PLUS ULTRA

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Abstract. We develop some basic simplicity-theoretic facts for quasi-finitary ultraimaginaries, i.e. classes of finite tuples modulo $\emptyset$-invariant equivalence relations, in a supersimple theory. We also show feeble elimination of ultraimaginaries: If e is an ultraimaginary definable over a tuple a with $SU(a) < \omega^{\alpha+1}$, then e is eliminable up to rank $< \omega^a lpha$. Finally, we show some uniform versions of the weak canonical base property.

1. Introduction

This paper arose out of an attempt to understand and generalize [2, Proposition 1.14 and Lemma 1.15]. In doing so, we realized that certain stability-theoretic phenomena were best explained using ultraimaginaries. Although Ben Yaacov [1] has shown that no satisfactory independence theory can exist for all ultraimaginaries, as there are problems both with the finite character and with the extension axiom for independence, at least finite character can be salvaged if one restricts to quasi-finitary ultraimaginaries in a supersimple theory. This enables us to recover certain tools from simplicity theory, even though, due to the lack of extension, canonical bases are not available.

2. Ultraimaginaries

Definition 2.1. An *ultraimaginary* is the class a_E of a tuple a under an $\emptyset$-invariant equivalence relation E.

Clearly, we may assume that a is a countable tuple.

Definition 2.2. An ultraimaginary a_E is *definable* over a set A if any automorphism of the monster model fixing A stabilises the E-class of a. It is *bounded* over A if the orbit of a under the group of automorphisms

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of the monster model which fix A is contained in boundedly many E-classes. A *representative* of an ultraimaginary e is any tuple a such that e is definable over a.

Remark 2.3. As usual, if $E_A(x, y)$ is an A-invariant equivalence relation, one considers the $\emptyset$-invariant relation $E(xX, yY)$ given by

$$(X = Y \wedge X \equiv A \wedge E_X(x, y)) \vee (X = Y \wedge x = y).$$

This is an equivalence relation, and $(aA)_E$ is interdefinable over A with a_{E_A}.

Remark 2.4. As any $\emptyset$-invariant relation, E is given by a union of types over $\emptyset$.

Ultraimaginaries arise quite naturally in stability and simplicity theory.

Example 2.5. Let $p_A \in S(A)$ be a regular type in a simple theory. For $A', A'' \models \mathrm{tp}(A)$ put $E(A', A'')$ if $p_{A'} \not\perp p_{A''}$. Then E is an $\emptyset$-invariant equivalence relation, and A_E codes the non-orthogonality class of p_A.

The work with ultraimaginaries requires caution: Many basic properties become problematic, as we shall see below.

Example 2.6. [1] Let E be the $\emptyset$-invariant equivalence relation on infinite sequences which holds if they differ only on finitely many elements. Consider a sequence $I = (a_i : i < \omega)$ of elements such that no finite subtuple is bounded over the remaining elements. Then every finite tuple $\bar{a} \in I$ can be moved to a disjoint conjugate over I_E, but I cannot. Similarly, if I is a Morley sequence in a simple theory, then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}\ I_E$ for any finite $\bar{a} \in I$, but $I \not\mathop{\smile\!\!\!\!\!\!|}\ I_E$. If I is a sequence of independent realizations of pairwise orthogonal (or even perpendicular) regular types, then I_E is orthogonal (or perpendicular) to any finite subset of them, but not to all of them simultaneously. (We call two ultraimaginaries independent if they have representatives which are.)

Even in the ω-stable context, for classes of finite tuples, the theory is not smooth.

Example 2.7. Let T be the theory of a cycle-free graph (forest) of infinite valency, with predicates $P_n(x, y)$ for couples of points of distance n for all $< \omega$. It is easy to see by back-and-forth that T eliminates quantifiers and is ω-stable of rank ω; the formula $P_n(x, a)$ has rank n over a. Let E be the $\emptyset$-invariant equivalence relation of being in the same connected component. Then existence of non-forking extensions fails over a_E, as any two points in the connected component of a have some finite distance n, and hence rank n over one another, but rank $\geq k$ over a_E for all $k < \omega$, since a_E is definable over a point of distance k.

The same behaviour can be observed for any type p of rank $SU(p) = \omega$ in a simple theory, with the relation $E(x,y)$ on p which holds if $SU(x/y) < \omega$ and $SU(y/x) < \omega$ (actually, one follows from the other by Lascar's inequalities).

The behaviour of Example 2.7 is inconvenient and signifies that we shall avoid working *over* an ultraimaginary. The behaviour of Example 2.6 is outright vexatious; we shall restrict the class of ultraimaginaries under consideration in order to preserve the finite character of independence.

Definition 2.8. Let T be simple. An ultraimaginary e is *tame* if for all sets A, B of hyperimaginaries we have $e \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $e \mathop{\smile\!\!\!\!\!\!|}_A B_0$ for all finite subsets $B_0 \subseteq B$. It is *quasi-finitary* if there is a finite tuple a such that e is bounded over a. It is *supersimple* if it has a representative of ordinal SU-rank.

Remark 2.9. A supersimple ultraimaginary in a simple theory is quasi-finitary; in a supersimple theory the converse holds as well.

Proof: Suppose A is a representative for an ultraimaginary e with $SU(A) < \infty$, and let B be a real tuple with $A \in \mathrm{bdd}(B)$. Let $b \in B$ be a finite subtuple with $SU(A/b)$ minimal; it follows that $A \mathop{\smile\!\!\!\!\!\!|} bB$. Hence $A \subseteq \mathrm{bdd}(b)$ and $e \in \mathrm{bdd}^u(b)$, so e is quasi-finitary. In a supersimple theory the converse is obvious. □

We are really interested in the set of tame ultraimaginaries. However, we do not have a good criterion when an ultraimaginary is tame; moreover, an ultraimaginary definable over a tame ultraimaginary need not be tame itself. For instance, the sequence I in Example 2.6 is tame (since it is real), but I_E is not. We shall see in Corollary 3.5 that supersimple ultraimaginaries are tame. Clearly, an ultraimaginary definable (or even bounded) over a quasi-finitary / supersimple ultraimaginary is itself quasi-finitary / supersimple.

The set of all / all quasi-finitary ultraimaginaries definable over A will be denoted by $\mathrm{dcl}^u(A)$ / $\mathrm{dcl}^{qfu}(A)$, respectively. Similarly, $\mathrm{bdd}^u(A)$ / $\mathrm{bdd}^{qfu}(A)$ will denote the corresponding bounded closures. Recall that two tuples a and b have the same *Lascar strong type* over A, denoted $a \equiv_A^{lstp} b$ or $b \models \mathrm{lstp}(a/A)$, if they lie in the same class modulo all A-invariant equivalence relations with only boundedly many classes. This is the finest bounded A-invariant equivalence relation, so $\mathrm{bdd}^u(A)$ is bounded by the number of Lascar strong types over A.

Proposition 2.10. *The following are equivalent:*

(1) $\mathrm{bdd}^u(Aa) \cap \mathrm{bdd}^u(Ab) = \mathrm{bdd}^u(A)$.
(2) *For any* $a' \models \mathrm{lstp}(a/A)$ *there is* $n < \omega$ *and a sequence* $(a_i, b_i : i \leq n)$ *such that*
$$a_0 = a, \quad b_0 = b, \quad a_n = a'$$
and for each $i < n$
$$b_{i+1} \models \mathrm{lstp}(b_i/Aa_i) \quad \text{and} \quad a_{i+1} \models \mathrm{lstp}(a_i/Ab_{i+1}).$$

If a *and* b *are finite, this is also eqivalent to* $\mathrm{bdd}^{qfu}(Aa) \cap \mathrm{bdd}^{qfu}(Ab) = \mathrm{bdd}^{qfu}(A)$.

Proof: (1) $\Rightarrow$ (2) Suppose $\mathrm{bdd}^u(Aa) \cap \mathrm{bdd}^u(Ab) = \mathrm{bdd}^u(A)$, and define an A-invariant relation on $\mathrm{lstp}(ab/A)$ by $E(xy, x'y')$ if there is a sequence $(x_i, y_i : i \leq n)$ such that
$$ab \equiv^{lstp}_A x_0y_0, \quad x_0y_0 = xy, \quad x_ny_n = x'y'$$
and for each $i < n$
$$y_{i+1} \models \mathrm{lstp}(y_i/Ax_i) \quad \text{and} \quad x_{i+1} \models \mathrm{lstp}(x_i/Ay_{i+1}).$$
Clearly E is an equivalence relation. Now if $b' \models \mathrm{lstp}(b/Aa)$, then $\models E(ab, ab')$. Hence $(ab)_E \in \mathrm{bdd}^u(Aa)$. Similarly $(ab)_E \in \mathrm{bdd}^u(Ab)$, whence $(ab_E) \in \mathrm{bdd}^u(A)$. But for any $a' \models \mathrm{lstp}(a/A)$ there is b' with $ab \equiv^{lstp}_A a'b'$. Then $\models E(ab, a'b')$; in particular (2) holds.

(2) $\Rightarrow$ (1) Suppose not, and consider $e \in (\mathrm{bdd}^u(Aa) \cap \mathrm{bdd}^u(Ab)) \setminus \mathrm{bdd}^u(A)$. As $e \notin \mathrm{bdd}^u(A)$ there is $a' \models \mathrm{lstp}(a/A)$ with $e \notin \mathrm{bdd}^u(Aa')$. Consider a sequence $(a_i, b_i : i \leq n)$ as in (2). Since $b_{i+1} \models \mathrm{lstp}(b_i/Aa_i)$ and $a_{i+1} \models \mathrm{lstp}(a_i/Ab_{i+1})$ we have
$$\begin{aligned}\mathrm{bdd}^u(Aa_i) \cap \mathrm{bdd}^u(Ab_i) &= \mathrm{bdd}^u(Aa_i) \cap \mathrm{bdd}^u(Ab_{i+1}) \\ &= \mathrm{bdd}^u(Aa_{i+1}) \cap \mathrm{bdd}^u(Ab_{i+1}).\end{aligned}$$
In particular,
$$\begin{aligned}e \in \mathrm{bdd}^u(Aa) \cap \mathrm{bdd}^u(Ab) &= \mathrm{bdd}^u(Aa_0) \cap \mathrm{bdd}^u(Ab_0) \\ &\subseteq \mathrm{bdd}(Aa) \cap \mathrm{bdd}^u(Aa'),\end{aligned}$$
a contradiction.

The last assertion follows from the fact that for finite ab the ultra-imaginary $(ab)_E$ in the proof of (1) $\Rightarrow$ (2) is quasi-finitary. □

Remark 2.11. We shall see in Theorem 4.6 that supersimple theories of finite rank have weak elimination of quasi-finitary ultraimaginaries. We thus recover a Lemma of Lascar [3] (see also [4, Lemma 2.2]).

3. Ultraimaginaries in simple theories

From now on the ambient theory will be simple. Our notation is standard and follows [8]. We shall be working in a sufficiently saturated model of the ambient theory. Tuples are tuples of hyperimaginaries, and closures (definable, algebraic and bounded closures) will include hyperimaginaries.

Remark 3.1. Since in a simple theory Lascar strong type equals Kim-Pillay strong type, we have $\mathrm{bdd}^u(A) = \mathrm{dcl}^u(\mathrm{bdd}(A))$. But of course, as with real and imaginary algebraic closures, $\mathrm{bdd}(A) \cap \mathrm{bdd}(B) = \mathrm{bdd}(\emptyset)$ does not imply $\mathrm{bdd}^u(A) \cap \mathrm{bdd}^u(B) = \mathrm{bdd}^u(\emptyset)$.

We shall first see that ultraimaginary bounded closures of independent sets intersect trivially.

Lemma 3.2. *If* $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$, *then* $\mathrm{bdd}^u(A) \cap \mathrm{bdd}^u(C) \subseteq \mathrm{bdd}^u(B)$.

Proof: Consider $a_E \in \mathrm{bdd}^u(A) \cap \mathrm{bdd}^u(C)$. We may assume $a \mathop{\smile\!\!\!\!\!\!\mid}_{AB} C$, whence $Aa \mathop{\smile\!\!\!\!\!\!\mid}_B C$. Let $(a_i : i < \omega)$ be a Morley sequence in $\mathrm{lstp}(a/BC)$. Then $E(a_i, a_j)$ for all $i, j < \omega$. But $a_i \mathop{\smile\!\!\!\!\!\!\mid}_B a_j$ for $i \neq j$, so $\pi(x, a_j) = \mathrm{tp}(a_i/a_j)$ does not fork over B, and neither does $\pi(x, a)$. Note that $\pi(x, y)$ implies $E(x, y)$.

Now suppose $a_E \notin \mathrm{bdd}^u(B)$. We can then find a long sequence $(a'_i : i < \alpha)$ of B-conjugates of a such that $\neg E(a'_i, a'_j)$ for $i \neq j$. By the Erdös-Rado theorem there is an infinite B-indiscernible sequence $(a''_i : i < \omega)$ whose 2-type over B is among the 2-types of $(a'_i : i < \alpha)$. In particular $\neg E(a''_i, a''_j)$ for $i \neq j$, and $(\pi(x, a''_i) : i < \omega)$ is 2-inconsistent. Since $a''_0 \models \mathrm{tp}(a/B)$, we see that $\pi(x, a)$ divides over B, a contradiction. $\square$

Definition 3.3. We shall say that two ultraimaginaries e and e' are *independent* over A, denoted $e \mathop{\smile\!\!\!\!\!\!\mid}_A e'$, if they have representatives which are. (It e' is hyperimaginary, it will be its own representative.)

Even if not all ultraimaginaries satisfy finite character of independence, transitivity still holds.

Lemma 3.4. *Let* a *and* b *be hyperimaginary, and* e *ultraimaginary. If* $a \mathop{\smile\!\!\!\!\!\!\mid} e$ *and* $b \mathop{\smile\!\!\!\!\!\!\mid}_a e$, *then* $ab \mathop{\smile\!\!\!\!\!\!\mid} e$.

Proof: Suppose $e = c_E$. By hypothesis there are $c' \equiv c \equiv c''$ with $c_E = c'_E = c''_E$ and $a \mathop{\smile\!\!\!\!\!\!\;|}\, c'$ as well as $b \mathop{\smile\!\!\!\!\!\!\;|}_a c''$. Consider $c''' \equiv_{ac''} c'$ with $c''' \mathop{\smile\!\!\!\!\!\!\;|}_{ac''} b$. Then $b \mathop{\smile\!\!\!\!\!\!\;|}_a c''c'''$; as $c' \mathop{\smile\!\!\!\!\!\!\;|}\, a$ implies $c''' \mathop{\smile\!\!\!\!\!\!\;|}\, a$ we get $c''' \mathop{\smile\!\!\!\!\!\!\;|}\, ab$, and clearly $E(c', c'')$ yields $E(c''', c'')$, whence $c'''_E = c''_E = e$. □

Corollary 3.5. *A supersimple ultraimaginary is tame. In particular, quasi-finitary ultraimaginaries in a supersimple theory are tame.*

Proof: Let e be a supersimple ultraimaginary, and a a representative with $SU(a) < \infty$. Consider sets A and B. There is a finite $b \in B$ with $a \mathop{\smile\!\!\!\!\!\!\;|}_{Ab} B$. So $e \mathop{\smile\!\!\!\!\!\!\;|}_A B$ if and only if $e \mathop{\smile\!\!\!\!\!\!\;|}_A b$ by Lemma 3.4. Thus e is tame. □

In a supersimple theory quasi-finitary ultraimaginaries are the correct ones to consider: Due to elimination of hyperimaginaries all parameters consist of imaginaries of ordinal SU-rank; as canonical bases of such imaginaries are finite, we can always reduce to a quasi-finitary situation.

The following two Propositions tells us how to obtain invariant equivalence relations, and hence ultraimaginaries.

Proposition 3.6. *Let T be stable. For algebraically closed A and an $\emptyset$-invariant equivalence relation E on* $\mathrm{tp}(b)$, *consider the relation $R(X, Y)$ given by*

$$\exists xy\,[Xx \equiv Yy \equiv Ab \wedge x \mathop{\smile\!\!\!\!\!\!\;|}_X Y \wedge y \mathop{\smile\!\!\!\!\!\!\;|}_Y X \wedge E(x, y)].$$

Then R is an $\emptyset$-invariant equivalence relation on $\mathrm{tp}(A)$.

Proof: Clearly, R is $\emptyset$-invariant, reflexive and symmetric. So suppose that $R(A, A')$ and $R(A', A'')$ both hold, and let this be witnessed by b, b' and b^*, b''. Let $b_1 \models \mathrm{tp}(b'/A) = \mathrm{tp}(b^*/A')$ with $b_1 \mathop{\smile\!\!\!\!\!\!\;|}_{A'} AA''$. Since A' is algebraically closed, $b' \mathop{\smile\!\!\!\!\!\!\;|}_{A'} A$ and $b^* \mathop{\smile\!\!\!\!\!\!\;|}_{A'} A''$ we have $b_1 \equiv_{AA'} b'$ and $b_1 \equiv_{A'A''} b^*$. Hence there are b_0, b_2 with $bb' \equiv_{AA'} b_0b_1$ and $b^*b'' \equiv_{A'A''} b_1b_2$. In particular $E(b_0, b_1)$ and $E(b_1, b_2)$ hold, and so does $E(b_0, b_2)$. Moreover, we may assume $b_0 \mathop{\smile\!\!\!\!\!\!\;|}_{AA'b_1} A''$ and $b_2 \mathop{\smile\!\!\!\!\!\!\;|}_{A'A''b_1} A'$. Now $b_1 \mathop{\smile\!\!\!\!\!\!\;|}_{A'} AA''$ implies $b_0 \mathop{\smile\!\!\!\!\!\!\;|}_{AA'} A''$ and $b_2 \mathop{\smile\!\!\!\!\!\!\;|}_{A'A''} A$. Then $b_0 \mathop{\smile\!\!\!\!\!\!\;|}_A A'$ and $b_2 \mathop{\smile\!\!\!\!\!\!\;|}_{A''} A'$ imply $b_0 \mathop{\smile\!\!\!\!\!\!\;|}_A A''$ and $b_2 \mathop{\smile\!\!\!\!\!\!\;|}_{A''} A$, whence $R(A, A'')$ holds. So R is transitive. □

Recall that a reflexive and symmetric binary relation $R(x, y)$ on a partial type $\pi(x)$ is *generically transitive* if whenever $x, y, z \models \pi$, $x \mathop{\smile\!\!\!\!\!\!\;|}_y z$ and $R(x, y)$ and $R(y, z)$ both hold, then $R(x, z)$ holds as well.

Proposition 3.7. *Let T be simple. Suppose R is an $\emptyset$-invariant, reflexive, symmetric and generically transitive relation on* $\mathrm{lstp}(a)$*, and p is a regular type such that $SU_p(a)$ is finite. Let E be the transitive closure of R, and suppose $a_E \in \mathrm{bdd}^u(\mathrm{cl}_p(\emptyset))$. Then there is $a' \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} a$ with $R(a, a')$.*

Proof: Put $c = \mathrm{bdd}(a) \cap \mathrm{cl}_p(\emptyset)$. Then $a \mathop{\smile\!\!\!\!\!\mid}_c \mathrm{cl}_p(\emptyset)$, whence $a_E \in \mathrm{bdd}^u(c)$ by Lemma 3.2. Let $a' \equiv_c^{lstp} a$ with $a' \mathop{\smile\!\!\!\!\!\mid}_c a$. Then $a_E = a'_E$, so there is $n < \omega$ and a chain $a = a_0, a_1, \ldots, a_n = a'$ such that $R(a_i, a_{i+1})$ holds for all $i < n$. Put $a'_0 = a_0$, and for $0 < i < n$ let

$$a'_i \equiv_{a_i}^{lstp} a'_{i-1} \quad \text{with} \quad a'_i \mathop{\smile\!\!\!\!\!\mid}_{a_i} a_{i+1}.$$

Claim. $\mathrm{bdd}^u(a'_i) \cap \mathrm{bdd}^u(a_{i+1}) \subseteq \mathrm{bdd}^u(a_0)$.

Proof of Claim: For $i = 0$ this is trivial. For $i > 0$, as $a'_i \equiv_{a_i}^{lstp} a'_{i-1}$ and $\mathrm{bdd}^u(a_i) = \mathrm{dcl}^u(\mathrm{bdd}(a_i))$, we get

$$\mathrm{bdd}^u(a'_i) \cap \mathrm{bdd}^u(a_i) = \mathrm{bdd}^u(a'_{i-1}) \cap \mathrm{bdd}^u(a_i).$$

Next, $a'_i \mathop{\smile\!\!\!\!\!\mid}_{a_i} a_{i+1}$ implies

$$\mathrm{bdd}^u(a'_i a_i) \cap \mathrm{bdd}^u(a_i a_{i+1}) = \mathrm{bdd}^u(a_i)$$

by Lemma 3.2. Hence inductively

$$\begin{aligned} \mathrm{bdd}^u(a'_i) \cap \mathrm{bdd}^u(a_{i+1}) &\subseteq \mathrm{bdd}^u(a'_i) \cap \mathrm{bdd}^u(a_i) \\ &= \mathrm{bdd}^u(a'_{i-1}) \cap \mathrm{bdd}^u(a_i) \\ &\subseteq \mathrm{bdd}^u(a_0). \quad \square \end{aligned}$$

Now by generic transitivity and induction, $R(a'_i, a_{i+1})$ holds for all $i < n$. In particular $R(a'_{n-1}, a_n)$ holds, and by Lemma 3.2

$$\mathrm{bdd}^u(a'_{n-1}) \cap \mathrm{bdd}^u(a_n) \subseteq \mathrm{bdd}^u(a_0) \cap \mathrm{bdd}^u(a_n) = \mathrm{bdd}^u(c).$$

Choose a'' with $R(a'', a'_{n-1})$ such that $SU_p(a''/a'_{n-1})$ is maximal possible. We may choose it such that $a'' \mathop{\smile\!\!\!\!\!\mid}_{a'_{n-1}} a_n$. Then

$$\mathrm{bdd}^u(a'') \cap \mathrm{bdd}^u(a_n) \subseteq \mathrm{bdd}^u(a_n) \cap \mathrm{bdd}^u(a'_{n-1}) \subseteq \mathrm{bdd}^u(c)$$

and

$$SU_p(a''/a_n) \geq SU_p(a''/a'_{n-1}a_n) = SU_p(a''/a'_{n-1}).$$

Rename $a''a_n$ as $a_1 a_2$, and note that $\mathrm{bdd}^u(a_1) \cap \mathrm{bdd}(a_2) \subseteq \mathrm{bdd}^u(c)$, $c \subseteq \mathrm{bdd}(a_2)$, and $SU_p(a_1/a_2)$ is maximal possible among tuples (x, y) with $R(x, y)$. Moreover,

$$SU_p(a_2/a_1) = SU_p(a_1a_2) - SU_p(a_1) = SU_p(a_1a_2) - SU_p(a_2) = SU_p(a_1/a_2),$$

so this is also maximal.

Choose $a_3 \mathop{\smile\!\!\!\!\!\!\mid}_{a_2} a_1$ with $a_3 \equiv^{lstp}_{a_2} a_1$. By generic transitivity $R(a_1, a_3)$ holds. Moreover,

$$SU_p(a_3/a_1) \geq SU_p(a_3/a_1a_2) = SU_p(a_3/a_2),$$

so equality holds. Similarly,

$$SU_p(a_1/a_3) = SU_p(a_1/a_2a_3) = SU_p(a_1/a_2).$$

For a set A let

$$\mathrm{cl}_p(A) = \{a : SU_p(a/A) = 0\}$$

denote the p-closure of A. Then $SU_p(a_i/a_j) = SU_p(a_i/a_ja_k)$ for $\{i, j, k\} = \{1, 2, 3\}$ means that

$$\mathrm{cl}_p(a_i) \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(a_j)} \mathrm{cl}_p(a_k).$$

In particular,

$$\mathrm{cl}_p(a_i) \cap \mathrm{cl}_p(a_k) = \mathrm{cl}_p(a_1) \cap \mathrm{cl}_p(a_2) \cap \mathrm{cl}_p(a_3).$$

Let $b = \mathrm{cl}_p(a_1) \cap \mathrm{cl}_p(a_2) \cap \mathrm{bdd}(a_1a_2)$. Then $\mathrm{cl}_p(a_1) \cap \mathrm{cl}_p(a_2) = \mathrm{cl}_p(b)$ by [5, Lemma 3.18]. Let $F(x, y)$ be the $\emptyset$-invariant equivalence relation on $\mathrm{lstp}(b)$ given by $\mathrm{cl}_p(x) = \mathrm{cl}_p(y)$. As b_F is fixed by the $\mathrm{bdd}(a_2)$-automorphism moving a_1 to a_3 and $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_{a_2} a_3$, we get $b_F \in \mathrm{bdd}^u(a_2)$ by Lemma 3.2. Similarly, considering an $a_3' \mathop{\smile\!\!\!\!\!\!\mid}_{a_1} a_2$ with $a_3' \equiv^{lstp}_{a_1} a_2$ we obtain $b_F \in \mathrm{bdd}^u(a_1)$, whence

$$b_F \in \mathrm{bdd}^u(a_1) \cap \mathrm{bdd}^u(a_2) \subseteq \mathrm{bdd}^u(c).$$

So if $b' \mathop{\smile\!\!\!\!\!\!\mid}_c b$ satisfies $\mathrm{lstp}(b/c)$, then $b'_F = b_F$ and

$$\mathrm{cl}_p(b') = \mathrm{cl}_p(b) = \mathrm{cl}_p(c) = \mathrm{cl}_p(\emptyset).$$

But now

$$\mathrm{Cb}(a_3/\mathrm{cl}_p(a_1)\mathrm{cl}_p(a_2)) \subseteq \mathrm{cl}_p(a_1) \cap \mathrm{cl}_p(a_2) = \mathrm{cl}_p(b) = \mathrm{cl}_p(\emptyset),$$

so $a_3 \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} a_2$, as required. □

Remark 3.8. We cannot generalize [8, Lemma 3.3.1] and strengthen Proposition 3.7 to say that if R is $\emptyset$-invariant, reflexive, symmetric and generically transitive on a Lascar strong type, then the transitive closure E of R equals the 2-step iteration of R. Consider on the forest of Example 2.7 the relation $R(a, b)$ which holds if 3 divides the distance between a and b. This is generically transitive, as for $a' \mathop{\smile\!\!\!\!\!\!\mid}_a a''$ the distance between a' and a'' is the sum of the distances between a' and a and between a and a''. However, two points of distance 2 are easily seen to be R^2-related, so the transitive closure E of R is just the relation of being in the same connected component. But no two points of distance 1 are R^2-related.

Definition 3.9. We shall say that an ultraimaginary e is (*almost*) Σ-*internal*, or is Σ-*analysable*, if it has a representative which is. Similarly, e is orthogonal over A to some type p if for all $B \mathop{\smile\!\!\!\!\!\!|}_A e$ such that p is over B and for any realization $b \models p|B$ we have $e \mathop{\smile\!\!\!\!\!\!|}_A Bb$.

Remark 3.10. This definition does not imply that we define the notion of an analysis of an ultraimaginary. Moreover, e orthogonal to p over A does not imply that e has a representative which is orthogonal to p.

Let Σ be an $\emptyset$-invariant family of partial types, and recall that the first Σ-level of a over A is the set

$$\ell_1^\Sigma(a/A) = \{b \in \mathrm{bdd}(aA) : \mathrm{tp}(b/A) \text{ is almost } \Sigma\text{-internal}\}.$$

Lemma 3.11. *If $c = \ell_1^\Sigma(b)$ and $\mathrm{tp}(c'/A)$ is almost Σ-internal, where $A \mathop{\smile\!\!\!\!\!\!|} b$, then $b \mathop{\smile\!\!\!\!\!\!|}_c Ac'$.*

Proof: $\mathrm{Cb}(Acc'/b)$ is definable over a Morley sequence in $\mathrm{lstp}(Acc'/b)$ and thus almost Σ-internal, as $A \mathop{\smile\!\!\!\!\!\!|} b$. So $\mathrm{Cb}(Acc'/b) \subseteq c$ by definition. $\square$

Proposition 3.12. *Let T be simple. Suppose b_E is an ultraimaginary non-orthogonal to some regular type p, and $SU_p(\ell_1^p(b)) < \omega$. Then there is almost p-internal $e \in \mathrm{bdd}^u(b_E) \setminus \mathrm{bdd}^u(\mathrm{cl}_p(\emptyset))$. Moreover, $e \in \mathrm{bdd}^u(\ell_1^p(b))$.*

Proof: Let $c = \ell_1^p(b)$. Define an $\emptyset$-invariant relation R on $\mathrm{tp}(c)$ by

$$R(c', c'') \quad \Leftrightarrow \quad \exists b'b''\ b'c' \equiv b''c'' \equiv bc \wedge E(b', b'').$$

This is reflexive and symmetric; moreover for $c' \mathop{\smile\!\!\!\!\!\!|}_{c''} c'''$ with $R(c', c'')$ and $R(c'', c''')$ we can find b', b'', b^*, b''' with

$$b'c' \equiv b''c'' \equiv b^*c'' \equiv b'''c''' \equiv bc,$$

such that $E(b', b'')$ and $E(b^*, b''')$ hold. Since c'' is boundedly closed, $b'' \equiv_{c''}^{lstp} b^*$, and $b'' \mathop{\smile\!\!\!\!\!\!|}_{c''} c'$ and $b^* \mathop{\smile\!\!\!\!\!\!|}_{c''} c'''$ by Lemma 3.11. By the Independence Theorem we can assume $b'' = b^*$, so $E(b', b''')$ and $R(c', c''')$ hold. Hence R is generically transitive; let F be its transitive closure. The class c_F is clearly almost p-internal. Moreover, if $\models E(b', b)$ and $b'c' \equiv bc$, then $\models F(c', c)$, so c_F is bounded over b_E.

Finally, suppose $c_F \in \mathrm{bdd}^u(\mathrm{cl}_p(\emptyset))$. By Proposition 3.7 there is $c' \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}_p(\emptyset)} c$ with $\models R(c', c)$. Hence there are b', b^* with $b'c' \equiv b^*c \equiv bc$

and $\models E(b', b^*)$. Applying a c-automorphism (and moving c'), we may assume $b = b^*$. Let $A \mathop{\smile\!\!\!\!\!\!\mid}\, b$ be some parameters and a some realizations of p over A with $a \not\mathop{\smile\!\!\!\!\!\!\mid}_A b_E$; we may assume $cb'c' \mathop{\smile\!\!\!\!\!\!\mid}_b Aa$, whence $A \mathop{\smile\!\!\!\!\!\!\mid}\, bcb'c'$. Then $b \mathop{\smile\!\!\!\!\!\!\mid}_c Aa$ by Lemma 3.11, whence $b'c' \mathop{\smile\!\!\!\!\!\!\mid}_{Ac} a$ and $b'c' \mathop{\smile\!\!\!\!\!\!\mid}_c Aa$. Thus $b'c' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(c)} Aa$. Now $c' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} c$ yields $c' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} \mathrm{cl}_p(c)$, and hence $c' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} Aa$. As $a \mathop{\smile\!\!\!\!\!\!\mid}_A \mathrm{cl}_p(\emptyset)$ we get $a \mathop{\smile\!\!\!\!\!\!\mid}_A c'$. Now $b' \mathop{\smile\!\!\!\!\!\!\mid}_{c'} Aa$ by Lemma 3.11, whence $b' \mathop{\smile\!\!\!\!\!\!\mid}_A a$. As $b_E = b'_E$ we obtain $a \mathop{\smile\!\!\!\!\!\!\mid}_A b_E$, a contradiction. □

Corollary 3.13. *Let e be a supersimple ultraimaginary. Suppose e is non-orthogonal to some regular type p over some set B. Then there is an almost p-internal supersimple $e' \in \mathrm{bdd}^{qfu}(Be) \setminus \mathrm{bdd}^{qfu}(\mathrm{cl}_p(B))$.*

Proof: Let a be a representative of e with $SU(a) < \infty$ and put $b = \mathrm{Cb}(a/B)$. Then $SU(b) < \infty$, as b is bounded over a finite initial segment of a Morley sequence in $\mathrm{lstp}(a/B)$. Now $e \mathop{\smile\!\!\!\!\!\!\mid}_b B$, so $\mathrm{tp}(e/b)$ is non-orthogonal to p. Note that $SU_p(\ell_1^p(a/b)/b)$ is finite by supersimplicity. By Proposition 3.12 applied over b there is an almost p-internal ultraimaginary $e' \in \mathrm{bdd}^u(be) \setminus \mathrm{bdd}^u(\mathrm{cl}_p(b))$; moreover $e' \in \mathrm{bdd}^u(\ell_1^p(a/b)) \subseteq \mathrm{bdd}(ab)$. Thus e' is supersimple, almost p-internal over b and thus over B; it is quasi-finitary by Remark 2.9. □

Remark 3.14. For hyperimaginary e in a simple theory, the proof of Corollary 3.13 uses the canonical base of some type over e. As we cannot consider types over ultraimaginaries, this does not make sense in our context.

Proposition 3.15. *Let T be supersimple. If $AB \mathop{\smile\!\!\!\!\!\!\mid}\, D$ and $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset)$, then $\mathrm{bdd}^{qfu}(AD) \cap \mathrm{bdd}^{qfu}(BD) = \mathrm{bdd}^{qfu}(D)$.*

Proof: We may assume that A, B and D are boundedly closed. Consider

$$e \in (\mathrm{bdd}^{qfu}(AD) \cap \mathrm{bdd}^{qfu}(BD)) \setminus \mathrm{bdd}^{qfu}(D).$$

Let p be a regular type of least SU-rank non-orthogonal to e over D. This exists by transitivity since e is tame. By Corollary 3.13 we may assume that e is almost p-internal of finite SU_p-rank over D; let a' be a representative which is almost p-internal over D. Put $a = \mathrm{Cb}(a'D/A)$. As $a \mathop{\smile\!\!\!\!\!\!\mid}\, D$ we obtain that $\mathrm{tp}(a)$ is almost p-internal; note that $SU(a) < \infty$. Since $e \mathop{\smile\!\!\!\!\!\!\mid}_{aD} A$, Lemma 3.2 implies $e \in \mathrm{bdd}^{qfu}(aD)$. So we may assume that $A = \mathrm{bdd}(a)$ and $SU_p(A) < \omega$. Similarly, we may assume that $D = \mathrm{bdd}(\mathrm{Cb}(aa'/D))$ is the bounded closure of a finite set.

Let $(A_i : i < \omega)$ be a Morley sequence in $\mathrm{lstp}(A/BD)$ with $A_0 = A$, and put $B' = \mathrm{bdd}(A_1A_2)$. Then B' is almost p-internal of finite SU_p-rank. Since $e \in \mathrm{bdd}^{qfu}(BD)$ we have $e \in \mathrm{bdd}^{qfu}(A_iD)$ for all $i < \omega$. Let e' be the set of $B'D$-conjugates of e, again a quasi-finitary ultraimaginary. Since any $B'D$-conjugate of e is again in

$$\begin{aligned}\mathrm{bdd}^{qfu}(A_1D) \cap \mathrm{bdd}^{qfu}(A_2D) &= \mathrm{bdd}^{qfu}(BD) \cap \mathrm{bdd}^{qfu}(A_1D)\\ &= \mathrm{bdd}^{qfu}(BD) \cap \mathrm{bdd}^{qfu}(AD),\end{aligned}$$

we have $e' \in \mathrm{dcl}^{qfu}(B'D) \cap \mathrm{bdd}^{qfu}(AD)$. Moreover, $B' \mathop{\smile\!\!\!\!\!\!\mid}_{BD} A$, whence $B' \mathop{\smile\!\!\!\!\!\!\mid}_{B} A$ and

$$\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B') \subseteq \mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset).$$

Choose $A' \equiv^{lstp}_{AD} B'$ with $A' \mathop{\smile\!\!\!\!\!\!\mid}_{AD} B'$. Then $e' \in \mathrm{dcl}^{qfu}(A'D) \cap \mathrm{dcl}^{qfu}(B'D)$. Furthermore, $D \mathop{\smile\!\!\!\!\!\!\mid}_{B} A$ implies $D \mathop{\smile\!\!\!\!\!\!\mid}_{B} AB'$; as $D \mathop{\smile\!\!\!\!\!\!\mid} B$ we get $D \mathop{\smile\!\!\!\!\!\!\mid} ABB'$ and $B' \mathop{\smile\!\!\!\!\!\!\mid}_{A} D$. Therefore $A' \mathop{\smile\!\!\!\!\!\!\mid}_{A} D$, whence $A' \mathop{\smile\!\!\!\!\!\!\mid}_{A} B'$ and

$$\mathrm{bdd}^{qfu}(A') \cap \mathrm{bdd}^{qfu}(B') \subseteq \mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B') = \mathrm{bdd}^{qfu}(\emptyset).$$

We may assume $e' = (A'D)_E$ for some $\emptyset$-invariant equivalence relation E. Define a $\emptyset$-invariant reflexive and symmetric relation R on $\mathrm{lstp}(A')$ by

$$R(X, Y) \Leftrightarrow \exists Z\ [XZ \equiv YZ \equiv A'D \wedge Z \mathop{\smile\!\!\!\!\!\!\mid} XY \wedge E(XZ, YZ)].$$

By the independence theorem, if $A_1 \mathop{\smile\!\!\!\!\!\!\mid}_{A_2} A_3$ such that $R(A_1, A_2)$ and $R(A_2, A_3)$ hold, we have $R(A_1, A_3)$. Hence R is generically transitive; let E' be the transitive closure of R. Clearly $A'_{E'}$ is quasi-finitary.

Next, consider $A'' \equiv_{B'} A'$ with $A'' \mathop{\smile\!\!\!\!\!\!\mid}_{B'} A'$. By the independence theorem there is D' with $A'D \equiv_{B'} A'D' \equiv_{B'} A''D'$ and $D' \mathop{\smile\!\!\!\!\!\!\mid}_{B'} A'A''$. Then $D' \mathop{\smile\!\!\!\!\!\!\mid} B'$, whence $D' \mathop{\smile\!\!\!\!\!\!\mid} A'A''$ and $(A'D')_E = (A''D')_E \in \mathrm{dcl}^{qfu}(B'D')$. Therefore $E'(A', A'')$ holds and $A'_{E'} \in \mathrm{dcl}^{qfu}(B')$. Thus

$$A'_{E'} \in \mathrm{dcl}^{qfu}(A') \cap \mathrm{dcl}^{qfu}(B') \subseteq \mathrm{bdd}^{qfu}(\emptyset).$$

By Proposition 3.7 there is $A'' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} A'$ with $R(A', A'')$. Let D' witness $R(A', A'')$. Then $D' \equiv_{A'} D$, so we may assume $D' = D$. Since $\mathrm{cl}_p(D) \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} \mathrm{cl}_p(A'A'')$ and $\mathrm{cl}_p(A') \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} \mathrm{cl}_p(A'')$ we obtain

$$\mathrm{cl}_p(A') \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(\emptyset)} \mathrm{cl}_p(A'')\mathrm{cl}_p(D)$$

and hence $A' \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{cl}_p(D)} A''$. But now

$$e' = (A'D)_E = (A''D)_E \in \mathrm{dcl}^{qfu}(A'D) \cap \mathrm{dcl}^{qfu}(A''D) \subseteq \mathrm{bdd}^{qfu}(\mathrm{cl}_p(D))$$

by Lemma 3.2. Since $e \in \mathrm{bdd}^{qfu}(e')$, this contradicts non-orthogonality of e to p over D. □

Remark 3.16. Again, the proof of the hyperimaginary analogue of Proposition 3.15 for simple theories uses canonical bases and does not generalize.

4. Elimination of ultraimaginaries

On cannot avoid the non-tame ultraimaginaries of Example 2.6 which do not satisfy finite character and hence cannot be eliminated. Similarly, on a type of rank ω we cannot eliminate the relation of having mutually finite rank over each other (example 2.7), since the rank over a class modulo such a relation is not defined. We thus content ourselves with elimination of ultraimaginaries of ordinal SU-rank in a simple theory (and in particular of quasi-finitary ultraimaginaries in a supersimple theory) up to rank of lower order of magnitude. This seems to be optimal, given the examples cited.

Definition 4.1. Let e be ultraimaginary. We shall say that $SU(a/e) < \omega^\alpha$ if for all representatives b of e we have $SU(a/b) < \omega^\alpha$. Conversely, $SU(e/a) < \alpha^\omega$ if there is a representative b with $SU(b/a) < \omega^\alpha$.

Remark 4.2. This does not mean that we define the value of $SU(a/e)$ or of $SU(e/a)$. In fact, one might define

$$SU(e/a) = \min\{SU(b/a) : \ b \text{ a representative of } e\},$$

but this suggests a precision I am not sure exists.

Lemma 4.3. *$SU(e/a) < \omega^0$ if and only if $e \in \mathrm{bdd}^u(a)$, and $SU(a/e) < \omega^0$ if and only if $a \in \mathrm{bdd}(e)$.*

Proof: If b is a representative of e with $SU(b/a) < \omega^0$, then $b \in \mathrm{bdd}(a)$, so $e \in \mathrm{bdd}^u(a)$. If $e \in \mathrm{bdd}^u(a)$, then $e \in \mathrm{dcl}^u(\mathrm{bdd}(a))$, so $b = \mathrm{bdd}(a)$ is a representative of e with $SU(b/a) < \omega^0$.

If $a \notin \mathrm{bdd}(e)$, then there are arbitrarily many e-conjugates of a. Then for any representative b of e there is some e-conjugate a' of a which is not in $\mathrm{bdd}(b)$. Let b' be the image of b under an e-automorphism mapping a' to a. Then b' is a representative of e, and $SU(a/b') \geq \omega^0$. On the other hand, if $a \in \mathrm{bdd}(e)$, then $a \in \mathrm{bdd}(b)$ for any representative b of e, whence $SU(a/b) < \omega^0$. □

Definition 4.4. An ultraimaginary e can be *α-eliminated* if there is a representative a with $SU(a/e) < \omega^\alpha$. A supersimple theory has *feeble* *elimination of ultraimaginaries* if for all ordinals α, all quasi-finitary ultraimaginaries of rank $< \omega^{\alpha+1}$ can be α-eliminated.

Remark 4.5. 0-elimination is usually called *weak* elimination; in the presence of imaginaries this equals full elimination. I do not know what the definition of feeble elimination of ultraimaginaries should be in general for simple theories — but then their whole theory is much more problematic.

Theorem 4.6. *If e is ultraimaginary with $SU(e) < \omega^{\alpha+1}$, then e can be α-eliminated. A supersimple theory has feeble elimination of quasi-finitary ultraimaginaries; a supersimple theory of finite rank has elimination of quasi-finitary ultraimaginaries.*

Proof: Let a be a representative of e of minimal rank. Since $SU(e) < \omega^{\alpha+1}$ we have $SU(a) < \omega^{\alpha+1}$. Suppose $SU(a/e) \geq \omega^\alpha$. Then there is some representative b of e with $SU(a/b) \geq \omega^\alpha$; we choose it such that $SU(a/b) \geq \omega^\alpha \cdot n$ for some maximal $n \geq 1$. Consider $a' \equiv_b^{lstp} a$ with $a' \perp_b a$. Since $e \in \mathrm{dcl}^u(b)$ we have $e \in \mathrm{dcl}^u(a')$, and

$$SU(a/a') \geq SU(a/a'b) = SU(a/b) \geq \omega^\alpha \cdot n.$$

By maximality of n we get

$$SU(a/a') < SU(a/a'b) + \omega^\alpha.$$

Hence, if

$$\mathrm{cl}_\alpha(A) = \{c : SU(c/A) < \omega^\alpha\}$$

denotes the α-closure of A, we have

$$a \underset{\mathrm{cl}_\alpha(a')}{\perp} \mathrm{cl}_\alpha(b).$$

On the other hand, $a \perp_b a'$ implies

$$a \underset{\mathrm{cl}_\alpha(b)}{\perp} \mathrm{cl}_\alpha(a'),$$

so

$$c = \mathrm{Cb}(a/\mathrm{cl}_\alpha(b)\mathrm{cl}_\alpha(a')) \subseteq \mathrm{cl}_\alpha(b) \cap \mathrm{cl}_\alpha(a').$$

Then $a \perp_c b$, so $e \in \mathrm{bdd}^u(c)$ by Lemma 3.2. On the other hand, $SU(c/a') < \omega^\alpha$, and $SU(a'/c) \geq SU(a'/cb) \geq \omega^\alpha$ since $SU(a'/b) \geq \omega^\alpha$ and $SU(c/b) < \omega^\alpha$. It follows that

$$SU(a) = SU(a') \geq SU(c) + \omega^\alpha.$$

In particular $\mathrm{bdd}(c)$ is a representative for e of lower rank, a contradiction. □

Remark 4.7. Let p be a regular type (or type of weight 1). Then two realizations a and b of p are independent if and only if $\mathrm{bdd}^{qfu}(a) \cap \mathrm{bdd}^{qfu}(b) = \mathrm{bdd}^{qfu}(\emptyset)$: One direction is Lemma 3.2, the other follows from the observation that dependence is an invariant equivalence relation on realizations of p. However, this does not always hold: By elimination of quasifinite ultraimaginaries, it is in particular false in non one-based theories of finite rank.

5. Decomposition

In this section Σ will be an $\emptyset$-invariant family of partial types in a simple theory. The following lemma is folklore, but we give a proof for completeness.

Lemma 5.1.
1. *Suppose $a \,\underline{\square}\, b$. If $c \perp a$ and $c \perp b$, then $a \,\underline{\square}_c\, b$.*
2. *Suppose $a \,\underline{\square}_c\, b$. If $c \perp ab$ then $a \,\underline{\square}\, b$.*
3. *Suppose $a \,\underline{\square}_c\, b$. If $\mathrm{tp}(a)$ and $\mathrm{tp}(b)$ are foreign to $\mathrm{tp}(c)$, then $a \,\underline{\square}\, b$.*

Proof:

1. Consider any d with $d \not\perp_c a$. Then $cd \not\perp a$, whence $cd \not\perp b$. Now $b \perp c$ implies $b \not\perp_c d$. The converse follows by symmetry.
2. Consider any d with $d \not\perp a$. Clearly we may assume $d \perp_{ab} c$, whence $abd \perp c$. Since $a \perp c$ we get $d \not\perp_c a$, whence $d \not\perp_c b$ and $cd \not\perp b$. But $c \perp_d b$, so $d \not\perp b$; the converse follows by symmetry.
3. Consider any d with $d \not\perp a$. Since $a \perp c$ we get $d \not\perp_c a$, whence $d \not\perp_c b$ and $cd \not\perp b$. If $b \perp d$, then $b \perp_d c$ by foreigness, whence $b \perp cd$, a contradiction. So $b \not\perp d$; the converse follows by symmetry. $\square$

Here are two versions of [2, Lemma 1.15].

Proposition 5.2. *Let A, B, a, b be (hyperimaginary) sets, such that a is quasi-finitary, $\mathrm{bdd}^{qfu}(Aa) \cap \mathrm{bdd}^{qfu}(Bb) = \mathrm{bdd}^{qfu}(\emptyset)$, and a and b are domination-equivalent over AB. Suppose AB is Σ-analysable, and $\mathrm{tp}(a/A)$ and $\mathrm{tp}(b/B)$ are foreign to Σ. Then $a \in \mathrm{bdd}(A)$ and $b \in \mathrm{bdd}(B)$.*

Proof: Suppose otherwise. Put $A_0 = \mathrm{Cb}(A/a)$. Then A_0 is Σ-analysable, so $a \perp_A A_0$. It follows that $A_0 \subseteq \mathrm{bdd}(A) \cap \mathrm{bdd}(a)$. Lemma 5.1(3) yields that a and b are domination-equivalent over A_0B. We may thus assume that Aa is quasi-finitary.

Define an $\emptyset$-invariant relation E on $\mathrm{lstp}(Aa)$ by

$$E(A'a', A''a'') \quad \Leftrightarrow \quad a' \,\underline{\square}_{A'A''}\, a''.$$

Clearly, this is reflexive and symmetric. Suppose $E(A'a', A''a'')$ and $E(A''a'', A'''a''')$. By Lemma 5.1(1)

$$a' \,\underline{\square}_{A'A''A'''}\, a'' \quad \text{and} \quad a'' \,\underline{\square}_{A'A''A'''}\, a''',$$

whence $a' \,\underline{\square}_{A'A''A'''}\, a'''$. Now $a' \,\underline{\square}_{A'A'''}\, a'''$ by Lemma 5.1(3). Thus $E(A'a', A'''a''')$ holds and E is transitive.

Let $A'a' \equiv^{lstp}_{Bb} Aa$ with $A'a' \mathop{\smile\hskip-0.9em ^| \ }_{Bb} Aa$. Again by Lemma 5.1(1)

$$a \,\underline{\square}_{AA'B}\, b \,\underline{\square}_{AA'B}\, a',$$

and $a \,\underline{\square}_{AA'}\, a'$ by Lemma 5.1(3). Thus $E(Aa, A'a')$ holds. But

$$\mathrm{bdd}^{qfu}(Aa) \cap \mathrm{bdd}^{qfu}(A'a') \subseteq \mathrm{bdd}^{qfu}(Aa) \cap \mathrm{bdd}^{qfu}(Bb) = \mathrm{bdd}^{qfu}(\emptyset).$$

Hence $(Aa)_E = (A'a')_E \in \mathrm{bdd}^{qfu}(\emptyset)$, and there is $A''a'' \mathop{\smile\hskip-0.9em ^| \ } Aa$ with $E(Aa, A''a'')$. But then $a \,\underline{\square}_{AA''}\, a''$ and $a \mathop{\smile\hskip-0.9em ^| \ }_{AA''} a''$ yield $a \mathop{\smile\hskip-0.9em ^| \ }_{AA''} a$, whence $a \in \mathrm{bdd}(AA'')$ and finally $a \in \mathrm{bdd}(A)$ as $a \mathop{\smile\hskip-0.9em ^| \ }_A A''$. Similarly, $b \in \mathrm{bdd}(B)$. □

Remark 5.3. If a is not quasi-finitary, the conclusion still holds if we assume $\mathrm{bdd}^u(Aa) \cap \mathrm{bdd}^u(Bb) = \mathrm{bdd}^u(\emptyset)$.

Corollary 5.4. *Let A, B, a, b be (hyperimaginary) sets, such that a is quasi-finitary, $\mathrm{bdd}^{qfu}(Aa) \cap \mathrm{bdd}^{qfu}(Bb) = \mathrm{bdd}^{qfu}(\emptyset)$, and a and b are interbounded over AB. Suppose AB is Σ-analysable. Then Aa and Bb are Σ-analysable.*

Proof: Clearly we may assume that $A \subseteq \mathrm{bdd}(Aa)$ and $B \subseteq \mathrm{bdd}(Bb)$ are maximal Σ-analysable subsets. Hence $\mathrm{tp}(a/A)$ and $\mathrm{tp}(b/B)$ are foreign to Σ. Since a and b are interbounded over AB, they are domination-equivalent, contradicting Proposition 5.2. □

Remark 5.5. By Theorem 4.6, if $SU(Aa)$ or $SU(Bb)$ is finite, then $\mathrm{bdd}(Aa) \cap \mathrm{bdd}(Bb) = \mathrm{bdd}(\emptyset)$ implies $\mathrm{bdd}^{qfu}(Aa) \cap \mathrm{bdd}^{qfu}(Bb) = \mathrm{bdd}^{qfu}(\emptyset)$, and we recover [2, Lemma 1.15].

Fact 5.6. [5, Theorem 3.4(3)] *Let Σ' be an $\emptyset$-invariant subfamily of Σ. Suppose* $\mathrm{tp}(a)$ *is Σ-analysable, but foreign to $\Sigma \setminus \Sigma'$. Then a and $\ell_1^{\Sigma'}(a)$ are domination-equivalent.*

Corollary 5.7. *Let $A \subseteq \mathrm{bdd}(\mathrm{Cb}(B/A))$ consist of quasi-finitary hyperimaginaries, with $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset)$. If A is Σ-analysable and Σ' is the subset of one-based partial types in Σ, then A is analysable in $\Sigma \setminus \Sigma'$.*

Proof: Suppose A is not analysable in $\Sigma \setminus \Sigma'$. For every finite tuple $\bar a \in A$ put $c_{\bar a} = \mathrm{Cb}(B/\bar a)$, and let $C = \bigcup\{c_{\bar a} : \bar a \in A\}$. Then $A \mathop{\perp\!\!\!\perp}_C B$, as for any $\bar a \in A$ and C-indiscernible sequence $(B_i : i < \omega)$ in $\mathrm{tp}(B/C)$ the set $\{\pi(\bar x, B_i) : i < \omega\}$ is consistent, where $\pi(\bar x, B) = \mathrm{tp}(\bar a/B)$, since $\pi(\bar x, B)$ does not fork over $c_{\bar a} \subseteq C$. So $A \subseteq \mathrm{bdd}(C)$; as A is not analysable in $\Sigma \setminus \Sigma'$, neither is C, and there is $\bar a \in A$ such that $c = c_{\bar a}$ is not analysable in $\Sigma \setminus \Sigma'$. Clearly $c \subseteq \mathrm{bdd}(\bar a)$ is quasi-finitary and $c = \mathrm{Cb}(B/c)$. Replacing A by c we may thus assume that A is quasi-finitary.

Let $A' \subseteq \mathrm{bdd}(A)$ and $B' \subseteq \mathrm{bdd}(B)$ be maximally analysable in $\Sigma \setminus \Sigma'$. So $\mathrm{tp}(A/A')$ and $\mathrm{tp}(B/B')$ are foreign to $\Sigma \setminus \Sigma'$, and $A \not\subseteq A'$. Since $A = \mathrm{Cb}(B/A)$ we get $A \not\mathop{\perp\!\!\!\perp}_{A'} B$; as $A \mathop{\perp\!\!\!\perp}_{A'} B'$ by foreignness to $\Sigma \setminus \Sigma'$, we obtain $A \not\mathop{\perp\!\!\!\perp}_{A'B'} B$. In particular $B \not\subseteq B'$.

By Fact 5.6 the first Σ'-levels $a = \ell_1^{\Sigma'}(A/A')$ and $b = \ell_1^{\Sigma'}(B/B')$ are non-trivial, one-based, and

$$a \mathrel{\underline{\square}}_{A'} A \quad \text{and} \quad b \mathrel{\underline{\square}}_{B'} B.$$

Since $\mathrm{tp}(Aa/A')$ is foreign to $\Sigma \setminus \Sigma'$, we have $Aa \mathop{\perp\!\!\!\perp}_{A'} B'$, whence $a \mathrel{\underline{\square}}_{A'B'} A$ by Lemma 5.1(1). Similarly $b \mathrel{\underline{\square}}_{A'B'} B$. But $A \not\mathop{\perp\!\!\!\perp}_{A'B'} B$, and thus $a \not\mathop{\perp\!\!\!\perp}_{A'B'} b$. Let $a_0 = \mathrm{bdd}(A'a) \cap \mathrm{bdd}(A'B'b)$ and $b_0 = \mathrm{bdd}(B'b) \cap \mathrm{bdd}(A'B'a)$. By one-basedness of $\mathrm{tp}(a/A')$ and $\mathrm{tp}(b/B')$,

$$A'a \mathop{\perp\!\!\!\perp}_{a_0} A'B'b \qquad \text{and} \qquad B'b \mathop{\perp\!\!\!\perp}_{b_0} A'B'a.$$

Hence

$$A'B'a \mathop{\perp\!\!\!\perp}_{A'B'a_0} b_0 \qquad \text{and} \qquad A'B'b \mathop{\perp\!\!\!\perp}_{A'B'b_0} a_0.$$

It follows that a_0 and b_0 are interbounded over $A'B'$. We can now apply Corollary 5.4 to see that a_0 is analysable in $\Sigma \setminus \Sigma'$, whence $a_0 \in A'$. But then $a \mathop{\perp\!\!\!\perp}_{A'B'} b$, a contradiction. $\square$

Remark 5.8. In a theory of finite SU-rank, due to weak elimination of quasi-finitary ultraimaginaries, we obtain that for any A, B

$$\mathrm{tp}(\mathrm{Cb}(A/B)/\mathrm{bdd}(A) \cap \mathrm{bdd}(B))$$

is analysable in the collection of non one-based types of SU-rank 1.

The following Theorem generalizes [2, Proposition 1.14] to supersimple theories of infinite rank, at the price of demanding that the quasifinite ultraimaginary bounded closures intersect trivially, rather than just the bounded closures. The proof is essentially the same, but we have to work with ultraimaginaries at key steps. Of course, in finite rank this is equivalent, due to elimination of quasifinite hyperimaginaries; moreover, the families Σ_i in the Theorem are just different orthogonality classes of regular types of rank 1.

Theorem 5.9. *Let T be supersimple. Suppose $A \subseteq \mathrm{bdd}(\mathrm{Cb}(B/A))$ and $B \subseteq \mathrm{bdd}(\mathrm{Cb}(A/B))$, with $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset)$. Let $(\Sigma_i : i \in I)$ be a family of pairwise perpendicular $\emptyset$-invariant families of partial types such that A is analysable in $\bigcup_{i\in I} \Sigma_i$. For $i \in I$ let A_i and B_i be the maximal Σ_i-analysable subset of $\mathrm{bdd}(A)$ and $\mathrm{bdd}(B)$, respectively. Then $A \subseteq \mathrm{bdd}(A_i : i < \alpha)$ and $B \subseteq \mathrm{bdd}(B_i : i < \alpha)$; moreover $A_i = \mathrm{bdd}(\mathrm{Cb}(B_i/A))$ and $B_i = \mathrm{bdd}(\mathrm{Cb}(A_i/B))$. If Σ_i is one-based, then $A_i = B_i = \mathrm{bdd}(\emptyset)$.*

Remark 5.10. If $C \perp\!\!\!\perp AB$, then $\mathrm{bdd}^{qfu}(AC) \cap \mathrm{bdd}^{qfu}(BC) = \mathrm{bdd}^{qfu}(C)$ by Lemma 3.15 and Theorem 5.9 applies over C. This can serve to refine the decomposition.

Proof: Since $\mathrm{Cb}(A_i/B)$ is $\mathrm{tp}(A_i)$-analysable and hence Σ_i-analysable, we have $\mathrm{Cb}(A_i/B) \subseteq B_i$; similarly $\mathrm{Cb}(B_i/A) \subseteq A_i$. As the families in $(\Sigma_i : i \in I)$ are perpendicular, we obtain

$$(A_i : i \in I) \mathop{\perp\!\!\!\perp}_{(B_i : i \in I)} B \quad \text{and} \quad (B_i : i \in I) \mathop{\perp\!\!\!\perp}_{(A_i : i \in I)} A.$$

Suppose $A \subseteq \mathrm{bdd}(A_i : i \in I)$. Then $B = \mathrm{Cb}(A/B) \subseteq \mathrm{bdd}(B_i : i \in I)$; moreover

$$\begin{aligned} \mathrm{bdd}(A) &= \mathrm{bdd}(\mathrm{Cb}(B/A)) = \mathrm{bdd}(\mathrm{Cb}(B_i/A) : i \in I) \\ &= \mathrm{bdd}(\mathrm{Cb}(B_i/A_i) : i \in I) \subseteq \mathrm{bdd}(A_i : i \in I) = \mathrm{bdd}(A) \end{aligned}$$

again by perpendicularily. Hence $\mathrm{bdd}(\mathrm{Cb}(B_i/A_i)) = A_i$, and similarly $\mathrm{bdd}(\mathrm{Cb}(A_i/B_i)) = B_i$. But if Σ_i is one-based, then

$$B_i = \mathrm{bdd}(\mathrm{Cb}(A_i/B_i)) \subseteq \mathrm{bdd}(A_i) \cap \mathrm{bdd}(B_i) = \mathrm{bdd}(\emptyset) \ ;$$

similarly $A_i = \mathrm{bdd}(\emptyset)$.

Put $\bar{A} = \mathrm{bdd}(A_i : i \in I)$ and $\bar{B} = \mathrm{bdd}(B_i : i \in I)$. It remains to show that $A \subseteq \bar{A}$. So suppose not. As in the proof of Corollary 5.7 put $c_{\bar{a}} = \mathrm{Cb}(B/\bar{a})$ for every finite tuple $\bar{a} \in A$, and let $C = \bigcup\{c_{\bar{a}} : \bar{a} \in A\}$. Then again $A \perp\!\!\!\perp_C B$ and $A \subseteq \mathrm{bdd}(C)$; moreover $c_{\bar{a}} = \mathrm{Cb}(B/_{\bar{a}})$. Since A is not contained in $\bar{A}$, neither is C. Hence there is $\bar{a} \in A$ such that $c = c_{\bar{a}} \notin \bar{A}$. As the maximal Σ_i-analysable subset of $\mathrm{bdd}(c)$ is equal to $\mathrm{bdd}(c) \cap A_i$ we may replace A by c and thus assume that A is quasi-finitary. Similarly, we may assume that B is quasi-finitary.

Since $A = \mathrm{Cb}(B/A) \not\subseteq \bar{A}$, we have $A \not\perp\!\!\!\perp_{\bar{A}} B$; as $A \perp\!\!\!\perp_{\bar{A}} \bar{B}$ we obtain $A \not\perp\!\!\!\perp_{\bar{A}\bar{B}} B$. Let $(b_j : j < \alpha)$ be an analysis of B over $\bar{B}$ such that for every $j < \alpha$ the type $\text{tp}(b_j/\bar{B}, b_\ell : \ell < j)$ is Σ_{i_j}-analysable for some $i_j \in I$. Let k be minimal with $A \not\downarrow_{\bar{A}\bar{B}} (b_j : j \leq k)$. Then $A \downarrow_{\bar{A}} \bar{B}, (b_j : j < k)$ and $\text{Cb}(\bar{B}, (b_j : j \leq k)/A)$ is almost Σ_{i_k}-internal over $\bar{A}$. Put $A' = \ell_1^{\Sigma_{i_k}}(A/\bar{A})$ and $B' = \ell_1^{\Sigma_{i_k}}(B/\bar{B})$. Then $A' \not\subseteq \bar{A}$, and $\text{Cb}(A'/B) \subseteq B'$ since $\bar{A} \downarrow_{\bar{B}} B$. Similarly $\text{Cb}(B'/A) \subseteq A'$. Moreover $A' \not\downarrow_{\bar{A}\bar{B}} B$, whence $A' \not\downarrow_{\bar{A}\bar{B}} B'$. Replacing A by $\text{Cb}(B'/A) = \text{Cb}(B'/A')$ and B by $\text{Cb}(A'/B) = \text{Cb}(A'/B')$ we may assume that $\text{tp}(A/\bar{A})$ and $\text{tp}(B/\bar{B})$ are both almost Σ_k-internal (where we write k instead of i_k for ease of notation).

Claim. $\text{bdd}^{qfu}(AB_k) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(B_k)$.

Proof of Claim: Suppose not. As B is analysable in $\bigcup_{i\in I} \Sigma_i$, Corollary 3.13 yields some $i \in I$ and

$$d \in (\text{bdd}^{qfu}(AB_k) \cap \text{bdd}^{qfu}(B)) \setminus \text{bdd}^{qfu}(B_k)$$

such that d is almost Σ_i-internal over B_k; since $\text{tp}(B/B_k)$ is foreign to Σ_k we have $i \neq k$. Hence $A \downarrow_{\bar{A}B_k} d$, whence $d \in \text{bdd}^{qfu}(\bar{A}B_k)$ by Lemma 3.2. But $\bar{A} = \text{bdd}(A_i : i \in I)$ and $d \downarrow_{B_kA_i} \bar{A}$ by almost Σ_i-internality of d over B_k, whence $d \in \text{bdd}^{qfu}(B_kA_i)$. If $B_kd \downarrow A_i$, then $d \downarrow_{B_k} A_i$ and $d \in \text{bdd}^{qfu}B_k$ by Lemma 3.2, contradicting the choice of d. Therefore $B_kd \not\downarrow A_i$; by Corollary 3.13 there is almost Σ_i-internal

$$d' \in \text{bdd}^{qfu}(B_kd) \setminus \text{bdd}^{qfu}(\emptyset).$$

Note that $d' \in \text{bdd}^{qfu}(A_iB_k) \cap \text{bdd}^{qfu}(B)$. But then $d'A_i \downarrow B_k$, whence $d' \downarrow_{A_i} B_k$ and

$$d' \in \text{bdd}^{qfu}(A_i) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset),$$

a contradiction. □

Claim. *We may assume* $B_k = \text{bdd}(\emptyset)$.

Proof of Claim: Put $A' = \text{Cb}(B/AB_k)$. Then $B_k \subset A' = \text{Cb}(B/A')$, and $\text{bdd}(A')^{qfu} \cap \text{bdd}(B)^{qfu} = \text{bdd}^{qfu}(B_k)$. If $B' = \text{Cb}(A'/B) = \text{Cb}(A'/B')$, then $A' \downarrow_{B'} B$ and $A \downarrow_{A'} B$ yield $B \downarrow_{B'} A$ by transitivity, since $B' \subseteq \text{bdd}(B)$. Thus $B \subset \text{bdd}(B')$. We add B_k to the language, and have to show that A', B' is still a counterexample over B_k.

So suppose not, and let $\text{bdd}(A') = \text{bdd}(A'_i : i \in I)$ and $\text{bdd}(B') = \text{bdd}(B'_i : i \in I)$ be decompositions, where A'_i and B'_i are maximally Σ_i-analysable over B_k in $\text{bdd}(A')$ and $\text{bdd}(B')$, respectively. So B'_k is Σ_k-analysable, whence $B'_k = B_k \subseteq \bar{B}$ by maximality. Since $B \subseteq \text{bdd}(B')$

is almost Σ_k-internal over $\bar{B}$ and $(B'_i : i \neq k)$ is foreign to Σ_k, we get $B \mathop{\smile\!\!\!\!\!\!\;|}_{\bar{B}} (B'_i : i \neq k)$, whence $B \subset \bar{B}$, a contradiction. $\square$

By symmetry, we may also assume $A_k = \mathrm{bdd}(\emptyset)$.

Put $B' = \mathrm{Cb}(B/A\bar{B})$. Then $\bar{B} \subseteq \mathrm{bdd}(B')$, and since B is almost Σ_k-internal over $\bar{B}$, so is B'. If $A' = \mathrm{Cb}(B'/A)$, then $B' \mathop{\smile\!\!\!\!\!\!\;|}_{A'} A$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_{B'} B$ yield $A \mathop{\smile\!\!\!\!\!\!\;|}_{A'} B$, since $A' \subseteq \mathrm{bdd}(A)$. Thus $A \subseteq \mathrm{bdd}(A')$. Put $B'' = \mathrm{Cb}(A/B') = \mathrm{Cb}(A/B'')$. Then

$$B'' \subseteq \mathrm{bdd}(B') \subseteq \mathrm{bdd}(A\bar{B}),$$

and B'' is almost Σ_k-internal over $\bar{B}$. Moreover, $A \mathop{\smile\!\!\!\!\!\!\;|}_{B''} B'$ implies

$$A \subseteq \mathrm{bdd}(\mathrm{Cb}(B'/A)) \subseteq \mathrm{bdd}(\mathrm{Cb}(B''/A)).$$

Claim. $\mathrm{bdd}^{qfu}(A') \cap \mathrm{bdd}^{qfu}(B'') = \mathrm{bdd}^{qfu}(\emptyset)$.

Proof of Claim: Suppose not. By Corollary 3.13 there is $i \in I$ and

$$d \in (\mathrm{bdd}^{qfu}(A') \cap \mathrm{bdd}^{qfu}(B'')) \setminus \mathrm{bdd}^{qfu}(\emptyset)$$

which is almost Σ_i-internal; since B'' is foreign to Σ_k we have $i \neq k$. As B'' is almost Σ_k-internal over $\bar{B}$ we have

$$d \in \mathrm{bdd}^{qfu}(\bar{B}) \cap \mathrm{bdd}^{qfu}(A') \subseteq \mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu}(\emptyset),$$

a contradiction. $\square$

Thus A', B'' is another counterexample; by induction on $SU(AB)$ we may assume $\mathrm{bdd}(A'B'') = \mathrm{bdd}(AB)$. But then

$$B \subset \mathrm{bdd}(A'B'') \subseteq \mathrm{bdd}(AB') \subseteq \mathrm{bdd}(A\bar{B}).$$

By symmetry $A \subset \mathrm{bdd}(B\bar{A})$. Since $\bar{A}\bar{B}$ are analysable in $\bigcup_{i\neq k} \Sigma_i$, so are A and B by Corollary 5.4. But $\mathrm{tp}(A/\bar{A})$ is almost Σ_k-internal, whence foreign to $\bigcup_{i\neq k} \Sigma_i$, yielding the final contradiction. $\square$

Remark 5.11. In the finite rank context, it is easy to achieve the hypothesis of Theorem 5.9, as it suffices work over $\mathrm{bdd}(A) \cap \mathrm{bdd}(B)$. In general, however, since $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) = \mathrm{bdd}^{qfu} C$ implies $\mathrm{bdd}(C) = \mathrm{bdd}(A) \cap \mathrm{bdd}(B)$. Thus, if

$$\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) \supsetneq \mathrm{bdd}^{qfu}(\mathrm{bdd}(A) \cap \mathrm{bdd}(B)),$$

we do not authorize ourselves to work over $\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B)$, which is not eliminable. Feeble elimination nevertheless yields

$$\mathrm{bdd}^{qfu}(A) \cap \mathrm{bdd}^{qfu}(B) \subset \mathrm{bdd}^{qfu}(\mathrm{cl}_\alpha(A) \cap \mathrm{cl}_\alpha(B))$$

if $SU(A/\mathrm{bdd}(A) \cap \mathrm{bdd}(B)) < \omega^{\alpha+1}$, so we can work over α-closed sets, as is done in [5, Theorem 5.4].

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