Iterated monotonic nonexpansive operators and asymptotic properties of zero-sum stochastic games.

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We consider an operator $\mathbf{\Psi}$ defined on a set of real valued functions and satisfying two properties of monotonicity and additive homogeneity. This is motivated by the case of zero sum stochastic games, for which the Shapley operator is monotone and additively homogeneous. We study the asymptotic of the trajectories defined by $v_n = \frac{\mathbf{\Psi}^n(0)}{n}$ $(n \in N, n \to \infty)$ and $v_\lambda = \lambda \mathbf{\Psi}\left(\frac{1-\lambda}{\lambda} v_\lambda\right)$ $(\lambda \in (0,1], \lambda \to 0)$. Examining the iterates of $\mathbf{\Psi}$, we exhibit analytical conditions on the operator that imply that v_n and v_λ have at most one accumulation point for the uniform norm. In particular this establishes the uniform convergence of v_n and v_λ to the same limit for a large subclass of the class of games where only one player control the transitions. We also study the general case of two players controlling the transitions, giving a sufficient condition for convergence.

Key words: zero-sum stochastic games ; asymptotic properties ; nonexpansive mappings ; Shapley operator ; dynamic programming

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1. Introduction. An important topic in the theory of two-player zero-sum repeated games is the asymptotic behavior of the values of finitely repeated (resp. discounted) games when the number of stage becomes large (resp. when the discount factor tends to 0). Three main questions in this framework are: existence of the limits, their equality, and their identification. Positive results have been obtained for different class of games, for example absorbing games [10], recursive games [6], games with incomplete information [1, 14], finite stochastic games [2, 3], and Markov chain games [17] but the original proofs in each case are specific.

In this paper we follow the operator approach based of the recursive structure of repeated games. This was used in [21] and [24] to give new proofs in the case of both absorbing games and games with incomplete information. The idea is to split the problem in two parts: on the first hand, one can derive from some characteristics of a repeated zero-sum game (*e.g.* number of states, topology of the sets of action, nature of the transition function) analytical properties of its Shapley operator. On the other hand, the asymptotic behavior of some trajectories defined by such an operator can be inferred by these analytical properties. Following an idea evoked in the appendix of [24], we extend methods of [21] to study iterates of Shapley operators.

Section 2 is devoted to the first definitions and results.

In section 3 we consider two class of stochastic games: the case where one player controls the transition; as well as the case of a bounded payoff function. We prove that the Shapley operator of a game in which one player controls the transition (resp. in which the payoff is bounded) satisfy a convexity inequality (resp. a Lipschitz condition). In the finite dimensional case, there are inverse properties: an operator satisfying this convex inequality (resp. this Lipschitz condition) is the Shapley operator of a game in which one player controls the transition (resp. in which the payoff is bounded).

In section 4 we establish that some properties of Shapley operators induce the same properties on their iterates.

In section 5 we use this to prove the convergence of v_n and v_λ for some particular operators, hence for some classes of games. We establish that for any game where only one player control the transitions, both family v_n and v_λ have at most one accumulation point for the uniform norm. In particular when the state space is precompact and when the v_n and v_λ are uniformly equicontinuous there is convergence of both families to the same limit.

Section 6 is an appendix consisting of technical results.

2. Definitions.

2.1 MHa Operators. Given a set Ω, define $\mathcal{F}_0$ as the set of bounded real functions on Ω endowed with the uniform norm $\|\cdot\|_\infty$, and with the usual partial order: $f_1 \le f_2$ if and only if $f(\omega) \le g(\omega)$ for all ω in Ω.

From now on $\mathcal{F}$ is a closed (hence complete) convex cone in $\mathcal{F}_0$ containing the constants. A mapping $\boldsymbol{\Psi}$ from $\mathcal{F}$ to itself is a MHa (Monotone Homogeneous additively) operator if it satisfies both property:

(M) Monotonicity : $f \le g \Longrightarrow \boldsymbol{\Psi}(f) \le \boldsymbol{\Psi}(g)$

(Ha) Homogeneous additivity $\boldsymbol{\Psi}(f + c) = \boldsymbol{\Psi}(f) + c$ for $c \in \mathbb{R}$

It is immediate to check that any MHa operator satisfies the following additional property:

(Ne) Nonexpansiveness $\|\boldsymbol{\Psi}(x) - \boldsymbol{\Psi}(y)\|_\infty \le \|x - y\|_\infty \ \forall (x, y) \in X^2$

We consider, for $n \in \mathbb{N}$ and $\lambda \in]0, 1]$, the following iterates and fixed points:

$$V_n = \boldsymbol{\Psi}(V_{n-1}) = \boldsymbol{\Psi}^n(0) \tag{1}$$
$$V_\lambda = \boldsymbol{\Psi}((1-\lambda)V_\lambda) \tag{2}$$

Notice that V_λ is well-defined since the nonexpansiveness of $\boldsymbol{\Psi}$ implies that $\boldsymbol{\Psi}((1-\lambda)\cdot)$ is strictly contracting on the complete set $\mathcal{F}$, hence has a unique fixed point

Example 2.1 *For any $c \in \mathbb{R}$, the mapping J from $\mathbb{R}$ to itself defined by $J(x) = x + c$ is nonexpansive. In that case, $V_n = nc$ and $V_\lambda = \frac{c}{\lambda}$.*

These quantities being unbounded in general (see previous example), we also introduce their normalized versions

$$v_n = \frac{V_n}{n} \tag{3}$$
$$v_\lambda = \lambda V_\lambda \tag{4}$$

In the previous example, one gets $v_n = v_\lambda = c$ for all n and λ. In general it is easy to prove that these normalized quantities are bounded:

Lemma 2.1 *For any nonexpansive operator $\boldsymbol{\Psi}$, sequences v_n and v_λ are bounded by $\|\boldsymbol{\Psi}(0)\|$.*

Proof. Since $\boldsymbol{\Psi}$ is non expansive,

$$\|V_n - V_{n-1}\| = \|\boldsymbol{\Psi}(V_{n-1}) - \boldsymbol{\Psi}(V_{n-2})\| \le \|V_{n-1} - V_{n-2}\|.$$

By induction this implies that

$$\|V_n\| \le n\|V_1\| = n\|\boldsymbol{\Psi}(0)\|.$$

On the other hand, again using the fact that $\boldsymbol{\Psi}$ is nonexpansive,

$$\begin{aligned} \|V_\lambda\| - \|\boldsymbol{\Psi}(0)\| &\le \|V_\lambda - \boldsymbol{\Psi}(0)\| \\ &= \|\boldsymbol{\Psi}((1-\lambda)V_\lambda) - \boldsymbol{\Psi}(0)\| \\ &\le (1-\lambda)\|V_\lambda\| \end{aligned}$$

and so

$$\|v_\lambda\| = \lambda\|V_\lambda\| \le \|\boldsymbol{\Psi}(0)\|.$$

□

A consequence is that the family v_n is of slow variation:

Lemma 2.2 *If $\boldsymbol{\Psi}$ satisfies (Ne), then $\|v_{n+1} - v_n\| = \mathrm{O}(1/n)$*

Proof.
$$\left\|\frac{\mathbf{\Psi}^{n+1}(0)}{n+1}-\frac{\mathbf{\Psi}^{n}(0)}{n}\right\|_\infty \leq \frac{\|\mathbf{\Psi}^{n+1}(0)\|_\infty}{n(n+1)}+\frac{1}{n}\|\mathbf{\Psi}^{n+1}(0)-\mathbf{\Psi}^{n}(0)\|_\infty \leq \frac{2}{n}\|\mathbf{\Psi}(0)\|_\infty$$

□

To point out the link between the families $\{v_n\}_{n\in\mathbb{N}}$ and $\{v_\lambda\}_{\lambda\in]0,1]}$ it is also of interest to introduce the family of strictly contracting operators $\mathbf{\Phi}(\alpha,\cdot)$, $\alpha\in]0,1]$, defined by

$$\mathbf{\Phi}(\alpha,x)=\alpha\mathbf{\Psi}\left(\frac{1-\alpha}{\alpha}x\right). \tag{5}$$

Then v_n and v_λ satisfy the relations

$$v_n = \mathbf{\Phi}\left(\frac{1}{n},v_{n-1}\right)\ ;\ v_0=0 \tag{6}$$
$$v_\lambda = \mathbf{\Phi}(\lambda,v_\lambda) \tag{7}$$

2.2 Shapley operators. A typical example of such an operator is obtained in the framework of zero-sum two player stochastic games[22]: Ω is a metric space of states, A and B are compact metric action sets for player 1 and 2 respectively, g is the continous payoff from $A\times B\times\Omega$ to $\mathbb{R}$, and ρ is the continuous transition probability from $A\times B\times\Omega$ to $\Delta_f(\Omega)$, the set of probabilities on Ω with finite support . Denote by $\mathcal{F}$ the set of continuous bounded real functions on Ω. We suppose in addition that for each $f\in\mathcal{F}$ and each $b\in B$ (resp. and each $a\in A$), $\int_\Omega f(\cdot)d\rho(\cdot|a,b,w)$ is jointly continuous on $A\times\Omega$ (resp. on $B\times\Omega$).

The game is played as follow: an initial stage $\omega_1\in\Omega$ is given, known by each player. At each stage m, knowing past history and current state ω_m, player 1 (resp. player 2) chooses $\sigma\in\Delta(A)$ (resp. $\tau\in\Delta(B)$). A move a_m of player 1 (resp. b_m of player 2) is drawn accordingly to σ (resp. τ). The payoff g_m at stage m is then $g(a_m,b_m,\omega_m)$ and ω_{m+1}, the state at stage $m+1$, is drawn accordingly to $\rho(a_m,b_m,\omega_m)$.

There are several ways of evaluating a payoff for a given infinite history:

(i)) $\frac{1}{n}\sum_{m=1}^{n} g_m$ is the payoff of the $n-$stage game

(ii)) $\lambda\sum_{m=1}^{+\infty}(1-\lambda)^{i-1}g_m$ is the payoff of the $\lambda-$discounted game.

For a given initial state ω, we denote the values of those games by $v_n(\omega)$ and $v_\lambda(\omega)$ respectively; v_n and v_λ are thus functions from Ω into $\mathbb{R}$.

The Shapley operator $\mathbf{\Psi}$ of this stochastic game is the mapping from $\mathcal{F}$ to itself defined by, for each $f\in\mathcal{F}$ and $\omega\in\Omega$,

$$\mathbf{\Psi}(f)(\omega) = \sup_{\sigma\in\Delta(A)}\inf_{\tau\in\Delta(B)}\left\{g(\sigma,\tau,\omega)+\int_\Omega f(\omega')d\rho(\omega'|\sigma,\tau,\omega)\right\} \tag{8}$$
$$= \inf_{\tau\in\Delta(B)}\sup_{\sigma\in\Delta(A)}\left\{g(\sigma,\tau,\omega)+\int_\Omega f(\omega')d\rho(\omega'|\sigma,\tau,\omega)\right\} \tag{9}$$

where we still denote by g and ρ the multilinear extensions from $A\times B$ to $\Delta(A)\times\Delta(B)$ of the corresponding function.

Such a Shapley operator is a MHA operator, and the value v_n of the n-stage game (resp. the value v_λ of the λ-discounted game) satisfy relation (6) (resp. (7)), see [23] for example.

This recursive structure holds in a wide class of games [15] ; more generally we will thus say that an operator from $\mathscr{F}$ to itself is a Shapley operator if there are two sets X and Y, and two functions $g:X\times Y\times\Omega\to\mathbb{R}$ and $\rho:X\times Y\times\Omega\to\Delta_f\Omega$ such that for any $f\in\mathscr{F}$ and $\omega\in\Omega$,

$$\mathbf{\Psi}(f)(\omega) = \sup_{\sigma\in X}\inf_{\tau\in Y}\left\{g(\sigma,\tau,\omega)+\int_\Omega f(\omega')d\rho(\omega'|\sigma,\tau,\omega)\right\} \tag{10}$$
$$= \inf_{\tau\in Y}\sup_{\sigma\in X}\left\{g(\sigma,\tau,\omega)+\int_\Omega f(\omega')d\rho(\omega'|\sigma,\tau,\omega)\right\}. \tag{11}$$

Notice that any such operator is MHa.

The asymptotic behavior of v_n (resp. v_λ) as n tends to $+\infty$ (resp. as λ tends to 0) is a major topic in game theory: one study properties of a stochastic game through a family of games with expected duration converging to infinity. This approach is also called the compact case since we can interpret v_n and v_λ as values of some time-discretizations of a game played in continuous time between time 0 and 1 [23]. In that framework letting n go to infinity or λ to 0 gives a finer and finer discretization of the continuous interval $[0, 1]$, hence it is natural to expect convergence of those values.

2.3 The finite dimension case. When Ω is finite with cardinal k, we identify it to $\{1, \cdots, k\}$ and $\mathcal{F}$ to $\mathbb{R}^k$. To simplify the notations in that case, we will write Δ_k instead of $\Delta_f(\Omega)$, and we will write the state as an index in $\mathbf{\Psi}$, g and ρ. For example formula (8) will be written as

$$\mathbf{\Psi}_i(x) = \sup_{\sigma\in\Delta(A)} \inf_{\tau\in\Delta(B)} \{g_i(\sigma, \tau) + \langle\rho_i(\sigma, \tau), x\rangle\} \tag{12}$$

where $\langle\ ,\ \rangle$ is the usual scalar product.

In that finite dimensional case MHa operators are known as topical operators and have been widely studied [7, 8, 9]. In particular it is known that properties (M) and (Ha) provide a characterization of Shapley operators [12]:

Proposition 2.1 *An operator $\mathbf{\Psi}$ from $\mathbb{R}^k$ to itself is the Shapley Operator of a k-states stochastic game if and only if it is a MHa operator.*

Even in that finite dimensional case, there are examples of nonexpansive operators [11] or MHa operators [9] such that neither v_n nor v_λ does converge.

3. Axiomatic Approach. The aim of this part is to link the characteristics of a game to some properties of its Shapley operator.

3.1 Property satisfied by Player 1-controlled games.

Definition 3.1 *A stochastic game is Player 1-controlled if the first player controls the transitions, that is if the transition function ρ does not depend on the action $v \in V$ of the second player.*

We will characterize Player 1-controlled games by introducing a new property of operators:

Definition 3.2 *An operator $\mathbf{\Psi}$ from $\mathcal{F}$ to itself satisfies property (C) if for every ω in Ω the function $\mathbf{\Psi}(\cdot)(\omega)$ is a convex mapping from $\mathcal{F}$ to $\mathbb{R}$: for any f_1, f_2 in $\mathcal{F}$ and $t \in [0, 1]$,*

$$\mathbf{\Psi}(tf_1 + (1 - t)f_2) \leq t\mathbf{\Psi}(f_1) + (1 - t)\mathbf{\Psi}(f_2)$$

A MHaC operator is an operator satisfynig properties (M) (Ha) and (C) ; we prove the following Proposition:

Proposition 3.1 *Any Shapley operator of a Player 1-controlled game is MHaC.*

Proof. Since $\mathbf{\Psi}$ is a Shapley operator, we already know that it is MHa. Since the first player controls the transitions, then for every $\omega \in \Omega$:

$$\begin{aligned}
\mathbf{\Psi}(tx+(1-t)y) &= \sup_{u\in\Delta(A)}\inf_{v\in\Delta(B)}\{g(u,v,\omega)+E_{\rho(u,\omega)}(tx+(1-t)y)\}\\
&= \sup_{u\in\Delta(A)}\left\{t\left[\inf_{v\in\Delta(B)}\{g(u,v,\omega)\}+E_{\rho(u,\omega)}x\right]\right.\\
&\quad \left.+\,(1-t)\left[\inf_{v\in\Delta(B)}\{g(u,v,\omega)\}+E_{\rho(u,\omega)}y\right]\right\}\\
&\leq t\sup_{u\in\Delta(A)}\inf_{v\in\Delta(B)}\{g(u,v,\omega)+E_{\rho(u,\omega)}x\}\\
&\quad +(1-t)\sup_{u\in\Delta(A)}\inf_{v\in\Delta(B)}\{g(u,v,\omega)+E_{\rho(u,\omega)}y\}\\
&= t\mathbf{\Psi}(x)+(1-t)\mathbf{\Psi}(y)
\end{aligned}$$

□

When Ω is finite the reverse holds:

PROPOSITION 3.2 *An operator $\mathbf{\Psi}$ from $\mathbb{R}^k$ to itself is the Shapley operator of a player 1-controlled game iff it is MHaC.*

PROOF. Let $\mathbf{\Psi}$ be a MHaC operator from $\mathbb{R}^k$ to itself, and let $\mathbf{\Psi}_i$ be the i-th coordinate of $\mathbf{\Psi}$. Let $\mathcal{D}\subset\mathbb{R}^k$ be the set on which every $\mathbf{\Psi}_i$ is differentiable and let $\mathcal{P}_i=\{\nabla\mathbf{\Psi}_i(x),x\in\mathcal{D}\}$. Since any MHa operator is nonexpansive, Rademacher's theorem implies that $\overline{\mathcal{D}}=\mathbb{R}^k$.

For any $f\in\mathcal{D}$, monotonicity of $\mathbf{\Psi}$ implies that

$$\mathbf{\Psi}_i(f+te_j)\geq\mathbf{\Psi}_i(f)$$

for all $1\leq i,j\leq k$ and $t>0$, hence that $\frac{\partial\Psi_i}{\partial j}(f)\geq 0$. Homogeneous additivity implies that for any $c\in\mathbb{R}$,

$$\mathbf{\Psi}_i(f+c)=\mathbf{\Psi}_i(f)+c$$

hence that $\sum_{j=1}^k\frac{\partial\Psi_i}{\partial j}(f)=1$, which prove that $\mathcal{P}_i\subset\Delta_k$ for every i.

Moreover, Property (C) implies that for any i,

$$\forall(x,y)\in\mathbb{R}^k\times\mathcal{D},\quad\langle\nabla\mathbf{\Psi}_i(y),x-y\rangle\leq\mathbf{\Psi}_i(x)-\mathbf{\Psi}_i(y). \tag{13}$$

For any $y\in\mathcal{D}$ let $g_i(y)=\mathbf{\Psi}_i(y)-\langle\nabla\mathbf{\Psi}_i(y),y\rangle\}$. Let us prove that

$$\mathbf{\Psi}_i(x)=\sup_{y\in\mathcal{D}}\{g_i(y)+\langle\nabla\mathbf{\Psi}_i(y),x\rangle\}. \tag{14}$$

Notice first that inequality (13) is

$$\mathbf{\Psi}_i(x)\geq\sup_{y\in\mathcal{D}}\{g_i(y)+\langle\nabla\mathbf{\Psi}_i(y),x\rangle\}.$$

On the other hand,

$$\begin{aligned}
\sup_{y\in\mathcal{D}}\{\mathbf{\Psi}_i(y)+\langle\nabla\mathbf{\Psi}_i(y),x-y\rangle\} &= \mathbf{\Psi}_i(x)+\sup_{y\in\mathcal{D}}\{\mathbf{\Psi}_i(y)-\mathbf{\Psi}_i(x)+\langle\nabla\mathbf{\Psi}_i(y),x-y\rangle\}\\
&\geq \mathbf{\Psi}_i(x)+\sup_{y\in\mathcal{D}}\{-2\|y-x\|_\infty\}
\end{aligned}$$

because $\mathbf{\Psi}$ is nonexpansive and $\mathbf{\Psi}_i(y)\in\Delta_k$. Since $\overline{\mathcal{D}}=\mathbb{R}^k$ this proves (14).

Equation (14) establish that $\mathbf{\Psi}$ is the Shapley operator of a Player 1-controlled game where the action set of the first player is $\mathcal{D}$, payoff is g, and transitions are given by $\rho_i(y)=\nabla\mathbf{\Psi}_i(y)$. □

3.2 Property satisfied by games with a bounded payoff. Let us introduce a new property of operators:

(L) Lipschitz $\exists K>0,\,\forall x\in\mathcal{F},\,\forall t,t'>0,\,\left\|t\mathbf{\Psi}\left(\frac{x}{t}\right)-t'\mathbf{\Psi}\left(\frac{x}{t'}\right)\right\|_\infty\leq K|t-t'|$

An operator is MHaL if it satisfies properties (M) (Ha) and (L) ; we show that assumption (L) characterizes Shapley operators of games with bounded payoff:

Lemma 3.1 *If $\mathbf{\Psi}$ is the Shapley operator of a game with bounded payoff, then $\mathbf{\Psi}$ is MHaL.*

Proof. It is enough to prove that if $\mathbf{\Psi}$ is a Shapley Operator it satisfies (L), which is a consequence of the fact that Val is nonexpansive for the supremum norm: if g_1 and g_2 are two functions from $A \times B$ to $\mathbb{R}$ then

$$\left| \underset{(\sigma,\tau)\in A\times B}{\mathrm{Val}} \{g_1(\sigma,\tau)\} - \underset{(\sigma,\tau)\in A\times B}{\mathrm{Val}} \{g_2(\sigma,\tau)\} \right| \leq \sup_{(\sigma,\tau)\in A\times B} |g_1(\sigma,\tau) - g_2(\sigma,\tau)|.$$

For any ω, applying this inequality to $g_1(\sigma,\tau) = tg(\sigma,\tau,\omega) + E_{p(\sigma,\tau,\omega)}x$ and $g_2(\sigma,\tau) = t'g(\sigma,\tau,\omega) + E_{p(\sigma,\tau,\omega)}x$ gives:

$$\left| t\mathbf{\Psi}\left(\frac{x}{t}\right)(\omega) - t'\mathbf{\Psi}\left(\frac{x}{t'}\right)(\omega) \right| \leq \sup_{\Delta(A)\times\Delta(B)} |g(\sigma,\tau,\omega)| \cdot |t - t'|$$

By hypothesis there is a $K > 0$ such that $|g(a,b,\omega)| \leq K$ for all $(a,b,\omega) \in U \times V \times \Omega$, so $\mathbf{\Psi}$ satisfies (L).

□

We now prove a weak version of the reverse of Lemma 3.1. First define:

Definition 3.3 *A stochastic game with a Shapley operator $\mathbf{\Psi}$ defined in (10) has a weakly bounded payoff if there exists $K > 0$ such that for every $\varepsilon > 0$, $\omega \in \Omega$ and $f \in \mathcal{F}$, there exists an action $u_\varepsilon \in X$ (resp. $v_\varepsilon \in Y$) of Player 1 (resp. Player 2) which is ε-optimal in $\mathbf{\Psi}(f)(\omega)$, and such that $|g(u_\varepsilon, v_\varepsilon, \omega)| \leq K$.*

In other terms, in a game with weakly bounded payoff there is a uniform bound K, such that for any state and for any evaluation of the future, there are almost optimal actions for each player which yields an expected payoff bounded by K.

With this new definition we can prove:

Proposition 3.3 *If $\mathbf{\Psi}$ is MHaL from $\mathbb{R}^k$ to itself, then $\mathbf{\Psi}$ is the Shapley operator of a game with weakly bounded payoff.*

We first prove a preliminary result. As in the proof of Proposition 3.2 define $\mathcal{D} \subset \mathbb{R}^k$ as the set on which every $\mathbf{\Psi}_i$ is differentiable. Then

Lemma 3.2 *If $\mathbf{\Psi} : \mathbb{R}^k \to \mathbb{R}^k$ is MHaL for a constant K and if $x \in \mathcal{D}$, then for any i:*

$$\|\mathbf{\Psi}_i(x) - \langle \nabla \mathbf{\Psi}_i(x), x\rangle\| \leq K$$

Proof. For any x and i, $t \to t\mathbf{\Psi}_i\left(\frac{x}{t}\right)$ is K-Lipschitz. If $x \in \mathcal{D}$, this function is differentiable at $t = 1$ and

$$\left| \left(t\mathbf{\Psi}_i\left(\frac{x}{t}\right)\right)'_{t=1} \right| \leq K$$

which implies the result. □

We now prove Proposition 3.3:

Proof of Proposition 3.3. Suppose $\mathbf{\Psi}$ MHaL for a constant K and let $1 \leq i \leq k$. For every a and b in $\mathcal{D}$ define $\Pi_i(a,b) = \int_0^1 \nabla \mathbf{\Psi}_i(ta + (1-t)b)dt$. $\mathbf{\Psi}_i$ being differentiable almost everywhere on any segment (because of the nonexpansiveness of $\mathbf{\Psi}$), it implies that Π_i is well defined as soon as $a \neq b$, and also when $a = b$ since we supposed that $u \in \mathcal{D}$.

Define $g_i(a,b) = \mathbf{\Psi}_i(a) - \langle \Pi_i(a,b), a\rangle$.

We prove that for any $x \in \mathbb{R}^k$

$$\mathbf{\Psi}_i(x) = \mathrm{Val}_{a\in\mathcal{D},b\in\mathcal{D}}\{g_i(a,b) + \langle \Pi_i(a,b), x\rangle\}$$

and that (a,a) is a couple of ε-optimal strategies as soon as $a \in \mathcal{D} \bigcap \mathcal{B}(x, \varepsilon/2)$.

Let $a \in \mathcal{D} \bigcap \mathcal{B}(x, \varepsilon/2)$ and $b \in \mathcal{D}$. We have

$$\begin{aligned} g_i(a,b) + \langle \Pi_i(a,b), x\rangle &= \mathbf{\Psi}_i(a) + \langle \Pi_i(a,b), x-a\rangle && (15)\\ &= \mathbf{\Psi}_i(x) + (\mathbf{\Psi}_i(a) - \mathbf{\Psi}_i(x)) + \langle \Pi_i(a,b), x-a\rangle && (16)\end{aligned}$$

so

$$\mathbf{\Psi}_i(x) - \varepsilon \leq g_i(a,b) + \langle \Pi_i(a,b), x\rangle \leq \mathbf{\Psi}_i(x) + \varepsilon.$$

On the other hand $\mathbf{\Psi}_i(x) - \mathbf{\Psi}_i(y) = \left\langle \int_0^1 \nabla \mathbf{\Psi}_i(tx + (1-t)y)dt, x-y \right\rangle$, so $g_i(x,y) = g_i(y,x)$, which implies

$$\mathbf{\Psi}_i(x) - \varepsilon \leq g_i(a,b) + \langle \Pi_i(a,b), x\rangle \leq \mathbf{\Psi}_i(x) + \varepsilon.$$

for any $u \in \mathcal{D}$ as soon as $b \in \mathcal{D} \bigcap \mathcal{B}(x, \varepsilon/2)$. The couple (a,a) is thus a couple of $\varepsilon-$optimal strategies.

This prove that $\mathbf{\Psi}$ is the Shapley operator of a game with weakly bounded payoff since $\overline{\mathcal{D}} = \mathbb{R}^k$ and $|g_i(a,a)| \leq K$ for every $a \in \mathcal{D}$ according to Lemma 3.2. □

Remark 3.1 *The interest of property (L) lies in the fact that one doesn't know if there are stochastic games with finite number of states and bounded payoff where the sequence v_n does not converge. By Lemma 3.1 and Proposition 3.3 this relates heavily to the study of existence of a cycle time for MHaL Operators defined on a finite dimensional space.*

3.3 Recession Operator. Let us define a new property of operators: we say that $\mathbf{\Psi}$ satisfies property (R) if $\mathbf{\Psi}$ has a recession operator $\mathbf{R}(\mathbf{\Psi})$, that is

$$\exists\, \mathbf{R}(\mathbf{\Psi}) : \mathcal{F} \to \mathcal{F},\ \forall x \in \mathcal{F},\ \lim_{t \to 0+} \left\| t\mathbf{\Psi}\left(\frac{x}{t}\right) - \mathbf{R}(\mathbf{\Psi})(x) \right\|_\infty = 0$$

Remark 3.2 *Since $\left\| t\mathbf{\Psi}\left(\frac{x}{t}\right) - \mathbf{\Phi}(t,x)\right\|_\infty \leq t\|x\|_\infty$, condition (R) is satisfied if and only if for any $x \in \mathscr{F}$, $\mathbf{\Phi}(\lambda, x)$ converges as λ goes to 0.*

In particular, the Shapley operator of a game with bounded payoff satisfies condition (R) and

$$\mathbf{R}(\mathbf{\Psi})(\cdot) = \mathbf{\Phi}(0, \cdot).$$

Remark 3.3 *The recession operator $\mathbf{R}(\mathbf{\Psi})$ is additively homogeneous: for any $c \in \mathbb{R}$ $\mathbf{R}(\mathbf{\Psi})(x+c) = \mathbf{R}(\mathbf{\Psi})(x) + c$. It is also positively multiplicatively homogeneous : for any $c \in \mathbb{R}^+$ $\mathbf{R}(\mathbf{\Psi})(cx) = c\mathbf{R}(\mathbf{\Psi})(x)$.*

An interest of the Recession operator lies in the following lemma

Lemma 3.3 *Let $\mathbf{\Psi}$ be a MHAR operator, then any accumulation point for the uniform norm of either $\{v_n\}$ or $\{v_\lambda\}$ is a fixed point of the recession operator $\mathbf{R}(\mathbf{\Psi})$.*

Proof. Recall that

$$\begin{aligned} v_n &= \mathbf{\Phi}\left(\frac{1}{n}, v_{n-1}\right)\\ v_\lambda &= \mathbf{\Phi}(\lambda, v_\lambda)\end{aligned}$$

so the result follow immediately from Remark 3.2, and Lemma 2.2 in the case of v_n. □

Remark 3.4 *The set of fixed point of $\mathbf{R}(\mathbf{\Psi})$ may however be large, for example in the case of games with incomplete information[21].*

To conclude this section, we prove that either (L), or (C) in the finite dimension case, is a sufficient condition for (R) to hold:

Lemma 3.4 *If $\mathbf{\Psi}$ is MHaC from $\mathbb{R}^k$ to itself, then $\mathbf{\Psi}$ satisfies (R).*

Proof. Given $x \in \mathbb{R}^k$, let $f(t) := t\mathbf{\Psi}\left(\frac{x}{t}\right) - t\mathbf{\Psi}(0)$ be defined from $\mathbb{R}^{+*}$ to $\mathbb{R}^k$. Since $\mathbf{\Psi}$ is nonexpansive, $\|f(t)\|_\infty \leq t\|x/t\|_\infty = \|x\|_\infty$, which implies that f is bounded. Moreover,

$$\begin{aligned}\|f(t) - f(t')\|_\infty &\leq |t - t'| \cdot \|\mathbf{\Psi}(0)\|_\infty + |t - t'| \cdot \left\|\mathbf{\Psi}\left(\frac{x}{t}\right)\right\|_\infty + t' \cdot \left\|\mathbf{\Psi}\left(\frac{x}{t}\right) - \mathbf{\Psi}\left(\frac{x}{t'}\right)\right\|_\infty \\ &\leq |t - t'| \cdot \left(\left\|\mathbf{\Psi}\left(\frac{x}{t}\right)\right\|_\infty + \|\mathbf{\Psi}(0)\|_\infty + \frac{\|x\|_\infty}{t}\right)\end{aligned}$$

so f is continuous. Finally, $\mathbf{\Psi}$ satisfies (C), which implies that for $t' < t$:

$$\begin{aligned}f(t) &= t\mathbf{\Psi}\left(\frac{x}{t}\right) - t\mathbf{\Psi}(0) \\ &= t\mathbf{\Psi}\left(\frac{t'}{t} \cdot \frac{x}{t'} + \left(1 - \frac{t'}{t}\right) \cdot 0\right) - t\mathbf{\Psi}(0) \\ &\leq t'\mathbf{\Psi}\left(\frac{x}{t'}\right) + (t - t')\mathbf{\Psi}(0) - t\mathbf{\Psi}(0) \\ &= f(t').\end{aligned}$$

We have established that each coordinate f_i of f is nonincreasing and bounded, it follows that each $f_i(t)$ converges as t goes to 0, which implies that $\mathbf{\Psi}$ satisfies (R). □

Lemma 3.5 *Any MHAL Operator satisfies property (R).*

Proof. Suppose that $\mathbf{\Psi}$ is MHaL and let $x \in \mathcal{F}$. Let h be the function defined on $]0,1[$ by $h(t) = t\mathbf{\Psi}(\frac{x}{t})$. Since $\mathbf{\Psi}$ satisfies (L), for any sequence t_n in $]0,1[$ converging to 0 the corresponding sequence $h(t_n)$ is Cauchy, hence converges. This prove that the function h converges as t goes to 0, which implies that $\mathbf{\Psi}$ satisfies (R). □

The following corollary follows immediately from lemmas 3.1 and 3.5:

Corollary 3.1 *If $\mathbf{\Psi}$ is the Shapley operator of a game with bounded pay off, $\mathbf{\Psi}$ is MHaR.*

4. Iterated Operators.

First let us define the lim sup and lim inf of sequences in $\mathscr{F}_0$:

Definition 4.1 *For any bounded sequence x_n in $\mathscr{F}_0$,*

$$\limsup(x_n)(\omega) = \limsup(x_n(\omega))$$

for every ω.

Notice that

$$x \leq \limsup x_n$$

for any accumulation point x of x_n for the uniform norm, but that the lim sup of a sequence is not necessarily an accumulation point itself for the uniform norm.

The following proposition, proved in [24], gives a sufficient condition for an element of $\mathscr{F}$ to be greater than both $\limsup v_n$ and $\limsup v_\lambda$:

Proposition 4.1 *Let $\mathbf{\Psi} : \mathcal{F} \to \mathcal{F}$ be a MHa operator.*

If $x \in \mathcal{F}$ satisfies $\mathbf{\Psi}(Lx) \leq (L+1)x$ for every L large enough, then $\limsup v_n \leq x$ and $\limsup v_\lambda \leq x$.

If $x \in \mathcal{F}$ satisfies $\mathbf{\Psi}(Lx) \geq (L+1)x$ for every L large enough, then $\limsup v_n \geq x$ and $\limsup v_\lambda \geq x$.

An interesting consequence is a sufficient condition for v_n and v_λ to converge: define

$$\begin{aligned}\mathscr{L}^+ &= \{x \in \mathscr{F}, \mathbf{\Psi}(Lx) \leq (L+1)x \text{ for all } L \text{ large enough}\} \\ \mathscr{L}^- &= \{x \in \mathscr{F}, \mathbf{\Psi}(Lx) \geq (L+1)x \text{ for all } L \text{ large enough}\}.\end{aligned}$$

Then if the intersection $\overline{\mathscr{L}^+} \cap \overline{\mathscr{L}^+}$ is nonempty, it is a singleton $\{v\}$ and $v = \lim v_n = \lim v_\lambda$.

However, there are examples of simple games where Proposition 4.1 does not apply immediately. Consider the following 0-player game (meaning that each set action is reduced to one element): there are two states ω_0 and ω_1 with payoff 0 and 1 respectively, and a deterministic transition from each state to the other :

$$\begin{aligned} g(\omega_0) &= 0 \\ g(\omega_1) &= 1 \\ \rho(\omega_1|\omega_0) &= 1 \\ \rho(\omega_0|\omega_1) &= 1 \end{aligned}$$

The Shapley operator of such a game is given by

$$\mathbf{\Psi}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a+1 \end{pmatrix}$$

and one verifies that

$$\begin{aligned} v_{2n} &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ v_{2n+1} &= \begin{pmatrix} \frac{n}{2n+1} \\ \frac{n+1}{2n+1} \end{pmatrix} \\ v_\lambda &= \begin{pmatrix} \frac{1-\lambda}{2-\lambda} \\ \frac{1}{2-\lambda} \end{pmatrix} \end{aligned}$$

hence both v_n and v_λ converges to $\left(\frac{1}{2}, \frac{1}{2}\right)$. In that case one checks that

$$\begin{aligned} \mathscr{L}^+ &= \{(x,x),\ x \geq 1\} \\ \mathscr{L}^- &= \{(x,x),\ x \leq 0\} \end{aligned}$$

hence the intersection is empty. However, let us consider the game played by blocks of two stages. Its Shapley operator is given by:

$$\widetilde{\mathbf{\Psi}}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+1 \\ b+1 \end{pmatrix}$$

and Proposition 4.1 immediately imply that in this new game both v_n and v_λ converge to $(1,1)$, hence that in the initial game both v_n and v_λ converge to $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Given a stochastic game Γ with a Shapley operator $\mathbf{\Psi}$ it is thus worthwhile to study the game Γ^m played by blocks of stages of a fixed length m, with payoff at stage l equals to $\sum_{k=1}^m g_{lm+k}$. This leads to a new Shapley operator equals to $\mathbf{\Psi}^m$. Since the payoff function in Γ^m is not bounded as m goes to $+\infty$, it is convenient to also introduce the game Γ_m, with payoff at stage l equals to $\frac{1}{m}\sum_{k=1}^m g_{lm+k}$. The Shapley operator of this normalized game is given by $x \to \frac{1}{m}\mathbf{\Psi}^m(mx)$.

This motivates us to define, for a general MHA operator:

Definition 4.2 *If $\mathbf{\Psi}$ is an MHa operator and $m \in \mathbb{N}^*$, we define operators $\mathbf{\Psi}^m(\cdot)$ $\mathbf{\Phi}^m(\lambda,\cdot)$, $\mathbf{\Psi}_m(\cdot)$ and $\mathbf{\Phi}_m(\lambda,\cdot)$by:*

$$\mathbf{\Psi}^m(x) = \mathbf{\Psi}\left(\mathbf{\Psi}^{m-1}(x)\right) \tag{17}$$

$$\mathbf{\Phi}^m(\lambda,x) = \mathbf{\Phi}(\lambda, \mathbf{\Phi}^{m-1}(\lambda,x)) \tag{18}$$

$$\mathbf{\Psi}_m(x) = \frac{1}{m}\mathbf{\Psi}^m(mx) \tag{19}$$

$$\mathbf{\Phi}_m(\lambda,x) = \lambda\mathbf{\Psi}_m\left(\frac{1-\lambda}{\lambda}x\right) \tag{20}$$

$$v_{n,m} = \frac{(\mathbf{\Psi}_m)^n(0)}{n} \tag{21}$$

$\mathbf{\Phi}_m(\lambda,\cdot)$ being $1-\lambda$ contracting, one can also define

$$v_{\lambda,m} = \mathbf{\Phi}_m\left(\lambda, v_{\lambda,m}\right) \tag{22}$$

In the two following propositions, we show that if an operator $\mathbf{\Psi}$ satisfies some properties, it is also the case for operators $\mathbf{\Psi}^m$ and $\mathbf{\Psi}_m$:

Lemma 4.1 *If $\mathbf{\Psi}$ is MHa, then so does $\mathbf{\Psi}^m$ for any $m \in \mathbb{N}$. Moreover,*

a) *If $\mathbf{\Psi}$ is MHaC, so does $\mathbf{\Psi}^m$.*

b) *If $\mathbf{\Psi}$ is MHaR, so does $\mathbf{\Psi}^m$ and $\mathbf{R}(\mathbf{\Psi}^m) = (\mathbf{R}(\mathbf{\Psi}))^m$*

c) *If $\mathbf{\Psi}$ is MHaL for a constant K, so does $\mathbf{\Psi}^m$ for the constant mK.*

Proof. If it easy to check that if $\mathbf{\Psi}$ is MHa, then $\mathbf{\Psi}^m$ is also MHa.

To prove part a) of the lemma, we proceed by induction on m: if we assume that $\mathbf{\Psi}^m$ satisfies (C), then for any $t \in [0,1]$,

$$\begin{aligned}\mathbf{\Psi}^{m+1}(tx+(1-t)y) &= \mathbf{\Psi}(\mathbf{\Psi}^m(tx+(1-t)y)) \\ &\leq \mathbf{\Psi}(t\mathbf{\Psi}^m(x)+(1-t)\mathbf{\Psi}^m(y)) \\ &\leq t\mathbf{\Psi}^{m+1}(x)+(1-t)\mathbf{\Psi}^{m+1}(y))\end{aligned}$$

For assertion b) we also proceed by induction on m: assume that $\mathbf{\Psi}^m$ satisfies (R) and $\mathbf{R}(\mathbf{\Psi}^m) = (\mathbf{R}(\mathbf{\Psi}))^m$ and fix x in $\mathcal{F}$. Since $\mathbf{\Psi}^m$ satisfies (Ne), it follows that

$$\left\| t\mathbf{\Psi}^{m+1}\left(\frac{x}{t}\right) - t\mathbf{\Psi}^m\left(\frac{\mathbf{R}(\mathbf{\Psi})(x)}{t}\right)\right\|_\infty \leq \left\| t\mathbf{\Psi}\left(\frac{x}{t}\right) - \mathbf{R}(\mathbf{\Psi})(x)\right\|_\infty. \tag{23}$$

Since $\mathbf{\Psi}$ satisfies (R), the righthand member of (23) goes to 0 as t converges to 0. The induction hypothesis implies that

$$\left\| t\mathbf{\Psi}^m\left(\frac{\mathbf{R}(\mathbf{\Psi})(x)}{t}\right) - \mathbf{R}(\mathbf{\Psi})^{m+1}(x)\right\|_\infty$$

goes to 0 as well, and so we have established the convergence to 0 of

$$\left\| t\mathbf{\Psi}^{m+1}\left(\frac{x}{t}\right) - \mathbf{R}(\mathbf{\Psi})^{m+1}(x)\right\|_\infty$$

as required.

For part c) we proceed again by induction: assume the property true for m. Notice that

$$\begin{aligned}t\mathbf{\Psi}^{m+1}\left(\frac{x}{t}\right) - t'\mathbf{\Psi}^{m+1}\left(\frac{x}{t'}\right) &= \left[t\mathbf{\Psi}^{m+1}\left(\frac{x}{t}\right) - t'\mathbf{\Psi}^m\left(\frac{t\mathbf{\Psi}\left(\frac{x}{t}\right)}{t'}\right)\right] \\ &+ \left[t'\mathbf{\Psi}^m\left(\frac{t\mathbf{\Psi}\left(\frac{x}{t}\right)}{t'}\right) - t'\mathbf{\Psi}^{m+1}\left(\frac{x}{t'}\right)\right]\end{aligned}$$

By induction hypothesis,

$$\begin{aligned}\left\| t\mathbf{\Psi}^{m+1}\left(\frac{x}{t}\right) - t'\mathbf{\Psi}^m\left(\frac{t\mathbf{\Psi}\left(\frac{x}{t}\right)}{t'}\right)\right\|_\infty &= \left\| t\mathbf{\Psi}^m\left(\frac{t\mathbf{\Psi}\left(\frac{x}{t}\right)}{t}\right) - t'\mathbf{\Psi}^m\left(\frac{t\mathbf{\Psi}\left(\frac{x}{t}\right)}{t'}\right)\right\|_\infty \\ &\leq mK|t-t'|\end{aligned}$$

On the other hand, $\mathbf{\Psi}$ being nonexpansive,

$$\begin{aligned}\left\| t'\mathbf{\Psi}^m\left(\frac{t\mathbf{\Psi}\left(\frac{x}{t}\right)}{t'}\right) - t'\mathbf{\Psi}^{m+1}\left(\frac{x}{t'}\right)\right\|_\infty &\leq t'\left\|\frac{t\mathbf{\Psi}\left(\frac{x}{t}\right)}{t'} - \mathbf{\Psi}\left(\frac{x}{t'}\right)\right\|_\infty \\ &= \left\| t\mathbf{\Psi}\left(\frac{x}{t}\right) - t'\mathbf{\Psi}\left(\frac{x}{t'}\right)\right\|_\infty \\ &\leq K|t-t'|\end{aligned}$$

and we deduce that

$$\left\| t\mathbf{\Psi}^{m+1}\left(\frac{x}{t}\right) - t'\mathbf{\Psi}^{m+1}\left(\frac{x}{t'}\right)\right\|_\infty \leq (m+1)K|t-t'|.$$

$\square$

Proposition 4.2 *If $\mathbf{\Psi}$ is MHa, then so does $\mathbf{\Psi}_m$ for any $m \in \mathbb{N}$. Moreover,*

a) *If $\mathbf{\Psi}$ is MHaC, so does $\mathbf{\Psi}_m$.*

b) *If $\mathbf{\Psi}$ is MHaR, so does $\mathbf{\Psi}_m$.*

c) *If $\mathbf{\Psi}$ is MHaL, so does $\mathbf{\Psi}_m$ for the same constant.*

Proof. It follows directly from Lemma 4.1 (and for the second assertion, from the observation that any recession operator is multiplicatively homogeneous). □

The operator $\mathbf{\Psi}_m$ is a more natural operator to consider than $\mathbf{\Psi}^m$ for several reasons: first notice that property (L) is preserved for the same constant for operator $\mathbf{\Psi}_m$, but not for $\mathbf{\Psi}^m$. Another point is that in the case of a Shapley operator, the payoff of $\mathbf{\Psi}^m$ is given by a sum of the payoffs in m stages of the original game, hence $\mathbf{\Psi}^m$ gives a weight m to the present and 1 to the future. The operator $\mathbf{\Psi}_m$, which can be seen as a normalized version of $\mathbf{\Psi}^m$, is thus a more adequate description of the game played by blocks of m stages. This is emphasized by two following propositions:

Proposition 4.3 *Let $\mathbf{\Psi}$ be any MHa operator. Then the equality $v_{n,m} = v_{nm}$ holds for any $m \in \mathbb{N}^*$ and $n \in \mathbb{N}$. In particular, for any $m \in \mathbb{N}^*$,*

$$\limsup_{n\to+\infty} v_{n,m} = \limsup_{n\to+\infty} v_n.$$

Proof. To prove the first part of the proposition, just verify that

$$v_{n,m} = \frac{(\mathbf{\Psi}_m)^n(0)}{n} = \frac{1}{m}\frac{\mathbf{\Psi}^{mn}(0)}{n} = v_{nm}.$$

In particular, for any m,

$$\limsup_{n\to+\infty} v_{n,m} = \limsup_{n\to+\infty} v_{nm}.$$

Lemma 2.2 implies that

$$\limsup_{n\to+\infty} v_{nm} = \limsup_{n\to+\infty} v_n$$

hence the second part of the proposition. □

Proposition 4.4 *Let $\mathbf{\Psi}$ be an MHAL operator. Then for any $m \in \mathbb{N}^*$,*

$$\limsup_{\lambda\to 0} v_{\lambda,m} = \limsup_{\lambda\to 0} v_\lambda.$$

The proof of this proposition is technical and postponed to the appendix. The idea is that the operator $\mathbf{\Phi}^m(\lambda,\cdot)$ and $\mathbf{\Phi}_m(\mu,\cdot)$ are close to each other for $\mu = 1-(1-\lambda)^m$.

Take for example the case of a Shapley operator and $m = 2$. Then $\mathbf{\Phi}^2(\lambda,\cdot)(\omega_1)$ is the value of the one-shot game with payoff

$$\lambda g(a_1,b_1,\omega_1) + \lambda(1-\lambda)g(a_2,b_2,\omega_2) + (1-\lambda)^2 f(a_3,b_3,\omega_3)$$

while $\mathbf{\Phi}_2(2\lambda-\lambda^2,\cdot)(\omega_1)$ is the value of the one-shot game with payoff

$$\left(\lambda - \frac{\lambda^2}{2}\right) g(a_1,b_1,\omega_1) + \left(\lambda - \frac{\lambda^2}{2}\right) g(a_2,b_2,\omega_2) + (1-\lambda)^2 f(a_3,b_3,\omega_3).$$

Remark 4.1 *It is unclear whether Proposition 4.4 still holds for an operator which does not satisfy condition (L).*

5. Convergence of v_n and v_λ - Operator Approach.

5.1 MHaC operators : the case of v_n. This section study the asymptotics of v_n for MHAC operators. We prove the following proposition:

PROPOSITION 5.1 *If $\mathbf{\Psi}$ is MHaCR then v_n has at most one accumulation point for the uniform norm.*

which implies the following corollaries:

COROLLARY 5.1 *If $\mathbf{\Psi}$ is MHaC from $\mathbb{R}^k$ to itself then v_n converges.*

PROOF. Since the space is finite dimensional, any $MHaC$ operator satisfies also condition (R) by Lemma 3.4 ; and the bounded sequence v_n has at least one accumulation point. The result thus follows from Proposition 5.1. □

COROLLARY 5.2 *Suppose that a Player 1-controlled zero-sum stochastic game with a Shapley operator $\mathbf{\Psi}$ satisfies the two following hypotheses:*

(i) *its payoff g is bounded*

(ii) *the set $\{v_n,\ n \in \mathbb{N}\}$ is relatively compact in $\mathscr{F}$ for the uniform topology.*

Then the sequence v_n converges uniformly as n tends to infinity.

PROOF. The Shapley operator of such a game is MHaC by Proposition 3.1, and since the payoff g is bounded Lemma 3.1 implies that it satisfies also (R). So, by Proposition 5.1, v_n has at most one accumulation point, towards which it converges since the set $\{v_n,\ n \in \mathbb{N}\}$ is relatively compact. □

Notice that Corollary 5.2 and Ascoli's Theorem imply in particular

COROLLARY 5.3 *Suppose that a Player 1-controlled zero-sum stochastic game with a Shapley operator $\mathbf{\Psi}$ satisfies the two following hypotheses:*

(i) *its payoff g is bounded*

(ii) *the state space Ω is endowed with a distance d such that (Ω, d) is precompact and such that the sequence v_n is uniformly equicontinuous.*

Then the sequence v_n converges uniformly as n tends to infinity.

REMARK 5.1 *Classes of games that satisfy the assumptions of Corollary 5.3 include repeated games with incomplete information and standard signalling [1], as well as Markov chain games [17] and stochastic games with incomplete information on one side where the informed player controls the transitions [20].*

The remainder of this section is devoted to the proof of Proposition 5.1, for which we will need a basic lemma:

LEMMA 5.1 *If $\mathbf{\Psi}$ satisfies (C) et (R), then for every x and y,*

$$\mathbf{\Psi}(x+y) \leq \mathbf{\Psi}(x) + \mathbf{R}\,(\mathbf{\Psi})(y).$$

PROOF. Since $\mathbf{\Psi}$ satisfies (C), then for any $t \in]0,1]$,

$$\mathbf{\Psi}(y+(1-t)x) = \mathbf{\Psi}\left(t\frac{y}{t} + (1-t)x\right) \leq t\mathbf{\Psi}\left(\frac{y}{t}\right) + (1-t)\mathbf{\Psi}(x)$$

which gives as t tends to 0

$$\mathbf{\Psi}(x+y) \leq \mathbf{\Psi}(x) + \mathbf{R}\,(\mathbf{\Psi})(y)$$

since we assumed that $\mathbf{\Psi}$ satisfies (R).

□

Remark 5.2 *We have thus*

$$\mathbf{\Psi}(x+y)-\mathbf{\Psi}(x)\le y$$

as soon as $y=\mathbf{R}(\mathbf{\Psi})(y)$. *This gives insight on each coordinate and is stronger than just using the nonexpansive inequality*

$$\|\mathbf{\Psi}(x+y)-\mathbf{\Psi}(x)\|_\infty\le\|y\|_\infty$$

We now prove Proposition 5.1 :

Proof of Proposition 5.1. Let $\mathbf{\Psi}$ be an operator MHaCR and v an accumulation point of $\{v_n\}$. The aim of this proof is to get the inequality

$$\limsup_{n\to+\infty} v_n\le v.$$

Since the reverse inequality is true for any accumulation point v, this will establish that v is the only accumulation point of $\{v_n\}$.

Recall that $v=\mathbf{R}(\mathbf{\Psi})(v)$ by Lemma 3.3. Let $\varepsilon>0$ and m such that $\|v_m-v\|_\infty\le\varepsilon$. The operator $\mathbf{\Psi}_m$ is MHaCR according to Proposition 4.2, so for any $L>0$ one can apply Lemma 5.1 to $\mathbf{\Psi}_m$ and $x=L(v+\varepsilon)$. This gives:

$$\mathbf{\Psi}_m(L(v+\varepsilon))\quad\le\quad\mathbf{R}(\mathbf{\Psi}_m)(L(v+\varepsilon))+\mathbf{\Psi}_m(0)$$

According to Proposition 4.2, $\mathbf{R}(\mathbf{\Psi}_m)=(\mathbf{R}(\mathbf{\Psi}))^m$. Moreover, Remark 3.3 implies that $\mathbf{R}(\mathbf{\Psi})(L(v+\varepsilon))=L(v+\varepsilon)$. Since $\mathbf{\Psi}_m(0)=v_m\le v+\varepsilon$ we thus deduce that

$$\mathbf{\Psi}_m(L(v+\varepsilon))\le(L+1)(v+\varepsilon)$$

Apply Proposition 4.1 to $\mathbf{\Psi}_m$ and $v+\varepsilon$:

$$\limsup_{n\to+\infty} v_{n,m}\le(v+\varepsilon)$$

hence, by Proposition 4.3,

$$\limsup_{n\to+\infty} v_n\le v+\varepsilon.$$

This inequality holds for any $\varepsilon>0$, which implies the result. □

Remark 5.3 *In the case of a finite number of states (Corollory 5.1), convergence of* v_n *can be proved in the same way with an hypothesis slightly weaker than C[8].*

5.2 MHaC operators : the case of v_λ. In this section we prove the analogous of Proposition 5.1 for the family $\{v_\lambda\}$:

Proposition 5.2 *If* $\mathbf{\Psi}$ *is MHaCL then* $\{v_\lambda\}$ *has at most one accumulation point for the uniform topology.*

This proposition has the following interesting corollary:

Corollary 5.4 *Suppose that a Player 1-controlled zero-sum stochastic game with a Shapley operator* $\mathbf{\Psi}$ *satisfies the two following hypotheses:*

(i) *its payoff* g *is bounded*

(ii) *the set* $\{v_\lambda,\ \lambda\in]0,1]\}$ *is relatively compact.*

Then the sequence v_λ *converges uniformly as* λ *tends to 0.*

Proof. The Shapley operator of such a game is MHaC by Lemma 3.1, and since the payoff g is bounded Lemma 3.1 implies that it satisfies also (L). So, by Proposition 5.2, v_λ has at most one accumulation point, towards which it converges since the set $\{v_\lambda,\ \lambda\in]0,1]\}$ is relatively compact. □

By Ascoli's Theorem we also have

Corollary 5.5 *Suppose that a Player 1-controlled zero-sum stochastic game with a Shapley operator* $\mathbf{\Psi}$ *satisfies the two following hypotheses:*

(i) *its payoff g is bounded*

(ii) *the state space Ω is endowed with a distance d such that (Ω, d) is precompact and such that the sequence v_λ is uniformly equicontinuous.*

Then the sequence v_λ converges uniformly as λ tends to 0.

As in the previous section, to prove Proposition 5.2 we study iterations of the operator. We use the following consequence of Proposition 4.1:

Lemma 5.2 *Assume that* $\mathbf{\Psi}$ *is MHaCR,* $\mathbf{R}(\mathbf{\Psi})(x) \le x$, *and* $\mathbf{\Phi}(\lambda_0, x) \le x$ *for some* $\lambda_0 > 0$. *Then* $\limsup v_\lambda \le x$.

Proof. According to Proposition 4.1, it is enough to show that $\mathbf{\Phi}(\lambda, x) \le x$ for every $\lambda \le \lambda_0$, and according to Lemma 5.1,

$$\begin{aligned}
\mathbf{\Phi}(\lambda, x) &= \lambda\mathbf{\Psi}\left(\left(\frac{1}{\lambda} - 1\right)x\right) && (24)\\
&= \lambda\mathbf{\Psi}\left(\left(\frac{1}{\lambda_0} - 1\right)x + \left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)x\right) && (25)\\
&\le \lambda\mathbf{\Psi}\left(\left(\frac{1}{\lambda_0} - 1\right)x\right) + \lambda\mathbf{R}(\mathbf{\Psi})\left(\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)x\right) && (26)\\
&= \frac{\lambda}{\lambda_0}\mathbf{\Phi}(\lambda_0, x) + \left(1 - \frac{\lambda}{\lambda_0}\right)\mathbf{R}(\mathbf{\Psi})(x) && (27)\\
&\le \frac{\lambda}{\lambda_0}x + \left(1 - \frac{\lambda}{\lambda_0}\right)x && (28)\\
&= x. && (29)
\end{aligned}$$

□

We can now prove Proposition 5.2 :

Proof of Proposition 5.2. Let v be an accumulation point of v_λ, which implies $v \le \limsup v_\lambda$. It is thus enough to prove that v satisfies $\limsup v_\lambda \le v$.

Fix m in $\mathbb{N}^*$ and $\lambda > 0$, and denote

$$\begin{aligned}
\mu &= 1 - (1-\lambda)^m\\
v' &= v + \frac{(2-\mu)\|v - v_\lambda\|_\infty + K(m\lambda - \mu)}{\mu}
\end{aligned}$$

where K is the constant of property (L). We want to apply Lemma 5.2 to $\mathbf{\Psi}_m$, $x = v'$ and $\lambda_0 = \mu$; we first verify that the hypotheses of this lemma are satisfied. The operator $\mathbf{\Psi}_m$ is MHaCR by Proposition 4.2. According to Lemma 3.3, $\mathbf{R}(\mathbf{\Psi})(v) = v$ so $\mathbf{R}(\mathbf{\Psi}_m)(v) = v$, which implies by additive homogeneity that $\mathbf{R}(\mathbf{\Psi}_m)(v') = v'$. It remains to show that $\mathbf{\Phi}_m(\mu, v') \le v'$. First we find an upper bound of $\mathbf{\Phi}^m(\lambda, v')$:

$$\begin{aligned}
\mathbf{\Phi}^m(\lambda, v') &= \mathbf{\Phi}^m\left(\lambda, v - \|v - v_\lambda\|_\infty + \frac{2\|v - v_\lambda\|_\infty + K(m\lambda - \mu)}{\mu}\right)\\
&\le \mathbf{\Phi}^m\left(\lambda, v_\lambda + \frac{2\|v - v_\lambda\|_\infty + K(m\lambda - \mu)}{\mu}\right)\\
&= v_\lambda + (1-\lambda)^m\frac{2\|v - v_\lambda\|_\infty + K(m\lambda - \mu)}{\mu}\\
&\le v + \|v - v_\lambda\|_\infty + (1-\mu)\frac{2\|v - v_\lambda\|_\infty + K(m\lambda - \mu)}{\mu}\\
&= v' - K(m\lambda - \mu)
\end{aligned}$$

According to Corollary A.1, $\mathbf{\Phi}_m(\mu, v') \leq \mathbf{\Phi}^m(\lambda, v') + K(m\lambda - \mu)$, so we deduce that indeed $\mathbf{\Phi}_m(\mu, v') \leq v'$.

We can thus apply Lemma 5.2 which implies that

$$\limsup_{\gamma\to 0} v_{\gamma,m} \leq v'$$

and so, according to Lemma 4.4, we have established that for every $m \in \mathbb{N}^*$ and $\lambda > 0$,

$$\begin{aligned}\limsup_{\gamma\to 0} v_\gamma &\leq v' \\ &\leq v + \frac{2\|v - v_\lambda\|_\infty + K(m\lambda - 1 + (1-\lambda)^m)}{1 - (1-\lambda)^m}\end{aligned} \tag{30}$$

Fix ε and $\lambda > 0$, and choose $m = \left[\frac{\varepsilon}{\lambda}\right]$ in (30). Since $(1-\lambda)^{\left[\frac{\varepsilon}{\lambda}\right]}$ converges to $e^{-\varepsilon}$ as λ tends to 0, we deduce that for λ small enough,

$$\limsup_{\gamma\to 0} v_\gamma \leq v + 2\frac{2\|v - v_\lambda\|_\infty + K(\varepsilon + e^{-\varepsilon} - 1)}{1 - e^{-\varepsilon}}$$

Since v is an accumulation point of the family v_λ, this implies by letting λ go to 0 that

$$\limsup_{\gamma\to 0} v_\gamma \leq v + 2K\frac{\varepsilon + e^{-\varepsilon} - 1}{1 - e^{-\varepsilon}}. \tag{31}$$

Letting ε tend to 0 in (31) finally prove

$$\limsup_{\gamma\to 0} v_\gamma \leq v$$

□

Remark 5.4 *The proof of Propositions 5.1 also establish that if v_n converges uniformly and $\mathbf{\Psi}$ is MHACR, then $\lim_{n\to+\infty} v_n \geq \limsup_{\lambda\to 0} v_\lambda$.*
Similarly, the proof of Proposition 5.2 shows that if v_λ converges uniformly and $\mathbf{\Psi}$ is MHACL, then $\lim_{\lambda\to 0} v_\lambda \geq \limsup_{n\to+\infty} v_n$.

In the particular framework of dynamic programming, it is already known [13] that uniform convergence of either v_n or v_λ implies uniform convergence of the other to the same limit.

Remark 5.5 *In the special case of dynamic programming with a finite number of states, existence of the uniform value was proven in[4] for a finite action space. It was then established for compact action spaces and continuous payoff and transition functions[5, 23], and finally without any assumptions [19]. It was also extended to the case of two player games with an informed controller [18].*

In the framework of dynamic programming with a general state space one can also find an alternative proof of Corollary 5.2 in [19], as well as sufficient topological conditions for the existence of the uniform value. For similar results in a continuous-time framework, see [16]

5.3 A criterion for 2-player games Define a new property of operators:

(D) Derivability $\exists\, \varphi : \mathcal{F} \to \mathcal{F}, \forall x \in \mathcal{F}$,

$$\lim_{\lambda\to 0+} \left\| \frac{\mathbf{\Phi}(\lambda, x) - \mathbf{R}(\mathbf{\Psi})(x)}{\lambda} - \varphi(x) \right\|_\infty = 0.$$

This property of operators is interesting since it is proved in [21] that :

Lemma 5.3 *Let Γ be a stochastic game with a finite state space, compact actions spaces and continuous payoff and transition functions. Then its Shapley operator $\mathbf{\Psi}$ is MHaDL.*

In the case of MHaDL operators, Lemma 4.1 can be reformulated as :

Lemma 5.4 *Suppose* $\mathbf{\Psi}$ *MHaDL, and let* x *be such that* $\mathbf{R}(\mathbf{\Psi})(x) \leq x$ *and* $\varphi(x) \leq 0$. *Then* $\limsup v_n \leq x$ *and* $\limsup v_\lambda \leq x$.

We do not know yet how to use Lemma 5.4 to prove that for stochastic games satisfying the hypotheses of 5.3, both v_n and v_λ converge to the same limit.

However, let us introduce a new property slightly stronger than (D):

(Du) Uniform derivability $\forall m \in \mathbb{N}, \exists\, \varphi_m : \mathcal{F} \to \mathcal{F}, \forall x \in \mathcal{F}$,

$$\lim_{\lambda \to 0+} \left\| \frac{\mathbf{\Phi}_m(\lambda, x) - \mathbf{R}(\mathbf{\Psi}_m)(x)}{\lambda} - \varphi_m(x) \right\|_\infty = 0 \text{ uniformly in } m.$$

Note that we do not assume that the convergence is uniform in x.

We can now use Lemma 5.4 to prove:

Proposition 5.3 *Assume that* $\mathbf{\Psi}$ *is MHaDuL, then* v_λ *has at most one accumulation point for the uniform topology. Moreover, if* v_λ *converges uniformly, then so does* v_n *to the same limit.*

Proof. Let $\mathbf{\Psi}$ be MHaDuL and v an accumulation point of the family v_λ, which satisfies $\mathbf{R}(\mathbf{\Psi})(v) = v$ by Lemma 3.3. We prove that 0 is an accumulation point of the family $\varphi_m(v)$. By property (Du) there is a function f from $]0,1]$ to $\mathbb{R}^+$, nondecreasing and converging to 0 as t goes to 0, such that for any m and μ,

$$\left\| \frac{\mathbf{\Phi}_m(\mu, v) - v}{\mu} - \varphi_m(v) \right\|_\infty \leq f(\mu) \tag{32}$$

On the other hand, contractiveness implies that for any λ,

$$\begin{aligned} \|\mathbf{\Phi}^m(\lambda, v) - v\|_\infty &\leq \|\mathbf{\Phi}^m(\lambda, v) - v_\lambda\|_\infty + \|v - v_\lambda\|_\infty & (33)\\ &= \|\mathbf{\Phi}^m(\lambda, v) - \mathbf{\Phi}^m(\lambda, v_\lambda)\|_\infty + \|v - v_\lambda\|_\infty & (34)\\ &\leq (1-\lambda)^m \|v - v_\lambda\|_\infty + \|v - v_\lambda\|_\infty & (35)\\ &\leq 2\|v - v_\lambda\|_\infty. & (36) \end{aligned}$$

Combining inequalities (32) and (36) as well as Corollary A.1 one obtains that for any λ and m,

$$\|\varphi_m(v)\|_\infty \leq \frac{2\|v - v_\lambda\|_\infty + K(m\lambda - \mu)}{\mu} + f(\mu) \tag{37}$$

where $\mu = 1 - (1-\lambda)^m$.

Let $\varepsilon > 0$, and choose $m = \left[\frac{\varepsilon}{\lambda}\right]$ in (37). Since $(1-\lambda)^{\left[\frac{\varepsilon}{\lambda}\right]}$ tends to $e^{-\varepsilon}$ as λ goes to 0, we deduce that for every $\varepsilon > 0$, and every λ small enough,

$$\left\|\varphi_{\left[\frac{\varepsilon}{\lambda}\right]}(v)\right\|_\infty \leq \frac{4\|v - v_\lambda\|_\infty + 2K(\varepsilon + e^{-\varepsilon} - 1)}{1 - e^{-\varepsilon}} + f(1 - e^{-\varepsilon}). \tag{38}$$

Let $\varepsilon' > 0$; since $f(t)$ tends to 0 as t goes to 0 one can choose $\varepsilon > 0$ such that

$$f(1 - e^{-\varepsilon}) \leq \frac{\varepsilon'}{3} \tag{39}$$

and

$$2K\frac{(\varepsilon + e^{-\varepsilon} - 1)}{1 - e^{-\varepsilon}} \leq \frac{\varepsilon'}{3}. \tag{40}$$

Such ε being fixed, since v is an accumulation point of the family v_λ one can find $\lambda > 0$ such that

$$\frac{4\|v - v_\lambda\|_\infty}{1 - e^{-\varepsilon}} \leq \frac{\varepsilon'}{3}. \tag{41}$$

Inequalities (38) to (41) thus implies that for any $\varepsilon' > 0$, one can find an m such that $\|\varphi_m(v)\|_\infty \le \varepsilon'$. Such an m being fixed, denote $v' = v + \varepsilon'$. Since $\mathbf{R}(\mathbf{\Psi})(v') = v'$ and $\varphi_m(v') = \varphi_m(v) - \varepsilon' \le 0$, according to Lemma 5.4, we deduce that

$$\begin{aligned}\limsup_{\lambda\to 0} v_{\lambda,m} &\le v' \\ \limsup_{n\to+\infty} v_{n,m} &\le v'.\end{aligned}$$

According to Lemmas 4.4 and 4.3, we thus have

$$\begin{aligned}\limsup_{\lambda\to 0} v_{\lambda} &\le v + \varepsilon' \\ \limsup_{n\to+\infty} v_{n} &\le v + \varepsilon'.\end{aligned}$$

This is true for any $\varepsilon' > 0$, so

$$\begin{aligned}\limsup_{\lambda\to 0} v_{\lambda} &\le v \\ \limsup_{n\to+\infty} v_{n} &\le v.\end{aligned}$$

In the same way one prove that

$$\begin{aligned}\liminf_{\lambda\to 0} v_{\lambda} &\ge v \\ \liminf_{n\to+\infty} v_{n} &\ge v\end{aligned}$$

and we have proved the strong convergence of v_λ and v_n to v. □

In the same vein one can prove

PROPOSITION 5.4 *Assume that $\mathbf{\Psi}$ is MHaDu, and suppose that the sequence v_n satisfies*

$$v_{n+1} - v_n = o\left(\frac{1}{n}\right).$$

Then v_n has at most one accumulation point for the uniform topology. Moreover, if v_n converges uniformly, then so does v_λ to the same limit.

PROOF. Let v be an accumulation point of v_n ; it satisfies $\mathbf{R}(\mathbf{\Psi})(v) = v$ by Lemma 3.3. By property (Du) there is a nonincreasing function h from $\mathbb{R}^+$ to itself such that $\lim_{L\to+\infty} h(L) = 0$ and that for every $m \in \mathbb{N}^*$

$$\|\mathbf{\Psi}_m(Lv) - (L+1)v - \varphi_m(v)\|_\infty \le h(L). \tag{42}$$

Let $\varepsilon > 0$, and $L > 1$ such that $h(L) \le \varepsilon$. By hypothesis, there exists $N \in \mathbb{N}^*$ such that

$$n \ge N \implies \|v_{n+1} - v_n\|_\infty \le \frac{\varepsilon}{n}. \tag{43}$$

Moreover, since v is an accumulation point of the sequence v_n, there exist $n \ge N$ such that

$$\|v_n - v\|_\infty \le \frac{\varepsilon}{L+1} \tag{44}$$

Denote $m = \left[\frac{n}{L}\right]$, $L' = \frac{n}{m}$ and observe that $L \le L' \le L\frac{m+1}{m} \le 2L$.

By (42),

$$\begin{aligned}\|\varphi_m(v)\|_\infty &\le h(L') + \|\mathbf{\Psi}_m(L'v) - (L'+1)v\|_\infty \\ &\le h(L) + \frac{1}{m}\|\mathbf{\Psi}^m(mL'v) - m(L'+1)v\|_\infty \\ &\le \varepsilon + \frac{1}{m}\left\|\mathbf{\Psi}^m(mL'v) - \mathbf{\Psi}^{m(L'+1)}(0)\right\|_\infty + (L'+1)\|v_{n+m} - v\|_\infty \\ &\le \varepsilon + \frac{1}{m}\left\|mL'v - \mathbf{\Psi}^{mL'(0)}\right\|_\infty + (L'+1)\|v_n - v\|_\infty + (L'+1)\|v_{n+m} - v_n\|_\infty \\ &\le \varepsilon + (2L'+1)\|v_n - v\|_\infty + (L'+1)\|v_{n+m} - v_n\|_\infty.\end{aligned}$$

Using (43) and (44), this implies

$$\begin{aligned} \|\varphi_m(v)\|_\infty &\le \varepsilon + \frac{2L'+1}{L+1}\varepsilon + \frac{(L'+1)m}{n}\varepsilon \\ &\le \varepsilon + \frac{4L+1}{L+1}\varepsilon + \frac{(L'+1)}{L'}\varepsilon \\ &\le 7\varepsilon. \end{aligned}$$

We conclude as in the proof of Proposition 5.3. □

REMARK 5.6 *While it is true that* $v_{n+1} - v_n = O\left(\frac{1}{n}\right)$ *for any operator MHa (see Lemma 2.2), it is unclear whether the stronger hypothesis of Proposition 5.4 is always satisfied.*

Appendix A. Appendix. This section is devoted to the proof of Proposition 4.4. The first lemma relates the operators $\mathbf{\Psi}^m$ and $\mathbf{\Phi}^m$:

LEMMA A.1 *If* $\mathbf{\Psi}$ *is MHaL for a constant* K*, then for any* $t > 0$*,* $\lambda \in]0,1[$*,* $m \in \mathbb{N}^*$ *and* $x \in \mathcal{F}$ *the following inequality holds:*

$$\left\| \mathbf{\Psi}^m\left(\frac{x}{t}\right) - \frac{\mathbf{\Phi}^m(\lambda,x)}{t(1-\lambda)^m} \right\|_\infty \le K \sum_{i=1}^m \left| 1 - \frac{\lambda}{t(1-\lambda)^i} \right|$$

PROOF. Proceed by induction on m. For $m = 1$, $\mathbf{\Psi}$ satisfies (L) so

$$\left\| t\mathbf{\Psi}\left(\frac{x}{t}\right) - \frac{\lambda}{1-\lambda}\mathbf{\Psi}\left(\frac{1-\lambda}{\lambda}x\right) \right\|_\infty \le K\left| t - \frac{\lambda}{1-\lambda} \right|$$

so dividing by t,

$$\left\| \mathbf{\Psi}\left(\frac{x}{t}\right) - \frac{\mathbf{\Phi}(\lambda,x)}{t(1-\lambda)} \right\|_\infty \le K\left| 1 - \frac{\lambda}{t(1-\lambda)} \right|. \tag{45}$$

Assume the result holds for $m-1$, that is

$$\left\| \mathbf{\Psi}^{m-1}\left(\frac{x}{t}\right) - \frac{\mathbf{\Phi}^{m-1}(\lambda,x)}{t(1-\lambda)^{m-1}} \right\|_\infty \le K \sum_{i=1}^{m-1} \left| 1 - \frac{\lambda}{t(1-\lambda)^i} \right|$$

which implies, $\mathbf{\Psi}$ being nonexpansive, that

$$\left\| \mathbf{\Psi}^m\left(\frac{x}{t}\right) - \mathbf{\Psi}\left(\frac{\mathbf{\Phi}^{m-1}(\lambda,x)}{t(1-\lambda)^{m-1}}\right) \right\|_\infty \le K \sum_{i=1}^{m-1} \left| 1 - \frac{\lambda}{t(1-\lambda)^i} \right|$$

On the other hand, $\mathbf{\Psi}$ satisfies (L), thus one can apply equation (45) with $x' = \mathbf{\Phi}^{m-1}(\lambda,x)$ and $t' = t(1-\lambda)^{m-1}$, which gives

$$\left\| \mathbf{\Psi}\left(\frac{\mathbf{\Phi}^{m-1}(\lambda,x)}{t(1-\lambda)^{m-1}}\right) - \frac{\mathbf{\Phi}(\lambda,\mathbf{\Phi}^{m-1}(\lambda,x))}{t(1-\lambda)^{m-1}(1-\lambda)} \right\|_\infty \le K\left| 1 - \frac{\lambda}{t(1-\lambda)^{m-1}(1-\lambda)} \right|$$

Triangular inequality implies that

$$\begin{aligned} \left\| \mathbf{\Psi}^m\left(\frac{x}{t}\right) - \frac{\mathbf{\Phi}^m(\lambda,x)}{t(1-\lambda)^m} \right\|_\infty &\le K \sum_{i=1}^{m-1} \left| 1 - \frac{\lambda}{t(1-\lambda)^i} \right| + K\left| 1 - \frac{\lambda}{t(1-\lambda)^m} \right| \\ &= K \sum_{i=1}^{m} \left| 1 - \frac{\lambda}{t(1-\lambda)^i} \right|. \end{aligned}$$

□

A consequence of this lemma is the following comparison between the operators $\mathbf{\Phi}_m$ and $\mathbf{\Phi}^m$:

COROLLARY A.1 *If* $\mathbf{\Psi}$ *is MHaL for a constant* K*, then for any* $\lambda \in]0,1[$*,* $m \in \mathbb{N}^*$ *and* $x \in \mathcal{F}$ *the following inequality holds:*

$$\|\mathbf{\Phi}_m(\mu,x) - \mathbf{\Phi}^m(\lambda,x)\|_\infty \le K(m\lambda - \mu)$$

where $\mu = 1 - (1-\lambda)^m$.

PROOF. Applying Proposition A.1 with $t = \frac{\mu}{m(1-\mu)}$ gives

$$\left\| \mathbf{\Psi}^m \left(\frac{1-\mu}{\mu} mx \right) - \frac{m(1-\mu)\mathbf{\Phi}^m(\lambda, x)}{\mu(1-\lambda)^m} \right\|_\infty \leq K \sum_{i=1}^{m} \left| 1 - \frac{m\lambda(1-\mu)}{\mu(1-\lambda)^i} \right|$$

Multiplying by $\frac{\mu}{m}$,

$$\left\| \mathbf{\Phi}_m(\mu, x) - \frac{(1-\mu)\mathbf{\Phi}^m(\lambda, x)}{(1-\lambda)^m} \right\|_\infty \leq K \sum_{i=1}^{m} \left| \frac{\mu}{m} - \frac{\lambda(1-\mu)}{(1-\lambda)^i} \right|$$

and thus, since $1 - \mu = (1-\lambda)^m$,

$$\|\mathbf{\Phi}_m(\mu, x) - \mathbf{\Phi}^m(\lambda, x)\|_\infty \leq K \sum_{i=1}^{m} \left| \frac{\mu}{m} - \lambda(1-\lambda)^{i-1} \right|.$$

Next, use the fact that for any i between 1 and m, $(1-\lambda)^{m-1} \leq (1-\lambda)^{i-1} \leq 1$, which implies

$$\sum_{i=1}^{m} \left| \frac{\mu}{m} - \lambda(1-\lambda)^{i-1} \right| \leq \max\left(|\mu - m\lambda|, |\mu - m\lambda(1-\lambda)^{m-1}| \right).$$

It remains to establish that $0 \leq \mu - m\lambda(1-\lambda)^{m-1} \leq m\lambda - \mu$. The first inequality is equivalent to

$$(1 + (m-1)\lambda)(1-\lambda)^{m-1} \leq 1$$

which can be shown by taking the logarithmic derivative of the left-hand side and noticing that it is negative on [0,1].

The second inequality can be written as

$$m\lambda + (2 + (m-2)\lambda)(1-\lambda)^{m-1} \geq 2$$

Both sides are equal for $\lambda = 0$, so it is enough to show that the derivative of the left-hand side

$$m \left[1 - (1 + (m-2)\lambda)(1-\lambda)^{m-2} \right]$$

is nonnegative. This can be shown by taking the logarithmic derivative of $(1 + (m-2)\lambda)(1-\lambda)^{m-2}$ and noticing that it is negative on [0,1].

□

We can finally prove Proposition 4.4:

PROOF OF PROPOSITION 4.4. Fix $m \in \mathbb{N}^*$ and $\lambda \in]0, 1]$, denote $\mu = 1 - (1-\lambda)^m$ and notice that, since $\mathbf{\Phi}_m(\mu, \cdot)$ is $1 - \mu$ contracting,

$$\begin{aligned} \mu\|v_\lambda - v_{\mu,m}\|_\infty &\leq \|v_\lambda - v_{\mu,m}\|_\infty - \|\mathbf{\Phi}_m(\mu, v_\lambda) - \mathbf{\Phi}_m(\mu, v_{\mu,m})\|_\infty \\ &\leq \|\mathbf{\Phi}_m(\mu, v_\lambda) - v_\lambda\|_\infty \\ &= \|\mathbf{\Phi}_m(\mu, v_\lambda) - \mathbf{\Phi}^m(\lambda, v_\lambda)\|_\infty. \end{aligned}$$

Corollary A.1 thus implies that

$$\|v_\lambda - v_{\mu,m}\|_\infty \leq K \left(\frac{m\lambda}{\mu} - 1 \right).$$

For a fixed m, notice that $\mu \sim m\lambda$ as λ goes to 0, which implies that $\|v_\lambda - v_{\mu,m}\|_\infty$ converges to 0 as λ goes to 0, so a fortiori $\limsup v_{\lambda,m} = \limsup v_\lambda$. □

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