A Poincaré cone condition in the Poincaré group

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Abstract

In [7], Ben Arous and Gradinaru described the singularity of the Green function of a general sub-elliptic diffusion. In this article we first adapt their proof to the more general context of a hypoelliptic diffusion. In a second time, we deduce a Wiener criterion and a Poincaré cone condition for a relativistic diffusion with values in the Poincaré group (i.e the group of affine direct isometries of the Minkowski space-time).

Key words: Green function. Wiener test. Poincaré cone condition. Relativistic diffusion. Hypoelliptic operator.

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1 Introduction

In [12], Dudley introduced a relativistic Brownian motion whose trajectories are time-like and which is invariant in law under Lorentz transformations. The space of states of this relativistic Brownian motion is the unitary tangent bundle of the Minkowski space-time, and the diffusion consists in a Brownian motion in $\mathbb{H}^d$ (the hyperboloid model of the hyperbolic space) and its time integral in the Minkowski space-time $\mathbb{R}^{1,d}$. By considering the Eells-Elworthy construction of the Brownian motion on $\mathbb{H}^d$, we obtain Dudley's diffusion by projecting a diffusion in the Poincaré group $G : SO_0(1,d) \ltimes \mathbb{R}^{1,d}$ onto $\mathbb{H}^d \times \mathbb{R}^{1,d}$. The group G is identified with the orthonormal frame bundle of the Minkowski space-time and an element (g,ξ) of G is made of a matrix $g \in SO_0(1,d)$ seen as an orthonormal frame above a point $\xi \in \mathbb{R}^{1,d}$. The diffusion $\{(g_t,\xi_t)\}_{t\geq 0}$ in G that we consider in this paper has generator

$$\mathcal{L} = \frac{\sigma^2}{2}\sum_{i=1}^{d} V_i^2 + H_0,$$

which satisfies the weak Hörmander hypoellipticity condition. This means, in particular, that the drift H_0 is needed to obtain the full rank of $Lie(G)$.

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Intuitively, $\{(g_t, \xi_t)\}_{t\geq 0}$ describes the time-like trajectory of a small rigid object in the Minkowski space-time and consists in a stochastic perturbation, at the velocities level, of a geodesic trajectory. This G-valued diffusion, considered as a Lorentzian analogue to the Euclidian Brownian motion, was studied by Bailleul in [5] where he determined its Poisson boundary by providing a comprehensive description of the asymptotic behavior which, afterwards, was completed in [26] by the study of the Lyapunov spectrum. In the present work we are interested in understanding the infinitesimal small-time behavior and we aim at describing, by means of a (local) Wiener criterion, the thinness of sets with respect to this diffusion. This kind of result is based on the knowledge of the geometry of the level sets of the Green function, and more generally on the knowledge of its behavior near the singularities. Since we dispose (see [5]) of a geometric description of the Poisson boundary of Dudley's diffusion, we can expect a Wiener criterion describing the thinness at the boundary. This question is in the spirit of the works of Ancona who provides, for example in [1], a answer in the case of a Brownian motion in a Cartan-Hadamard manifold. Unfortunately the method used in this paper cannot provide such asymptotic result and we look, here, only at the infinitesimal behavior.

Let us briefly perform a short survey on this subject. In the elliptic case, see for example [6], the Wiener criterion is classical and the fine topology does not depend on the particular elliptic operator. For sub-elliptic diffusions one finds analytic proofs in [24] and probabilistic methods in [10] (for the Heisenberg Laplacian) and in [7] (for a general sub-elliptic diffusion under the strong Hörmander condition). The proofs rely on either explicit expression of Green functions or estimates given in [23] or [25].

In the parabolic context the situation is more delicate due to the difficulty to understand the geometry of level sets of Green functions; moreover the results are less generic. For the heat operator, a probabilistic proof of Wiener criterion can be found in [27] and an analytic one in [13]. The result is extended to the case of variable coefficients in [17] and to the case of the heat operator on the Heisenberg group in [19]. The proofs are based on either explicit expression of Green functions or strong Gaussian estimates.

Here the diffusion $\{(g_t, \xi_t)\}_{t\geq 0}$ is neither sub-elliptic nor (purely) parabolic and the Green function is not known explicitly. Nevertheless, considering the Dudley's diffusion (i.e the projection of $\{g_t\}_{t\geq 0}$ onto $\mathbb{H}^d \times \mathbb{R}^{1,d}$), by means of some time-changes, we are able to recover to a parabolic situation with a generator of type

$$\sum_{i,j=1}^{d} a^{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + a^i(x)\frac{\partial}{\partial x_i} + b^i(x)\frac{\partial}{\partial y_i} + \frac{\partial}{\partial t},$$

at (x, y, t) in a local chart $U \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ of $\mathbb{H}^d \times \mathbb{R}^{1,d}$, where $a^{i,j}$ (definite positive), a^i and b^i are given explicitely.

This generator is a kind of generalization of a Kolmogorov parabolic type operators (for which $b^i(x) = x_i$). In the recents works [20], [11] and [22] one can find some local Gaussian estimates for the Green functions of those operators. But, even in the simplest case where $a^{i,j}$ are constant and the Green function is explicit, no Wiener criterion is known. However let us mention [18] where the authors studied level sets of the Green function and prove an Harnack inequality in the case of Kolmogorov operators with constant coefficients.

Thus, it seems that we are far from reaching a full Wiener criterion for $\{(g_t, \xi_t)\}_{t\geq 0}$, or even for Dudley's diffusion. In this article, we nevertheless provide a weak Wiener criterion for $\{(g_t, \xi_t)\}_{t\geq 0}$, which concerns sets into some homogeneous cones and we deduce a Poincaré cone sufficient condition for thinness. We use the same technics as in [7], extracting the information contained in the infinitesimal homogeneity, by comparing the diffusion to a scale invariant "tangent process". In Section 2, using the estimates of [23] and the stochastic Taylor formula of [9], we adapt the proof of [7] to the context of a general diffusion satisfying the weak Hörmander condition. We obtain Theorem 1, which provides some information about the singularity of the Green function. There are not more technical difficulties than in [7], and the drift X_0 is hidden in the notation. Even if this result is far from giving a full description of the behavior of the Green function on the diagonal for a general hypoelliptic diffusion, it is nevertheless sufficient to obtain, in the Section 3, a weak Wiener criterion and a Poincaré cone condition for $\{(g_t, \xi_t)\}_{t\geq 0}$ in the Poincaré group.

2 Estimates of the Green function for a general hypoelliptic diffusion.

This section is written in a general setting and we consider a diffusion in an open set of a smooth manifold $\mathcal{M}^n$ of dimension n, solution of the SDE

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + X_0(x_t)dt, \tag{1}$$

and generated by

$$\frac{1}{2}\sum_{i=1}^{m} X_i^2 + X_0.$$

The vector fields $(X_0, X_1, \cdots, X_m)$ will satisfy the weak Hörmander conditions for hypoellipticity:

$$\forall x \in \mathcal{M}^n, \quad Lie(X_1, \ldots, X_m, X_0)|_x = T_x\mathcal{M}^n, \tag{H}$$

This insures the local existence of a smooth Green function, denoted by $G(x, y)$.

We shall show the following result, which is the theorem (1.9) of [7] written for a general hypoelliptic diffusion (non necessarily sub-Riemanian):

Theorem 1. *Let x be fixed. We have*

$$\lim_{\varepsilon\to 0} \sup_{|y|_x<\varepsilon} \left| G(x,y)|y|_x^{Q(x)-2} - \frac{1}{J_x(0)} g^{(x)}\big(0, \theta_x(y)\big)\right| = 0.$$

Here $Q(x)$ is some homogeneous dimension and $|\cdot|_x$ some homogeneous norm associated to $(X_0, X_1, \cdots, X_m)$ at x. The angular variable $\theta_x(y)$ is the projection of y onto the unit homogenous sphere centered at x and the non null Jacobian $J_x(0)$ appears with a change of variable. The function $g^x(0, \cdot)$ is the Green function of some "tangent process" associated to the diffusion. When the diffusion satisfies the strong Hörmander condition, as in [7], this function is strictly positive but in the general case it can vanish on a part of the unit homogeneous sphere. This is the case for the diffusion in the Poincaré group.

Let us briefly sketch an idea of the proof. The way how the $\{X_i\}_{i=0\cdots m}$ generate $T_x\mathcal{M}^n$ yields a family of dilatations of $T_x\mathcal{M}^n$, denoted by T_λ for $\lambda > 0$. Then we consider some rescaled diffusions in $T_x\mathcal{M}^n \simeq \mathbb{R}^n$ defined by $v_t^{(x,\varepsilon)} := T_{1/\varepsilon}(x_{\varepsilon^2 t})$. This amounts to zooming x_t more and more as $\varepsilon \to 0$. Then, using the stochastic Taylor formula of Castell ([9]), a "tangent process" $u_t^{(x)}$ appears in the first term of the Taylor expansion at x; we have indeed $v_t^{(x,\varepsilon)} = u_t^{(x)} + \varepsilon R(\varepsilon, t)$ where the term $R(\varepsilon, t)$ is bounded in probability when $\varepsilon \to 0$. We finish by using the estimates obtained by Nagel, Stein and Wainger in [23] (Corollary p 114), to show that the Green functions $G^{(x,\varepsilon)}$ of the rescaled diffusions $v^{(x,\varepsilon)}$ converge uniformly on compact set, when $\varepsilon \to 0$, towards the Green function g^x of $u_t^{(x)}$:

Proposition 1. *For all compact set $K \subset \mathbb{R}^n \setminus \{0\}$ we have:*

$$\sup_{u\in K} \left| G^{(x,\varepsilon)}(0,u) - g^{(x)}(0,u) \right| \xrightarrow[\varepsilon\to 0]{} 0.$$

In fine, Theorem 1 is obtained by taking $\varepsilon = |y|_x$ and expressing $G^{(x,\varepsilon)}(0, u)$ in terms of $G(x, y)$.

2.1 Notation and hypothesis.

We fix the notation and provide assumptions on the geometry induced by the $\{X_i\}_{i=0\cdots m}$.

For every multi-index $J \in \{0, \ldots, m\}^l$, we denote by:

- $|J|$ the length l of J.
- $\|J\|$ the weight of J:

$$\|J\| := |J| + \text{Number of zeros in } J.$$

- $X^J := [X_{j_1}, [X_{j_2}, [\dots, [X_{j_{l-1}}, X_{j_l}]] \dots]$ et $X^{(j)} = X_j$ (for $|J| = 1$) .
- For $B_t = (B_t^1, \dots, B_t^l)$ a $\mathbb{R}^l$-Brownian motion, we set $B_t^0 = t$ and denote by B_t^J the Stratonovich iterated integral:

$$B_t^J := \int_{\Delta_t^l} \circ dB_{t_1}^{j_1} \cdots \circ dB_{t_l}^{j_l},$$

where $\Delta_t^l = \{(t_1, \dots, t_l);\ 0 < t_1 < \cdots < t_l < t\}$.
- For σ a permutation of $\{1, \cdots, l\}$, we set $J \circ \sigma = (j_{\sigma(1)}, \dots, j_{\sigma(l)})$ and denote by $e(\sigma) = \mathrm{Card}\{j \in \{1, \dots, l-1\}; \sigma(j) > \sigma(j+1)\}$ the number of errors in ordering σ.
- We denote by c_t^J **the linear combination of Stratonovich iterated integrals**:

$$c_t^J := \sum_{\sigma \in \mathfrak{S}_{|J|}} \frac{(-1)^{e(\sigma)}}{|J|^2 C_{|J|-1}^{e(\sigma)}} B_t^{J \circ \sigma^{-1}}.$$

- For some smooth vector field X, we denote by $\exp(sX)(x_0)$ the solution at time s of the ordinary differential equation :

$$\begin{cases} \dfrac{du}{ds} = X(u(s)) \\ u(0) = x_0. \end{cases}$$

We need to consider:

$$C_i(x) = \mathrm{vect}\left\{X^J(x);\ \|J\| \le i\right\}.$$

Since the $\{X_i\}_{i=0\cdots m}$ satisfy the Hörmander condition (H), we denote by $r(x)$ the smaller integer such that $C_{r(x)}(x) = T_x\mathcal{M}^n$.

Denote by $\mathcal{B} = (J_1, \dots, J_n)$ a family of multi-indexes such that $(X^J)_{J \in \mathcal{B}}$ is a triangular basis of $T_x\mathcal{M}^n$. This means that for $j \le r$, $\{X^J;\ J \in \mathcal{B},\ \|J\| \le j\}$ is a basis of C_j and thus $\dim C_j = \mathrm{Card}\{k, \|J_k\| \le j\}$.

For any multi-index L there exist smooth functions $a_J^L(x)$ such that:

$$X^L = \sum_{J \in \mathcal{B}} a_J^L X^J.$$

We denote by $Q(x)$ **the graded dimension at** x:

$$\begin{aligned} Q(x) &= \sum_{i=1}^{r(x)} i \times (\dim C_i(x) - \dim C_{i-1}(x)) \\ &= \sum_{i=1}^{r(x)} i \times \mathrm{Card}\{k, \|J_k\| = i\}. \end{aligned}$$

We will make the following assumptions on the family $\{X_i\}$:

i) We assume that the geometry of the Lie brackets is constant, this means that the $\dim C_i(x)$ are locally constant for $i \in \mathbb{N}^*$. Thus r et Q are locally constant too.

ii) $r \ge 2$

iii) $\dim C_i - \dim C_{i-1} (= \mathrm{Card}\{k;\ \|J_k\| = i\}) \ge 1,\ \forall i = 2, \cdots, r$

iv) $\dim C_1 (= \mathrm{Card}\{k;\ \|J_k\| = 1\}) \ge 2$.

Hypothesis ii) iii) and iv) are technical and ensure that the dimensions are large enough so that x_t leaves any bounded domain within an almost surely finite time. This is needed to justify the finiteness of integrals which appear in the proof of Proposition 1. Moreover, it is easy to check that this is satisfied for the relativistic diffusion in the Poincaré group.

There exists a neighborhood W of 0 in $\mathbb{R}^n$ such that the map:

$$\varphi_x :\ u \mapsto \exp\left(\sum_{i=1}^n u_i X^{J_i}\right)(x)$$

is a smooth diffeomorphism from W onto $\varphi_x(W)$.

Let be $U \subset V \cap \varphi_x(W)$ a neighborhood of x. For $y \in U$ we define by

$$|y|_x := \left[\sum_{k=1}^{r} \left(\sum_{i, \|J_i\|=k} u_i^2 \right)^{\frac{Q}{2k}} \right]^{\frac{1}{Q}},$$

the homogeneous norm at x of $y = \varphi_x(u)$. For $(u_i)_{i=1\cdots n} \in \mathbb{R}^n$, we set $|u|_n := |\varphi_x(u)|_x$, and denote by $\|u\|$ is the euclidian norm.

During the proof we will use homogenous norm at different points near of the reference point x. For this, we use that φ_x depends continuously on x, and we can take U small enough so that every $\varphi_y^{-1} : U \to \varphi_y^{-1}(U)$, where $y \in U$, be a diffeomorphism.

Thus, $|z|_y$ is well defined for $z, y \in U$.

This norm is homogenous with respect to the dilatations $T_\varepsilon : u \mapsto (\varepsilon^{\|J_i\|} u_i)_{i=1\dots n}$; this means that:

$$|\varphi_x \circ T_\varepsilon(u)|_x = \varepsilon |\varphi_x(u)|_x.$$

2.2 Homogeneous norm and estimations of [23]

We briefly recall the results obtained by Nagel, Stein and Wainger in [23] relating to the estimation of the Green function, in terms of the pseudo-metric ρ associated to the X_i defined by:

$$\rho(y, z) := \inf\{\delta > 0; \quad \exists \varphi \in C(\delta) \text{ tq } \varphi(0) = y \text{ et } \varphi(1) = z\}, \tag{2}$$

where $C(\delta)$ is the set of absolutely continous functions such that almost everywhere:

$$\varphi'(t) = \sum_{i=1}^{d} a_i(t) X^{J_i}(\varphi(t)),$$

with $|a_i(t)| < \delta^{\|J_i\|}$. The Corollary p 114 of [23] provides that, when $n \geq 2$, for any K compact set in $U \times U$, there exists $C > 0$ such that :

$$\forall y, z \in K, \quad |G(y, z)| \leq C \frac{\rho^2(y, z)}{\mathrm{Vol}\,(B(y, \rho(y, z)))} \tag{3}$$

and for $J = (j_1, \cdots, j_k)$:

$$\forall y, z \in K, \quad |X_{j_1} \cdots X_{j_k} G(x, y)| \leq \frac{\rho^{2-\|J\|}(y, z)}{\mathrm{Vol}\,(B(y, \rho(y, z)))}. \tag{4}$$

We have the following property:

Proposition 2 (Equivalence between homogenous norm and pseudo-metric)**.** *There exists a neighborhood $\tilde{U} \subset U$ of x such that for every compact set $K \subset \tilde{U}$, we can find $C_1, C_2 > 0$ such that:*

$$\forall y, z \in K, \quad C_1 \rho(y, z) \leq |z|_y \leq C_2 \rho(y, z). \tag{5}$$

Proof. Set:

$$\rho_2(x, y) = \inf\{\delta > 0; \quad \exists \varphi \in C_2(\delta) \text{ tq } \varphi(0) = x \text{ et} \varphi(1) = y\}, \tag{6}$$

where $C_2(\delta)$ is the set of smooth functions φ such that:

$$\varphi'(t) = \sum_{i=1}^{n} a_i X^{J_i}(\varphi(t)), \tag{7}$$

with constants a_i such that $|a_i| \leq \delta^{\|J_i\|}$.

Let $z \in U$; there exists a unique vector $(u_i)_{i=1\cdots n}$ such that:

$$z = \exp\left(\sum_{i=1}^{n} u_i X^{J_i}\right)(y),$$

and, if $\varphi(t) = \exp\left(\sum_{i=1}^{n} t u_i X^{J_i}\right)(y)$ we have:

$$\varphi'(t) = \sum_{i=1}^{n} u_i X^{J_i}(\varphi(t))$$

and $|u_i| \leq |z|_y^{\|J_i\|}$. So $\rho_2(y,z) \leq |z|_y$.

Moreover, for some $z, y \in U$, there is a unique function φ such that $\varphi'(t) = \sum_{i=1}^{n} a_i X^{J_i}(\varphi(t))$ with $\varphi(0) = y$ and $\varphi(1) = z$: this is $t \mapsto \exp(\sum_{i=1}^{n} t u_i X^{J_i})(y)$.

Thus, if $\delta < |z|_y$, then $\delta^{\|J_i\|} < |u_i|$ and $C_2(\delta)$ is empty and we have $\rho_2(y,z) = |z|_y$. To finish the proof, we use Theorem 2 of [23] which shows that ρ and ρ_2 are locally equivalent. $\square$

Using (3) and the previous proposition we obtain:

Proposition 3 (*a priori* estimates)**.** *For any small enough compact set U, we can find $C > 0$ such that:*

$$\forall y \neq z \in U,\ |G(y,z)| \leq \frac{C}{|z|_y^{Q-2}}.$$

Moreover for $J = (j_1, \cdots, j_k)$ we have:

$$\forall y \neq z \in U,\ |X_{j_1} \cdots X_{j_k} G(y,z)| \leq \frac{C}{|z|_y^{Q-2+\|J\|}}$$

Proof. This is immediate, remarking that:

$$\mathrm{Vol}(B(y,|z|_y)) = \int_{|u|_d < |z|_y} du = |z|_y^Q \int_{|u|_d < 1} du.$$

$\square$

Proposition 4 (Triangular inequality and comparaison with a euclidian norm)**.** *For any small enough compact set U, we can find $c_0 > 0$ such that for any $t, y, z \in U$:*

$$|y|_t \leq c_0 \left(|z|_t + |z|_y\right). \tag{8}$$

Moreover, there exist constant $c', c'' > 0$ such that for any $y, z \in U$:

$$c' \|\varphi_y^{-1}(z)\| \leq |z|_y \leq c'' \|\varphi_y^{-1}(z)\|^{1/r}. \tag{9}$$

Proof. This is a consequence of the local equivalence between ρ and $|\cdot|_x$, and of the proposition (1.1) p 107 and of (iii') p 109 in [23]. $\square$

2.3 Rescaled diffusions and tangent process

Zooming on the trajectory x_t in a neighborhood U of x we introduce the **rescaled diffusions** $v_t^{(x,\varepsilon)} := T_{1/\varepsilon} \circ \varphi_x^{-1}(x_{\varepsilon^2 t})$. It is a $\mathbb{R}^n$-diffusion which lies in a neighborhood $\tilde{U}^\varepsilon := T_{1/\varepsilon} \circ \varphi_x^{-1}(U)$ of 0 and is defined up to the time $\tau_\varepsilon := \tau/\varepsilon$, τ being the exit time for x_t from U. **We denote by** $G^{(x,\varepsilon)}(u,v)$ **the Green function of** $v_t^{(x,\varepsilon)}$, defined for $u \neq v \in \tilde{U}^\varepsilon$. We denote by $J_x := |\mathrm{Jac}(\varphi_x)|$ the Jacobian of φ_x on $\tilde{U}^\varepsilon$. A direct computation shows that:

$$G^{(x,\varepsilon)}(u,v) = \varepsilon^{Q-2} J_x(T_\varepsilon(v)) G(\varphi_x \circ T_\varepsilon(u), \varphi_x \circ T_\varepsilon(v)). \tag{10}$$

This can be rewriten for $y \in U$ as:

$$G(x,y) = \frac{1}{J_x(\varphi_x^{-1}(y))\varepsilon^{Q-2}} G^{(x,\varepsilon)}(0, T_{1/\varepsilon} \circ \varphi_x^{-1}(y)). \tag{11}$$

We can take U small enough so that it be a chart of $\mathcal{M}^n$ and we consider, via some choice of coordinates, x_t and X_i as vectors in $\mathbb{R}^n$. We can use Theorem 4.1 of [9] and write the Taylor expansion:

$$\begin{aligned}
x_{\varepsilon^2 t} &= \exp\left(\sum_{k=1}^{r} \sum_{L, \|L\|=k} \varepsilon^k c_t^L X^L\right) + \varepsilon^{r+1}\tilde{R}(\varepsilon,t) \\
&= \exp\left(\sum_{k=1}^{r} \sum_{L, \|L\|=k} \varepsilon^k c_t^L \sum_{i=1}^{d} a_{J_i}^L X^{J_i}\right) + \varepsilon^{r+1}\tilde{R}(\varepsilon,t) \\
&= \exp\left(\sum_{i=1}^{n} \left(\sum_{L, \|J_i\| \le \|L\| \le r} \varepsilon^{\|L\|} a_{J_i}^L c_t^L\right) X^{J_i}\right) + \varepsilon^{r+1}\tilde{R}(\varepsilon,t),^1 \qquad (12)
\end{aligned}$$

where the remainder term $\tilde{R}(\varepsilon,t)$ is bounded in probability. Then, composing by $T_{1/\varepsilon} \circ \varphi_x^{-1}$ we obtain:

$$\begin{aligned}
v_t^{(x,\varepsilon)} &= T_{1/\varepsilon} \circ \varphi_x^{-1}(x_{\varepsilon^2 t}) &(13)\\
&= \left(\sum_{L, r \ge \|L\| \ge \|J_i\|} \varepsilon^{\|L\|-\|J_i\|} a_{J_i}^L c_t^L\right)_{i=1\cdots n} + \varepsilon \bar{R}(\varepsilon,t) &(14)\\
&= u_t^{(x)} + \varepsilon\left(\left(\sum_{L, r \ge \|L\| > \|J_i\|} \varepsilon^{\|L\|-\|J_i\|-1} a_{J_i}^L c_t^L\right)_{i=1\cdots n} + \bar{R}(\varepsilon,t)\right), &(15)
\end{aligned}$$

where $u_t^{(x)}$ is a $\mathbb{R}^d$-valued process, called **tangent process**, defined by:

$$u_t^{(x)} := \left(\sum_{L, \|L\|=\|J_i\|} a_{J_i}^L c_t^L\right)_{i=1\cdots n}.$$

Considering the free r-nilpotent Lie algebra with m-generators we can show, as in Proposition (3.2) of [7], that $u_t^{(x)}$ is a linear projection of an hypoelliptic diffusion in the associated nilpotent Lie group. In [7] the authors deduce (corollary (3.6)) that $u_t^{(x)}$ admits a smooth density with respect to the Lebesgue measure, but this does not necessarily hold in our context. Nevertheless we can prove, using the linear projection, that $u_t^{(x)}$ admits a smooth Green function, denoted by $g^{(x)}(0,u)$. This means that we have, for every $\phi \in C_b^0(\mathbb{R}^n)$:

$$\mathbb{E}_0\left[\int_0^{+\infty} \phi(u_t^{(x)})dt\right] = \int_{u \in \mathbb{R}^d} \phi(u) g^{(x)}(0,u)du.$$

Moreover since $u_{\varepsilon^2 t}^{(x)}$ has the same law as $T_\varepsilon(u_t^{(x)})$ we deduce that (since $v = T_\varepsilon(u) \Rightarrow dv = \varepsilon^Q du$):

$$g^{(x)}\left(0, T_{1/\varepsilon}(u)\right) = \varepsilon^{Q-2} g^{(x)}(0,u). \tag{16}$$

For $\varepsilon = 1/|y|_x$ we obtain:

$$g^{(x)}(0, \varphi_x^{-1}(y)) = \frac{1}{|y|_x^{Q-2}} g^{(x)}(0, \theta_x(y)). \tag{17}$$

Here $\theta_{\mathbf{x}}(\mathbf{y}) := \mathbf{T}_{\mathbf{1}/|\mathbf{y}|_{\mathbf{x}}}(\varphi_{\mathbf{x}}^{-\mathbf{1}}(\mathbf{y}))$ is an angular variable in the unit sphere S_x of $T_x\mathcal{M}^n$ for the homogenous norm $|\cdot|_n$.

[1] Recall that $(X^J)_{J \in \mathcal{B}}$ is a triangular basis and that we have $a_J^L = 0$ when $\|L\| < \|J\|$.

2.4 Proof of Theorem 1

Considering (17) and (11) and taking $K = S_x$ and $\varepsilon = |y|_x$, Theorem 1 follows easily from Proposition 1.

Now we present the proof of Proposition 1. As in [7] we begin by showing a weak convergence of $G^{(x,\varepsilon)}(0,\cdot)$ to $g^x(0,\cdot)$:

Proposition 5. *Let K be a compact set in $\mathbb{R}^n \setminus \{0\}$ and f be a smooth function supported on K. We have:*

$$\lim_{\varepsilon\to 0}\left|\int f(u)G^{(x,\varepsilon)}(0,u)du - \int f(u)g^{(x)}(0,u)du\right| = 0.$$

Proof. By definition of the Green functions we have:

$$\left|\int f(u)G^{(x,\varepsilon)}(0,u)du - \int f(u)g^{(x)}(0,u)du\right| = \left|\mathbb{E}\left[\int_0^{\tau_\varepsilon} f(v_t^{(x,\varepsilon)})dt - \int_0^{+\infty} f(u_t^{(x)})dt\right]\right|.$$

Now fixing $T > 0$, we can decompose the term $\int_0^{\tau_\varepsilon} f(v_t^{(x,\varepsilon)})dt - \int_0^{+\infty} f(u_t^{(x)})dt$ to:

$$\begin{aligned}\mathbf{1}_{T\le\tau_\varepsilon}\int_0^T \left(f(v_t^{(x,\varepsilon)}) - f(u_t^{(x)})\right)dt + \mathbf{1}_{T\le\tau_\varepsilon}\int_T^{\tau_\varepsilon} f(v_t^{(x,\varepsilon)})dt - \int_T^{+\infty} f(u_t^{(x)})dt \\ + \mathbf{1}_{T>\tau_\varepsilon}\int_0^{\tau_\varepsilon} f(v_t^{(x,\varepsilon)})dt - \mathbf{1}_{T>\tau_\varepsilon}\int_0^T f(u_t^{(x)})dt.\end{aligned}$$

Thus we have the inequality:

$$\begin{aligned}\left|\mathbb{E}\left[\int_0^{\tau_\varepsilon} f(v_t^{(x,\varepsilon)})dt - \int_0^{+\infty} f(u_t^{(x)})dt\right]\right| \le \mathbb{E}\left[\mathbf{1}_{T\le\tau_\varepsilon}\int_0^T |f(v_t^{(x,\varepsilon)}) - f(u_t^{(x)})|dt\right] \\ + \mathbb{E}\left[\mathbf{1}_{T\le\tau_\varepsilon}\int_T^{\tau_\varepsilon} |f(v_t^{(x,\varepsilon)})|dt\right] + \mathbb{E}\left[\int_T^{+\infty} |f(u_t^{(x)})|dt\right] + 2\|f\|_\infty T\mathbb{P}(T \ge \tau_\varepsilon).\end{aligned}$$

Now, taking first the $\limsup_{\varepsilon\to 0}$ and secondly the $\limsup_{T\to\infty}$, Proposition 5 follows from the following facts:

Fact 1. *For $T > 0$ fixed,*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\mathbf{1}_{T\le\tau_\varepsilon}\int_0^T |f(v_t^{(x,\varepsilon)}) - f(u_t^{(x)})|dt\right] = 0.$$

Fact 2.

$$\lim_{T\to+\infty}\mathbb{E}\left[\int_T^{+\infty} |f(u_t^{(x)})|dt\right] = 0.$$

Fact 3.

$$\liminf_{T\to+\infty}\limsup_{\varepsilon\to 0}\mathbb{E}\left[\mathbf{1}_{T\le\tau_\varepsilon}\int_T^{\tau_\varepsilon} |f(v_t^{(x,\varepsilon)})|dt\right] = 0.$$

- The proof of the fact 1 is very similar to the proof of Proposition (2.12) in [7].
  We use the Taylor expansion (15) which can be written $v_t^{(x,\varepsilon)} = u_t^{(x)} + \varepsilon R^{(x)}(\varepsilon,t)$, where:

  $$R^{(x)}(\varepsilon,t) := \left(\sum_{L,r\ge\|L\|>\|J_i\|} \varepsilon^{\|L\|-\|J_i\|-1} a_{J_i}^L c_t^L\right)_{i=1\cdots n} + \bar{R}(\varepsilon,t).$$

  Theorem 4.1 and Proposition P_1 and P_2 in [9] ensure that there exist $\alpha > 0$ and $c > 0$ such that for all $R > c$:

  $$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{0\le s\le T}\|R^{(x)}(\varepsilon,s)\| > R,\ T < \tau_\varepsilon\right) \le \exp\left(-\frac{R^\alpha}{cT}\right).$$

Hence for $\eta > 0$ we can find $\varepsilon_0 > 0$ and $R > 0$ such that for all $\varepsilon < \varepsilon_0$

$$\mathbb{P}\left(\sup_{0\le s\le T}\|R^{(x)}(\varepsilon,s)\| > R,\ T<\tau_\varepsilon\right) \le \frac{\eta}{2T\|f\|}.$$

So, decomposing the expectation on the event $\left\{\sup_{0\le s\le T}\|R^{(x)}(\varepsilon,s)\| > R,\ T\le\tau_\varepsilon\right\}$ we obtain that:

$$\mathbb{E}\left[\mathbf{1}_{T\le\tau_\varepsilon}\int_0^T |f(v_t^{(x,\varepsilon)}) - f(u_t^{(x)})|dt\right] \le \eta + \varepsilon TR\|Df\|_\infty.$$

Hence for all $\eta > 0$ we have:

$$\limsup_{\varepsilon\to 0}\mathbb{E}\left[\mathbf{1}_{T\le\tau_\varepsilon}\int_0^T |f(v_t^{(x,\varepsilon)}) - f(u_t^{(x)})|dt\right] \le \eta,$$

and Fact 1 is proved.

- To prove Fact 2, it suffices to show that:

$$\mathbb{E}\left[\int_0^{+\infty}\mathbf{1}_{B(0,\rho)}(u_t^{(x)})dt\right] < \infty, \tag{18}$$

where $B(0,\rho)$ is the ball of radius ρ for the homogenous norm $|\cdot|_n$. The proof of (18) is similar to the proof of Proposition (3.11) of [7] and it remains to show that $\int_{p_x^{-1}(B(0,\rho))}\frac{1}{|\tilde u|_N^{\tilde Q-2}}d\tilde u < \infty$ where N is the dimension of the free r-nilpotent Lie algebra with m-generators $\mathcal{G}(m,r)$, and p_x the linear projection which maps the diffusion in $\mathcal{G}(m,r)$ to the tangent process and $\tilde Q$ the homogenous dimension of the diffusion in $\mathcal{G}(m,r)$. The hypotheses i), ii) and iii) on the dimension ensure the finiteness of the integral, using Lemma (A.7) of [7].

- The proof of Fact 3 is very similar to the proof of Proposition (4.1) of [7], and we prove two lemmas to conclude. Denote by $\mu_T^{(x,\varepsilon)}$ the measure whose density is $\mathbf{1}_{T<\tau_\varepsilon}$ with respect to the law of $v_T^{(x,\varepsilon)}$. By the Markov property we have:

$$\mathbb{E}_0\left[\mathbf{1}_{T<\tau_\varepsilon}\int_T^{\tau_\varepsilon}|f(v_t^{(x,\varepsilon)})|dt\right] = \int_{T_{1/\varepsilon}\circ\varphi_x^{-1}(U)} d\mu_T^{(x,\varepsilon)}(u)\int_{T_{1/\varepsilon}\circ\varphi_x^{-1}(U)} G^{(x,\varepsilon)}(u,v)|f(v)|dv.$$

For $u,v\in T_{1/\varepsilon}\circ\varphi_x^{-1}(U)$, we set $u_\varepsilon^x := \varphi_x\circ T_\varepsilon(u)$ and $v_\varepsilon^x := \varphi_x\circ T_\varepsilon(v)$. [2]
By (10) we have:

$$G^{(x,\varepsilon)}(u,v) = \varepsilon^{Q-2}J_x(T_\varepsilon(v))G(u_\varepsilon^x, v_\varepsilon^x),$$

and using Proposition 3 we can find $C > 0$ such that for all $\varepsilon > 0$:

$$\int_{T_{1/\varepsilon}\circ\varphi_x^{-1}(U)} G^{(x,\varepsilon)}(u,v)|f(v)|dv \le \int_{B(0,\rho)}\frac{C\varepsilon^{Q-2}}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}}dv,$$

where ρ is large enough, so that the support of f be included in $B(0,\rho)$.
As in [7] we show:

Lemma 1. *For all $R > 0$ there exist $\varepsilon_0 > 0$ and $c > 0$ such that for all $\varepsilon < \varepsilon_0$ and $u\in\mathbb{R}^n$ such that $\|u\|\ge R$ we have:*

$$\int_{B(0,\rho)}\frac{\varepsilon^{Q-2}}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}}dv \le c.$$

Moreover,

$$\lim_{\|u\|\to\infty}\int_{B(0,\rho)}\frac{\varepsilon^{Q-2}}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}}dv = 0,$$

uniformly with respect to ε.

[2]Be carefull: u_ε^x is a vector of $\mathbb{R}^n$ and $u_\varepsilon^{(x)}$ denotes the tangent process at time ε.

Proof of Lemma 1. For $u=0$, this means $u^x_\varepsilon = x$, $\frac{\varepsilon^{Q-2}}{|v^x_\varepsilon|^{Q-2}_{u^x_\varepsilon}} = \frac{1}{|\varphi_x(v)|^{Q-2}_x} = \frac{1}{|v|^{Q-2}_n}$. Lemma (A-1) of [7] ensures that $\int_{B(0,\rho)} \frac{dv}{|v|^{Q-2}_d}$ is bounded by a constant depending only on ρ. The family of diffeomorphisms φ_y^{-1} depends smoothly on $y \in U$. So we can find a neighborhood U_x of x, a constant $C>0$ and some $\varepsilon_0>0$ such that for all $y \in U_x$ and for all $\varepsilon < \varepsilon_0$, $B(0,\varepsilon\rho) \subset \varphi_x^{-1}(U) \cap \varphi_y^{-1}(U)$ and for all $w \in B(0,\varepsilon\rho)$, $|\varphi_y^{-1}\circ\varphi_x(w)|_n \geq C|w|_n$.
So we have:

$$\sup_{y\in U_x}\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{|v^x_\varepsilon|^{Q-2}_y}dv = \sup_{y\in U_x}\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{|\varphi_y^{-1}(v^x_\varepsilon)|^{Q-2}_n}dv = \sup_{y\in U_x}\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{|\varphi_y^{-1}\circ\varphi_x\circ T_\varepsilon(v)|^{Q-2}_n}dv$$
$$\leq \tilde{C}\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{|T_\varepsilon(v)|^{Q-2}_n}dv \leq \tilde{C}\int_{B(0,\rho)} \frac{dv}{|v|^{Q-2}_n}. \tag{19}$$

Now choose $\varepsilon_1>0$ such that for all $\varepsilon<\varepsilon_1$ and for all $u\in\mathbb{R}^n$ such that $\|u\|\leq R$ we have $u^x_\varepsilon \in U_x$. Using (19), there is a constant c such that for all $\varepsilon < \min(\varepsilon_0,\varepsilon_1)$ and for all $u\in\mathbb{R}^n$ with $\|u\|\leq R$:

$$\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{|v^x_\varepsilon|^{Q-2}_{u^x_\varepsilon}}dv \leq c.$$

This is the first point of the lemma. The second point will follow from (8) and (9) of Proposition 4. For $\|u\|$ large enough such that $\frac{1}{c_0}|u^x_\varepsilon|_x - |v^x_\varepsilon|_x > 0$ we have:

$$\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{|v^x_\varepsilon|^{Q-2}_{u^x_\varepsilon}}dv \leq \int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{\left(\frac{1}{c_0}|u^x_\varepsilon|_x - |v^x_\varepsilon|_x\right)^{Q-2}}dv \leq \int_{B(0,\rho)} \frac{1}{\left(\frac{1}{c_0}|\varphi^{-1}(u)|_x - |\varphi^{-1}(v)|_x\right)^{Q-2}}dv$$
$$\leq \int_{B(0,\rho)} \frac{1}{\left(\frac{c'}{c_0}\|u\|-\rho\right)^{Q-2}}dv.$$

This inequality ensures that the convergence is uniform in ε. ☐

Let return to the proof of Fact 3.
By the previous lemma we obtain:

$$\mathbb{E}_0\left[\mathbf{1}_{T<\tau_\varepsilon}\int_T^{\tau_\varepsilon}|f(v_t^{(x,\varepsilon)})|dt\right] \leq c\mu_T^{(x,\varepsilon)}(B(0,R)) + \underbrace{\sup_{u,\|u\|\geq R}\int_{B(0,\rho)}\frac{C\varepsilon^{Q-2}}{|v^x_\varepsilon|^{Q-2}_{u^x_\varepsilon}}dv}_{\underset{R\to+\infty}{\longrightarrow}0}.$$

To end the proof of Fact 3 it remains to prove:

Lemma 2. *For all $R>0$,*

$$\liminf_{T\to+\infty}\limsup_{\varepsilon\to0}\mu_T^{(x,\varepsilon)}(B(0,R)) = 0.$$

Proof of Lemma 2. By definition of $\mu_T^{(x,\varepsilon)}$ we have,

$$\mu_T^{(x,\varepsilon)}(B(0,R)) = \mathbb{E}_0\left[\mathbf{1}_{T<\tau_\varepsilon}\mathbf{1}_{B(0,R)}(v_T^{(x,\varepsilon)})\right].$$

Let χ be a smooth function which is equals to 1 on $B(0,R)$ and which is supported on $B(0,R+1)$. We have

$$\mu_T^{(x,\varepsilon)}(B(0,R)) \leq \mathbb{E}_0\left[\mathbf{1}_{T<\tau_\varepsilon}\chi(v_T^{(x,\varepsilon)})\right].$$

Using the Taylor expansion of $v_T^{(x,\varepsilon)}$, as in the proof of the fact 1, we obtain:

$$\mathbb{E}_0\left[\mathbf{1}_{T<\tau_\varepsilon}\chi(v_T^{(x,\varepsilon)})\right] \underset{\varepsilon\to0}{\longrightarrow} \mathbb{E}_0\left[\chi(u_T^{(x)})\right].$$

Moreover $\mathbb{E}_0\left[\chi(u_T^{(x)})\right] \leq \mathbb{E}_0\left[\mathbf{1}_{B(0,R+1)}(u_T^{(x)})\right]$ and $t \mapsto \mathbb{E}_0\left[\mathbf{1}_{B(0,R+1)}(u_t^{(x)})\right]$ is a positive bounded function which is integrable according to (18). Thus we obtain

$$\liminf_{T\to+\infty} \mathbb{E}_0\left[\mathbf{1}_{B(0,R+1)}(u_T^{(x)})\right] = 0.$$

□

□

As for the proof of (5.4) in [7], we deduce easily from (10) and Proposition 3 the following estimates:

Proposition 6.

$$\limsup_{\varepsilon>0} \sup_{u\in K} |G^{(x,\varepsilon)}(0,u)| < +\infty.$$

For $i = 1\cdots n$ we have:

$$\limsup_{\varepsilon>0} \sup_{u\in K} |\frac{\partial}{\partial u_i} G^{(x,\varepsilon)}(0,u)| < +\infty;$$

By the previous proposition and Ascoli's Theorem, the family $\{G^{(x,\varepsilon)}(0,\cdot)\}_{\varepsilon>0}$ is relatively compact for the uniform norm on K. Proposition 5 ensures that there is only one limit point, thus Proposition 1 is proved.

3 A Wiener criterion and a Poincaré cone condition in the Poincaré group.

3.1 Geometric framework.

We denote by $\mathbb{R}^{1,d}$ the space $\mathbb{R}^{d+1}$ endowed with the Minkowski's quadratic form q:

$$q(\xi) = \left(\xi^0\right)^2 - \sum_{i=1}^{d} \left(\xi^i\right)^2.$$

We denote by $SO(1,d)$ the sub-group of $SL(\mathbb{R}^{d+1})$ made of direct q-isometries, and by $SO_0(1,d)$ the connected component of identity in $SO(1,d)$. The Poincaré group is the affine group $G := SO_0(1,d) \ltimes \mathbb{R}^{1,d}$ where the law is defined by:

$$(g,\xi)(g',\xi') = (gg', \xi + g\xi').$$

G is seen as the following matrix sub-group of $SL(d+2)$:

$$G = \left\{ \begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix};\ g \in SO_0(1,d),\ \xi \in \mathbb{R}^{(1,d)} \right\}.$$

The Lie algebra of $SO(1,d)$ is:

$$so(1,d) = \left\{ \begin{pmatrix} 0 & {}^t(u_i) \\ (u_i) & (u_{ij}) \end{pmatrix};\ (u_i) \in \mathbb{R}^d, (u_{ij}) \in \mathbb{R}^{d\times d};\ (u_{ji}) = -(u_{ij}) \right\}.$$

Thus, $SO_0(1,d) = \exp(so_{(1,d)})$ and $G = \exp(\mathfrak{g})$ where:

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & (u_{i0}) \\ 0 & 0 & 0 \end{pmatrix}; (u_i)_{i=1\cdots d} \in \mathbb{R}^d,\ (u_{ji}) = -(u_{ij}),\ (u_0, (u_{i0})_{i=1\cdots d}) \in \mathbb{R}^{1,d} \right\}.$$

Denote by $\mathbb{H}^d$ the half-unit sphere of $\mathbb{R}^{1,d}$:

$$\mathbb{H}^d := \left\{\xi \in \mathbb{R}^{1,d} |\quad q(\xi) = 1 \text{ and } \xi^0 > 0\right\}.$$

Endowing $T\mathbb{H}^d$ with the metric $q|_{T\mathbb{H}^d}$, $\mathbb{H}^d$ is a Riemaniann manifold of constant negative curvature. This is the hyperboloid model of the hyperbolic space.

3.2 Relativistic diffusion.

We introduce a left invariant diffusion on G which is the lift of the relativistic diffusion on $\mathbb{H}^d\times\mathbb{R}^{1,d}$ introduced by Dudley in [12]. The asymptotic behavior of G-valued diffusion was studied in [5]. Relativistic diffusion can be seen as a stochastic perturbation of the geodesic flow on a Lorentzian manifold. For more informations on this subject see [4].

For $i = 1\cdots d$, we denote by X_i the left invariant vector field on G defined by:

$$X_i(g,\xi) = (g,\xi)\underbrace{\begin{pmatrix} 0 & {}^te_i & 0 \\ e_i & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:=E_i\in\mathfrak{g}}.$$

We denote by X_0 the left invariant vector field on G defined by:

$$X_0(g,\xi) = (g,\xi)\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:=E_0\in\mathfrak{g}}.$$

We denote by $\{(g_s,\xi_s)\}_{s\geq 0}$ the diffusion on G solution of:

$$d(g_s,\xi_s) = \sum_{i=1}^d X_i(g_s,\xi_s)\circ dB_s^i + X_0(g_s,\xi_s)ds.$$

Let $\mathcal{L}$ denote the generator associated to $(g_s)_s$:

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^d X_i^2 + X_0.$$

Denote by $(e_0,\cdots,e_d)$ the canonical basis of $\mathbb{R}^{1,d}$. We have the following proposition (cf [4], [5], [14])

Proposition 7. *The process $(g_t(e_0),\xi_t)$ in $\mathbb{H}^d\times\mathbb{R}^{1,d}$ is the relativistic Dudley diffusion and is generated by*

$$\frac{1}{2}\Delta_{\mathbb{H}^d} + H_0$$

where the vector field H_0 is the geodesic flow in $T^1\mathbb{R}^{1,d}\equiv\mathbb{H}^d\times\mathbb{R}^{1,d}$. In other words $g_t(e_0)$ is a Brownian motion in $\mathbb{H}^d$ and ξ_t its time integral: $\xi_t = \int_0^t g_s(e_0)ds$.

A simple computation shows that, for $i,j = 1\cdots d$:

$$[X_i,X_j](g,\xi) = (g,\xi)\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_i\otimes e_j - e_j\otimes e_i & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:=E_{ij}\in\mathfrak{g}}$$

$$[X_i,X_0](g,\xi) = (g,\xi)\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{0} & e_i \\ 0 & 0 & 0 \end{pmatrix}}_{:=E_{i0}\in\mathfrak{g}}$$

Thus at every point (g,ξ) of G we have

$$\mathrm{vect}\{X_0, X_i, [X_i,X_j], [X_i,X_0],\ i,j = 1\cdots d\} = T_{(g,\xi)}G,$$

and, by Hörmander's Theorem, $\mathcal{L}$ is hypoelliptic.

Moreover we have

$$[X_i, [X_i, X_0]] = X_0, \tag{20}$$

and then,

$$\mathrm{vect}\{X_i, [X_i, X_j], [X_i, X_0], [X_i, [X_i, X_0]] \ i, j = 1 \cdots d\} = T_{(g,\xi)}G.$$

By Hörmander's Theorem g_t has a smooth density with respect to the *Haar* measure of G. With the notation of Section 2 the vectors fields X_i induce the following graduation of $\mathfrak{g}$:

$$C_1(g,\xi) = \left\{ (g,\xi) \begin{pmatrix} 0 & {}^t(u_i) & 0 \\ (u_i) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; (u_i) \in \mathbb{R}^d \right\} \tag{21}$$

$$C_2(g,\xi) = \left\{ (g,\xi) \begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & 0 \\ 0 & 0 & 0 \end{pmatrix} ; u_0 \in \mathbb{R}, (u_i) \in \mathbb{R}^d, u_{ji} = -u_{ji} \right\} \tag{22}$$

$$C_3(g,\xi) = T_{(g,\xi)}G. \tag{23}$$

Thus we have $r(g,\xi) = 3$. We choose a triangular basis $\beta := (X_i, X_0, [X_i, X_j], [X_i, X_0])_{i<j=1\cdots d}$ for $T_{(g,\xi)}G = (g,\xi)\mathfrak{g}$. The graded dimension Q is constant on G and is computed explicitly by:

$$Q(g,\xi) = d + 2(d(d-1)/2 + 1) + 3d = d^2 + 3d + 2.$$

The family of dilations on $\mathfrak{g}$ is:

$$T_\varepsilon \begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & (u_{i0}) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon\, {}^t(u_i) & \varepsilon^2 u_0 \\ \varepsilon(u_i) & \varepsilon^2(u_{ij}) & \varepsilon^3(u_{i0}) \\ 0 & 0 & 0 \end{pmatrix} \tag{24}$$

The homogeneous norm is given by (at $\mathbf{e} = (id, 0)$):

$$\left| \exp\left(\begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & (u_{i0}) \\ 0 & 0 & 0 \end{pmatrix} \right) \right|_{\mathbf{e}} = \left((\sum_{i=1}^d u_i^2)^{Q/2} + (u_0^2 + \sum_{1=i<j=d} u_{ij}^2)^{Q/4} + (\sum_{i=1}^d u_{i0}^2)^{Q/6} \right)^{1/Q} \tag{25}$$

The tangent process, which lies in $\mathfrak{g}$, is given by:

$$u_t^{\mathbf{e}} = \begin{pmatrix} 0 & B_t^1 & \cdots & B_t^d & t \\ B_t^1 & & & & \frac{1}{2}\left(\int_0^t B_s^1 ds - \int_0^t s \circ dB_s^1\right) \\ \vdots & & \left(\frac{1}{2}\left(\int_0^t B_s^i \circ dB_s^j - \int_0^t B_s^j \circ dB_s^i\right)\right)_{i=1\cdots d}^{j=1\cdots d} & & \vdots \\ B_t^d & & & & \frac{1}{2}\left(\int_0^t B_s^d ds - \int_0^t s \circ dB_s^d\right) \\ 0 & & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

Proposition 8 of [3] establishes that the support of $\{(g_t, \xi_t)\}_{t\geq 0}$ coincides with the future half cone at (g_0, ξ_0). Thus the Green function is strictly positive on this domain:

$$G(\mathbf{e}, (g,\xi)) > 0 \Longleftrightarrow q(\xi) > 0 \text{ and } \xi^0 > 0.$$

By scaling, the support of $T_{1/\varepsilon} \circ \exp^{-1}(g_{\varepsilon^2 s}, \xi_{\varepsilon^2 s})$ is the half cone

$$\{(g,\xi);\ (\varepsilon^2\xi^0)^2 - \|\varepsilon^3\vec{\xi}\|^2 > 0, \xi^0 > 0\}.$$

When ε goes to 0 this half cone becomes eventually the half space $\{(g,\xi) \in G | \xi^0 > 0\}$. From Proposition 1 we deduce that the support of the tangent process coincides with this half space.

We call a **homogeneous cone** with vertex (g_0, ξ_0), a subset $C_h(g_0, \xi_0)$ of G which is invariant under the dilatations T_ε at (g_0, ξ_0) such that the "sole"

$$\left\{(g, \xi) \in C_h(g_0, \xi_0);\ |(g, \xi)|_{(g_0,\xi_0)} = 1\right\}$$

be a compact subset of the half space:

$$\left\{(g, \xi) \in G;\ (\xi - \xi_0)^0 > 0\right\}.$$

By Theorem 1 we can find a neighborhood of (g_0, ξ_0) such that for every set B in a homogeneous cone $C_h(g_0, \xi_0)$, there exists $\alpha > 0,\ \beta > 0$ such that:

$$\forall (g, \xi) \in B, \quad \frac{\alpha}{|(g, \xi)|^{Q-2}_{(g_0,\xi_0)}} \leq G\left((g_0, \xi_0), (g, \xi)\right) \leq \frac{\beta}{|(g, \xi)|^{Q-2}_{(g_0,\xi_0)}}. \tag{27}$$

Recall: $Q = d(d + 3) - 2$.

Remark 1. *The right inequality in* (27) *remains true even if the sole is not compact.*

3.3 A Wiener criterion for thinness.

We say that a point (g, ξ) is regular with respect to a set B when $\mathbb{P}_{(g,\xi)}(T_B = 0) = 1$, where $T_B = \inf\{s > 0; (g_s, \xi_s) \in B\}$ is the entrance time in B.

We denote by B^r the set of regular points for B. By continuity of trajectories we have $\mathring{B} \subset B^r \subset \bar{B}$.

Proposition 8. *There exists only one measure μ_B supported by B^r such that:*

$$\mathbb{P}_{(g,\xi)}[T_B < +\infty] = \int G\left((g, \xi), (g', \xi')\right) \mu_B(d(g', \xi')).$$

Proof. It is shown in [3] that $\mathcal{L}$ admits an adjoint (with respect to the *Haar* measure of G) without zero-order term. Dual processes theory developped in [8] can be applied and ensures the existence and unicity of μ_B. □

We call capacity of B, and denote by $C(B)$, the total mass of μ_B. We also have

$$C(B) = \sup\{\mu(B); \mu \in \mathcal{M}(B), G\mu \leq 1\}. \tag{28}$$

Here $\mathcal{M}(B)$ is the set of finite positive measures supported in B. Let fix $\lambda < 1$. We denote by

$$B_n := \left\{(g, \xi) \in B;\ \lambda^{n+1} \leq |(g, \xi)|_{(g_0,\xi_0)} < \lambda^n\right\}$$

the homogeneous slices of B. We obtain the following Wiener criterion:

Proposition 9 (Wiener criterion)**.** *Let B a subset of G which is included in a homogeneous cone $C_h(g_0, \xi_0)$. Then (g_0, ξ_0) is regular for B if and only if $\sum_n \lambda^{d(d+3)n} C(B_n) = \infty$.*

Proof. The sketch of the proof is classical and can be found in [6] for the Brownian motion in $\mathbb{R}^n$ and in [16] in the case of a nilpotent group. Let us just recall that the key point is the double inequality (27), which allows to use a Borel-Cantelli lemma due to J. Lamperti in [21]. □

3.4 Capacities of small compact sets and Poincaré's cone condition.

Let H be the closure of a domain of $\mathfrak{g}$ included in

$$\left\{\begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & (u_{i0}) \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g};\ u_0 > 0\right\}.$$

We denote by $H_\varepsilon := \exp(T_\varepsilon(H))$ a "small" compacts set in the future of e. To study the behavior of $C(H_\varepsilon)$ when $\varepsilon \to 0$ we need the following lemma, where u_ε and v_ε are two points of H_ε and $\theta_{u_\varepsilon}(v_\varepsilon) := T_{1/|v_\varepsilon|_{u_\varepsilon}}(\exp^{-1}_{u_\varepsilon}(v_\varepsilon))$ is the solid angle, with respect to the homogeneous norm of v_ε (seen from u_ε).

Lemma 3. *The following limits exist:*

$$\lim_{\varepsilon\to 0}\frac{|v_\varepsilon|_{u_\varepsilon}}{\varepsilon} := \alpha(u,v) > 0.$$

$$\lim_{\varepsilon\to 0}\theta_{u_\varepsilon}(v_\varepsilon) := \beta(u,v) \in \mathfrak{g} - \{0\}.$$

Proof. Let $u, v \in \mathfrak{g}$ be such that $u_\varepsilon = \exp(T_\varepsilon(u))$ and $v_\varepsilon = \exp(T_\varepsilon(v))$. We want to find $w \in \mathfrak{g}$ such that $v_\varepsilon = u_\varepsilon \times \exp(w)$, *i.e*, $\exp(w) = \exp(-T_\varepsilon(u))\exp(T_\varepsilon(v))$.

By the Campell-Hausdorff formula we obtain:

$$\begin{aligned} w = T_\varepsilon(v) - T_\varepsilon(u) - \frac{1}{2}[T_\varepsilon(u), T_\varepsilon(v)] \\ + \frac{1}{12}([-T_\varepsilon(u), [-T_\varepsilon(u), T_\varepsilon(v)]] + [T_\varepsilon(v), [T_\varepsilon(v), -T_\varepsilon(u)]]) + \cdots \end{aligned} \tag{29}$$

A simple computation gives:

$$\begin{aligned} [T_\varepsilon(v), T_\varepsilon(u)] = \sum_{i=1}^{d} o(\varepsilon)X_i + \sum_{1\le i<j\le d}\left(\varepsilon^2(u_iv_j - u_jv_i) + o(\varepsilon^2)\right)[X_i, X_j] + \\ \sum_{i=1}^{d}\left(\varepsilon^3(v_iu_o - v_0u_i) + o(\varepsilon^3)\right)[X_i, X_0] + o(\varepsilon^2)X_0. \end{aligned}$$

With (29) we get:

$$\begin{aligned} w_i = \varepsilon(v_i - u_i) + o(\varepsilon), \quad w_0 = \varepsilon^2(v_0 - u_0) + o(\varepsilon^2), \\ w_{ij} = \varepsilon^2(v_{ij} - u_{ij} - \frac{1}{2}(v_iu_j - u_iv_j)) + o(\varepsilon^2), \\ w_{i0} = \varepsilon^3(v_{i0} - u_{i0} - \frac{1}{2}(u_iv_0 - u_0v_i)) + o(\varepsilon^3). \end{aligned}$$

By the definition of the homogeneous norm we have $\frac{|v_\varepsilon|_{u_\varepsilon}}{\varepsilon} = \frac{|w|_e}{\varepsilon}$ and this quantity converges, when ε goes to 0, to the homogeneous norm $\alpha(u,v) \neq 0$ of:

$$\begin{pmatrix} 0 & {}^t(v_i - u_i) & (v_0 - u_0) \\ (v_i - u_i) & (v_{ij} - u_{ij} - \frac{1}{2}(v_iu_j - u_iv_j)) & (v_{i0} - u_{i0} - \frac{1}{2}(u_iv_0 - u_0v_i)) \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Moreover, since by definition $\theta_{u_\varepsilon}(v_\varepsilon) = T_{1/|w|_e}(w)$ we have

$$\lim_{\varepsilon\to 0}\theta_{u_\varepsilon}(v_\varepsilon) = \beta(u,v),$$

where

$$\beta(u,v) := \frac{1}{\alpha(u,v)}\begin{pmatrix} 0 & {}^t(v_i - u_i) & (v_0 - u_0) \\ (v_i - u_i) & (v_{ij} - u_{ij} - \frac{1}{2}(v_iu_j - u_iv_j)) & (v_{i0} - u_{i0} - \frac{1}{2}(u_iv_0 - u_0v_i)) \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$$

□

Set

$$q(H) := \frac{m(H)}{\max_{u\in\partial H}\int_H \frac{\psi_e(\beta(u,v))}{\alpha(u,v)^{Q-2}}m(dv)}.$$

Recall: $\psi_x(\cdot) := \frac{1}{J_x(0)}g^{(x)}(0,\cdot)$ and $g^{(x)}$ is the Green function of the tangent process at x and we denote by m the image measure of $Haar$, on $\mathfrak{g}$, under exp^{-1}.

Proposition 10 (Capacities of small compact sets)**.**

$$\liminf_{\varepsilon\to 0}\frac{C(H_\varepsilon)}{\varepsilon^{Q-2}} \geq q(H).$$

Proof. Denote by $\nu_\varepsilon := \mathbf{1}_{H_\varepsilon} Haar$ the image of the measure $\nu := \mathbf{1}_H m$ under $\exp \circ T_\varepsilon$.

By (28), to obtain a lower bound of $C(H_\varepsilon)$ it is sufficient to get an upper bound for $G\nu_\varepsilon$. If τ denotes the entrance time in H_ε, we have, for $(g,\xi) \in G$:

$$\begin{aligned} G\nu_\varepsilon(g,\xi) &= \int_{H_\varepsilon} G((g,\xi),(g',\xi'))Haar(d(g',\xi')) \\ &= \mathbb{E}_{(g,\xi)}\left[\int_\tau^{+\infty} \mathbf{1}_{H_\varepsilon}(g_t,\xi_t)dt\right] \\ &= \mathbb{E}_{(g,\xi)}\left[\mathbb{E}_{(g_\tau,\xi_\tau)}\left[\int_0^{+\infty} \mathbf{1}_{H_\varepsilon}(g_t,\xi_t)dt\right]\right], \end{aligned}$$

hence

$$G\nu_\varepsilon(g,\xi) = \mathbb{E}_{(g,\xi)}\left[G\nu_\varepsilon(g_\tau,\xi_\tau)\right]. \tag{30}$$

Thus it is sufficient to find an upper bound for $G\nu_\varepsilon$ on ∂H_ε. For some $u_\varepsilon \in \partial H_\varepsilon$, by definition of ν_ε, we have:

$$G\nu_\varepsilon(u_\varepsilon) = \int_H G(u_\varepsilon, v_\varepsilon)m(dv),$$

where $v_\varepsilon := \exp(T_\varepsilon(v))$.

For any $\eta > 0$, Theorem 1 provides some $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$:

$$\forall u_\varepsilon, v_\varepsilon \in H_\varepsilon \quad G(u_\varepsilon, v_\varepsilon) \leq \frac{\eta + \psi_{u_\varepsilon}(\theta_{u_\varepsilon}(v_\varepsilon))}{|v_\varepsilon|_{u_\varepsilon}^{Q-2}}.$$

Moreover, by Lemma 3, we can find $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$:

$$\forall u_\varepsilon, v_\varepsilon \in H_\varepsilon \quad \varepsilon^{Q-2}\frac{\eta + \psi_{u_\varepsilon}(\theta_{u_\varepsilon}(v_\varepsilon))}{|v_\varepsilon|_{u_\varepsilon}^{Q-2}} \leq \frac{\eta + \psi_e(\beta(u,v))}{\alpha(u,v)^{Q-2}} + \eta$$

Thus,

$$\begin{aligned} \forall u_\varepsilon \in \partial H_\varepsilon \quad \varepsilon^{Q-2}G\nu_\varepsilon(u_\varepsilon) &\leq \int_H \left(\frac{\eta + \psi_e(\beta(u,v))}{\alpha(u,v)^{Q-2}} + \eta\right) m(dv). \\ &\leq \eta \times m(H) + \max_{u \in \partial H} \int_H \frac{\eta + \psi_e(\beta(u,v))}{\alpha(u,v)^{Q-2}} m(dv). \end{aligned}$$

By (30) we have

$$\|G\nu_\varepsilon\|_\infty \leq \frac{1}{\varepsilon^{Q-2}}\left(\eta \times m(H) + \max_{u \in \partial H} \int_H \frac{\eta + \psi_e(\beta(u,v))}{\alpha(u,v)^{Q-2}} m(dv)\right).$$

By (28), the previous upper bound provides the following lower bound for the capacity:

$$C(H_\varepsilon) \geq \frac{\varepsilon^{Q-2}\nu_\varepsilon(H_\varepsilon)}{\eta \times m(H) + \max_{u \in \partial H} \int_H \frac{\eta + \psi_e(\beta(u,v))}{\alpha(u,v)^{Q-2}} m(dv)}.$$

Since $\nu_\varepsilon(H_\varepsilon) = m(H)$,letting η go to 0 we finally obtain

$$\liminf_{\varepsilon \to 0} \frac{C(H_\varepsilon)}{\varepsilon^{Q-2}} \geq q(H) := \frac{m(H)}{\max_{u \in \partial H} \int_H \frac{\psi_e(\beta(u,v))}{\alpha(u,v)^{Q-2}} m(dv)}.$$

□

Proposition 9 and 10 above provide a Poincaré's cone condition of regularity:

Corollary 1 (Poincaré's cone condition of regularity)**.** *If a subset B contains a homogeneous cone based at (g_0, ξ_0) then this point is regular for B.*

Proof. It suffices to remark that the vertex of an homogeneous cone is regular for it. Indeed, the slices B_n are such that $B_n = T^{(g_0,\xi_0)}_{\lambda^n}(B_0)$ and the capacities $C(B_n)$ are, by Proposition 10, of order $\lambda^{n(2-Q)}$. Thus $\sum_n \lambda^{(Q-2)n} C(B_n)$ diverges and we conclude by the Wiener criterion of Proposition 9. □

Remark 2. *In [15] Franchi and Le Jan defined relativistic diffusions with value in the unitary tangent bundle, $T^1\mathcal{M}^{d+1}$, over some generic Lorentz manifold $\mathcal{M}^{d+1}$. Roughly speaking, their diffusions are obtained by "rolling without slipping" the space $\mathbb{H}^d \times \mathbb{R}^{1,d}$ over $T^1\mathcal{M}^{d+1}$ along some trajectory of Dudley's diffusion. The asymptotic behavior of such diffusions in Robertson-Walker space-times was studied in [2]. These diffusions are projections of diffusions r_t in the orthonormal frame bundle $\mathcal{OM}^{d+1}$ which are solution of the SDE*

$$dr_t = \sigma \sum_{i=1}^{d} V_i(r_t) \circ dB^i_t + H_0(r_t)dt,$$

where the V_i are vertical vector fields on $\mathcal{OM}^{d+1}$ corresponding to the infinitesimal action on the fibre of the infinitesimal boost $e_i \otimes e_0 + e_0 \otimes e_i \in so(1,d)$. The horizontal vector field H_0 denotes the infinitesimal parallel displacement of the frame along the geodesic started in the direction of the timelike vector of the frame.

The tangent process associated to r_t is the same that the one associated to g_t and we obtain a Wiener criterion and a Poincaré cone condition for r_t as well.

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