Numerical radius and distance from unitary operators

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Abstract

Denote by $w(A)$ the numerical radius of a bounded linear operator A acting on Hilbert space. Suppose that A is invertible and that $w(A) \le 1+\varepsilon$ and $w(A^{-1}) \le 1+\varepsilon$ for some $\varepsilon \ge 0$. It is shown that $\inf\{\|A-U\| : U \text{ unitary}\} \le c\varepsilon^{1/4}$ for some constant $c > 0$. This generalizes a result due to J.G. Stampfli, which is obtained for $\varepsilon = 0$. An example is given showing that the exponent $1/4$ is optimal. The more general case of the operator ρ-radius $w_\rho(\cdot)$ is discussed for $1 \le \rho \le 2$.

1 Introduction and statement of the results

Let H be a complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. We denote by $\mathcal{B}(H)$ the C*-algebra of all bounded linear operators on H equipped with the operator norm

$$\|A\| = \sup\{\|Ah\| : h \in H, \|h\| = 1\}.$$

It is easy to see that unitary operators can be characterized as invertible contractions with contractive inverses, i.e. as operators A with $\|A\| \le 1$ and $\|A^{-1}\| \le 1$. More generally, if $A \in \mathcal{B}(H)$ is invertible then

$$\inf\{\|A-U\| : U \text{ unitary }\} = \max\left(\|A\| - 1, 1 - \frac{1}{\|A^{-1}\|}\right).$$

We refer to [6, Theorem 1.3] and [9, Theorem 1] for a proof of this equality using the polar decomposition of bounded operators. It also follows from this proof that if $A \in \mathcal{B}(H)$ is an invertible operator satisfying $\|A\| \le r$ and $\|A^{-1}\| \le r$ for some $r \ge 1$, then there exists a unitary operator $U \in \mathcal{B}(H)$ such that $\|A-U\| \le r-1$.

The numerical radius of the operator A is defined by

$$w(A) = \sup\{|\langle Ah, h\rangle| : h \in H, \|h\| = 1\}.$$

Stampfli has proved in [8] that numerical radius contractivity of A and of its inverse A^{-1}, that is $w(A) \le 1$ and $w(A^{-1}) \le 1$, imply that A is unitary. We define a function $\psi(r)$ for $r \ge 1$ by

$$\psi(r) = \sup\{\|A\| : A \in \mathcal{B}(H), w(A) \le r,\ w(A^{-1}) \le r\},$$

the supremum being also considered over all Hilbert spaces H. Then the conditions $w(A) \le r$ and $w(A^{-1}) \le r$ imply $\max(\|A\|-1, 1-\|A^{-1}\|^{-1}) \le \max(\|A\|-1, \|A^{-1}\|-1) \le \psi(r)-1$, hence the existence of a unitary operator U such that $\|A-U\| \le \psi(r)-1$. We have the two-sided estimate

$$r+\sqrt{r^2-1} \le \psi(r) \le 2r.$$

The upper bound follows from the well-known inequalities $w(A) \le \|A\| \le 2w(A)$, while the lower bound is obtained by choosing $H = \mathbb{C}^2$ and

$$A = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad y = \sqrt{r^2-1},$$

in the definition of ψ. Indeed, we have $A = A^{-1}$, $w(A) = \sqrt{1+y^2} = r$, and $\|A\| = y + \sqrt{1+y^2} = r + \sqrt{r^2-1}$.

Our first aim is to improve the upper estimate.

Theorem 1.1. *Let $r \geq 1$. Then*

$$\psi(r) \leq X(r) + \sqrt{X(r)^2-1}, \quad \text{with} \quad X(r) = r + \sqrt{r^2-1}. \tag{1}$$

The estimate given in Theorem 1.1 is more accurate than $\psi(r) \leq 2r$ for r close to 1, more precisely for $1 \leq r \leq 1.0290855\ldots$. It also gives $\psi(1) = 1$ (leading to Stampfli's result) and the following asymptotic estimate.

Corollary 1.2. *We have*

$$\psi(1+\varepsilon) \leq 1 + \sqrt[4]{8\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \to 0.$$

Our second aim is to prove that the exponent $1/4$ in Corollary 1.2 is optimal. This is a consequence of the following result.

Theorem 1.3. *Let n be a positive integer of the form $n = 8k+4$. There exists a $n \times n$ invertible matrix A_n with complex entries such that*

$$w(A_n) \leq \frac{1}{\cos\frac{\pi}{n}}, \quad w(A_n^{-1}) \leq \frac{1}{\cos\frac{\pi}{n}}, \quad \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.$$

Indeed, Theorem 1.3 implies that

$$\psi\Big(\frac{1}{\cos\frac{\pi}{n}}\Big) \geq \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.$$

Taking $1+\varepsilon = 1/\cos\frac{\pi}{n} = 1 + \frac{\pi^2}{2n^2} + O(\frac{1}{n^4})$, we see that the exponent $\frac{1}{4}$ cannot be improved.

More generally, we can consider for $\rho \geq 1$ the ρ-radius $w_\rho(A)$ introduced by Sz.-Nagy and Foiaş (see [5, Chapter 1] and the references therein). Consider the class $\mathcal{C}_\rho$ of operators $T \in \mathcal{B}(H)$ which admit unitary ρ-dilations, i.e. there exist a super-space $\mathcal{H} \supset H$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$T^n = \rho P U^n P^*, \qquad \text{for } n = 1, 2, \ldots.$$

Here P denotes the orthogonal projection from $\mathcal{H}$ onto H. Then the operator ρ-radius is defined by

$$w_\rho(A) = \inf\{\lambda > 0\,;\, \lambda^{-1}A \in \mathcal{C}_\rho\}.$$

From this definition it is easily seen that $r(A) \leq w_\rho(A) \leq \rho\|A\|$, where $r(A)$ denotes the spectral radius of A. Also, $w_\rho(A)$ is a non-increasing function of ρ. Another equivalent definition follows from [5, Theorem 11.1]:

$$w_\rho(A) = \sup_{h \in \mathcal{E}_\rho} \Big\{(1-\tfrac{1}{\rho})\,|\langle Ah,h\rangle| + \sqrt{(1-\tfrac{1}{\rho})^2|\langle Ah,h\rangle|^2 + (\tfrac{2}{\rho}-1)\,\|Ah\|^2}\Big\}, \quad \text{with}$$

$$\mathcal{E}_\rho = \{h \in H\,;\, \|h\| = 1 \text{ and} (1-\tfrac{1}{\rho})^2|\langle Ah,h\rangle|^2 - (1-\tfrac{2}{\rho})\,\|Ah\|^2 \geq 0\}.$$

Notice that $\mathcal{E}_\rho = \{h \in H\,;\, \|h\| = 1\}$ whenever $1 \leq \rho \leq 2$. This shows that $w_1(A) = \|A\|$, $w_2(A) = w(A)$ and $w_\rho(A)$ is a convex function of A if $1 \leq \rho \leq 2$.

We now define a function $\psi_\rho(r)$ for $r \geq 1$ by

$$\psi_\rho(r) = \sup\{\|A\|\,;\, A \in \mathcal{B}(H), w_\rho(A) \leq r,\ w_\rho(A^{-1}) \leq r\}.$$

As before, the conditions $w_\rho(A) \leq r$ and $w_\rho(A^{-1}) \leq r$ imply the existence of a unitary operator U such that $\|A-U\| \leq \psi_\rho(r)-1$, and we have $\psi_\rho(r) \leq \rho r$. We will generalize the estimate (1) from Theorem 1.1 by proving, for $1 \leq \rho \leq 2$, the following result.

Theorem 1.4. *For* $1 \le \rho \le 2$ *we have*

$$\psi_\rho(r) \le X_\rho(r) + \sqrt{X_\rho(r)^2 - 1}, \tag{2}$$
$$\text{with} \quad X_\rho(r) = \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2r}.$$

Corollary 1.5. *For* $1 \le \rho \le 2$ *we have*

$$\psi_\rho(1+\varepsilon) \le 1 + \sqrt[4]{8(\rho - 1)\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \to 0.$$

We recover in this way for $1 \le \rho \le 2$ the recent result of Ando and Li [2, Theorem 2.3], namely that $w_\rho(A) \le 1$ and $w_\rho(A^{-1}) \le 1$ imply A is unitary. The range $1 \le \rho \le 2$ coincides with the range of those $\rho \ge 1$ for which $w_\rho(\cdot)$ is a norm. Contrarily to [2], we have not been able to treat the case $\rho > 2$.

The organization of the paper is as follows. In Section 2 we prove Theorem 1.4, which reduces to Theorem 1.1 in the case $\rho = 2$. The proof of Theorem 1.3 which shows the optimality of the exponent 1/4 in Corollary 1.2 is given in Section 3.

As a concluding remark, we would like to mention that the present developments have been influenced by the recent work of Sano/Uchiyama [7] and Ando/Li [2]. In [3], inspired by the paper of Stampfli [8], we have developed another (more complicated) approach in the case $\rho = 2$.

2 Proof of Theorem 1.4 about ψ_ρ

Let us consider $M = \frac{1}{2}(A + (A^*)^{-1})$; then

$$M^*M - 1 = \tfrac{1}{4}(A^*A + (A^*A)^{-1} - 2) \ge 0.$$

This implies $\|M^{-1}\| \le 1$. In what follows $C^{1/2}$ will denote the positive square root of the self-adjoint positive operator C. The relation $(A^*A - 2M^*M + 1)^2 = 4M^*M(M^*M - 1)$ yields

$$A^*A - 2M^*M + 1 \le 2(M^*M)^{1/2}(M^*M - 1)^{1/2},$$
$$\text{whence} \quad A^*A \le ((M^*M)^{1/2} + (M^*M - 1)^{1/2})^2.$$

Therefore $\|A\| \le \|M\| + \sqrt{\|M\|^2 - 1}$.

We now assume $1 \le \rho \le 2$. Then $w_\rho(.)$ is a norm and the two conditions $w_\rho(A) \le r$ and $w_\rho(A^{-1}) \le r$ imply $w_\rho(M) \le r$. The desired estimate of $\psi_\rho(r)$ will follow from the following auxiliary result.

Lemma 2.1. *Assume* $\rho \ge 1$. *Then the assumptions* $w_\rho(M) \le r$ *and* $\|M^{-1}\| \le 1$ *imply* $\|M\| \le X_\rho(r)$.

Proof. The contractivity of M^{-1} implies

$$\|u\| \le \|Mu\|, \quad (\forall u \in H). \tag{3}$$

As $w_\rho(M) \le r$, it follows from a generalization by Durszt [4] of a decomposition due to Ando [1], that the operator M can be decomposed as

$$M = \rho r \, B^{1/2} U C^{1/2},$$

with U unitary, C selfadjoint satisfying $0 < C < 1$, and $B = f(C)$ with $f(x) = (1-x)/(1-\rho(2-\rho)x)^{-1}$. Notice that f is a decreasing function on the segment $[0, 1]$ and an involution: $f(f(x)) = x$. Let $[\alpha, \beta]$ be the smallest segment containing the spectrum of C. Then $[\sqrt{\alpha}, \sqrt{\beta}]$ is the smallest segment containing the spectrum of $C^{1/2}$ and $[\sqrt{f(\beta)},\sqrt{f(\alpha)}]$ is the smallest segment containing the spectrum of $B^{1/2}$. We have

$$\|u\|\leq\|Mu\|\leq\rho r\,\sqrt{f(\alpha)}\|C^{1/2}u\|,\quad (\forall u\in H).$$

Choosing a sequence u_n of norm-one vectors ($\|u_n\|=1$) such that $\|C^{1/2}u_n\|$ tends to $\sqrt{\alpha}$, we first get $1\leq\rho r\sqrt{\alpha f(\alpha)}$, i.e. $1-(2{+}\rho r^2{-}\rho)\rho\alpha+\rho^2r^2\alpha^2\leq 0$. Consequently we have

$$\frac{2{+}\rho r^2{-}\rho-\sqrt{(2{+}\rho r^2{-}\rho)^2{-}4r^2}}{2\,\rho\,r^2}\leq\alpha\leq\frac{2{+}\rho r^2{-}\rho+\sqrt{(2{+}\rho r^2{-}\rho)^2{-}4r^2}}{2\,\rho\,r^2},$$

and by $\alpha=f(f(\alpha))$

$$\frac{2{+}\rho r^2{-}\rho-\sqrt{(2{+}\rho r^2{-}\rho)^2{-}4r^2}}{2\,\rho\,r^2}\leq f(\alpha)\leq\frac{2{+}\rho r^2{-}\rho+\sqrt{(2{+}\rho r^2{-}\rho)^2{-}4r^2}}{2\,\rho\,r^2}.$$

Similarly, noticing that $\|(M^*)^{-1}\|\leq 1$, $M^*=\rho r\,C^{1/2}U^*B^{1/2}$ and $C=f(B)$, we obtain

$$\frac{2{+}\rho r^2{-}\rho-\sqrt{(2{+}\rho r^2{-}\rho)^2{-}4r^2}}{2\,\rho\,r^2}\leq\beta\leq\frac{2{+}\rho r^2{-}\rho+\sqrt{(2{+}\rho r^2{-}\rho)^2{-}4r^2}}{2\,\rho\,r^2}.$$

Therefore

$$\|M\|\leq\rho r\,\|B^{1/2}\|\,\|C^{1/2}\|=\rho r\,\sqrt{f(\alpha)\beta}\leq\frac{2{+}\rho r^2{-}\rho+\sqrt{(2{+}\rho r^2{-}\rho)^2{-}4r^2}}{2\,r}.$$

This shows that $\|M\|\leq X_\rho(r)$. □

3 The exponent $1/4$ is optimal (Proof of Theorem 1.3)

Consider the family of $n\times n$ matrices $A=DBD$, defined for $n=8k+4$, by

$$D=\mathrm{diag}(e^{i\pi/2n},\dots,e^{(2\ell-1)i\pi/2n},\dots,e^{(2n-1)i\pi/2n}),$$
$$B=I+\tfrac{1}{2\,n^{3/2}}E,\quad \text{where } E \text{ is a matrix whose entries are defined as}$$
$$e_{ij}=1\ \text{ if } 3k+2\leq|i-j|\leq 5k+2,\qquad e_{ij}=0\ \text{ otherwise.}$$

We first remark that $\|A\|=\|B\|=1+\frac{1}{8\sqrt{n}}$. Indeed, B is a symmetric matrix with non negative entries, $Be=(1{+}\frac{1}{8\sqrt{n}})e$ with $e^T=(1,1,1\dots,1)$. Thus $\|B\|=r(B)=1{+}\frac{1}{8\sqrt{n}}$ by the Perron-Frobenius theorem.

Consider now the permutation matrix P defined by $p_{ij}=1$ if $i=j{+}1$ modulo n and $p_{ij}=0$ otherwise and the diagonal matrix $\Delta=\mathrm{diag}(1,\dots,1,-1)$. Then $P^{-1}DP=e^{i\pi/n}\Delta D$ and $P^{-1}EP=E$, whence $(P\Delta)^{-1}AP\Delta=e^{2i\pi/n}A$. Since $P\Delta$ is a unitary matrix, the numerical range $W(A)=\{\langle Au,u\rangle,;\|u\|=1\}$ of A satisfies $W(A)=W((P\Delta)^{-1}AP\Delta)=e^{2i\pi/n}W(A)$. This shows that the numerical range of A is invariant by the rotation of angle $2\pi/n$ centered in 0, and the same property also holds for the numerical range of A^{-1}.

We postpone the proof of the estimates $\big\|\frac12(A{+}A^*)\big\|\leq 1$ and $\big\|\frac12(A^{-1}{+}(A^{-1})^*)\big\|\leq 1$ to later sections. Using these estimates, we obtain that the numerical range $W(A)$ is contained in the half-plane $\{z\,;\mathrm{Re}\,z\leq 1\}$, whence in the regular n-sided polygon given by the intersection of the half-planes $\{z\,;\mathrm{Re}(e^{2i\pi k/n}z)\leq 1\}$, $k=1,\dots,n$. Consequently $w(A)\leq 1/\cos(\pi/n)$. The proof of $w(A^{-1})\leq 1/\cos(\pi/n)$ is similar.

3.1 Proof of $\left\|\frac{1}{2}(A+A^*)\right\| \le 1$.

Since the (ℓ, j)-entry of A is $e^{(\ell+j-1)i\frac{\pi}{n}}\Big(\delta_{\ell,j} + \frac{e_{\ell,j}}{2n^{3/2}}\Big)$, the matrix $\frac{1}{2}(A + A^*)$ is a real symmetric matrix whose $(i, j)-$entry is $\cos\big((i{+}j{-}1)\frac{\pi}{n}\big)\Big(\delta_{i,j} + \frac{e_{i,j}}{2n^{3/2}}\Big)$. It suffices to show that, for every $u = (u_1, \cdots, u_n)^T \in \mathbb{R}^n$, we have $\|u\|^2 - \mathrm{Re}\langle Au, u\rangle \ge 0$. Let $\mathcal{E} = \{(i, j)\,; 1 \le i, j \le n, 3k + 2 \le |i - j| \le 5k + 2\}$. The inequality which has to be proved is equivalent to

$$\sum_{i=1}^{n} 2\sin^2((i{-}\tfrac{1}{2})\tfrac{\pi}{n})\, u_i^2 - \tfrac{1}{2\,n^{3/2}} \sum_{i,j\in\mathcal{E}} \cos((i{+}j{-}1)\tfrac{\pi}{n}) u_i\, u_j \ge 0.$$

Setting $v_j = u_j \sin((j{-}\frac{1}{2})\frac{\pi}{n})$, this may be also written as follows

$$2\|v\|^2 - \langle Mv, v\rangle + \tfrac{1}{2n^{3/2}}\,\langle Ev, v\rangle \ge 0, \qquad (v \in \mathbb{R}^n). \tag{4}$$

Here M is the matrix whose entries are defined by

$$m_{ij} = \tfrac{1}{2n^{3/2}}\, \cot((i{-}\tfrac{1}{2})\tfrac{\pi}{n})\, \cot((j{-}\tfrac{1}{2})\tfrac{\pi}{n}), \quad \text{if } (i, j) \in \mathcal{E}, \quad m_{ij} = 0 \quad \text{otherwise.}$$

We will see that the Frobenius (or Hilbert-Schmidt) norm of M satisfies $\|M\|_F \le \sqrt{9/32} < 3/4$. A fortiori, the operator norm of M satisfies $\|M\| \le \frac{3}{4}$. Together with $\|E\| = n/4$, this shows that $\|M\| + \frac{1}{2n^{3/2}}\|E\| \le \frac{7}{8}$. Property (4) is now verified.

It remains to show that $\|M\|_F^2 \le \frac{9}{32}$. First we notice that $m_{ij} = m_{ji} = m_{n+1-i,n+1-j}$, and $m_{ii} = 0$. Hence, with $\mathcal{E}' = \{(i, j) \in \mathcal{E}\,; i < j \text{ and } i + j \le n + 1\}$,

$$\|M\|_F^2 = 2\sum_{i<j} |m_{ij}|^2 \le 4 \sum_{(i,j)\in\mathcal{E}'} |m_{ij}|^2.$$

We have, for $(i, j) \in \mathcal{E}'$,

$$\begin{aligned} 2j \le i + j + 5k + 2 \le n + 5k + 3 = 13k + 7, &\quad \text{thus} \quad 3k + 3 \le j \le \tfrac{13k+7}{2},\\ 2i \le i + j - 3k - 2 \le n - 3k - 1 = 5k + 3, &\quad \text{thus} \quad 1 \le i \le \tfrac{5k+3}{2}. \end{aligned}$$

This shows that

$$\tfrac{3\pi}{16} \le \tfrac{3k+2}{16k+8}\pi \le (j{-}\tfrac{1}{2})\tfrac{\pi}{n} \le \tfrac{13k+6}{16k+8}\pi \le \pi - \tfrac{3\pi}{16}, \quad \text{hence} \quad |\cot((j{-}\tfrac{1}{2})\tfrac{\pi}{n})| \le \cot\tfrac{3\pi}{16} \le \tfrac{3}{2}.$$

We also use the estimate $\cot((i{-}\frac{1}{2})\frac{\pi}{n}) \le n/(\pi(i{-}\frac{1}{2}))$ and the classical relation $\sum_{i\ge1}(i - 1/2)^{-2} = \pi^2/2$ to obtain

$$\|M\|_F^2 \le 4 \sum_{(i,j)\in\mathcal{E}'} |m_{ij}|^2 \le \frac{4}{4n^3}\, \frac{n^2}{\pi^2} \sum_{i\ge1} \frac{1}{(i - 1/2)^2}\, (2k{+}1)\, \frac{9}{4} = \frac{9}{32}.$$

3.2 Proof of $\left\|\frac{1}{2}(A^{-1}{+}(A^{-1})^*)\right\| \le 1$.

We start from

$$\begin{aligned}(A^{-1})^* &= D(1 + \tfrac{1}{2n^{3/2}}E)^{-1}D\\ &= D^2 - \tfrac{1}{2n^{3/2}}DED + \tfrac{1}{4n^3}DE^2(1 + \tfrac{1}{2n^{3/2}}E)^{-1}D,\end{aligned}$$

and we want to show that $\|u\|^2 - \mathrm{Re}\langle A^{-1}u, u\rangle \ge 0$. As previously, we set $v_j = u_j \sin((j{-}\frac{1}{2})\frac{\pi}{n})$. The inequality $\left\|\frac{1}{2}(A^{-1}{+}(A^{-1})^*)\right\| \le 1$ is equivalent to

$$2\|v\|^2 - \langle(M_1 + M_2 + M_3 + M_4)v, v\rangle \ge 0, \qquad (v \in \mathbb{R}^n).$$

Here the entries of the matrices M_p, $1 \le p \le 4$, are given by

$$
\begin{aligned}
(m_1)_{ij} &= -\tfrac{1}{2n^{3/2}} \cot((i-\tfrac{1}{2})\tfrac{\pi}{n}) \cot((j-\tfrac{1}{2})\tfrac{\pi}{n})e_{ij},\\
(m_2)_{ij} &= \tfrac{1}{2n^{3/2}}\, e_{ij},\\
(m_3)_{ij} &= \tfrac{1}{4n^3} \cot((i-\tfrac{1}{2})\tfrac{\pi}{n}) \cot((j-\tfrac{1}{2})\tfrac{\pi}{n})f_{ij},\\
(m_4)_{ij} &= -\tfrac{1}{4n^3}\, f_{ij},
\end{aligned}
$$

e_{ij} and f_{ij} respectively denoting the entries of the matrices E and $F = E^2(1+\frac{1}{2n^{3/2}}E)^{-1}$. Noticing that $M_1 = -M$, we have $\|M_1\| \le \frac{3}{4}$, $\|M_2\| = \frac{1}{8\sqrt{n}}$, $\|F\| \le \frac{n^2/16}{1-1/(8\sqrt{n})} \le \frac{n^2}{14}$ and $\|M_4\| = \frac{1}{4n^3}\|F\|$. Now we use

$$\|M_3\|^2 \le \|M_3\|_F^2 \le \tfrac{1}{16n^6} \max_{ij} |f_{ij}|^2 \sum_{i,j} |\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})|^2\, |\cot((j-\tfrac{1}{2})\tfrac{\pi}{n})|^2,$$

together with

$$
\begin{aligned}
\sum_{i,j} |\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})|^2\, |\cot((j-\tfrac{1}{2})\tfrac{\pi}{n})|^2 &= \Big(\sum_{i=1}^{n} |\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})|^2\Big)^2\\
&\le 4\Big(\sum_{i=1}^{n/2} |\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})|^2\Big)^2 \le n^4,
\end{aligned}
$$

to obtain

$$\|M_3\| \le \tfrac{1}{4n} \max_{ij} |f_{ij}|.$$

Using the notation $\|E\|_\infty := \max\{\|Eu\|_\infty\,; u \in \mathbb{C}^n, \|u\|_\infty \le 1\}$ for the operator norm induced by the maximum norm in $\mathbb{C}^d$, it holds $\|E\|_\infty = n/4$, whence $\|\frac{1}{2n^{3/2}}E\|_\infty \le 1/8$ and thus $\|(1+\frac{1}{2n^{3/2}}E)^{-1}\|_\infty \le \frac{1}{1-1/8} = \frac{8}{7}$. This shows that

$$\max_{ij} |f_{ij}| \le \|(1+\tfrac{1}{2n^{3/2}}E)^{-1}\|_\infty \max_{ij} |e^2_{ij}| \le \frac{2n}{7},$$

by denoting e^2_{ij} the entries of the matrix $E^2$and noticing that $\max_{i,j} |e^2_{ij}| = n/4$. Finally, we obtain $\|M_3\| \le \frac{1}{14}$ and $\|M_1+M_2+M_3+M_4\| \le \frac{3}{4}+\frac{1}{8}+\frac{1}{14}+\frac{1}{56} < 1$.

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