Optimal control of the convergence rate of Global-in-time Schwarz algorithms

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Abstract In this study we present a *global-in-time non-overlapping Schwarz method* applied to the one dimensional unsteady diffusion equation. We derive efficient interface conditions using an *optimal control* approach once the problem is discretized. Those conditions are compared to the usual optimized conditions derived at the PDE level by solving a *min-max problem.*

1 Introduction

Schwarz-like domain decomposition methods are very popular in mathematics, computational sciences and engineering notably for the implementation of coupling strategies. They are based on a separation of a given original problem into subproblems easier to solve. The connection between subproblems is done iteratively with information exchange leading, at convergence, to the solution of the original problem. In order to accelerate the convergence speed of the iterative process it has been originally suggested by Tan and Borsboom [1994] to design the interface conditions by solving an optimization problem related to the convergence rate of the method. There are mainly

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two ways to optimize the convergence speed: either by trying to find the best local approximation of the absorbing boundary conditions, or by adding relaxation parameters in the transmission conditions between the subproblems (*e.g.* Hadjidimos et al. [2000]).

In this study we specifically address the optimization problem arising from the use of *Robin* type transmission conditions in the framework of a *global-in-time Schwarz method* (sometimes called *Schwarz waveform relaxation*). For this type of problem the existing work has been achieved mainly at the *PDE level* [Gander et al., 1999, Gander and Halpern, 2007]. The aim here is to use the *optimal control theory* paradigm [Lions, 1968] to find parameters optimized at the *discrete level* and thus to systematically make a comparison with the parameters determined at the PDE level. This paper is organized as follows : in section 2 we briefly recall the basics of optimized Schwarz methods in the framework of a time evolution problem. Section 3 is dedicated to the determination of the *optimal control problem* that we intend to address. Finally in section 4 we apply our approach to a diffusion problem.

2 Optimization of the convergence at the PDE level

2.1 Model problem and Optimized Schwarz Methods

Let us consider a bounded domain $\Omega \subset \mathcal{R}^n$. The problem is to find u such that u satisfies

$$\mathcal{L}u = f \qquad \text{in } \Omega \times [0,T] \tag{1}$$

$$\mathcal{B}u = g \qquad \text{on } \partial\Omega \times [0,T] \tag{2}$$

where $\mathcal{L}$ and $\mathcal{B}$ are two partial differential operators. This problem is complemented by an initial condition

$$u(x,0) = u_0(x) \qquad x \in \Omega \tag{3}$$

We consider a splitting of our domain Ω into two *non-overlapping domains* Ω_1 and Ω_2 communicating through their common interface Γ. The operator $\mathcal{L}$ introduced previously is split into two operators $\mathcal{L}_j$ restricted to Ω_j ($j = 1, 2$). By noting $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{G}_1$ and $\mathcal{G}_2$ the operators defining the interface conditions, the multiplicative form of the *Global-in-time Schwarz algorithm* reads

$$\begin{cases} \mathcal{L}_1 u_1^k = f_1 & \text{in } \Omega_1 \times [0,T] \\ u_1^k(x,0) = u_o(x) & x \in \Omega_1 \\ \mathcal{B}_1 u_1^k(x,t) = g_1 & \text{in } [0,T] \times \partial\Omega_1 \\ \mathcal{F}_1 u_1^k(0,t) = \mathcal{F}_2 u_2^{k-1}(0,t) & \text{in } \Gamma \times [0,T] \end{cases} \quad \begin{cases} \mathcal{L}_2 u_2^k = f_2 & \text{in } \Omega_2 \times [0,T] \\ u_2^k(x,0) = u_o(x) & x \in \Omega_2 \\ \mathcal{B}_2 u_2^k(x,t) = g_2 & \text{in } [0,T] \times \partial\Omega_2 \\ \mathcal{G}_2 u_2^k(0,t) = \mathcal{G}_1 u_1^k(0,t) & \text{in } \Gamma \times [0,T] \end{cases} \tag{4}$$

where the initial guess $u_2^0(0,t)$ must be given. The operators $\mathcal{F}_j$ and $\mathcal{G}_j$ must be chosen to impose the desired consistency of the solution on the interface Γ. We consider here the one-dimensional diffusion equation with constant (possibly discontinuous) diffusion coefficients $\kappa_j (\kappa_j > 0, j = 1, 2)$. We thus define $\mathcal{L}_j = \partial_t - \kappa_j \partial_x^2$ and $\Gamma = \{x = 0\}$. In this context we require the equality of the subproblems solution and of their normal fluxes on the interface Γ. To obtain such a consistency we use mixed boundary conditions of *Robin* type

$$\mathcal{F}_j = \kappa_j \partial_x + p_1 \qquad \mathcal{G}_j = \kappa_j \partial_x + p_2 \qquad (j = 1, 2) \tag{5}$$

where p_1 and p_2 are two parameters that can be optimally chosen to improve the convergence speed of algorithm (4).

2.2 Optimization of the convergence factor

To demonstrate the convergence of algorithm (4) a classical approach is to define the error e_j^k between the exact solution $u^\star$ and the iterates u_j^k. A Fourier analysis enables the transformation of the original PDEs into ODEs that can be solved analytically. The analytical solution on each subdomain is then used to define a convergence factor ρ of the corresponding *Schwarz algorithm*. For a diffusion problem, defined on subdomains of infinite size, one finds

$$\rho(p_1, p_2, \omega) = \left| \frac{(p_2 - \sqrt{i\omega\kappa_2})}{(p_2 + \sqrt{i\omega\kappa_1})} \frac{(p_1 - \sqrt{i\omega\kappa_1})}{(p_1 + \sqrt{i\omega\kappa_2})} \right| \tag{6}$$

where p_1 and p_2 are two degrees of freedom that can be tuned to accelerate the convergence speed. A general approach to choose the Robin parameter is to solve a minimax problem [Gander et al., 1999]

$$\min_{p_1, p_2 \in \mathcal{R}} \left(\max_{\omega \in [\omega_{\min}, \omega_{\max}]} \rho(p_1, p_2, \omega) \right) \tag{7}$$

For the diffusion problem under consideration here, the analytical solution of the optimization problem (7) has been derived in Lemarié et al. [2010] in a general *two-sided* case (i.e. with $p_1 \neq p_2$) with discontinuous coefficients $\kappa_1 \neq \kappa_2$. For the sake of simplicity, we consider in the present study the continuous case ($\kappa_1 = \kappa_2 = \kappa$) and we recall the result found in Lemarié et al. [2010] in this case.

Theorem 1. *Under the assumption $\kappa_1 = \kappa_2 = \kappa$, the optimal parameters $p_1^\star$ and $p_2^\star$ of the minmax problem (7) are given by*

$$p_1^\star = \frac{\alpha\sqrt{2\kappa}}{4}\left[\sqrt{8+v^2} - v\right] \qquad p_2^\star = \frac{\alpha\sqrt{2\kappa}}{4}\left[\sqrt{8+v^2} + v\right]$$

where $\alpha = (\omega_{\min}\omega_{\max})^{1/4}$, $\beta = \alpha^{-1}(\sqrt{\omega_{\min}} + \sqrt{\omega_{\max}})$ and

$$v = \begin{cases} 2\sqrt{\beta - 1} & \text{if } \beta \geq 1+\sqrt{5} \\ \sqrt{2\beta^2 - 12} & \text{if } \sqrt{6} \leq \beta < 1+\sqrt{5} \\ 0 & \text{if } 2 < \beta < \sqrt{6} \end{cases}$$

Even if the diffusion coefficients are continuous the *two-sided* case provides a faster convergence than the *one-sided* case studied by Gander and Halpern [2003] (Fig. 1).

General remarks :

- The usual methodology to optimize the convergence at the continuous level comes with a few assumptions that may lead to inaccuracies once the problem is discretized. For example, as discussed in Lemarié et al. [2008] (Sec. 5), the *infinite domain assumption* used to determine the convergence factor (6) may lead to appreciable differences in the optimized parameters compared to an approach taking the finiteness of the subdomains into account. We numerically found that the *infinite domain assumption* is valid as long as the *dimensionless Fourier number* $\text{Fo} = \kappa_j/(L_j^2\omega)$ (with L_j the size of subdomain Ω_j) of the problem does not exceed a critical value $\text{Fo}_c = 0.02$.
- The optimization problem (7) aims at minimizing the maximum value of $\rho(p_1, p_2, \omega)$ over the entire interval $[\omega_{\min}, \omega_{\max}]$. This provides a very robust method general enough to deal with the worst case scenario when all the temporal frequencies are present in the error. An even more efficient way to proceed would be to adjust the values of p_1 and p_2 at each iteration so that those parameters are efficiently chosen to "fight" the remaining frequencies in the error.

3 Optimal control of the *Robin* parameters

To investigate the robustness of the optimized parameters once the problem is discretized, the use of the *optimal control theory* appears as a natural choice. We aim at controlling the *Robin* parameter in order to get the best possible convergence speed in the sense of a given cost function $\mathcal{J}$. Moreover, following

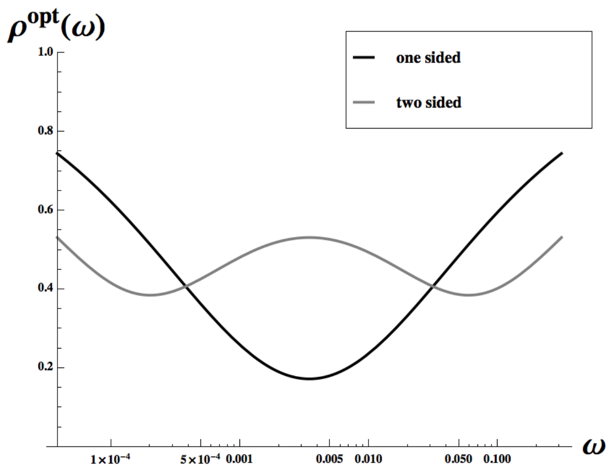

Fig. 1 Convergence factor optimized at the PDE level in the *two-sided* [Lemarié et al., 2010] and *one-sided* case [Gander and Halpern, 2003].

the approach of Gander and Golub [2003] and the previous discussion, we consider the possibility to use different parameters p_j for different steps of the iterative process. It is easy to check that by choosing different parameters at each iteration we still converge to the solution of the global problem. A first way to choose the parameters is to look, at each iteration k, for p_1^k and p_2^k minimizing the error at the interface. In this case the cost function that we intend to minimize at each iteration would be

$$\begin{aligned}\mathcal{J}(p_1^k, p_2^k) = \frac{1}{2}\int_0^T \left(u_1^k(0,t) - u_2^k(0,t)\right)^2 dt \\ + \frac{\mathcal{W}}{2}\int_0^T \left(\kappa_1 \partial_x u_1^k(0,t) - \kappa_2 \partial_x u_2^k(0,t)\right)^2 dt\end{aligned} \tag{8}$$

The constant $\mathcal{W}$ must be chosen to balance both terms. The cost function (8) is designed in agreement with the consistency we want to impose at the interface between subdomains. An other strategy could be to minimize the error at a given iteration K. The cost function would thus be

$$\begin{aligned}\mathcal{J}\left((p_1^k, p_2^k)_{k=1,K}\right) = \frac{1}{2}\int_0^T \left(u_1^K(0,t) - u_2^K(0,t)\right)^2 dt \\ + \frac{\mathcal{W}}{2}\int_0^T \left(\kappa_1 \partial_x u_1^K(0,t) - \kappa_2 \partial_x u_2^K(0,t)\right)^2 dt\end{aligned} \tag{9}$$

leading to an optimization on $2K$ parameters. This latter approach is particularly interesting when we intend to obtain the best possible approximation of the exact solution after a number of iterations set in advance. We propose here to lead our study with this kind of approach with $K = 5$.

The *optimal control* approach does not *per se* reduces the computational cost of the algorithm because many evaluation of the cost function are required during the minimization process. We use this approach as a tool to improve our understanding of the behaviour of the Robin parameters in order to find new directions to further accelerate the convergence speed when Robin-type interface conditions are used. We denote by $\mathbf{p_1^{\star,num}}$ and $\mathbf{p_2^{\star,num}}$ the parameters found numerically by solving the optimal control problem. Those parameters correspond to two vectors of size K. Similarly we will denote by $\mathrm{p}_1^{\star,\mathrm{ana}}$ and $\mathrm{p}_2^{\star,\mathrm{ana}}$ the parameters found analytically (cf Theorem 1).

4 Numerical experiments

We discretized problem (4) using a *backward Euler* scheme in time and a second order scheme defined on a staggered grid in space. We decompose the domain Ω into two non-overlapping subdomains $\Omega_1 = [-500\mathrm{m}, 0]$ and $\Omega_2 = [0, 500\mathrm{m}]$. The diffusion coefficient is $\kappa = 10^{-2}\mathrm{m}^2\ \mathrm{s}^{-1}$ and the total simulation time is $T = 81920\mathrm{s}$ with $\Delta t = 10\mathrm{s}$. The parameter values lead to a *dimensionless Fourier number* smaller than 0.02 so that the *infinite domain assumption* is valid. We simulate directly the error equations, *i.e.* $f_1 = f_2 = 0$ in (4) and $u_0(x) = 0$. We start the iteration with a random initial guess $u_2^0(0,t)$ ($t \in [0,T]$) so that it contains a wide range of the temporal frequencies that can be resolved by the computational grid. We first perform the Optimized non-overlapping Schwarz Method (referred as OSM case) using $\mathrm{p}_1^{\star,\mathrm{ana}}$ and $\mathrm{p}_2^{\star,\mathrm{ana}}$ and then using an optimal control of the *Robin* parameters with $K = 5$ (referred as OptCon case). We first check that the minimization of cost function $\mathcal{J}$ consistently implies the reduction of the errors $\|e_j\|_\infty$ of the associated algorithm (Fig. 2). We also notice that in the OptCon case the convergence speed is significantly improved compared to the OSM case. Indeed, 9 iterations of the OSM are required to obtain the same accuracy than the OptCon case after only 5 iterations. In order to have more insight on the way the parameters $\mathbf{p_1^{\star,num}}$ and $\mathbf{p_2^{\star,num}}$ evolve throughout the iterations we plot, in Fig. 3, the corresponding convergence factor (6) at each iteration. It is striking to realize that the optimal convergence is obtained through a combination of 2-point (equivalent to the *one-sided* case) and 3-point (equivalent to the *two-sided* case) equioscillations sometimes shifted along the ω-axis to adapt to the temporal frequencies still present in the error. The first two iterations aim at working mainly on the high-frequency components while the last three iterations are optimized to work on the low-frequency component. The adaptivity of the *Robin* parameters from one iteration to the other brings more flexibility to the method enabling more scale selectivity.

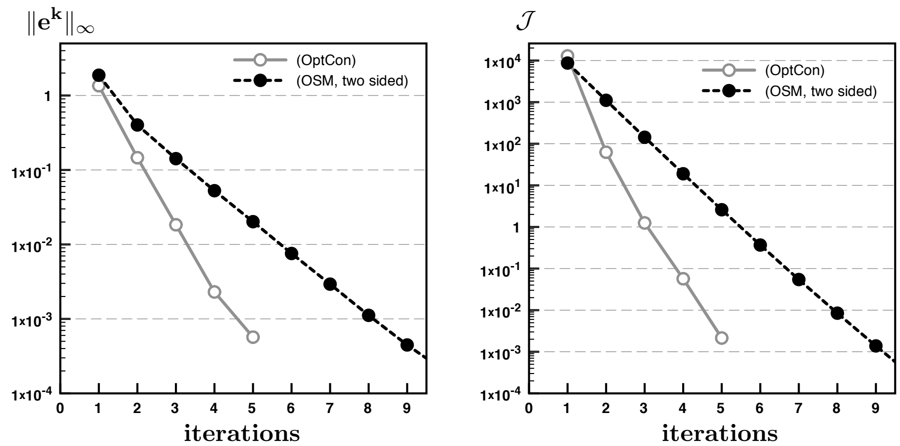

Fig. 2 Evolution of the $\mathcal{L}^{\infty}$-norm of the error (left) and of the cost function $\mathcal{J}$ (right) with respect to the iterates k in the OSM and OptCon cases.

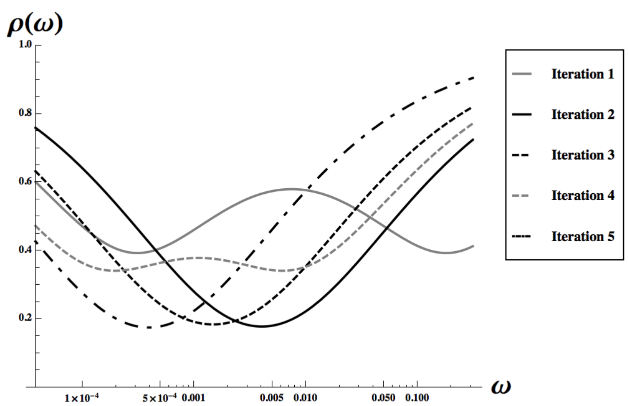

Fig. 3 Sequence of convergence factors $\rho(\omega)$ resulting from the optimal control of the Robin parameters determined to get the best possible convergence after $K = 5$ iterations.

5 Conclusion

Due to its simplicity, the use of *Robin-type transmission conditions* is very attractive when one wants to couple unsteady problems defined on non-overlapping subdomains. Once the *Robin* parameters are properly chosen one can achieve a fast convergence [Gander et al., 1999]. In the present study we showed that there is still room for improvement in the design of the Robin conditions. If the *Robin* parameters are adjusted from one iteration to the other we showed, thanks to an optimal control approach, that we can significantly improve the convergence speed. It is important to emphasize the fact that the *optimal control* paradigm proposed in this study is general enough to be used with any type of PDE and an arbitrary number of subdomains.

Acknowledgements This research was partially supported by the ANR project COMMA (COupling in Multi-physics and multi-scale problems: Models and Algorithms) and by the INRIA project-team MOISE (Modelling, Observation and Identification for Environmental Sciences). We are thankful to Héloïse Pelen (ENS Lyon) for her contribution during her masters internship.

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