = \Sigma_{\pm}(\kappa, R(\kappa))$, then $\varphi_{\pm} = \Phi_{\pm} \circ I$, where

$$\Phi_{\pm}(w) = w - A \log_{\pm}(w) + C_{\pm} + o(1) \tag{3}$$

as $w \rightarrow \infty$ within $\Sigma_{\pm}(\kappa)$ respectively. We specify the constants C_{\pm} and the log-branches in such a way, that Φ_{\pm} are real for real w . Namely, we choose $C_- = 0$ and \log_- to be the standard log-branch in the slit plane $\mathbf{C} \setminus \{\mathbf{x} \leq \mathbf{0}\}$. In turn, let $C_+ = iA\pi$ and \log_+ to be a branch of \log in $\mathbf{C} \setminus \{\mathbf{x} \geq \mathbf{0}\}$ so that $\log_+(w) = \log|w| + i\pi$ for $w < 0$.

Now, we extend φ_{\pm} in the following way. Since

$$\varphi_-(F(z)) = T_1(\varphi_-(z)) \tag{4}$$

where T_{σ} denotes the translation by σ , and since every orbit converging to α passes through U_- exactly once, φ_- extends uniquely to Δ to an holomorphic function still satisfying (4).

It is seen from (4), that the map $\varphi_- : \Delta \rightarrow \mathbf{C}$ is a branched covering, with the critical points at 0 and all its preimages in Δ by F^n , $n > 0$, and the critical values at the real numbers $\varphi_-(0) - n$, $n \geq 0$. In particular, there exists a simply-connected domain $\Omega_- \subset \Delta \cap \mathbf{H}^+$, where \mathbf{H}^+ is the upper half-plane, such that $\varphi_- : \Omega_- \rightarrow \mathbf{H}^+$ is a holomorphic homeomorphism. Moreover, the intersection of the boundary of Ω_- with \mathbf{R} is the interval $(0, \alpha)$.

Concerning the repelling Fatou coordinate it is best to consider $\psi_+ = \varphi_-^{-1} : V_+ \rightarrow U_+$. The functional relation is now

$$\psi_+(T_1(z)) = F(\psi_+(z)) \quad (5)$$

and we can extend ψ_+ to an entire function by putting, for $n \in \mathbf{Z}$, $\psi_+(T_n(z)) = F^n(\psi_+(z))$ for $z \in V_+$. There exists a simply connected domain $\Omega_+ \subset \mathbf{H}^+$, such that $\psi_+ : \Omega_+ \rightarrow \mathbf{H}^+$ is a homeomorphism.

Let now σ be a real number. We define the Lavaurs map g_σ on the component Δ of the interior of the filled-in Julia set of f_{c_0} by

$$g_\sigma = \psi_+ \circ T_\sigma \circ \varphi_-.$$

The "raison d'être" of this definition is the following theorem due to Douady and Lavaurs ([2]) :

Theorem 2 : *There exists a sequence of positive ϵ_n converging to 0 and a sequence of positive integers N_n such that*

$$g_\sigma(z) = \lim_{n \rightarrow \infty} f_{c_0 + \epsilon_n}^{kN_n}(z) \quad (6)$$

uniformly on compact sets of Δ .

Using (3) with the constants C_\pm and the log-branches specified as above, it is easy to get, that, for every κ , if w tends to ∞ in $\Sigma(\kappa) := \Sigma_-(\kappa) \cap \Sigma_+(\kappa) \cap \mathbf{H}^+$, then $g_\infty(w) := I \circ g_\sigma \circ I^{-1}(w) = w + (\sigma - iA\pi) + O(\frac{1}{|w|})$, where A is real and positive.

Therefore, for every real σ and every $\kappa > \kappa(\sigma)$, the inverse map g_∞^{-1} leaves the sector $\Sigma(\kappa)$ invariant and $g_\infty^{-n} \rightarrow \infty$ as $n \rightarrow \infty$. Coming back to the z -plane, we conclude that the branch $G = I^{-1} \circ g_\infty^{-1} \circ I$ of g_σ^{-1} leaves the set $S(\kappa) = I^{-1}(\Sigma(\kappa))$ invariant, and $G^n(z) \rightarrow \alpha$ as $n \rightarrow \infty$, for $z \in S(\kappa)$ and every $\kappa > \kappa(\sigma)$. We have, for $w \in \Sigma(\kappa)$:

$$G_\infty(w) := I \circ G \circ I^{-1}(w) = w + (-\sigma + iA\pi) + O(\frac{1}{|w|}), \quad (7)$$

Now, from the definition of $g_\sigma(z)$ and the global properties of the maps φ_- and ψ_+ , it follows the existence of a simply-connected domain $\Omega_0 \subset \Omega_-$, which is mapped by g_σ homeomorphically onto \mathbf{H}^+ and such that $\alpha \in \bar{\Omega}_0$. Moreover, from the above description, $S(\kappa) \subset \Omega_0$, for every $\kappa > \kappa(\sigma)$. Therefore, the branch G of g_σ^{-1} which is defined above, extends to a global univalent branch $G : \mathbf{H}^+ \rightarrow \Omega_0$ of g_σ^{-1} . Since $\Omega_0 \subset \mathbf{H}^+$, the iterates $G^n : \mathbf{H}^+ \rightarrow \Omega_0$, $n > 0$, converge uniformly on compact sets in \mathbf{H}^+ to a unique fixed point in $\bar{\Omega}_0$, which must be α .

Let us consider the continuous map $\sigma \mapsto g_\sigma(0) = \psi_+(\varphi_-(0) + \sigma)$: if σ runs in the interval $I = \{\varphi_+(x) - \varphi_-(0) : x \in U_+ \cap \mathbf{R}\}$, then $g_\sigma(0)$ runs over $U_+ \cap \mathbf{R}$. It is thus clear, and this is the key point in the proof, that we can choose σ in such a way that $g_\sigma(0)$ is a preimage of α : there is $j \geq 1$, such that $f_{c_0}^j \circ g_\sigma(0) = \alpha$. Since 0 is a critical point for g_σ , taking the inverse image by $f_{c_0}^j \circ g_\sigma$ has the same effect as multiplying the number of petals by d and we may state :

Lemma 2.1 *There exists an infinite iterated function system defined on a small compact neighborhood B_0 of zero and generated by some holomorphic branches of $f_{c_0}^{-1}$ and g_σ^{-1} such that its limit set has Hausdorff dimension bigger than $2d/(d+1)$.*

Comment 3 *In fact, the limit set is a subset of a so-called Julia-Lavours set denoted by $J_{c_0,\sigma}$. It is defined as follows. The map g_σ can be extended in a natural way from Δ to the interior of the filled-in Fatou set of f_{c_0} : if $f_{c_0}^k(z) \in \Delta$, set $g_\sigma(z) = g_\sigma \circ f_{c_0}^k(z)$. Then $J_{c_0,\sigma}$ is simply the closure of the set of points z for which there exists $m \in \mathbf{N}$ such that $g_\sigma^m(z)$ is defined and belongs to $J(f_0)$.*

This lemma together with the above discussion implies the theorem. Indeed, by a general property of iterated function systems [7], there exists its finite subsystem with the Hausdorff dimension of its limit set bigger than $2d/(d+1)$. On the other hand, the finite iterated function system persists for $f_{c_0+\epsilon}$ by (6). To be more precise, if $\{I_j : B_0 \rightarrow X_j, 1 \leq j \leq j_0\}$ is this finite iterated function system, then each I_j can be extended to a univalent map to a fixed neighborhood Y of B_0 as $I_j : Y \rightarrow Y_j$. Consider the inverse univalent map $I_j^{-1} : Y_j \rightarrow Y$. Since the convergence in (6) is uniform on compacts in Δ , for every ϵ_n small enough there is some integer $N_j > 0$ and a compact set $X_{j,n}$, so that $f_{c_0+\epsilon_n}^{N_j} : X_{j,n} \rightarrow B_0$ is univalent, too. Now it is clear, that, for every ϵ_n small enough, the non-escaping set K_n of the dynamical system which consists of a finitely many maps $f_{c_0+\epsilon_n}^{N_j} : X_{j,n} \rightarrow B_0, 1 \leq j \leq j_0$, has the Hausdorff dimension which is bigger than $2d/(d+1)$. On the other hand, K_n must lie in the Julia set of $f_{c_0+\epsilon_n}$ because some iterate of the map $f_{c_0+\epsilon_n}^{N_1 N_2 \dots N_{j_0}}$ leaves the set K_n invariant and is expanding on it.

Proof of Lemma 2.1. As the first step, let us fix a small enough closed ball B_0 around zero, so that it does not contain points of the postcritical set of f_{c_0} . There exists its preimage B' by F^{-1} in $\Delta \cap \mathbf{H}^+$. Then we can apply to B' the maps $G^n, n > 0$. By the above, $B'_n = G^n(B')$ are pairwise disjoint, compactly contained in Δ , and $B'_n \rightarrow \alpha$ as $n \rightarrow \infty$. Now, for every $n \geq n_0$, so that B'_n lies in a small enough neighborhood U of α , we make ‘‘clones’’ of B'_n in $U \cap \Delta$ applying to it $F^r, r \in \mathbf{Z}$, where F^r for $r < 0$ is a well-defined in $U \cap \Delta$ branch which fixes α . We obtain the sets $B'_{n,r} = F^r(B'_n)$. On the second step, we consider the map $f_{c_0}^j \circ g_\sigma$ from a neighborhood V of 0 onto U , This map is a ramified cover with the only ramification point at 0 of order d . Let $U^* = U \setminus \{x \geq \alpha\}$, and $V^* = V \cap \{z : \text{Arg}(z) \in (0, 2\pi/d)\}$. Denote by h a branch of $(f_{c_0}^j \circ g_\sigma)^{-1}$ from U^* onto V^* . Let $B_{n,r} = h(B'_{n,r})$. We obtain a system of holomorphic maps $\Psi = \{\psi_{n,r} : B_0 \rightarrow B_{n,r}\}$, where $\psi_{n,r} = h \circ F^r \circ G^n \circ F^{-1}, r \in \mathbf{Z}, n \geq n_0$. If the neighborhood U is chosen small enough, the maps $\psi_{n,r}$ extend to univalent maps in a fixed neighborhood \tilde{B} of B_0 into itself. In particular, the compact sets $B_{n,r}$ are pairwise disjoint and compactly contained in B_0 . Now, it is quite standard to check that Ψ form a conformal infinite iterated function system in the sense of [7] (strictly speaking, in the hyperbolic metric of \tilde{B} , which is equivalent to the Euclidian one

on B_0 though). Let us calculate the parameter $\theta = \inf\{t : \psi(t) < \infty\}$ of Ψ , where $\psi(t) = \sum_{(n,r)} \max_{z \in B_0} |\psi'_{n,r}(z)|^t$. The map $\psi_{n,r} = h \circ I^{-1} \circ F_\infty^r \circ G_\infty^n \circ I \circ F^{-1}$. Here F^{-1} is a univalent map of a neighborhood \tilde{B} of B_0 into Δ . Now, routine and well-know calculations based on (2)-(7) show (see e.g. [10]), that, for all $r \in \mathbf{Z}, n \geq n_0$ and some C , which depends only on a compact set in $\mathbf{H}^+ \cap \Delta$, from which w is taken, $C^{-1}|r + (-\sigma + i\pi A)n| \leq |F_\infty^r \circ G_\infty^n(w)| \leq C|r + (-\sigma + i\pi A)n|$, and $C^{-1} \leq |(F_\infty^r \circ G_\infty^n)'(w)| \leq C$. On the other hand, the map h is a composition of a univalent map with an inverse branch of $z^{1/d}$. This gives us : $C_1^{-1}|r + (-\sigma + i\pi A)n|^{-1-1/d} \leq |\psi'_{n,r}(z)| \leq C_1|r + (-\sigma + i\pi A)n|^{-1-1/d}$, for some C_1 and every $z \in B_0$. It follows that the series for $\psi(t)$ converges if and only if $t > \theta = 2d/(d+1)$, and $\psi(\theta) = \infty$. Hence [7], the Hausdorff dimension of the limit set of Ψ is strictly bigger than $2d/(d+1)$.

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Références

- [1] A.Avila and M.Lyubich. Examples of Feigenbaum Julia sets with small Hausdorff dimension. *Dynamics on the Riemann Sphere, in honor of B.Branner*, 1 :71–87, 2006.
- [2] A.Douady. Does a Julia set depend continuously on the Polynomial? *Proceedings of Symposia in Applied Mathematics*, 49 :91–135, 1994.
- [3] A.Douady and J.Hubbard. Etude dynamique des Polynômes complexes : Première partie. *Publications mathématiques d'Orsay*, 84-02 :1–75, 1984.
- [4] G.Levin and G.Swiatek. Hausdorff dimension of Julia sets of Feigenbaum type. *Communications in Mathematical Physics*, 258 :135–148, 2005.
- [5] G.Levin and G.Swiatek. Measure of the Julia set of the Feigenbaum map with high criticality. *Ergodic Theory and Dynamical Systems*, 30 :855–875, 2009.
- [6] J.Graczyk and G.Swiatek. Generic Hyperbolicity in the Logistic Family. *Annals of Mathematics*, 146 :1–52, 1997.
- [7] R.D. Mauldin and M. Urbanski. Dimension and measure in infinite iterated function systems. *Proc. London Math. Soc.*, 73 :105–154, 1996.
- [8] M.Lyubich. Dynamics of quadratic polynomials. *Acta Mathematica*, 178 :185–297, 2007.
- [9] M.Shishikura. The Hausdorff Dimension of the Boundary of the Mandelbrot set and Julia Sets. *Annals of Mathematics*, 147 :225–267, 1998.
- [10] M.Zinsmeister. Fleur de Leau-Fatou et dimension de Hausdorff. *Comptes-rendus de l'Académie des Sciences*, 326 :1227–1232, 1998.

- [11] O.Kozlovsky, W.Shen, and S. van Strien. Density of Hyperbolicity in Dimension one. *Annals of mathematics*, 166 :145–182, 2007.
- [12] X.Buff and A.Chéritat. Ensembles de Julia quadratiques de mesure strictement positive. *Comptes-rendus de l'Académie des Sciences*, 341 :669–674, 2005.
- [13] X.Buff, R.Devaney, A.Douady, and P.Sentenac. Baby Mandelbrot sets are born in cauliflowers. *London Mathematical Society Lecture Notes Series*, 274 :19–36, 2000.
- [14] Anna Zdunik. Parabolic orbifolds and the dimension of the maximal measure for rational maps. *Inventiones Mathematicae*, 99 :627–649, 1990.