Smoothing effect for the regularized Schrdinger equation with non controlled orbits

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Abstract

We prove that the geometric control condition is not necessary to obtain the smoothing effect and the uniform stabilization for the strongly dissipative Schrödinger equation.

Key words: Smoothing effect, Resolvent estimates, Stabilization and Geometric Control.

1 Introduction and statement of results

It is well known that the Schrödinger equation enjoys some smoothing properties. One of them says that if $u_0 \in L^2(\mathbb{R}^d)$ with compact support, then the solution of the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0 \text{ in } \mathbb{R} \times \mathbb{R}^d \\ u(0,.) = u_0 \qquad \text{in } \mathbb{R}^d, \end{cases} \tag{1}$$

satisfies

$$u \in C^\infty(\mathbb{R} \setminus \{0\} \times \mathbb{R}^d).$$

We say that the Schrödinger propagator has an infinite speed. Another type of gain of regularity for system (1) is the Kato-1/2 smoothing effect (see [8], [15], [16]), namely any solution of (1) satisfies

$$\int_{\mathbb{R}} \int_{|x|<R} |(1-\Delta)^{\frac{1}{4}} u(t,x)|^2 dx dt \le C_R \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{2}$$

In particular, this result implies that for a.e. $t \in \mathbb{R}$, $u(t,.)$ is locally smoother than u_0 and this happens despite the fact that (1) conserves the global L^2 norm. The Kato-effect has been extended to variable coefficients operators with non trapping metric by Doi ([9], [10])) and to non trapping exterior domains by Burq, Gerard and Tzvetkov [4]. On the other hand, Burq [3] proved that the nontrapping assumption is necessary for the $H^{1/2}$ smoothing effect. Moreover, using Ikawa's result [11], he showed, in the case of several convex bodies satisfying certain assumptions, that the smoothing effect with an $\varepsilon > 0$ loss still holds.

Recently, the first author [1,2] has introduced the forced smoothing effect for Schrödinger equation. The idea is inspired from the stabilization problem and it consists of acting on the equation in order to produce some smoothing effects. More precisely, in [2] the following regularized Schrödinger equation on a bounded domain $\Omega \subset \mathbb{R}^d$ is considered:

$$\begin{cases} i\partial_t u - \Delta_D u + ia(x)(-\Delta_D)^{\frac{1}{2}} a(x) u = 0 \text{ in } \mathbb{R} \times \Omega, \\ u(0,.) = f \qquad \text{in } \Omega, \\ u|_{\mathbb{R}\times\partial\Omega} = 0, \end{cases} \tag{3}$$

where Δ_D denotes the Dirichlet realization of the Laplace operator on Ω and $a(x)$ is a smooth real-valued function. Under the geometric control condition (GCC) on the set $w = \{a \neq 0\}$, it is proved in [2] that any solution with initial data in $H^s_D(\Omega)$ belongs to $L^2_{loc}((0,\infty), H^{s+1}_D(\Omega))$, where $s \in [-s_0, s_0]$ and $s_0 \ge 1$ depends on the behavior of $a(x)$ near the boundary. When the function a is constant near each component of the boundary, we have $s_0 = \infty$. Then by iteration of the last result, a C^∞-smoothing effect is proved in [2]. Note that these smoothing effects hold away from $t = 0$ and they seem strong compared with the Kato effect for which the GCC is necessary. Therefore the case when $w = \{a \neq 0\}$ does not control geometrically Ω is very interesting.

In this work we give an example of geometry where the geometric control condition is not satisfied but the C^∞ smoothing effect holds. More precisely, let $O = \cup_{i=1}^N O_i \subset \mathbb{R}^d$ be the union of a finite number of bounded strictly convex bodies, O_i, satisfying the conditions of [11], namely:

- For any $1 \le i, j, k \le N$, $i \ne j$, $j \ne k$, $k \ne i$, one has
$$\text{Convex Hull}(O_i \cup O_j) \cap O_k = \emptyset. \tag{4}$$
- Denote by κ the infimum of the principal curvatures of the boundaries of the bodies O_i, and L the infimum of the distances between two bodies. Then if $N > 2$ we assume that $\kappa L > N$ (no assumption if $N = 2$).

Let B be a bounded domain containing O with smooth boundary and such that $\Omega_0 = O^c \cap B$ is connected, where $O^c = \mathbb{R}^d \setminus O$. In the present paper we will consider the regularized Schrödinger equation (3) in Ω_0. For a bounded domain Ω of $\mathbb{R}^d$ and any $s \in \mathbb{R}$, we denote by $H_D^s(\Omega)$ the Hilbert space

$$H_D^s(\Omega) = \{u = \sum_j a_j e_j, \quad \sum_j \gamma_j^{2s} |a_j|^2 < \infty\},$$

where $\{\gamma_j^2\}$ are the eigenvalues of $-\Delta_D$ and $\{e_j\}$ is the corresponding orthonormal basis of $L^2(\Omega)$. We have the following interpolation inequalities:

$$\|g\|_{H_D^s(\Omega_0)} \le \|g\|_{H_D^t(\Omega_0)}^{\frac{s}{t}} \|g\|_{L^2(\Omega_0)}^{1-\frac{s}{t}} \text{ for all } g \in H_D^t(\Omega_0), \quad 0 \le s \le t. \tag{5}$$

Clearly, $H_D^s(\Omega)$ and $H_D^{-s}(\Omega)$ are in duality and $H_D^s(\Omega)$ is the domain of $(-\Delta_D)^{\frac{s}{2}}$. Remark also that $H_D^s(\Omega) = H^s(\Omega)$ is the usual Sobolev space for $0 \le s < \dfrac{1}{2}$ and $H_D^s(\Omega) = \{u \in H^s(\Omega),\ \Delta^j u|_{\partial\Omega} = 0, 2j \le s - \frac{1}{2}\}$ for $s \ge \dfrac{1}{2}$. Throughout this paper $a \in C^\infty(\Omega_0)$ will be a real-valued function such that $\operatorname{supp} a \subset \{x \in \overline{\Omega}_0 : \text{dist}(\text{x}, \partial\text{B}) \le 2\varepsilon_0\}$ and $a = Const \ne 0$ on $\{x \in \overline{\Omega}_0 : \text{dist}(\text{x}, \partial\text{B}) \le \varepsilon_0\}$, where $0 < \varepsilon_0 \ll 1$ is a constant. Under this assumption the following properties hold for all $s \in \mathbb{R}$ and $n \in \mathbb{N}$:

($\mathcal{P}_\mathbf{s}$) the multiplication by a maps $H_D^s(\Omega_0)$ into itself,
($\mathcal{Q}_{\mathbf{s},\mathbf{n}}$) the commutator $[a, (-\Delta_D)^n]$ maps $H_D^s(\Omega_0)$ into $H_D^{s-2n+1}(\Omega_0)$.

Set $B_a = a(x)(-\Delta_D)^{\frac{1}{2}} a(x)$ and define the operator $A_a = -\Delta_D + iB_a$ on $L^2(\Omega_0)$ with domain

$$D(A_a) = \{f \in L^2(\Omega_0); A_a f \in L^2(\Omega_0), \quad f = 0 \quad \text{on} \quad \partial\Omega_0\}.$$

Since the properties ($\mathcal{P}_\mathbf{s}$) and ($\mathcal{Q}_{\mathbf{s},\mathbf{n}}$) hold for all $s \in \mathbb{R}$ and $n \in \mathbb{N}$, the problem (3) is well posed in $H_D^s(\Omega_0)$ for all $s \in \mathbb{R}$. Moreover the operator A_a generates a semi-group, $U(t)$, such that for $f \in H_D^s(\Omega_0)$, $U(t)f \in C([0, +\infty[, H_D^s(\Omega_0))$

is the unique solution of (3). It is easy to see that the spectrum, $\mathrm{sp}(A_a)$, of A_a consists of complex numbers, τ_j, satisfying $|\tau_j| \to \infty$. Furthermore, since $a(x)$ is not identically zero, we have

$$\mathrm{sp}(A_a) \subset \{\tau \in \mathbb{C}, \mathrm{Im}\,\tau > 0\}.$$

The resolvent

$$(A_a - \tau)^{-1} : L^2(\Omega_0) \to L^2(\Omega_0)$$

is holomorphic on $\{\mathrm{Im}\,\tau < 0\}$ and can be extended to a meromorphic operator on $\mathbb{C}$. Our main result is the following

Theorem 1 *If the function a is as above, there exist positive constants σ_0 and C such that for $|\mathrm{Im}\,\tau| < \sigma_0$ we have*

$$\left\|(A_a - \tau)^{-1}\right\|_{L^2(\Omega_0)\to L^2(\Omega_0)} \leq C\langle\tau\rangle^{-\frac{1}{2}} \log^2\langle\tau\rangle, \tag{6}$$

where $\langle\tau\rangle = \sqrt{1+|\tau|^2}$.

A similar bound has been recently proved in [5] for the Laplace operator in Ω_0 with strong dissipative boundary conditions on ∂B, provided B is strictly convex (viewed from the exterior). Note also that a better bound (with log instead of $\log^2$) was obtained in [6], [7] in the case of the damped wave equation on compact manifolds without boundary under the assumption that there is only one closed hyperbolic orbit which does not pass through the support of the dissipative term. This has been recently improved in [14] for a class of compact manifolds with negative curvature, where a strip free of eigenvalues has been obtained under a pressure condition.

As an application of this resolvent estimate we obtain the following smoothing result for the associated Schrödinger propagator.

Theorem 2 *Let $s \in \mathbb{R}$. Under the assumptions of Theorem 1, we have*

(i) For each $\varepsilon > 0$ there is a constant $C > 0$ such that the function

$$u(t) = \int_0^t e^{i(t-\tau)A_a} f(\tau)d\tau$$

satisfies

$$\|u\|_{L^2_T H^{s+1-\varepsilon}_D(\Omega_0)} \leq C\,\|f\|_{L^2_T H^s_D(\Omega_0)} \tag{7}$$

for all $T > 0$ and $f \in L^2_T H^s_D(\Omega_0)$.

(ii)If $v_0 \in H^s_D(\Omega_0)$, then

$$v \in C^\infty((0,+\infty) \times \Omega_0) \tag{8}$$

where v is the solution of (3) with initial data v_0.

Theorem 1 also implies the following stabilization result.

Theorem 3 *Under the assumptions of Theorem 1, there exist $\alpha, c > 0$ such that for the solution u of (3) with initial data u_0 in $L^2(\Omega_0)$, we have*

$$\|u\|_{L^2(\Omega_0)} \leq ce^{-\alpha t}\|u_0\|_{L^2(\Omega_0)}, \ \forall\ t > 1.$$

This result shows that we can stabilize the Schrödinger equation by a (strongly) dissipative term that does not satisfy the geometric control condition of [12]. In fact, to have the exponential decay above it suffices to have the estimate (6) with a constant in the right-hand side.

2 Resolvent estimates

This section is devoted to the proof of Theorem 1. Since the resolvent $(A_a - \tau)^{-1}$ is meromorphic on $\mathbb{C}$ and has no poles on the real axes, it suffices to prove (6) for $|\tau| \gg 1$. Let u be a solution of the following equation

$$\begin{cases} (\Delta_D + \lambda^2 - ia(x)(-\Delta_D)^{\frac{1}{2}}a(x))u = v \text{ in } \Omega_0, \\ u|_{\partial\Omega_0} = 0, \end{cases} \tag{9}$$

with $v \in L^2(\Omega_0)$. Clearly, it suffices to prove the following

Proposition 4 *Under the assumptions of Theorem 1, there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ and every solution u of (9) we have*

$$\|u\|_{L^2(\Omega_0)} \lesssim \frac{\log^2 \lambda}{\lambda}\|v\|_{L^2(\Omega_0)}. \tag{10}$$

Proof. Let $\chi \in C_0^\infty(B)$ be such that $\chi = 0$ on $\{x \in B : \text{dist}(x, \partial B) \leq \varepsilon_0/3\}$, $\chi = 1$ on $\{x \in B : \text{dist}(x, \partial B) \geq \varepsilon_0/2\}$. Clearly we have

$$\|(1-\chi)u\|_{L^2(\Omega_0)} \lesssim \|au\|_{L^2(\Omega_0)}. \tag{11}$$

On the other hand, the function χu satisfies the equation

$$\begin{cases} (\Delta_D + \lambda^2)\chi u = \chi v + i\chi a(x)(-\Delta_D)^{\frac{1}{2}}a(x)u + [\Delta_D, \chi]u \quad \text{ in } \mathbb{R}^d \setminus O, \\ u|_{\partial O} = 0. \end{cases} \tag{12}$$

Hence, according to Proposition 4.8 of [3], it follows that

$$\|\chi u\|_{L^2(\Omega_0)} \lesssim \frac{\log \lambda}{\lambda}(\|v\|_{L^2(\Omega_0)} + \|au\|_{H^1(\Omega_0)}). \tag{13}$$

By (11) and (13),

$$\|u\|_{L^2(\Omega_0)} \lesssim \frac{\log\lambda}{\lambda}\|v\|_{L^2(\Omega_0)} + \frac{\log\lambda}{\lambda}\|au\|_{H^1(\Omega_0)} + \|au\|_{L^2(\Omega_0)}. \tag{14}$$

To estimate the second and the third terms in the right hand side of (14), we need the following

Lemma 5 *Let $s \in [0,1]$ and $\psi \in C^\infty(\Omega_0)$. Then we have, for $\lambda \gg 1$,*

$$\|\psi u\|_{H^{s+1}} \lesssim \lambda\|\psi u\|_{H^s} + \|v\|_{L^2} + \lambda^{1/2}\|u\|_{L^2}, \tag{15}$$

$$\|\psi u\|_{H^s} \lesssim \lambda^{-1}\|\psi u\|_{H^{s+1}} + \lambda^{-1}\|v\|_{L^2} + \lambda^{-1/2}\|u\|_{L^2}. \tag{16}$$

Proof. The function $w = (-\Delta_D)^{s/2}\psi u := P_s u$ satisfies the equation

$$\begin{cases} (-\Delta_D - \lambda^2 + iB_a)w = P_s v + [\Delta_D, P_s]u + i[B_a, P_s]u \text{ in } \Omega_0, \\ w|_{\partial O} = 0. \end{cases} \tag{17}$$

Multiplying equation (17) by $\overline{w}$, integrating by parts and taking the real part, we obtain

$$\|(-\Delta_D)^{1/2}w\|^2_{L^2(\Omega_0)} - \lambda^2\|w\|^2_{L^2(\Omega_0)} = \operatorname{Re}\langle P_s v + [\Delta_D, P_s]u + i[B_a, P_s]u, w\rangle. \tag{18}$$

Using that $[\Delta_D, P_s] = (-\Delta_D)^{s/2}[\Delta_D, \psi]$, we deduce from (18)

$$\|(-\Delta_D)^{1/2}w\|^2_{L^2} \lesssim \lambda^2\|w\|^2_{L^2} + \|\psi v\|_{L^2}\|P_{2s}u\|_{L^2} + \|u\|_{H^{s+1}}\|w\|_{L^2}. \tag{19}$$

This implies

$$\begin{aligned} &\|\psi u\|^2_{H^{s+1}} \\ &\lesssim \lambda^2\|\psi u\|^2_{H^s} + \|v\|_{L^2}\|\psi u\|_{H^{2s}} + \|u\|_{H^{s+1}}\|\psi u\|_{H^s} \\ &\lesssim \lambda^2\|\psi u\|^2_{H^s} + \|v\|^2_{L^2} + \varepsilon\|\psi u\|^2_{H^{2s}} + \|u\|_{H^{s+1}}\|\psi u\|_{H^s} \\ &\lesssim \lambda^2\|\psi u\|^2_{H^s} + \|v\|^2_{L^2} + \varepsilon\|\psi u\|^2_{H^{s+1}} + \varepsilon\|\psi u\|^2_{H^s} + \lambda^{-2}\|u\|^2_{H^{s+1}} + \lambda^2\|\psi u\|^2_{H^s}. \end{aligned} \tag{20}$$

Choosing ε small enough, we obtain

$$\|\psi u\|^2_{H^{s+1}} \lesssim \lambda^2\|\psi u\|^2_{H^s} + \|v\|^2_{L^2} + \lambda^{-2}\|u\|^2_{H^{s+1}}. \tag{21}$$

Taking $\psi = 1$ in the previous estimate, we get for $\lambda \gg 1$

$$\|u\|^2_{H^{s+1}} \lesssim \lambda^2\|u\|^2_{H^s} + \|v\|^2_{L^2}. \tag{22}$$

Inserting (22) in (21), we obtain

$$\|\psi u\|^2_{H^{s+1}} \lesssim \lambda^2\|\psi u\|^2_{H^s} + \|v\|^2_{L^2} + \|u\|^2_{H^s}. \tag{23}$$

On the other hand, since $s \in [0,1]$, by interpolation we have

$$\|u\|_{H^s}^2 \le \|u\|_{H^1}\|u\|_{L^2} \le \lambda^{-1}\|u\|_{H^1}^2 + \lambda\|u\|_{L^2}^2. \tag{24}$$

Choosing $s = 0$ in (22), we get

$$\|u\|_{H^1}^2 \lesssim \lambda^2\|u\|_{L^2}^2 + \|v\|_{L^2}^2. \tag{25}$$

Combining (24) and (25), we obtain

$$\|u\|_{H^s}^2 \lesssim \lambda\|u\|_{L^2}^2 + \lambda^{-1}\|v\|_{L^2}^2. \tag{26}$$

Inserting this estimate in (23) we get

$$\|\psi u\|_{H^{s+1}}^2 \lesssim \lambda^2\|\psi u\|_{H^s}^2 + \|v\|_{L^2}^2 + \lambda\|u\|_{L^2}^2. \tag{27}$$

This completes the proof of (15). Clearly, (16) can be proved in the same way.

We deduce from Lemma 5 the following

Lemma 6 *Let u be a solution of (9) and $\psi \in C^\infty(\Omega_0)$. Then we have, for $\lambda \gg 1$,*

$$\|\psi u\|_{L^2} \lesssim \lambda^{-1/2}\|\psi u\|_{H^{1/2}} + \lambda^{-1}\|v\|_{L^2} + \lambda^{-1/2}\|u\|_{L^2}, \tag{28}$$

$$\|\psi u\|_{H^1} \lesssim \lambda^{1/2}\|\psi u\|_{H^{1/2}} + \lambda^{-1/2}\|v\|_{L^2} + \|u\|_{L^2}. \tag{29}$$

Proof. Using Lemma 5 together with an interpolation argument, we get

$$\begin{aligned}\|\psi u\|_{H^1}^2 &\lesssim \lambda^{-1}\|\psi u\|_{H^{3/2}}^2 + \lambda\|\psi u\|_{H^{1/2}}^2 \\ &\lesssim \lambda^{-1}(\lambda^2\|\psi u\|_{H^{1/2}}^2 + \|v\|_{L^2}^2 + \lambda\|u\|_{L^2}^2) + \lambda\|\psi u\|_{H^{1/2}}^2 \\ &\lesssim \lambda\|\psi u\|_{H^{1/2}}^2 + \lambda^{-1}\|v\|_{L^2}^2 + \|u\|_{L^2}^2,\end{aligned} \tag{30}$$

which proves (29). Using (18) and (29) we obtain

$$\begin{aligned}\|\psi u\|_{L^2}^2 &\lesssim \lambda^{-2}\|\psi u\|_{H^1}^2 + \lambda^{-2}\|v\|_{L^2}^2 + \lambda^{-1}\|u\|_{L^2}^2 \\ &\lesssim \lambda^{-1}\|\psi u\|_{H^{1/2}}^2 + \lambda^{-2}\|v\|_{L^2}^2 + \lambda^{-1}\|u\|_{L^2}^2,\end{aligned} \tag{31}$$

which proves (28).

We now return to the proof of Proposition 4. Using Lemma 6 and the estimate (14), we get

$$\|u\|_{L^2} \lesssim \frac{\log\lambda}{\lambda}\|v\|_{L^2} + \frac{\log\lambda}{\sqrt{\lambda}}\|au\|_{H^{1/2}} \text{ for } \lambda \gg 1. \tag{32}$$

Let's now estimate the $H^{1/2}$ term. Multiplying equation (9) by $\overline{u}$, integrating by parts and taking the imaginary part, we obtain

$$\|au\|_{H^{1/2}}^2 = \langle (-\Delta_D)^{\frac{1}{2}} au, au\rangle = \mathrm{Im}\,\langle v, u\rangle \le \|v\|_{L^2}\|u\|_{L^2}. \tag{33}$$

By (32) and (33), we get

$$\|u\|_{L^2} \lesssim \frac{\log\lambda}{\lambda}\|v\|_{L^2} + \frac{\log\lambda}{\sqrt{\lambda}}\|v\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}. \tag{34}$$

This implies

$$\|u\|_{L^2} \lesssim \frac{\log\lambda}{\lambda}\|v\|_{L^2} + \frac{\log\lambda}{\sqrt{\lambda}}\Big(\frac{\log\lambda}{\sqrt{\lambda}}\|v\|_{L^2} + \varepsilon\frac{\sqrt{\lambda}}{\log\lambda}\|u\|_{L^2}\Big), \tag{35}$$

for any $\varepsilon > 0$. Choosing ε small enough, we get for large λ

$$\|u\|_{L^2} \lesssim \frac{\log^2\lambda}{\lambda}\|v\|_{L^2}, \tag{36}$$

which is the desired result.

3 Smoothing effect

We will first prove the following

Proposition 7 *If $a(x)$ is as in the introduction, then for every $s \in \mathbb{R}$, $\varepsilon > 0$ there exist positive constants C and σ_0 such that*

$$\|(A_a - \tau)^{-1}\|_{H_D^s(\Omega_0)\to H_D^{s+1-\varepsilon}(\Omega_0)} \le C \tag{37}$$

holds for $|\mathrm{Im}\,\tau| < \sigma_0$.

Proof. Let u and f satisfy the equation

$$\begin{cases} (-\Delta_D - \tau + iB_a)u = f & \text{in } \ \Omega_0, \\ u = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{38}$$

Let's see that the following estimate holds

$$\|u\|_{H_D^2(\Omega_0)} \le C\langle\tau\rangle^{\frac{1}{2}}\log^2\langle\tau\rangle\|f\|_{L^2(\Omega_0)}, \tag{39}$$

for $|\mathrm{Im}\,\tau| < \sigma_0$. Using Proposition 1 we get

$$\begin{aligned} \|u\|_{H^2_D(\Omega_0)} &= \|\Delta_D u\|_{L^2(\Omega_0)} \\ &= \|\tau u - iB_a u + f\|_{L^2(\Omega_0)} \\ &\leq C\langle\tau\rangle^{\frac{1}{2}} \log^2\langle\tau\rangle \|f\|_{L^2(\Omega_0)} + C\|u\|_{H^1_D(\Omega_0)} \\ &\leq C\langle\tau\rangle^{\frac{1}{2}} \log^2\langle\tau\rangle \|f\|_{L^2(\Omega_0)} + \varepsilon\|u\|_{H^2_D(\Omega_0)} + C_\varepsilon\|f\|_{L^2(\Omega_0)}. \end{aligned} \tag{40}$$

Choosing ε small enough, we obtain (39). Using (39) and (5) with $s = 1-\varepsilon$, $t = 2$, we obtain

$$\begin{aligned} \|u\|^2_{H^{1-\varepsilon}_D(\Omega_0)} &\leq \|u\|^{1-\varepsilon}_{H^2_D(\Omega_0)} \|u\|^{1+\varepsilon}_{L^2(\Omega_0)} \\ &\leq C(\langle\tau\rangle^{\frac{1}{2}} \log^2\langle\tau\rangle)^{1-\varepsilon} \left(\frac{\log^2\langle\tau\rangle}{\langle\tau\rangle^{\frac{1}{2}}}\right)^{1+\varepsilon} \|f\|^2_{L^2(\Omega_0)} \\ &\leq C\|f\|^2_{L^2(\Omega_0)}. \end{aligned} \tag{41}$$

Then we get (37) for $s = 0$, i.e.

$$\|(-\Delta_D - \tau + iB_a)^{-1}\|_{L^2(\Omega_0)\to H^{1-\varepsilon}_D(\Omega_0)} \leq C. \tag{42}$$

We will prove (37) for $s = 2N$ with $N \in \mathbb{N}$, namely

$$\|u\|_{H^{2N+1-\varepsilon}_D(\Omega_0)} \lesssim \|f\|_{H^{2N}_D(\Omega_0)}. \tag{43}$$

Let $f \in H^{2N}_D(\Omega_0)$ and let u be the corresponding solution of (38). The function $(-\Delta_D)^N u$ satisfies

$$(\Delta_D + \tau - iB_a)((-\Delta_D)^N u) = (-\Delta_D)^N f - i[B_a, (-\Delta_D)^N]u. \tag{44}$$

Using (42), we obtain

$$\|(-\Delta_D)^N u\|_{H^\gamma(\Omega_0)} \lesssim \|(-\Delta_D)^N f\|_{L^2(\Omega_0)} + \|[B_a, (-\Delta_D)^N]u\|_{L^2(\Omega_0)} \tag{45}$$

where $\gamma = 1-\varepsilon$. Since

$$\|u\|_{H^{2N+\gamma}_D(\Omega_0)} = \|(-\Delta)^N u\|_{H^\gamma(\Omega_0)}, \tag{46}$$

we obtain

$$\|u\|_{H^{2N+\gamma}(\Omega_0)} \lesssim \|(-\Delta_D)^N f\|_{L^2(\Omega_0)} + \|[B_a, (-\Delta_D)^N]u\|_{L^2(\Omega_0)}. \tag{47}$$

On the other hand, we have

$$[B_a, (-\Delta_D)^N] = a(-\Delta_D)^{\frac{1}{2}}[a, (-\Delta_D)^N] + [a, (-\Delta_D)^N](-\Delta_D)^{\frac{1}{2}} a. \tag{48}$$

Using the properties $(\mathcal{P}_{\mathbf{s}})$ and $(\mathcal{Q}_{\mathbf{s},\mathbf{n}})$ we get

$$\|[B_a, (-\Delta_D)^N]u\|_{L^2(\Omega_0)} \lesssim \|u\|_{H_D^{2N}(\Omega_0)}. \tag{49}$$

Consequently

$$\begin{aligned} \|u\|_{H_D^{2N+\gamma}(\Omega_0)} &\lesssim \|f\|_{H_D^{2N}(\Omega_0)} + \|u\|_{H_D^{2N}(\Omega_0)} \\ &\lesssim \|f\|_{H_D^{2N}(\Omega_0)} + \varepsilon\|u\|_{H_D^{2N+\gamma}(\Omega_0)} + C_\varepsilon\|u\|_{H_D^{\gamma}(\Omega_0)}. \end{aligned} \tag{50}$$

Choosing ε small enough and using (42) we obtain

$$\begin{aligned} \|u\|_{H_D^{2N+\gamma}(\Omega_0)} &\lesssim \|f\|_{H_D^{2N}(\Omega_0)} + \|u\|_{H_D^{\gamma}(\Omega_0)} \\ &\lesssim \|f\|_{H_D^{2N}(\Omega_0)} + \|f\|_{L^2(\Omega_0)} \\ &\lesssim \|f\|_{H_D^{2N}(\Omega_0)}, \end{aligned} \tag{51}$$

which proves (37) for $s = 2N$. Using the identity

$$[a, (-\Delta_D)^{-N}] = (-\Delta_D)^{-N}[(-\Delta_D)^N, a](-\Delta_D)^{-N}, \tag{52}$$

we can prove (37) for $s = -2N$ in the same way as in the case $s = 2N$. Finally, by an interpolation argument we obtain the result for $s \in \mathbb{R}$. This completes the proof of Proposition 7.

Proof of Theorem 2. We will first prove (7). Extend f by 0 for $t \in \mathbb{R}\backslash[0, T]$. The Fourier transforms (in t) of u and f are holomorphic in the domain $\operatorname{Im} z < 0$ and satisfy the equation

$$(-z - \Delta_D + iB_a)\widehat{u}(z, \cdot) = \widehat{f}(z, \cdot). \tag{53}$$

We take $z = \lambda - i\sigma$, $\lambda \in \mathbb{R}$, $\sigma > 0$, and we let σ tend to zero. Using Proposition 7, we get

$$\|\widehat{u}\|_{L^2(\mathbb{R}; H_D^{s+1-\varepsilon}(\Omega_0))} \lesssim \|\widehat{f}\|_{L^2(\mathbb{R}; H_D^s(\Omega_0))}, \qquad s \in \mathbb{R}. \tag{54}$$

The fact that the Fourier transform of any function from $\mathbb{R}$ to a Hilbert space H defines an isometry on $L^2(\mathbb{R}; H)$ completes the proof of (7).

Now we turn to the proof of (8). Let $\varphi \in C_0^\infty]0, +\infty[$, then the function $w(t, \cdot) = \varphi(t)v(t, \cdot)$ satisfies the equation

$$\begin{cases} i\partial_t w - \Delta_D w + iB_a w = i\varphi'(t)v & \text{in } \mathbb{R}_+ \times \Omega_0, \\ w(0, \cdot) = 0 & \text{in } \Omega_0, \\ w|_{\mathbb{R}\times\partial\Omega_0} = 0. \end{cases} \tag{55}$$

Using (7) with $\varepsilon = 1/2$ we obtain

$$\begin{aligned}\|w\|_{L^2(\mathbb{R}_+, H_D^{s+1/2}(\Omega_0))} &\lesssim \|\varphi' v\|_{L^2(\mathbb{R}_+, H_D^s(\Omega_0))} \\ &\lesssim \|v_0\|_{H_D^s(\Omega_0)}, \end{aligned} \tag{56}$$

which implies

$$v \in L^2_{loc}((0,\infty), H_D^{s+1/2}(\Omega_0)). \tag{57}$$

By iteration we obtain

$$v \in L^2_{loc}((0,\infty), H_D^{s+k}(\Omega_0)), \qquad \forall k \in \mathbb{N}. \tag{58}$$

Using the equation satisfied by v, we deduce that

$$v \in H^k_{loc}((0,\infty), H_D^{s+k}(\Omega_0)), \qquad \forall k \in \mathbb{N}. \tag{59}$$

This implies that $v \in C^\infty((0,\infty) \times \Omega_0)$ and the proof of Theorem 2 is completed.

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