ON THE ARITHMETIC CHERN CHARACTER

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Let X be a proper and flat scheme over $\mathbb{Z}$, with smooth generic fiber $X_{\mathbb{Q}}$. In [4] we attached to every hermitian vector bundle $\overline{E} = (E, \| \ \|)$ on X a Chern character class lying in the arithmetic Chow groups of X:

$$\widehat{\mathrm{ch}}(\overline{E}) \in \bigoplus_{p \geq 0} \widehat{\mathrm{CH}}^p(X) \otimes \mathbb{Q} \,.$$

Unlike the usual Chern character with values in the ordinary Chow groups, $\widehat{\mathrm{ch}}$ is not additive on exact sequences; indeed suppose that $\overline{E}_i$, $i = 0, 1, 2$ is a triple of hermitian vector bundles on X, and that we are give an exact sequence

$$0 \to E_0 \to E_1 \to E_2 \to 0$$

of the underlying vector bundles on X, (*i.e.* in which we ignore the hermitian metrics). Then the difference $\widehat{\mathrm{ch}}(\overline{E}_0) + \widehat{\mathrm{ch}}(\overline{E}_2) - \widehat{\mathrm{ch}}(\overline{E}_1)$, is represented by a secondary characteristic class $\widetilde{\mathrm{ch}}$ first introduced by Bott and Chern [1] and defined in general in [2]. These Bott-Chern forms measure the defect in additivity of the Chern forms associated by Chern-Weil theory to the hermitian bundles in the exact sequence.

Assume now that the sequence

$$0 \to E_0 \to E_1 \to E_2 \to 0 \qquad (*)$$

is exact on the generic fiber $X_{\mathbb{Q}}$ but not on the whole of X. We shall prove here (Theorem 1) that $\widehat{\mathrm{ch}}(\overline{E}_0) + \widehat{\mathrm{ch}}(\overline{E}_2) - \widehat{\mathrm{ch}}(\overline{E}_1)$ is the sum of the class of $\widetilde{\mathrm{ch}}$ and the localized Chern character of $(*)$ (see [3], 18.1). This result fits well with the idea that characteristic classes with support on the finite fibers of X are the non-archimedean analogs of Bott-Chern classes (see [6]).

In Theorem 2 we compute more explicitly these secondary characteristic classes in a situation encountered when proving a "Kodaira vanishing theorem" on arithmetic surfaces ([7], 3.3).

Notation. If A is an abelian group we let $A_{\mathbb{Q}} = A \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

1. A general formula

1.1. Let $S = \mathrm{Spec}(\mathbb{Z})$ and $f : X \to S$ a flat scheme of finite type over S. We assume that the generic fiber $X_{\mathbb{Q}}$ is smooth and equidimensional of dimension d. For every integer $p \geq 0$ we denote by $A^{pp}(X_{\mathbb{R}})$ the real vector space of smooth real differential forms α of type (p, p) on the complex manifold $X(\mathbb{C})$ such that

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$F_\infty^*(\alpha) = (-1)^p\,\alpha$, where F_∞ is the anti-holomorphic involution of $X(\mathbb{C})$ induced by complex conjugation. Let

$$A(X) = \underset{p\geq 0}{\oplus} A^{pp}(X_{\mathbb{R}})$$

and

$$\widetilde{A}(X) = \underset{p\geq 1}{\oplus} A^{p-1,p-1}(X_{\mathbb{R}})/(\mathrm{Im}(\partial)+\mathrm{Im}(\overline{\partial}))\,.$$

For every $p \geq 0$ we let $\widehat{\mathrm{CH}}_p(X)$ be the p-th *arithmetic Chow homology* group of X ([5], §2.1, Definition 2). Elements of $\widehat{\mathrm{CH}}_p(X)$ are represented by pairs (Z,g) consisting of a p-dimensional cycle Z on X and a Green current g for $Z(\mathbb{C})$ on $X(\mathbb{C})$. Recall that here a Green current for $Z(\mathbb{C})$ is a current (*i.e.* a form with distribution coefficients) of type $(p-1,p-1)$ such that $dd^c(g)+\delta_Z$ is C^∞, δ_Z being the current of integration on $Z(\mathbb{C})$). There are canonical morphisms ([5], 2.2.1):

$$\begin{aligned} z : \widehat{\mathrm{CH}}_p(X) &\rightarrow \mathrm{CH}_p(X)\\ (Z,g) &\mapsto Z \end{aligned}$$

and

$$\begin{aligned} \omega : \widehat{\mathrm{CH}}_p(X) &\rightarrow A^{pp}(X_{\mathbb{R}})\\ (Z,g) &\mapsto dd^c(g)+\delta_Z\,. \end{aligned}$$

Let $\mathrm{CH}_p^{\mathrm{fin}}(X)$ be the Chow homology group of cycles on X the support of which does not meet $X_{\mathbb{Q}}$. There is a canonical morphism

$$b : \mathrm{CH}_p^{\mathrm{fin}}(X) \rightarrow \widehat{\mathrm{CH}}_p(X)$$

mapping the class of Z to the class of $(Z,0)$. The composite morphism

$$z\circ b : \mathrm{CH}_p^{\mathrm{fin}}(X) \rightarrow \mathrm{CH}_p(X)$$

is the obvious map. Let

$$a : A^{d-p-1,d-p-1}(X_{\mathbb{R}}) \rightarrow \widehat{\mathrm{CH}}_p(X)$$

be the map sending η to the class of $(0,\eta)$. We have

$$\omega\circ a(\eta) = dd^c(\eta)\,.$$

1.2. We assume given a sequence

$$0 \rightarrow \overline{E}_0 \rightarrow \overline{E}_1 \rightarrow \overline{E}_2 \rightarrow 0$$

of hermitian vector bundles on X, the restriction of which to $X_{\mathbb{Q}}$ is exact. Let

$$\mathrm{ch}^{\mathrm{fin}}(E_\bullet)\cap[X] \in \mathrm{CH}_{\mathrm{fin}}(X)_{\mathbb{Q}} = \underset{p\geq 0}{\oplus} \mathrm{CH}_p^{\mathrm{fin}}(X)_{\mathbb{Q}}$$

be the *localized Chern character* of $E_\bullet$ ([3] 18.1), and

$$\widetilde{\mathrm{ch}}(\overline{E}_\bullet) \in \widetilde{A}(X)_{\mathbb{Q}}$$

the Bott-Chern secondary characteristic class [2], such that

$$dd^c\,\widetilde{\mathrm{ch}}(\overline{E}_\bullet) = \sum_{i=0}^{2}(-1)^i\,\mathrm{ch}(\overline{E}_{i,\mathbb{C}})\,,$$

where $\mathrm{ch}(\overline{E}_{i,\mathbb{C}}) \in A(X)$ is the differential form representing the Chern character of the restriction $E_{i,\mathbb{C}}$ of E_i to $X(\mathbb{C})$. Finally, if $i = 0, 1, 2$, we let

$$\widehat{\mathrm{ch}}(\overline{E}_i) \cap [X] \in \widehat{\mathrm{CH}}(X)_{\mathbb{Q}} = \underset{p \geq 0}{\oplus} \widehat{\mathrm{CH}}_p(X)_{\mathbb{Q}}$$

be the *arithmetic Chern character* of $\overline{E}_i$ ([4] 4.1, [5] Theorem 4).

Theorem 1. *The following equality holds in* $\widehat{\mathrm{CH}}(X)_{\mathbb{Q}}$*:*

$$\sum_{i=0}^{2} (-1)^i \, \widehat{\mathrm{ch}}(\overline{E}_i) \cap [X] = b\,(\mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X]) + a\,(\widetilde{\mathrm{ch}}(\overline{E}_\bullet))\,.$$

1.3. This theorem is a special case of Lemma 21 in [5], though this may not be immediately apparent. Therefore, for the sake of completeness, we give a proof here.

1.4. To prove Theorem 1 we consider the Grassmannian graph construction applied to $E_\bullet$ ([3] 18.1, [5] 1.1). It consists of a proper surjective map

$$\pi : W \to X \times \mathbb{P}^1$$

such that, if $\phi \subset X$ is the support of the homology of $E_\bullet$ (hence $\phi_{\mathbb{Q}}$ is empty), the restriction of π onto $(X - \phi) \times \mathbb{P}^1$ and $X \times \mathbb{A}^1$ is an isomorphism. The effective Cartier divisor

$$W_\infty = \pi^{-1}(X \times \{\infty\})$$

is the union of the Zariski closure $\widetilde{X}$ of $(X - \phi) \times \{\infty\}$ with $Y = \pi^{-1}(\phi \times \{\infty\})$. The sequence $E_\bullet$ extends to a complex

$$0 \to \widetilde{E}_0 \to \widetilde{E}_1 \to \widetilde{E}_2 \to 0\,,$$

which is isomorphic to the pull-back of $E_\bullet$ over $X \times \mathbb{A}^1$. The restriction of $\widetilde{E}_\bullet$ to $\widetilde{X}$ is canonically split exact. On $W_{\mathbb{Q}} = X_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ the sequence $\widetilde{E}_\bullet$ is exact; it coincides with $E_\bullet$ (resp. $0 \to E_0 \to E_0 \oplus E_2 \to E_2 \to 0$) when restricted to $X_{\mathbb{Q}} \times \{0\}$ (resp. $X_{\mathbb{Q}} \times \{\infty\}$). We choose a metric on $\widetilde{E}_\bullet$ for which these isomorphisms are isometries.

1.5. Let

$$x = \sum_{i=0}^{2} (-1)^i \, \widehat{\mathrm{ch}}(\overline{\widetilde{E}}_i)\,,$$

and denote by t the standard parameter of $\mathbb{A}^1$. In the arithmetic Chow homology of W we have

$$0 = x \cap (W_0 - W_\infty, -\log|t|^2)\,.$$

If x is the class of (Z, g), with Z meeting properly W_0 and W_∞, we get

$$x \cap (W_0 - W_\infty, -\log|t|^2) = (Z \cap (W_0 - W_\infty), g * (-\log|t|^2))\,,$$

where the $*$-product is equal to

$$g * (-\log|t|^2) = g(\delta_{W_0} - \delta_{W_\infty}) - \mathrm{ch}(\overline{\widetilde{E}}_\bullet) \log|t|^2\,.$$

Since $W_\infty = \widetilde{X} \cup Y$, with $Y_{\mathbb{Q}} = \emptyset$, we get

$$\begin{aligned} (1) \quad 0 &= x \cap (W_0 - W_\infty, -\log|t|^2) \\ &= (Z \cap W_0, g\,\delta_{W_0}) - (Z \cap \widetilde{X}, g\,\delta_{\widetilde{X}}) - (Z \cap Y, 0) - (0, \mathrm{ch}(\overline{\widetilde{E}}_\bullet) \log|t|^2)\,. \end{aligned}$$

The restriction of $\overline{\overline{E}}_\bullet$ to $\widetilde{X}$ is split exact, therefore

$$(Z \cap \widetilde{X}, g\,\delta_{\widetilde{X}}) = 0\,.$$

Applying π_* to (1) we get

$$0 = \widehat{\mathrm{ch}}(\overline{E}_\bullet) - \pi_*(Z \cap Y), 0) - (0, \pi_*(\mathrm{ch}(\overline{\overline{E}}_\bullet) \log |t|^2)\,. \tag{2}$$

By definition of the localized Chern character ([3], 18.1, (14))

$$\pi_*(Z \cap Y) = \mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X] \tag{3}$$

in $\mathrm{CH}^{\mathrm{fin}}(X)_{\mathbb{Q}}$. On the other hand we deduce from [4], (1.2.3.1), (1.2.3.2) that

$$-\,\pi_*(\mathrm{ch}(\overline{\overline{E}}_1) \log |t|^2) = \widetilde{\mathrm{ch}}(\overline{E}_\bullet)\,. \tag{4}$$

and upon replacing t by $1/t$, as in the proof of (1.3.2) in [4], we see that

$$\pi_*(\mathrm{ch}(\overline{\overline{E}}_\bullet) \log |t|^2) = -\,\pi_*(\mathrm{ch}(\overline{\overline{E}}_1) \log |t|^2)\;. \tag{5}$$

Theorem 1 follows from (2), (3), (4), (5).

2. A special case

2.1. We keep the hypotheses of the previous section, and we assume that X is normal, $d = 1$, E_0 and E_2 have rank one and the metrics on E_0 and E_2 are induced by the metric on E_1. Finally, we assume that there exists a closed subscheme ϕ in X which is 0-dimensional and such that there is an exact sequence of sheaves on X

$$0 \to E_0 \to E_1 \to E_2 \otimes I_\phi \to 0\,, \tag{6}$$

where I_ϕ is the ideal of definition of ϕ.

Let

$$\widetilde{c}_2 \in A^{1,1}(X_{\mathbb{R}})/(\mathrm{Im}(\partial) + \mathrm{Im}(\overline{\partial}))$$

be the second Bott-Chern class of (6), $\Gamma(\phi, \mathcal{O}_\phi)$ the finite ring of functions on ϕ and $\#\,\Gamma(\phi, \mathcal{O}_\phi)$ its order. Let

$$f_* : \widehat{\mathrm{CH}}_0(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}_0(S) = \mathbb{R}$$

be the direct image morphism.

Theorem 2. *We have an equality of real numbers*

$$f_*(\widehat{c}_2(\overline{E}_1) \cap [X]) = f_*(\widehat{c}_1(\overline{E}_0)\,\widehat{c}_1(\overline{E}_2) \cap [X]) - \int_{X(\mathbb{C})} \widetilde{c}_2 + \log \#\,\Gamma(\phi, \mathcal{O}_\phi)\,.$$

2.2. To prove Theorem 2 we remark first that

$$\widehat{c}_1(\overline{E}_1) = \widehat{c}_1(\overline{E}_0) + \widehat{c}_1(\overline{E}_2)\,,$$

because the metrics on E_0 and E_2 are induced from $\overline{E}_1$. Therefore, since $\mathrm{ch}_2 = -c_2 + \frac{c_1^2}{2}$, we get

$$\begin{aligned} \widehat{\mathrm{ch}}_2(\overline{E}_1) &= -\widehat{c}_2(\overline{E}_1) + \frac{(\widehat{c}_1(\overline{E}_0) + \widehat{c}_1(\overline{E}_2))^2}{2} \\ &= -\widehat{c}_2(\overline{E}_1) + c_1(\overline{E}_0)\,\widehat{c}_1(\overline{E}_2) + \widehat{\mathrm{ch}}_2(\overline{E}_0) + \widehat{\mathrm{ch}}_2(\overline{E}_2)\,. \end{aligned}$$

By Theorem 1, this implies that

$$\widehat{c}_2(\overline{E}_1) \cap [X] = \widehat{c}_1(\overline{E}_0)\,\widehat{c}_1(\overline{E}_2) \cap [X] + b\,(\mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X]) + a\,(\widetilde{\mathrm{ch}}(\overline{E}_\bullet))\,. \tag{7}$$

Since $\widetilde{\mathrm{ch}}_0(\overline{E}_\bullet)$ and $\widetilde{\mathrm{ch}}_1(\overline{E}_\bullet)$ vanish we have

$$\widetilde{\mathrm{ch}}(\overline{E}_\bullet) = -\,\widetilde{c}_2\,.$$

Therefore, if we apply f_* to (7), we get

$$f_*(\widehat{c}_2(\overline{E}_1) \cap [X]) = f_*(\widehat{c}_1(\overline{E}_0)\,\widehat{c}_1(\overline{E}_2) \cap [X]) - \int_{X(\mathbb{C})} \widetilde{c}_2 + f_*(b(\mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X]))\,,$$

and we are left with showing that

$$f_* \circ b\,(\mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X]) = \log \#\, \Gamma(\phi, \mathcal{O}_\phi)\,. \tag{8}$$

Let $|\phi| = \{P_1, \cdots, P_n\} \subset X$ be the support of ϕ and $\psi = f(|\phi|) \subset S$. The following diagram is commutative:

$$\begin{array}{ccc} \mathrm{CH}_0(\phi) & \xrightarrow{\ b\ } & \widehat{\mathrm{CH}}_0(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{CH}_0(\psi) & \xrightarrow{\ b\ } & \widehat{\mathrm{CH}}_0(S) = \mathbb{R}\,, \end{array}$$

where

$$b : \mathrm{CH}_0(\psi) = \mathbb{Z}^\psi \to \mathbb{R}$$

maps $(n_p)_{p\in\psi}$ to $\sum\limits_p n_p \log(p)$.

For any prime $p \in \psi$ we let $\mathbb{Z}_{(p)}$ be the local ring of S at p and we let $\ell_p = \ell_p(\phi) \geq 0$ be the length of the finite $\mathbb{Z}_{(p)}$-module $\Gamma(\phi, \mathcal{O}_\phi) \otimes \mathbb{Z}_{(p)}$. Clearly

$$\log \#\, \Gamma(\phi, \mathcal{O}_\phi) = \sum_{p\in\psi} \ell_p \log(p)\,,$$

hence it is enough to prove that

$$f_*(\mathrm{ch}^{\mathrm{fin}}(E_\bullet) \cap [X]) = (\ell_p) \in \mathrm{CH}_0(\psi)_{\mathbb{Q}} = \mathbb{Q}^\psi\,. \tag{9}$$

The complex $E_\bullet$ defines an element

$$[E_\bullet] = \sum_{i=1}^n [\mathcal{O}_{\phi,P_i}] \in K_0^\phi(X) = \bigoplus_{i=1}^n K_0^{P_i}(X)\,.$$

To prove (9), by replacing X by an affine neighbourhood of P, one can assume that $|\phi| = \{P\}$, and it is enough to show that, if $p = f(P)$,

$$f_*(\mathrm{ch}^{\mathrm{fin}}(\mathcal{O}_{\phi,P}) \cap [X]) = \ell_p(\mathcal{O}_{\phi,P})[p]\,.$$

Now recall that, if $\mathcal{F}$ is a coherent sheaf on a scheme X of finite type over S, supported on a finite set of closed points, the associated 0-cycle

$$[\mathcal{F}] = \sum_{P\in|\mathcal{F}|} \ell_p(\mathcal{F}_P)[P] \in Z_0(X)$$

is such that, if $f : X \to Y$ is a proper morphism of schemes of finite type over S,

$$f_*[\mathcal{F}] = [f_*(\mathcal{F})]$$

([3] , 15.1.5). Hence it is enough to show that

$$\mathrm{ch}^P(\mathcal{O}_{\phi,P}) = \ell_p(\mathcal{O}_{\phi,P})[P] \in \mathrm{CH}_0(P)_{\mathbb{Q}} \simeq \mathbb{Q}\,. \tag{10}$$

Replacing X by an affine neighbourhood of P, we may assume that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X^2 \xrightarrow{\beta} \mathcal{O}_X \longrightarrow \mathcal{O}_\phi \longrightarrow 0 \,. \tag{11}$$

Hence the ideal $I_\phi \subset \mathcal{O}_X(X)$ is generated by two elements β_1 and β_2. Since X is normal, its local rings satisfy Serre property S_2 and, as $\dim(X) = 2$, X is Cohen-Macaulay. Since β_1 and β_2 span an ideal of height two, (β_1, β_2) is a regular sequence and the sequence (11) is isomorphic to the Koszul resolution of $\mathcal{O}_\phi = \mathcal{O}_X/(\beta_1, \beta_2)$. Now (10) can be deduced from the following general fact:

Lemma 1. *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme and $Z \subset X$ a closed subset such that the ideal $I_Z = (x_1, \ldots, x_n)$ is generated by a regular sequence $(x_1, \ldots, x_n)$. Let $K_\bullet(x_1, \ldots, x_n)$ be the Koszul complex associated to $(x_1, \ldots, x_n)$. Then*

$$\mathrm{ch}_n^Z(K_\bullet(x_1, \ldots, x_n)) = [\mathcal{O}_Z] \in \mathrm{CH}_0(Z)_{\mathbb{Q}} \,.$$

Proof. The Grassmannian-graph construction on $K_\bullet(x_1, \ldots, x_n)$ coincides with the deformation to the normal bundle of Z in X. If W is defined as in 1.4,

$$W_\infty = \widetilde{X} \cup \widehat{\mathbb{P}}(N_{Z/X}) \,,$$

where $\widetilde{X}$ is the blow up of X along Z, and $\widehat{P}(N_{Z/X})$ is the projective completion of the normal bundle of Z in X. The pull back of the Koszul complex $K_\bullet(x_1, \ldots, x_n)$ to $W \backslash W_\infty$ extends to a complex $\widetilde{K}_\bullet(x_1, \ldots, x_n)$ on W. The restriction of $\widetilde{K}_\bullet(x_1, \ldots, x_n)$ to $\widetilde{X}$ is acyclic while the restriction of $\widetilde{K}_\bullet(x_1, \ldots, x_n)$ to $\widehat{\mathbb{P}}(N_{Z/X})$ is a resolution of the structure sheaf of the zero section $Z \subset N_{Z/X} \subset \widehat{\mathbb{P}}(N_{Z/X})$.

Now observe that $Z \subset \widehat{\mathbb{P}}(N_{Z/X})$ is an intersection of Cartier divisors $D_1, \ldots, D_n$, hence

$$\begin{aligned} & \mathrm{ch}(\widetilde{K}_\bullet(x_1, \ldots, x_n) \mid_{\widehat{\mathbb{P}}(N_{Z/X})}) \\ = & \prod_{i=1}^n \mathrm{ch}(\mathcal{O}(-D_i) \to \mathcal{O}_{\widehat{\mathbb{P}}(N_{Z/X})}) \\ = & \prod_{i=1}^n \mathrm{ch}(\mathcal{O}(D_i)) \,. \end{aligned}$$

Since

$$\mathrm{ch}(\mathcal{O}_{D_i}) = \mathrm{ch}_1(\mathcal{O}_{D_i}) + x_i = [D_i] + x_i$$

where x_i has degree ≥ 2, we get

$$\mathrm{ch}(\widetilde{K}_\bullet(x_1, \ldots, x_n) \mid_{\widehat{\mathbb{P}}(N_{Z/X})}) = [D_1] \ldots [D_n] = [Z] \,.$$

This ends the proof of Lemma 1 and Theorem 2.

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