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# Stability of linear switched systems with quadratic bounds and Observability of bilinear systems

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## Abstract

The aim of this paper is to give sufficient conditions for a switched linear system defined by a pair of Hurwitz matrices that share a common but not strict quadratic Lyapunov function to be GUAS.

We show that this property is equivalent to the uniform observability on  $[0, +\infty)$  of a bilinear system defined on a subspace whose dimension is in most cases much smaller than the dimension of the switched system.

Some sufficient conditions of uniform asymptotic stability are then deduced from the equivalence theorem, and illustrated by examples.

Keywords: Switched systems; Asymptotic stability; Quadratic Lyapunov functions; Observability; Bilinear systems.

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# 1 Introduction.

The aim of this paper is to give sufficient conditions for a switched linear system to be asymptotically stable for all measurable inputs, that is to be GUAS, in the case where the system is defined by a pair  $\{B_0, B_1\}$  of  $d \times d$  Hurwitz matrices that share a common, but not strict in general, quadratic Lyapunov function.

We can assume without loss of generality that the Lyapunov matrix is the identity, so that the matrices  $B_i$  verify  $B_i^T + B_i \leq 0$  for  $i = 0, 1$ .

In this setting the linear subspace

$$\mathcal{K} = \mathcal{K}_0 \cap \mathcal{K}_1, \quad \text{where } \mathcal{K}_i = \ker(B_i^T + B_i) \quad i = 0, 1,$$

turns out to be a fundamental object (see [3] and [7]).

In the previous paper [3] it was proved that the system is GUAS as soon as  $\mathcal{K} = \{0\}$ . However this condition is not necessary, and it is possible to build GUAS systems for which  $\dim \mathcal{K} = d - 1$  regardless of  $d$  (see Example 7.4).

Firstly we show that a necessary and sufficient condition for the switched system to be GUAS is that a certain bilinear system in  $\mathcal{K}$  is observable on  $[0, +\infty)$ . More accurately the convexified matrix  $B_\lambda = (1 - \lambda)B_0 + \lambda B_1$  writes according to the decomposition  $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$ :

$$B_\lambda = \begin{pmatrix} A_\lambda & -C_\lambda^T \\ C_\lambda & D_\lambda \end{pmatrix},$$

with  $A_\lambda^T + A_\lambda = 0$  for  $\lambda \in [0, 1]$ , and  $D_\lambda^T + D_\lambda < 0$  for  $\lambda \in (0, 1)$ .

Consider the bilinear system in  $\mathcal{K}$  with output in  $\mathcal{K}^\perp$ :

$$(\Sigma) = \begin{cases} \dot{x} = A_\lambda x \\ y = C_\lambda x \end{cases} \quad \lambda \in [0, 1], \quad x \in \mathcal{K}, \quad y \in \mathcal{K}^\perp.$$

Our main result is then:

## Theorem 2

*The switched system is GUAS if and only if  $(\Sigma)$  is observable for all inputs defined on  $[0, +\infty)$ .*

The interest of this equivalence is twofold: on the one hand the bilinear observed system is in  $\mathcal{K}$ , whose dimension is most often (but not always)

much smaller than  $d$ . On the other hand the observability of  $(\Sigma)$  is rather easier to deal with than the asymptotic properties of the switched system.

Thanks to Theorem 2 we obtain the following sufficient conditions of uniform asymptotic stability (the definition of the set  $G$  is rather technical and the reader is referred to Section 5):

### Theorem 3

*The switched system is GUAS as soon as the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ , and one of the following conditions holds:*

1. *the set  $G$  is discrete;*
2.  *$\dim \mathcal{K} \leq 2$ .*

*In particular the switched system is GUAS if  $\ker C_\lambda = \{0\}$  for  $\lambda \in [0, 1]$ .*

As we know of no system which is not GUAS and such that the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$  we make the following conjecture:

### Conjecture

*The switched system is GUAS if and only if the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ .*

From [4] we know that a GUAS switched system has always a common strict Lyapunov function. One might think that under the assumption of the existence of a common non strict quadratic Lyapunov function, a GUAS system has a common strict quadratic Lyapunov function. This conjecture is wrong, as shown by Example 7.5, due to Paolo Mason.

The paper is organized as follows: basic definitions and notations are stated in Section 2. Section 3 is devoted to a necessary and sufficient condition for a matrix  $B$  that verifies  $B^T + B$  to be Hurwitz. This condition is expressed in terms of observability of a linear system, and in our opinion enlightens the proof of the main theorem 2 which is stated and proved in Section 4. The observability of  $(\Sigma)$  is studied in the next one, that is Section 5, and the summarizing theorem 3 is stated in Section 6. The results are illustrated by some examples in Section 7.

## 2 Preliminaries

We will use the symbol  $X$  to denote the elements of  $\mathbb{R}^d$ . When  $\mathbb{R}^d$  will be decomposed as  $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$  we will write

$$X = x + y \approx \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{where } x \in \mathcal{K}, \quad y \in \mathcal{K}^\perp.$$

As explained in the introduction we deal with a pair  $\{B_0, B_1\}$  of  $d \times d$  Hurwitz matrices, assumed to share a common, but not strict in general, quadratic Lyapunov function. More accurately there exists a symmetric positive definite matrix  $P$  such that the symmetric matrices  $B_i^T P + P B_i$  are nonpositive ( $B^T$  stands for the transpose of  $B$ ). Since the Lyapunov matrix  $P$  is common to the  $B_i$ 's we can assume without loss of generality that  $P$  is the identity matrix, in other words that  $B_i^T + B_i$  is non positive for  $i = 0, 1$  :

$$\forall X \in \mathbb{R}^d, \quad X^T (B_i^T + B_i) X \leq 0 \quad \text{for } i = 0, 1. \quad (1)$$

**Norms.** The natural scalar product of  $\mathbb{R}^d$  in this context is the canonical one, defined by  $\langle X, Y \rangle = X^T Y$  (it would be  $X^T P Y$  if the Lyapunov matrix were  $P$ ). The norm of  $\mathbb{R}^d$  is consequently chosen to be  $\|X\| = \sqrt{X^T X}$ .

**The switched system.** We consider the switched system in  $\mathbb{R}^d$

$$\dot{X} = B_{u(t)} X \quad (2)$$

where the input, or switching law,  $t \mapsto u(t)$  is a measurable function from  $[0, +\infty)$  into the discrete set  $\{0, 1\}$ .

Such a switching signal being given, the solution of (2) for the initial condition  $X$  writes

$$t \mapsto \Phi_u(t) X,$$

where  $t \mapsto \Phi_u(t)$  is the solution of the matrix equation  $\dot{M} = B_{u(t)} M$ ,  $\Phi_u(0) = I_d$ , or in integral form:

$$\Phi_u(t) = I_d + \int_0^t B_{u(s)} \Phi_u(s) ds. \quad (3)$$

**The  $\omega$ -limit sets.** For  $X \in \mathbb{R}^d$  we denote by  $\Omega_u(X)$  the set of  $\omega$ -limit points of  $\{\Phi_u(t) X; t \geq 0\}$ , that is the set of limits of sequences  $(\Phi_u(t_j) X)_{j \geq 0}$ , where  $(t_j)_{j \geq 0}$  is strictly increasing to  $+\infty$ .

Thanks to Condition (1), the norm  $\|\Phi_u(t) X\|$  is nonincreasing, and  $\Omega_u(X)$  is a compact and connected subset of a sphere  $\mathcal{S}(r) = \{x \in \mathbb{R}^d; \|x\| = r\}$  for some  $r \geq 0$  (see Proposition 1 of [3]).

**Definition 1** *The switched system is said to be Globally Uniformly Asymptotically Stable, or GUAS in short, if for every switching law  $u$  the system (2) is globally asymptotically stable, that is*

$$\forall X \in \mathbb{R}^d \quad \Phi_u(t)X \xrightarrow{t \rightarrow +\infty} 0.$$

### 3 Hurwitz Property and Observability of linear systems

**Theorem 1** *Let  $B$  be a  $d \times d$ -matrix that satisfies  $B^T + B \leq 0$  and let  $\mathcal{K} = \ker(B^T + B)$ . Up to an orthogonal transformation and according to the orthogonal decomposition  $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$ ,  $B$  can be written as*

$$B = \begin{pmatrix} A & -C^T \\ C & D \end{pmatrix}$$

with  $A^T + A = 0$  and  $D^T + D < 0$ .

Then  $B$  is Hurwitz if and only if the pair  $(C, A)$  is observable.

*Proof.* Let  $(b_1, \dots, b_d)$  be an orthonormal basis of  $\mathbb{R}^d$  such that  $(b_1, \dots, b_k)$  span  $\mathcal{K}$ . In that basis  $B$  writes

$$B = \begin{pmatrix} A & C_1 \\ C & D \end{pmatrix}$$

according to the decomposition  $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$ . The condition  $B^T + B \leq 0$  being invariant under orthogonal transformations, and  $\mathcal{K}$  being equal to  $\ker(B^T + B)$ , we obtain at once  $A^T + A = 0$ ,  $C_1 = -C^T$  and  $D^T + D < 0$ .

Let us now consider the observed linear system in  $\mathcal{K}$  with output in  $\mathcal{K}^\perp$ :

$$(\Sigma) = \begin{cases} \dot{x} & = Ax \\ y & = Cx \end{cases}$$

If  $(\Sigma)$  is not observable, then there exists  $x \in \mathcal{K}$ ,  $x \neq 0$ , such that  $Ce^{tA}x = 0$  for all  $t \in \mathbb{R}$ . Therefore

$$\frac{d}{dt} \begin{pmatrix} e^{tA}x \\ 0 \end{pmatrix} = B \begin{pmatrix} e^{tA}x \\ 0 \end{pmatrix}, \quad \text{hence} \quad e^{tB} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} e^{tA}x \\ 0 \end{pmatrix}$$

does not tend to 0 as  $t$  goes to  $+\infty$ , since  $e^{tA}$  is a rotation matrix. This shows that  $B$  is not Hurwitz.

Conversely let us assume that  $B$  is not Hurwitz: there exists  $X \in \mathbb{R}^d$  such that  $e^{tB}X$  does not tend to 0 as  $t$  goes to  $+\infty$ . According to the assumption  $B^T + B \leq 0$  the function  $t \mapsto \|e^{tB}X\|$  is non increasing, and tends to a limit  $r > 0$ .

Let  $l$  be an  $\omega$ -limit point of this trajectory, that is  $l = \lim_{j \rightarrow +\infty} e^{t_j B} X$  for some sequence  $(t_j)_{j>0}$  increasing to  $+\infty$ . For all  $t \in \mathbb{R}$  the point  $e^{tB}l = \lim_{j \rightarrow +\infty} e^{(t_j+t)B} X$  is also an  $\omega$ -limit point. Consequently

$$\forall t \in \mathbb{R}, \quad \|e^{tB}l\| = \|l\| = r \quad \text{and} \quad 0 = \frac{d}{dt} \|e^{tB}l\|^2 = (e^{tB}l)^T (B^T + B) e^{tB}l$$

which shows that the trajectory  $e^{tB}l$ ,  $t \in \mathbb{R}$ , is contained in  $\mathcal{K}$ .

Finally let us write  $l = (x, 0)^T$  in  $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$ . Then

$$e^{tB}l = \begin{pmatrix} e^{tA}x \\ 0 \end{pmatrix} \quad \text{with} \quad C e^{tA}x = 0$$

showing that the output of  $(\Sigma)$  does not distinguish the states 0 and  $x$ . □

## 4 Main result

Let  $B_0$  and  $B_1$  be two  $d \times d$ -matrices that satisfy  $B_i^T + B_i \leq 0$ , and let  $\mathcal{K}_i = \ker(B_i^T + B_i)$  for  $i = 0, 1$ . Here and subsequently  $\mathcal{K}$  stands for

$$\mathcal{K} = \mathcal{K}_0 \cap \mathcal{K}_1 = \ker(B_0^T + B_0) \cap \ker(B_1^T + B_1).$$

For  $\lambda \in [0, 1]$  we write  $B_\lambda = (1 - \lambda)B_0 + \lambda B_1$ . Let us firstly state the following easy but useful lemma.

**Lemma 1** *For all  $\lambda \in (0, 1)$ ,  $\ker(B_\lambda^T + B_\lambda) = \mathcal{K}$ .*

*Moreover, up to an orthogonal transformation and according to the orthogonal decomposition  $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$ ,  $B_\lambda$  writes*

$$B_\lambda = \begin{pmatrix} A_\lambda & -C_\lambda^T \\ C_\lambda & D_\lambda \end{pmatrix},$$

*with  $A_\lambda^T + A_\lambda = 0$  for  $\lambda \in [0, 1]$ , and  $D_\lambda^T + D_\lambda < 0$  for  $\lambda \in (0, 1)$ .*

*Proof.*

We have only to prove that  $\ker(B_\lambda^T + B_\lambda) = \mathcal{K}$  for  $\lambda \in (0, 1)$ , the proof of the second assertion being similar to the beginning of the one of Theorem 1. If  $X^T(B_\lambda^T + B_\lambda)X = 0$  for some  $X \in \mathbb{R}^d$ , then

$$0 = (1 - \lambda)X^T(B_0^T + B_0)X + \lambda X^T(B_1^T + B_1)X.$$

But  $X^T(B_i^T + B_i)X \leq 0$  for  $i = 0, 1$ , and since  $\lambda \neq 0, 1$  we obtain  $X^T(B_0^T + B_0)X = X^T(B_1^T + B_1)X = 0$ , that is  $X \in \mathcal{K}_0 \cap \mathcal{K}_1$ .

The converse is straightforward. □

**Remark.** The strict inequality  $D_\lambda^T + D_\lambda < 0$  does not hold for  $\lambda = 0, 1$  whenever  $\mathcal{K}_0$  or  $\mathcal{K}_1$  is strictly larger than  $\mathcal{K}$ . However the non strict inequality  $D_\lambda^T + D_\lambda \leq 0$  holds for  $\lambda \in [0, 1]$ .

In the same way as in Section 3 we will consider the bilinear controlled and observed system:

$$(\Sigma) = \begin{cases} \dot{x} &= A_\lambda x \\ y &= C_\lambda x \end{cases}$$

where  $\lambda \in [0, 1]$ ,  $x \in \mathcal{K}$ , and  $y \in \mathcal{K}^\perp$ .

**Definition 2** *The system  $(\Sigma)$  is said to be uniformly observable on  $[0, +\infty)$  if for any measurable input  $t \mapsto \lambda(t)$  from  $[0, +\infty)$  into  $[0, 1]$ , the output distinguish any two different initial states, that is*

$$\forall x_1 \neq x_2 \in \mathcal{K} \quad m\{t \geq 0; C_{\lambda(t)}x_1(t) \neq C_{\lambda(t)}x_2(t)\} > 0,$$

where  $m$  stands for the Lebesgue measure on  $\mathbb{R}$ , and  $x_i(t)$  for the solution of  $\dot{x} = A_{\lambda(t)}x$  starting from  $x_i$ , for  $i = 1, 2$ .

### Remarks

1. As the output depends explicitly on the input, it is measurable but not necessarily continuous. It is the reason for which our definition of observability involves the Lebesgue measure.
2. The observability on  $[0, +\infty)$  is not equivalent to the observability on bounded time intervals (See Section 5.3 and Examples 7.3, 7.4).

3. The system being linear with respect to the state, it is clearly observable for a given input if and only if the output does not vanish for almost every  $t \in [0, +\infty)$  as soon as the initial state is different from 0.

We are now in a position to state our main result:

**Theorem 2** *The switched system is GUAS if and only if  $(\Sigma)$  is uniformly observable on  $[0, +\infty)$ .*

*Proof.*

Let us first assume that the switched system is not GUAS. There exist a measurable input  $t \mapsto u(t)$  from  $[0, +\infty)$  into  $\{0, 1\}$  and an initial state  $X \in \mathbb{R}^d$  for which the switched system does not converge to 0.

Let  $l$ , with  $\|l\| = r > 0$ , be a limit point for  $X$ , and  $(t_j)_{j \geq 0}$  a strictly increasing sequence such that

$$l = \lim_{j \rightarrow +\infty} \Phi_u(t_j)X.$$

Let  $\tau$  be an arbitrary positive number and let us define the sequence  $(\phi_j)_{j \geq 0}$  by  $\phi_j(t) = \Phi_u(t_j + t)X$  for  $t \in [0, \tau]$ . Each function  $\phi_j$  verifies

$$\phi_j(t) = \phi_j(0) + \int_0^t B_{u(t_j+s)} \phi_j(s) ds.$$

On the other hand the sequence  $(B^j)_{j \geq 0}$  of functions from  $[0, \tau]$  to  $\mathcal{M}(d; \mathbb{R})$ , defined by  $B^j(s) = B_{u(t_j+s)}$ , is bounded, and consequently converges weakly in  $L^\infty([0, \tau], \mathcal{M}(d; \mathbb{R}))$ , up to a subsequence that we continue to denote by  $(B^j)_{j \geq 0}$ .

Moreover the limit takes its values in the convexification of  $\{B_0, B_1\}$  (see [8], Lemme 10.1.3, page 424), and can be written

$$B_{\lambda(t)} = (1 - \lambda(t))B_0 + \lambda(t)B_1,$$

where  $t \mapsto \lambda(t)$  is a measurable function from  $[0, \tau]$  into  $[0, 1]$ .

Let us denote by  $\psi$  the absolutely continuous and  $\mathbb{R}^d$ -valued function defined on  $[0, \tau]$  by the equation

$$\psi(t) = l + \int_0^t B_{\lambda(s)} \psi(s) ds.$$

According to Theorem 1, page 157, of [8], the sequence  $(\Psi_j)_{j \geq 1}$  converges uniformly on  $[0, \tau]$  to  $\psi$ . Moreover this function takes its values in  $\Omega_u(X)$ , so that

$$\forall t \in [0, \tau] \quad \|\psi(t)\|^2 = \|\psi(0)\|^2 = \|l\|^2 = r^2 > 0.$$

Thus we have for almost every  $t \in [0, \tau]$

$$\frac{d}{dt} \|\psi(t)\|^2 = \psi(t)^T (B_{\lambda(t)}^T + B_{\lambda(t)}) \psi(t) = 0. \quad (4)$$

**Lemma 2** *For all  $t \in [0, \tau]$  the vector  $\psi(t)$  belongs to  $\mathcal{K} = \mathcal{K}_0 \cap \mathcal{K}_1$ . In particular  $l = \psi(0) \in \mathcal{K}$ .*

*Proof.* For almost every  $t \in [0, \tau]$  we have

$$\begin{aligned} \psi(t)^T (B_{\lambda(t)}^T + B_{\lambda(t)}) \psi(t) \\ = (1 - \lambda(t)) \psi(t)^T (B_0^T + B_0) \psi(t) + \lambda(t) \psi(t)^T (B_1^T + B_1) \psi(t). \end{aligned}$$

But according to (4) and

$$\psi(t)^T (B_i^T + B_i) \psi(t) \leq 0 \quad \text{for } i = 0, 1$$

we obtain for almost every  $t \in [0, \tau]$

$$\begin{aligned} \lambda(t) \neq 0 &\implies \psi(t) \in \mathcal{K}_1 \\ \lambda(t) \neq 1 &\implies \psi(t) \in \mathcal{K}_0 \end{aligned}$$

so that  $\psi(t) \in \mathcal{K}_0 \cup \mathcal{K}_1$ . Assume that  $l \in \mathcal{K}_0 \setminus \mathcal{K}_1$ . Then for some  $T$ ,  $0 < T \leq \tau$ , we have

$$\psi([0, T]) \cap \mathcal{K}_1 = \emptyset, \quad \text{hence} \quad \psi(t)^T (B_1^T + B_1) \psi(t) < 0$$

for all  $t \in [0, T]$ . Consequently  $\lambda(t) = 0$  for almost every  $t \in [0, T]$ , and

$$\psi(t) = e^{tB_0} \psi(0).$$

But according to the Hurwitz property of  $B_0$ , the norm  $\|\psi(t)\|$  would be strictly decreasing, in contradiction with its belonging to  $\Omega_u(X)$ . Consequently  $l \in \mathcal{K}$ , and in the same way  $\psi(t) \in \mathcal{K}$  for  $t \in [0, \tau]$ . □

*End of the proof of Theorem 2.*

As  $\psi(t)$  is in  $\mathcal{K}$  for all  $t$  it can be written according to the decomposition  $\mathbb{R}^d = \mathcal{K} \oplus \mathcal{K}^\perp$  as a column

$$\psi(t) = \begin{pmatrix} \phi(t) \\ 0 \end{pmatrix}.$$

Moreover the derivative of  $\psi(t)$  is also in  $\mathcal{K}$  for almost every  $t$ . This derivative is

$$\frac{d}{dt}\psi(t) = B(t)\psi(t) = \begin{pmatrix} A_{\lambda(t)}\phi(t) \\ C_{\lambda(t)}\phi(t) \end{pmatrix}$$

and the belonging of  $\frac{d}{dt}\psi(t)$  to  $\mathcal{K}$  turns out to be

$$C_{\lambda(t)}\phi(t) = 0 \quad \text{for almost every } t \in [0, \tau].$$

The conclusion is that  $\phi$  is a trajectory of

$$(\Sigma) = \begin{cases} \dot{x} &= A_\lambda x \\ y &= C_\lambda x \end{cases}$$

for which the output vanishes almost surely. Notice that  $\phi$  does not vanish since  $\|\phi(t)\|^2 = \|\phi(0)\|^2 = \|l\|^2 = r^2 > 0$  for all  $t \in [0, \tau]$ .

To conclude this part of the proof it remains to notice that  $\phi$  can be extended to  $[0, +\infty)$  with the same properties: starting from the final point  $\psi(\tau)$  we can obtain a similar limit trajectory on  $[\tau, \tau_1]$  for any  $\tau_1 > \tau$ .

This proves that  $(\Sigma)$  is not uniformly observable on  $[0, +\infty)$ .

Conversely assume the switched system to be GUAS. It is a well know fact that the convexified system is also GUAS (see [5]). If there exists for  $(\Sigma)$  an input defined on  $[0, +\infty)$  and with values in  $[0, 1]$  such that the trajectory  $\phi(t)$  for the initial condition  $\phi(0) \neq 0$  satisfies  $C_{\lambda(t)}x(t) = 0$  for  $t \geq 0$  then

$$\psi(t) = \begin{pmatrix} \phi(t) \\ 0 \end{pmatrix}$$

is, for the same input, a trajectory of the convexified switched system that does not converge to 0, a contradiction. □

## 5 Observability of the bilinear system

We consider now the controlled and observed bilinear system

$$(\Sigma) = \begin{cases} \dot{x} &= A_\lambda x \\ y &= C_\lambda x \end{cases}$$

where  $x \in \mathcal{K}$ ,  $y \in \mathcal{K}^\perp$  and  $\lambda \in [0, 1]$ . Notice that the matrices  $A_0$  and  $A_1$  are skew-symmetric, so that the trajectories of  $(\Sigma)$  are contained in spheres. We will denote by  $\mathcal{S}^{k-1}$ , where  $k = \dim \mathcal{K}$ , the unit sphere of  $\mathcal{K}$ .

A solution  $t \mapsto x(t)$  of  $(\Sigma)$  on  $I = [0, T]$  or  $I = [0, +\infty)$ , which is in  $\mathcal{S}^{k-1}$  and satisfies

$$C_{\lambda(t)}x(t) = 0 \quad \text{for almost every } t \in I,$$

will be called a **bad trajectory** on  $I$ .

The purpose is to find conditions for  $(\Sigma)$  to be uniformly observable on  $[0, +\infty)$ . An obvious necessary condition is that  $(\Sigma)$  is observable for every constant input, that is the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ . Since  $B_0$  and  $B_1$  are Hurwitz this property is guaranteed for  $\lambda = 0, 1$  (although  $\mathcal{K}$  is not necessarily the kernel of  $B_i^T + B_i$ ,  $i = 0, 1$ , this can be easily shown using the same kind of arguments as in the proof of Theorem 1). Notice that under this condition no bad trajectory can be constant. This remark is used in the forthcoming proofs.

A sufficient condition is that  $(\Sigma)$  is uniformly observable on every bounded interval  $[0, T]$ ,  $T > 0$ , that is  $(\Sigma)$  is uniformly observable in the usual meaning.

The first task is to characterize the locus where the output vanishes (Section 5.1). Then we will state some sufficient conditions of uniform observability (Section 5.2), and of uniform observability on  $[0, +\infty)$  (Section 5.3).

### 5.1 The bad locus

The condition

$$\exists \lambda \in [0, 1] \quad \text{such that} \quad C_\lambda x = (1 - \lambda)C_0x + \lambda C_1x = 0$$

is equivalent to saying that  $C_0x$  and  $C_1x$  are colinear and in opposite directions, that last condition being due to  $\lambda \in [0, 1]$ . Consequently the set of

points  $x \in \mathcal{K}$  for which there exists  $\lambda \in [0, 1]$  such that  $C_\lambda x = 0$  can be characterized in the following way:

$$\begin{aligned} \exists \lambda \in [0, 1] \text{ s.t. } C_\lambda x = 0 &\iff \langle C_0 x, C_1 x \rangle + \|C_0 x\| \|C_1 x\| = 0 \\ &\iff C_0 x \wedge C_1 x = 0 \text{ and } \langle C_0 x, C_1 x \rangle \leq 0. \end{aligned}$$

Here the scalar product and the norm are the restrictions to  $\mathcal{K}^\perp$  of the ones of  $\mathbb{R}^d$ . The exterior product  $C_0 x \wedge C_1 x$  is considered as a  $\frac{k'(k' - 1)}{2}$  vector, where  $k'$  is the dimension of  $\mathcal{K}^\perp$ .

This set is a cone that will be denoted by  $F$ .

We also write  $N = \ker C_0 \cap \ker C_1$  and  $F_0 = F \setminus N$ . For  $x \in F_0$  it is clear that  $C_0 x \neq C_1 x$  so that the unique  $\lambda$  such that  $C_\lambda x = 0$  is given by:

$$\lambda(x) = \frac{\langle C_0 x - C_1 x, C_0 x \rangle}{\|C_0 x - C_1 x\|^2}, \quad (5)$$

that is  $x \mapsto \lambda(x)$  is the restriction to  $F_0$  of an analytic function defined on  $\mathcal{K} \setminus \{C_0 x = C_1 x\}$ .

Any bad trajectory lies in the intersection of  $F$  with  $\mathcal{S}^{k-1}$ , and as long as it does not meet  $N$ , that is as long as it remains in  $F_0$ , Formula (5) shows that  $t \mapsto \lambda(t)$  and  $t \mapsto x(t)$  are analytic (more accurately  $t \mapsto \lambda(t)$  is almost everywhere equal to an analytic function).

## 5.2 Uniform observability

Let  $(\lambda(t), x(t))$  be a bad trajectory on  $[0, T]$  for some  $T > 0$ . The point  $x(t)$  belongs to  $F$  for all  $t$ , so that  $C_0 x(t) \wedge C_1 x(t) = 0$ , and by differentiation

$$\frac{d}{dt} C_0 x(t) \wedge C_1 x(t) = C_0 A_{\lambda(t)} x(t) \wedge C_1 x(t) + C_0 x(t) \wedge C_1 A_{\lambda(t)} x(t) = 0 \quad \text{a.e.}$$

that is  $A_{\lambda(t)} x(t)$  is tangent to  $F$  (in a weak sense because  $F$  need not be regular at every point).

Let  $G$  stand for the set of points  $x \in \mathcal{S}^{k-1} \cap F$  that verify:

$$\begin{cases} x \in N & \text{and } \exists \lambda \in [0, 1] \text{ s.t. } C_0 A_\lambda x \wedge C_1 x + C_0 x \wedge C_1 A_\lambda x = 0 \\ \text{or} \\ x \in F_0 & \text{and } C_0 A_{\lambda(x)} x \wedge C_1 x + C_0 x \wedge C_1 A_{\lambda(x)} x = 0 \end{cases}$$

It is clear that  $x(t) \in G$  for almost every  $t$ . Assume the set  $G$  to be discrete, then the trajectory is reduced to a point, and  $\lambda(t)$  is almost everywhere equal to a constant. We have proved the proposition:

**Proposition 1** *If the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$  and the set  $G$  is discrete then  $(\Sigma)$  is uniformly observable on  $[0, T]$  for all  $T > 0$ .*

*It is in particular true if  $\ker C_\lambda = \{0\}$  for  $\lambda \in [0, 1]$ .*

### 5.3 Uniform observability on $[0, +\infty)$

It may happen that  $G$  is not discrete, though the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ . For instance when  $\dim \mathcal{K}^\perp = 1$  the condition  $C_0x \wedge C_1x = 0$  is empty and  $F$  contains an open subset of  $\mathcal{K}$ . The interior of the set  $F_0$  is not empty either and the output of the analytic system

$$\begin{cases} \dot{x} &= A_{\lambda(x)}x \\ y &= C_{\lambda(x)}x \end{cases}$$

vanishes **as long as the trajectory remains in  $F_0$** . Consequently  $(\Sigma)$  cannot be uniformly observable on small time intervals. However it may happen that under the assumption that the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ , no trajectory remains in  $F$ . We present below a proof in the case where  $\dim \mathcal{K}$  is 1 or 2.

#### 5.3.1 $\dim \mathcal{K} = 1$

The sphere  $\mathcal{S}^{k-1}$  consists of two points and under the condition that the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ , no bad trajectory can exist.

#### 5.3.2 $\dim \mathcal{K} = 2$

We can assume without loss of generality that the rank of  $C_\lambda$  is equal to 1 for every  $\lambda \in [0, 1]$ . Indeed if it vanishes for some  $\lambda_0$ , then the pair  $(C_{\lambda_0}, A_{\lambda_0})$  is not observable. If it is greater than 1 for one  $\lambda$  then it is greater than 1 for all  $\lambda$  except for isolated values, and the bad trajectories are obtained for these constant inputs.

In this setting we have two cases:

1.  $\ker C_0 = \ker C_1$ . Then  $C_1 = \alpha C_0$  for some  $\alpha > 0$  (if  $\alpha \leq 0$  then  $C_\lambda$  vanishes for some  $\lambda \in [0, 1]$ ). A bad trajectory is contained in the one-dimensional space  $\ker C_0$  and is reduced to a point.
2.  $\ker C_0 \neq \ker C_1$ . The set  $F$  is the cone  $\{(C_0x)(C_1x) \leq 0\}$ . On the other hand the matrices  $\exp(tA_i)$  are rotation ones ( $i = 0, 1$ ). If they have

the same direction of rotation, all trajectories run through the whole circle and go out of  $F$ . If their directions of rotation are opposite or if one is zero, then  $A_\lambda$  vanishes for some  $\lambda$ . The pair  $(C_\lambda, A_\lambda)$  is not observable for that value.

We have proved

**Proposition 2** *If  $\dim \mathcal{K} \leq 2$  then  $(\Sigma)$  is uniformly observable on  $[0, +\infty)$  if and only if the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ .*

## 6 Concluding Theorem and Conjecture

We keep the notations of the previous sections. In view of Theorem 2 and Propositions 1 and 2, we can state:

**Theorem 3** *The switched system is GUAS as soon as the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ , and one of the following conditions holds:*

1. *the set  $G$  is discrete;*
2.  *$\dim \mathcal{K} \leq 2$ .*

*In particular the switched system is GUAS if  $\ker C_\lambda = \{0\}$  for  $\lambda \in [0, 1]$ .*

In this theorem, only sufficient conditions are stated. However we know of no system which is not GUAS and such that the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ . We therefore make the following conjecture:

### Conjecture

*The switched system is GUAS if and only if the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$ .*

## 7 Examples

### 7.1 Hurwitz matrices

Consider the matrix

$$\begin{pmatrix} A & -C^T \\ C & D \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the first line of  $C$  is  $(1 \ 0)$ . According to Theorem 1 this matrix is Hurwitz as soon as  $D$  satisfies  $D^T + D < 0$ .

## 7.2 Two general examples

Let us choose a skew-symmetric  $k \times k$  matrix  $A$ , and a  $k' \times k$  matrix  $C$  such that the pair  $(C, A)$  is observable. Then for any matrices  $D_0$  and  $D_1$  such that  $D_i^T + D_i < 0$  the system  $\{B_0, B_1\}$  is GUAS, where:

$$B_0 = \begin{pmatrix} A & -C^T \\ C & D_0 \end{pmatrix} \quad B_1 = \begin{pmatrix} A & -C^T \\ C & D_1 \end{pmatrix}$$

Indeed the system is in the canonical form of Lemma 1, and  $(\Sigma)$  does not depend on  $\lambda$ . It is therefore uniformly observable.

In the same way, and as a direct application of Proposition 1, we can consider the case where the dynamics of  $(\Sigma)$  is null, that is the system defined by

$$B_0 = \begin{pmatrix} 0 & -C_0^T \\ C_0 & D_0 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & -C_1^T \\ C_1 & D_1 \end{pmatrix} \quad \text{with } D_i^T + D_i < 0.$$

It is GUAS if and only if  $C_\lambda$  is one-to-one for all  $\lambda \in [0, 1]$ .

## 7.3 An example with $\dim \mathcal{K} = 2$

Consider the case where

$$A_0 = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad \text{the first line of } C_0 \text{ is } (1 \ 0),$$

$$A_1 = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad \text{the first line of } C_1 \text{ is } (0 \ 1),$$

and  $D_0, D_1$ , are  $k' \times k'$  matrices,  $k' \geq 1$ , with  $D_i^T + D_i < 0$ ,  $i = 1, 2$ .

A straightforward computation shows that the pair  $(C_\lambda, A_\lambda)$  is observable for every  $\lambda \in [0, 1]$  if and only if  $a$  and  $b$  are both positive or both negative: the determinant of the observability matrix is equal to  $(2\lambda^2 - 2\lambda + 1)((1 - \lambda)a + \lambda b)$ .

The cone  $F$  is here the set  $\{(x_1, x_2) \in \mathbb{R}^2; x_1 x_2 \leq 0\}$ , that is the union of the two orthants  $\{x_1 \geq 0; x_2 \leq 0\}$  and  $\{x_1 \leq 0; x_2 \geq 0\}$ , and  $F_0$  is equal to  $F$  minus the origin. For  $x = (x_1, x_2) \in F_0$  we can define

$$\lambda(x) = \frac{-x_2}{x_1 - x_2}.$$

This system cannot be uniformly observable on small time intervals since for the feedback  $x \mapsto \lambda(x)$  the output vanishes as long as  $x(t)$  belongs to  $F_0$  whose interior is not empty.

However  $(\Sigma)$  is uniformly observable on  $[0, +\infty)$ , under the condition  $a$  and  $b$  both positive or both negative: indeed a trajectory starting at  $x \neq 0$  runs through the whole circle with radius  $\|x\|$ , hence goes out of  $F$ .

Finally the switched system is GUAS if and only if  $ab > 0$ .

## 7.4 The $\dim \mathcal{K} = d - 1$ case

Let us begin by a very simple example. Let  $A$  be a  $(d - 1) \times (d - 1)$  skew-symmetric matrix and  $C$  a  $1 \times (d - 1)$  matrix such that the pair  $(C, A)$  is observable, and let  $d_0$  and  $d_1$  be two different positive numbers. The matrices

$$B_0 = \begin{pmatrix} A & -C^T \\ C & -d_0 \end{pmatrix} \quad B_1 = \begin{pmatrix} A & -C^T \\ C & -d_1 \end{pmatrix}$$

define a GUAS switched system with  $\dim \mathcal{K} = d - 1$ .

For a less trivial example consider the skew-symmetric  $2q \times 2q$  matrix  $A$  which has  $q$  blocks

$$\begin{pmatrix} 0 & -a_j \\ a_j & 0 \end{pmatrix}$$

on the diagonal and vanishes elsewhere, and

$$C_0 = (1 \ 0 \ 1 \ 0 \ \dots \ 1 \ 0), \quad C_1 = (0 \ 1 \ 0 \ 1 \ \dots \ 0 \ 1).$$

Assume  $(a_1, \dots, a_q)$  to be rationally independent. Then the orbit of  $\dot{x} = Ax$  for a non zero initial state  $(x_1^0, \dots, x_{2q}^0)$  is dense in the torus

$$x_{2j-1}^2 + x_{2j}^2 = (x_{2j-1}^0)^2 + (x_{2j}^0)^2 = T_j^2 \quad j = 1, \dots, q$$

where at least one  $T_j$  does not vanish.

Therefore this orbit meets the subset of the orthant  $\{x_i \geq 0; i = 1, \dots, 2q\}$  where  $x_{2j-1} > 0$  and  $x_{2j} > 0$  for at least one  $j$ . But in this subset we have  $(C_0 x)(C_1 x) > 0$ . This shows that every non zero orbit goes out of  $F$  and that the bilinear system defined by  $A_0 = A_1 = A$ ,  $C_0$  and  $C_1$  is uniformly observable on  $[0, +\infty)$ . Finally the switched system defined by the matrices

$$B_0 = \begin{pmatrix} A & -C_0^T \\ C_0 & -d_0 \end{pmatrix} \quad B_1 = \begin{pmatrix} A & -C_1^T \\ C_1 & -d_1 \end{pmatrix}$$

is GUAS for any choice of positive numbers  $d_0$  and  $d_1$ .

## 7.5 A singular case of the Dayawansa-Martin example

It is a well known fact that a GUAS switched system has always a common strict Lyapunov function, but not always a quadratic one: in [4] Dayawansa and Martin provide an example to show that even for planar switched linear systems GUAS does not imply the existence of a common strict quadratic Lyapunov function.

We give here an example, due to Paolo Mason, which shows that for a linear switched system, GUAS and the existence of a common non strict quadratic Lyapunov function do not either imply the existence of a common strict quadratic Lyapunov function.

Consider the  $2 \times 2$  switched system defined by the matrices

$$B_0 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} -1 & -3 - 2\sqrt{2} \\ 3 - 2\sqrt{2} & -1 \end{pmatrix}.$$

The symmetric positive matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 3 + 2\sqrt{2} \end{pmatrix}$$

defines a weak quadratic Lyapunov function for this system, that is  $B_i^T P + P B_i \leq 0$  for  $i = 0, 1$ .

On the other hand the switched system is GUAS: to see it, just apply Theorem 1 of [2]. Indeed our system is in the class satisfying the  $S_4$ -GUAS condition of the mentioned theorem. It remains to show that it admits no strict quadratic Lyapunov function.

We are seeking a positive definite symmetric matrix  $P$  in the form

$$\begin{pmatrix} 1 & q \\ q & r \end{pmatrix}$$

such that

$$M_i = B_i^T P + P B_i < 0, \quad i = 1, 2 \tag{6}$$

Equation (6) is satisfied if the interior of the ellipses in the  $(q, r)$  plan given by  $\det M_i = 0$  intersect. It is straightforward to check that those ellipses have the same major axis  $q = 0$  and have respectively the vertices

$$\{(0, 3 - 2\sqrt{2}), (0, 3 + 2\sqrt{2})\} \quad \text{and} \quad \{(0, 3 + 2\sqrt{2}), (0, 99 + 70\sqrt{2})\}.$$

Consequently their interiors do not intersect.

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