Energy transport through rare collisions

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Abstract

We study a one-dimensional hamiltonian chain of masses perturbed by an energy conserving noise. The dynamics is such that, according to its hamiltonian part, particles move freely in cells and interact with their neighbors through collisions, made possible by a small overlap of size $\epsilon > 0$ between near cells. The noise only randomly flips the velocity of the particles. If $\epsilon \to 0$, and if time is rescaled by a factor $1/\epsilon$, we show that energy evolves autonomously according to a stochastic equation, which hydrodynamic limit is known in some cases. In particular, if only two different energies are present, the limiting process coincides with the simple symmetric exclusion process.

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1 Introduction

Fourier's law asserts that the flux of energy in a bulk of material is proportional to the gradient of temperature:

$$J \;=\; -\kappa(T)\,\nabla T,$$

where κ is the conductivity of the material. While this phenomenological law is widely verified in practice, its derivation from a microscopic hamiltonian dynamics still remains a very challenging question [2].

One actually knows that integrable hamiltonian systems usually violate Fourier's law, as it is the case for ordered harmonic chains [9], for the Toda lattice with equal masses [10][1], or for a system of one-dimensional identical particles interacting only through collisions (see Remark 1 in Subsection 2.3). In these three examples, ballistic transport of energy is observed. Moreover, the conductivity of one-dimensional chains of oscillators conserving the total momentum (no pinning) is generally expected to diverge [7].

On an other hand, Fourier's law have been derived starting from some purely stochastic dynamics, called lattice gases. In these models, particles have a fixed position on a lattice (pinning) and exchange their energy with their neighbors according to the value of some random variable. The KMP model [6] or the simple symmetric exclusion process (SSEP) [5] constitute easy examples where the heat equation can be recovered from a microscopic dynamics. To obtain a first hamiltonian derivation of Fourier's law, it seems thus desirable to start with hamiltonian systems which might look as close as possible to such stochastic models. In the recent years, many progress have been made in this direction.

Let us consider a periodic lattice of N masses, each of them being confined in a region of the space by a pinning potential. We assume that the strength of the interaction between near atoms is controlled by a small parameter $\epsilon \geq 0$. When $\epsilon = 0$, there is no interaction, and we suppose that the dynamics of each single atom has good mixing properties (hyperbolic dynamics). Then, when the interaction is turned on ($\epsilon > 0$) but is still very small, energy should flow between atoms at a much slower rate than the rate at which every atom reaches its own equilibrium for a given energy. One expects therefore the evolution of energy to be similar to the one of a stochastic process.

Starting from there, it has been proposed to derive Fourier's law in two steps, as described by Gaspard and Gilbert [4]. First, one tries to show that the energy of the atoms becomes an autonomous Markov process in the limit $\epsilon \to 0$, if time has been rescaled by a factor $\epsilon^{-\alpha}$ for some $\alpha > 0$, in order to obtain a non-trivial limit (but no space rescaling). The process obtained after this first step corresponds to a mesoscopic description of the material. Second, one hopes that the hydrodynamic limit of the mesoscopic process can be shown to coincide with the heat equation.

The first part of this program has by now been carried out rigorously when the interaction is of the form ϵV, where V is a smooth potential coupling nearest neighbor atoms. Liverani and Olla [8] first considered the case where the uncoupled dynamics is made of a hamiltonian part perturbed by a stochastic noise, modeling an hyperbolic dynamics. Dolgopyat and Liverani [3] were then able to obtain a similar result starting from a purely hamiltonian dynamics. One has however not yet been able to identify rigorously the hydrodynamic limit of the obtained mesoscopic processes.

The aim of this paper is to investigate the case where the interaction between atoms takes place through elastic collision rather than a smooth potential. In this context, the parameter ϵ represents the size of a small interaction zone where near particles can collide, and the limit $\epsilon \to 0$ corresponds thus to a limit of rare interactions, rather than weak interactions. The model we consider is one-dimensional, and the lack of hyperbolicity of the uncoupled dynamics is supplied by a stochastic noise (see Subsection 2.1).

The limiting process we obtain is described by (2.4-2.5) below. Its behavior in the hydrodynamic regime can be understood by means of the nowdays probabilistic tools, at least when the number of energies reachable by the system is finite. In the special case where only two different energies are present, we even recognize the generator of the SSEP. The mesoscopic description of the dynamics is thus so simple in this model, that it leaves us some hope to go beyond the slightly artificial two steps procedure followed here.

The rest of this paper is organized as follows. In Section 2, we describe the dynamics at a non technical level, we state our result, and give some notations to be used throughout the text. A detailed definition of the dynamics together with some elementary observations is to be found in Section 3. Section 4 contains the proof of Theorem 1, admitting Lemma 1, which proof is deferred to Section 5. The proof of this lemma constitutes actually the core of the argument. Section 6 deals with the convergence to equilibrium for the uncoupled dynamics. Finally, the two figures of the text are collected in Section 7.

2 Model and result

2.1 The model

We consider a system of N classical point particles of unit mass, each one evolving in a one-dimensional cell of size one (see figure 1 in Section 7). Let us first describe the dynamics when all the particles move independently from each others. For this, let us associate a Poisson process (a "Poisson clock") to each of them. The N Poisson processes are independent but have the same parameter $\lambda > 0$. A particle moves then freely in its cell, meaning that it travels at a constant speed and is elastically reflected at the boundaries. In addition, each time its Poisson clock rings, the sign of its velocity is reversed. So the Poisson processes serve to model a chaotic dynamics inside the cells. Observe that the kinetic energy of every particle is a conserved quantity under this uncoupled dynamics.

We then introduce some interaction between the particles. Let $0 < \epsilon < 1/2$, and put the cells in a one-dimensional row, as depicted in figure 1 in Section 7, so that there is an overlap of length ϵ between near cells. If two nearest neighbor particles hit each others, they undergo an elastic collision, resulting in a exchange of their momenta. This is the only interaction ; in particular, particles do not hit against the boundary of neighbor cells. This dynamics still preserves the total kinetic energy of the system.

To fully specify the dynamics, one finally needs to give the initial positions and velocities $x = \big(q_1, \dots, q_N, p_1, \dots, p_N\big)$. One supposes the initial positions to be such that particle k is at the left of particle $k+1$ for $1 \le k \le N-1$. The dynamics is so that this condition holds then at all further times. If $q_k \in [0,1]$ for $1 \le k \le N$, this means $q_{k+1} \ge q_k - 1 + \epsilon$ for $1 \le k \le N-1$. One assumes also that no initial velocity is zero. For $t \ge 0$, the state of the system is then entirely given by the value of the Markov process

$$X^\epsilon(x,t) = (q_1^\epsilon(x,t), \dots, q_N^\epsilon(x,t), p_1^\epsilon(x,t), \dots, p_N^\epsilon(x,t)),$$

where we have emphasized the dependence on ϵ.

2.2 The result

Let $0 < \mathsf{e}_1 < \dots < \mathsf{e}_{N'} < +\infty$ be the initial energies, where N' is an integer smaller or equal to N. These are thus numbers such that $\frac{1}{2}p_k^2 \in \{\mathsf{e}_1, \dots, \mathsf{e}_{N'}\}$ for $1 \le k \le N$. The dynamics is such that $\{\mathsf{e}_1, \dots, \mathsf{e}_{N'}\}$ constitute also the set of all energies reachable by the system for all times. One wants to describe how energy flows between the particles in the limit $\epsilon \to 0$. Since one suspects the rate of collisions to be proportional to ϵ (see Remark 3 in Subsection 2.3), one needs also to rescale time by a factor ϵ^{-1} in order to obtain a non-trivial limit. More precisely, we seek for the limit distribution of the time rescaled kinetic energy

$$\mathcal{E}^\epsilon(t) = \big(\mathcal{E}_1^\epsilon(t), \dots, \mathcal{E}_N^\epsilon(t)\big) = \Big(\frac{1}{2}(p_1^\epsilon)^2(\epsilon^{-1}t), \dots, \frac{1}{2}(p_N^\epsilon)^2(\epsilon^{-1}t)\Big)$$

as $\epsilon \to 0$.

We need a couple of extra definitions to express our result:

1. Let $\mathcal{D}([0,1], \mathrm{A})$ be the set of cad-lag functions on $[0,1]$ with value in $\mathrm{A} \subset \mathbb{R}^d$, and observe that actually $\mathcal{E}^\epsilon \in \mathcal{D}\big([0,1], \{\mathsf{e}_1, \dots, \mathsf{e}_{N'}\}^N\big)$.

2. Let γ be a function on $\mathbb{R}_+^2$ given by

$$\gamma(a,b) = \frac{1}{2}\max\{\sqrt{2a}, \sqrt{2b}\}. \tag{2.1}$$

3. Let $\mathbf{Q}^\epsilon$ be the physical space accessible to the particles:

$$\mathbf{Q}^\epsilon \;=\; \{(q_1, \dots, q_N) \in \; [0,1]^N \;:\; q_{k+1} \;\ge\; q_k - 1 + \epsilon \;\text{ for }\; 1 \le k \le N-1\}. \tag{2.2}$$

4. A number $\xi \in \mathbb{R}$ is called diophantine if there exist constants $\mathrm{C}, \beta < +\infty$ such that, for every $p/q \in \mathbb{Q}$, one has

$$\Big|\xi - \frac{p}{q}\Big| \;\ge\; \frac{\mathrm{C}}{q^\beta}. \tag{2.3}$$

With the definitions and notations introduced up to now, one has

Theorem 1. *Let $e_1(0), \dots, e_N(0) \in \{\mathsf{e}_1, \dots, \mathsf{e}_{N'}\}$. Let us assume that*

1. *for $1 \le j \ne k \le N'$, the numbers $\sqrt{\mathsf{e}_j/\mathsf{e}_k}$ are diophantine,*
2. *the law of $X^\epsilon(\,\cdot\,, 0)$ is the only probability measure which density is proportional to*

$$\chi_{\mathbf{Q}^\epsilon}(q_1, \dots, q_N) \cdot \chi_{(e_1(0), \dots, e_N(0))}(p_1^2/2, \dots, p_N^2/2).$$

Then

1. *as $\epsilon \to 0$, the distribution of the process $\mathcal{E}^\epsilon$ tends weakly to the distribution of a process*

$$\mathcal{E} = (\mathcal{E}_1, \dots, \mathcal{E}_N) \ \in \ \mathcal{D}\big([0,1], \{\mathsf{e}_1, \dots, \mathsf{e}_{N'}\}^N\big),$$

2. *$\mathcal{E}$ is the unique Markov process which law $\overline{\mathsf{P}}$ solves the equations*

$$\partial_t \overline{\mathsf{P}}(e_1, \dots, e_N, t) = \sum_{k=1}^{N-1} \gamma(e_k, e_{k+1}) \Big(\overline{\mathsf{P}}(\dots, e_{k+1}, e_k, \dots, t) - \overline{\mathsf{P}}(\dots, e_k, e_{k+1}, \dots, t) \Big), \tag{2.4}$$

$$\overline{\mathsf{P}}(e_1, \dots, e_N, 0) = \chi_{(e_1(0), \dots e_N(0))}(e_1, \dots, e_N). \tag{2.5}$$

2.3 Remarks

Remark 1. The dynamics, with or without noise, should actually becomes completely elementary if the particles were not confined into cells[1]. Indeed, in this case, the system can be seen as a collection of N independent particles, since they just exchange their momentum when colliding.

Remark 2. The condition on the initial measure is much more restrictive than needed. A close look at the proof shows that, with some extra work, one should be able to start from a measure with bounded density with respect to the positions. Our choice mainly allows us to avoid an extra initial step that should be needed to apply Lemma 1 below.

Remark 3. It is not so obvious that time has to be rescaled by a factor ϵ^{-1} in order to obtain a non-trivial result (the correct scaling factor is ϵ^{-2} in [3] and [8]). So let us try to motivate this a little bit, following the heuristic of [4]. It is enough to concentrate on the case of only two particles ($N = 2$), with two energies $\mathsf{e}_1 < \mathsf{e}_2$. When ϵ becomes very small, the rate of collisions between these particles should tend to zero as well. So, since the dynamics of uncoupled particles have good mixing properties, the distribution of position and velocity of a given particle with energy $e \in \{\mathsf{e}_1, \mathsf{e}_2\}$ should be well approximated by the equilibrium measure of an uncoupled particle of the same energy. It is easily checked that this measure is given by

$$\mu_e(u) \ = \ \frac{1}{2} \sum_{p = \pm\sqrt{2e}} \int_0^1 u(q,p)\, \mathrm{d}q.$$

[1] we thank J.L. Lebowitz for having recalled us this fact

Assuming that, at a given time, the energy of particle 1 is e_1, and the energy of particle 2 is e_2, it seems then reasonable to identify the instantaneous energy exchange rate as

$$\begin{aligned}\tilde{\gamma} \;&=\; \lim_{\eta\to 0}\frac{1}{\eta}\,\mathsf{P}\big(\text{particle 1 hits particle 2 in a time interval } \eta\big)\\ &\simeq\; \lim_{\eta\to 0}\frac{1}{\eta}\,\frac{1}{4}\sum_{\substack{p_1=\pm\sqrt{2\mathsf{e}_1}\\ p_2=\pm\sqrt{2\mathsf{e}_2}}}\int_0^1 \mathrm{d}q_1\int_{1-\epsilon}^{2-\epsilon}\mathrm{d}q_2\,\chi_{[0,(p_1-p_2)\eta]}(q_2-q_1)\,\chi_{\mathbb{R}_+}(p_1-p_2),\end{aligned}$$

where one has taken into account the fact that particle 1 should be at the left of particle 2. One computes that

$$\tilde{\gamma} \;=\; \frac{\epsilon\sqrt{2\mathsf{e}_2}}{2} \;=\; \frac{\epsilon}{2}\max\{\sqrt{2\mathsf{e}_1},\sqrt{2\mathsf{e}_2}\},$$

so that indeed $\tilde{\gamma}$ will coincide with γ as defined in (2.1) if time is rescaled by a factor ϵ^{-1}.

Remark 4. The presence of a diophantine condition on the velocities may appear as a surprise in the present context. To see where it comes from, let us again take the case $N = N' = 2$, and let us assume that one velocity has value 1, and the other 2, violating thus the diophantine condition. The dynamics of a single particle is such that, for any fixed time interval, in the microscopic time scale, there is a strictly positive probability, independent of ϵ, that the Poisson clock of the particle does not ring during this time, so that its trajectory is in fact deterministic. Now, let us suppose that the two particles collide, and let us take the "typical" case where the collision does not occur too close to the borders of the overlap region between the two near cells. It can then be seen that, no matter how small ϵ is taken, they will collide once more with each other, in a time interval of order 1, if they travel along the deterministic trajectory. This happens thus with a strictly positive probability.

However, the basic assumption behind our theorem is that successive shocks occur "randomly", and are typically spaced by time intervals of order ϵ^{-1}, in the microscopic time scale. This needs to be so for the coefficient γ to be given by (2.1) in (2.4). But we see that, if the quotients of velocities are rational like here, successive shocks are actually quite correlated, and tend to occur in clusters. It seems clear however, that these clusters should themselves occur at random and be typically spaced by time intervals of order ϵ^{-1}, so that we conjecture Theorem 1 to still remain valid without diophantine condition, but with a smaller coeffeicient γ since recollisions tend to cancel the transfer of energy. We have not been able to show this.

2.4 Notations

Constants. The number called constants in this text may depend on the following parameters of our problem : the number N of particles, the number N' of different energies, and the value of the smallest and the largest energy, namely e_1 and $\mathsf{e}_{N'}$. They never depend nor on ϵ, nor on time.

Time discretization (η). We will find convenient to divide time in small intervals. Throughout the text, we will denote these time steps by η. One always assume $\eta \leq \epsilon$. Its exact value matters few, and one can always think that $\eta << \epsilon$.

Deterministic dynamics (∼). Objects with a tilde refer to the deterministic dynamics. So e.g. $\tilde{X}^\epsilon$ is the trajectory of our process driven by the deterministic dynamics.

The phase space. Points of the phase space Φ are written $x = (\mathbf{q}, \mathbf{p}) = (q_1, \dots, q_N, p_1, \dots, p_N)$. Moreover, one writes $\mathbf{e} = (e_1, \dots, e_n) = (p_1^2/2, \dots, p_N^2/2)$.

Norms. Let G be the Gibbs measure defined in Subsection 3.3 below. The norm of the space $\mathrm{L}^p(\Phi, \mathsf{G})$ is written $\| \cdot \|_p$ $(1 \leq p \leq +\infty)$. If μ is a measure on Φ, one writes $\|\mu\|_1 = \sup_{u:\|u\|_\infty \leq 1} |\mu(u)|$.

The space $\mathcal{S}^\epsilon$. Functions u on Φ of the form $u(\mathbf{q}, \mathbf{p}) = \chi_{\mathbf{Q}^\epsilon}(\mathbf{q}) \,.\, v(\mathbf{e})$ are said to belong to the space $\mathcal{S}^\epsilon$. A measure μ on Φ which density w.r.t. G belongs to $\mathcal{S}^\epsilon$, is itself said to belong to $\mathcal{S}^\epsilon$.

3 The dynamics

3.1 Definition

Let $\epsilon \geq 0$, let $N \geq N' \geq 1$ be two integers, and let $0 < \mathsf{e}_1 < \dots < \mathsf{e}_{N'} < +\infty$ be given energies. Let

$$\mathbf{P} \;=\; \{-\sqrt{2\mathsf{e}_1}, \sqrt{2\mathsf{e}_1}, \dots, -\sqrt{2\mathsf{e}_{N'}}, \sqrt{2\mathsf{e}_{N'}}\}$$

be the set of velocities, let

$$\Phi \;=\; [0,1]^N \times \mathbf{P}^N$$

be the phase space, and let

$$\Phi^\epsilon \;=\; \mathbf{Q}^\epsilon \times \mathbf{P}^N,$$

with $\mathbf{Q}^\epsilon$ defined by (2.2), be the subset of Φ where the dynamics actually takes place. Let $x = (\mathbf{q}, \mathbf{p}) = (q_1, \dots, q_N, p_1, \dots, p_N) \in \Phi$ and, for $t \geq 0$, let

$$X^\epsilon(x,t) = (\mathbf{q}^\epsilon(x,t), \mathbf{p}^\epsilon(x,t)) = (q_1^\epsilon(x,t), \dots, q_N^\epsilon(x,t), p_1^\epsilon(x,t), \dots, p_N^\epsilon(x,t))$$

be the stochastic process on Φ such that $X^\epsilon(x,0) = x$ and that, dropping out the superscript ϵ for clarity,

$$\mathrm{d}q_k \;= p_k \,\mathrm{d}t, \tag{3.1}$$

$$\mathrm{d}p_k \;=\; -\,2p_k(\mathrm{dN}_k + \mathrm{dN}_k^{\mathrm{l}} + \mathrm{dN}_k^{\mathrm{r}}) + (p_{k-1} - p_k)\,\mathrm{dN}_{k-1}^{\mathrm{c}} - (p_k - p_{k+1})\,\mathrm{dN}_k^{\mathrm{c}} \tag{3.2}$$

for $1 \leq k \leq N$, where

1. the processes N_k are independent Poisson processes with identical parameter $\lambda > 0$,
2. the processes $\mathrm{N}_k^{\mathrm{l}}$ are cad-lag processes such that $\mathrm{N}_k^{\mathrm{l}}(t_+) - \mathrm{N}_k^{\mathrm{l}}(t_-) \in \{0,1\}$ for $t \geq 0$, with

$$\mathrm{N}_k^{\mathrm{l}}(t_+) - \mathrm{N}_k^{\mathrm{l}}(t_-) \;=\; 1 \quad \text{if and only if} \quad q_k(t_-) = 0 \;\text{ and }\; p_k(t_-) < 0,$$

3. the processes $\mathrm{N}_k^{\mathrm{r}}$ are cad-lag processes such that $\mathrm{N}_k^{\mathrm{r}}(t_+) - \mathrm{N}_k^{\mathrm{r}}(t_-) \in \{0,1\}$ for $t \geq 0$, with

$$\mathrm{N}_k^{\mathrm{r}}(t_+) - \mathrm{N}_k^{\mathrm{r}}(t_-) \;=\; 1 \quad \text{if and only if} \quad q_k(t_-) = 1 \;\text{ and }\; p_k(t_-) > 0,$$

4. the processes $\mathrm{N}_k^{\mathrm{c}}$ are cad-lag processes such that $\mathrm{N}_k^{\mathrm{c}}(t_+) - \mathrm{N}_k^{\mathrm{c}}(t_-) \in \{0,1\}$ for $t \geq 0$, with

$$\mathrm{N}_k^{\mathrm{c}}(t_+) - \mathrm{N}_k^{\mathrm{c}}(t_-) \;=\; 1 \quad \text{if and only if} \quad q_{k+1}(t_-) = q_k(t_-) - 1 + \epsilon \;\text{ and }\; p_{k+1}(t_-) < p_k(t_-)$$

for $1 \leq k \leq N-1$, and $\mathrm{N}_N^{\mathrm{c}} = 0$.

Here, $f(t_+)$ and $f(t_-)$ represent respectively the left and right limit of a cad-lag function f at time t.

Equations (3.1-3.2) describe the evolution of a particle in a N-dimensional billiard, which projection on the $(q_k, q_{k+1})-$plane is depicted on figure 2 in Section 7. It is checked that (3.1-3.2) admits a unique solution given almost any initial condition w.r.t. the measure G^0 defined in Subsection 3.3 below. Indeed, looking at figure 2, the deterministic solution (when none of the Poisson clock rings) can be constructed at hand for a fixed finite time interval. The general case can then be obtained, noting that the Poisson processes have almost surely finitely many jumps in any time interval.

Observe that the condition $p_{k+1}(t_-) < p_k(t_-)$ in the definition of $\mathrm{N}_k^{\mathrm{c}}$ is such that particles may move from $\Phi - \Phi^\epsilon$ to Φ^ϵ, but may not escape from Φ^ϵ. The definition of the dynamics outside Φ^ϵ is absolutely irrelevant for the statement of Theorem 1, but our choice turns out to be convenient for its proof.

3.2 Evolution semi-group and generator

Let $\epsilon \geq 0$. For $t \geq 0$, the operator $\mathcal{P}^{\epsilon,t}$ acts on bounded functions u on Φ according to the formula

$$\mathcal{P}^{\epsilon,t} u(x) \;=\; \mathsf{E}(u \circ X^\epsilon(x,t)).$$

Its adjoint $\mathcal{P}^{\epsilon *,t}$ acts on measures on Φ, and is defined by means of the relation

$$\mathcal{P}^{\epsilon *,t} \mu(u) \;=\; \mu(\mathcal{P}^{\epsilon,t} u).$$

Let now u be some smooth function on Φ, and let $v \equiv v(x,t) := \chi_{\Phi^\epsilon}(x) \,.\, \mathcal{P}^t u(x)$. The evolution equations imply that (3.1-3.2) that v satisfy the boundary conditions

$$v(\dots, q_j, \dots, p_j, \dots, t) = v(\dots, q_j, \dots, -p_j, \dots, t) \;\text{ if }\; q_j = 0 \text{ or } q_j = 0, \tag{3.3}$$

$$v(\dots, q_j, q_{j+1}, \dots, p_j, p_{j+1}, \dots, t) = v(\dots, q_j, q_{j+1}, \dots, p_{j+1}, p_j, \dots, t) \;\text{ if }\; q_{j+1} = q_j - 1 + \epsilon, \tag{3.4}$$

the first of these relations being satisfied for $1 \leq j \leq N$, and the second for $1 \leq j \leq N-1$. Moreover, it solves the following differential equation in the interior of Φ:

$$\partial_t v \;=\; \sum_{k=1}^{N} p_k \partial_{q_k} v + \lambda \sum_{k=1}^{N} \big(v(\dots, -p_k, \dots) - v(\dots, p_k, \dots) \big). \tag{3.5}$$

3.3 Invariant measure

The probability measure G^ϵ is defined as the uniform probability measure on Φ^ϵ:

$$\mathsf{G}^\epsilon(u) \equiv \int_\Phi u(x)\,\mathsf{G}^\epsilon(\mathrm{d}x) = \frac{1}{|\mathbf{P}^N|}\sum_{\mathbf{p}\in\mathbf{P}^N}\frac{1}{|\mathbf{Q}^\epsilon|}\int_{\mathbf{Q}^\epsilon} u(\mathbf{q},\mathbf{p})\,\mathrm{d}\mathbf{q},$$

where

$$|\mathbf{P}^N| = (2N')^N \quad \text{and} \quad |\mathbf{Q}^\epsilon| = \int_{\mathbf{Q}^\epsilon}\mathrm{d}\mathbf{q}.$$

When $\epsilon = 0$, we will just write G instead of G^0. One checks using (3.5) together with the boundary conditions (3.3-3.4) that the measure G^ϵ is invariant under the dynamics, meaning that $\mathcal{P}^{\epsilon *,t}\mathsf{G}^\epsilon = \mathsf{G}^\epsilon$ for every $t \geq 0$. Since energy is conserved, this is not the only invariant measure. In fact G^ϵ corresponds to a Gibbs measure at infinite temperature.

4 Proof of Theorem 1

Let $0 < \epsilon_0 < 1/2$. One first shows that the family $(\mathcal{E}^\epsilon)_{\epsilon\leq\epsilon_0}$ is tight in $\mathcal{D}([0,1])$. One then proves that the law of $\mathcal{E}^\epsilon(t)$ converges weakly for every $t \in [0,1]$ to the law $\overline{\mathsf{P}}(\,\cdot\,,t)$ characterized by (2.4-2.5). One finally establishes that the limit process $\mathcal{E}$ is markovian.

4.1 Tightness

We prove here that the family $(\mathcal{E}^\epsilon)_{\epsilon\leq\epsilon_0}$ is tight in $\mathcal{D}\big([0,1],\{\mathsf{e}_1,\dots,\mathsf{e}_{N'}\}^N\big)$, meaning that for every $\delta > 0$, there exists a compact set $K_\delta \subset \mathcal{D}\big([0,1],\{\mathsf{e}_1,\dots,\mathsf{e}_{N'}\}^N\big)$ such that $\mathsf{P}^\epsilon(K_\delta) \geq 1-\delta$ for every $\epsilon \in]0,\epsilon_0]$, where P^ϵ is the law of $\mathcal{E}^\epsilon$. For $n \geq 1$, let K_n be the set of functions in $\mathcal{D}\big([0,1],\{\mathsf{e}_1,\dots,\mathsf{e}_{N'}\}^N\big)$ having at most n jumps. Since $\{\mathsf{e}_1,\dots,\mathsf{e}_{N'}\}^N$ is finite, the sets K_n are compacts. Let now $\delta \in]0,1[$, and let us show that there exists $n(\delta) \geq 1$ such that $\mathsf{P}^\epsilon(K_{n(\delta)}) \geq 1-\delta$ for every $\epsilon \in]0,\epsilon_0]$.

The energy of the particles can only change due to a collision with a neighbor, and so it suffices to establish that there exists a sequence $(c_n)_{n\geq1}$, with $c_n \to 0$ as $n \to \infty$, such that, for every $\epsilon \in]0,\epsilon_0]$ and $1 \leq k \leq N-1$,

$$\mathsf{P}_\mu\Big(\int_0^{\epsilon^{-1}} \mathrm{dN}_k^{\mathrm{c}}(s) \;\geq\; n\Big) \;\leq\; c_n, \tag{4.1}$$

where $\mu \in \mathcal{S}^\epsilon$ is the law of $X^\epsilon(\,\cdot\,,0)$.

One would like to replace the integrand in this expression by an explicit function on the Markov chain X^ϵ. If the dynamics were deterministic (no Poisson processes), one should have

$$\int_0^{\epsilon^{-1}} \mathrm{dN}_k^{\mathrm{c}}(s) \;\leq\; \mathrm{C}\int_0^{\epsilon^{-1}} \eta^{-1}\cdot\chi_{\mathrm{Y}_k(\epsilon,\eta)}\circ X_s^\epsilon\,\mathrm{d}s \tag{4.2}$$

with $\eta > 0$ as small as we want, and

$$\mathrm{Y}_k(\epsilon,\eta) \;:=\; \{x\in\Phi : |q_{k+1}-q_k+1-\epsilon| \leq \eta\}.$$

But, for $\epsilon > 0$ one finds $\eta \equiv \eta(\epsilon)$ small enough such that the event

$$\Delta(\epsilon,\eta) \;:=\; \big\{\omega : \max_{t\in[0,\epsilon^{-1}]} \big(\mathrm{N}_k(t+\eta) - \mathrm{N}_k(t)\big) + \big(\mathrm{N}_{k+1}(t+\eta) - \mathrm{N}_{k+1}(t)\big) \;\leq\; 1\big\}$$

has a probability as close to 1 as one wishes.

Therefore, taking η small enough for a given ϵ, one may replace (4.1) by

$$\mathsf{P}_\mu\Big(\int_0^{\epsilon^{-1}} \mathrm{dN}^{\mathrm{c}}_k(s) \;\geq\; n \;\Big|\; \Delta(\epsilon,\eta)\Big) \;\leq\; c_n. \tag{4.3}$$

Doing so, inequality (4.2) may now be applied (with a larger constant) even in the presence of the Poisson processes, and, by Markov's inequality, one gets

$$\begin{aligned}\mathsf{P}_\mu\Big(\int_0^{\epsilon^{-1}} \mathrm{dN}^{\mathrm{c}}_k(s) \;\geq\; n \;\Big|\; \Delta(\epsilon,\eta)\Big) &\leq \mathrm{C}\, n^{-1} \,.\, \mathsf{E}_\mu \int_0^{\epsilon^{-1}} \eta^{-1}\,.\,\chi_{\mathrm{Y}_k(\epsilon,\eta)} \circ X^\epsilon_s \,\mathrm{d}s \\ &= \mathrm{C}\, n^{-1}\,.\int_0^{\epsilon^{-1}} \mathrm{d}s \int \mathcal{P}^{\epsilon,t}(\eta^{-1}\,.\,\chi_{\mathrm{Y}_k(\epsilon,\eta)})\,\mathrm{d}\mu \\ &\leq \mathrm{C}\, n^{-1}\epsilon^{-1}\,.\,\mathsf{G}^\epsilon(\eta^{-1}\,.\,\chi_{\mathrm{Y}_k(\epsilon,\eta)}) \;\leq\; \mathrm{C}\, n^{-1},\end{aligned}$$

where one has used the fact that $\mu \leq \mathrm{C}\,\mathsf{G}^\epsilon$ to get the second inequality, and the fact that $\mathsf{G}^\epsilon(\eta^{-1}\,.\,\chi_{\mathrm{Y}_k(\epsilon,\eta)}) = \mathcal{O}(\epsilon)$ for any value of $\eta \in]0,\epsilon]$ to get the last one. $\Box$

4.2 Convergence at a given time

We show here that the law of $\mathcal{E}^\epsilon(t)$ converges weakly for every $t \in [0,1]$ to the law $\overline{\mathsf{P}}(\,\cdot\,,t)$, characterized by (2.4-2.5). Observe that, for given $\epsilon > 0$, there is an obvious identification between functions on $\{\mathsf{e}_1,\dots \mathsf{e}_{N'}\}^N$ and measures in $\mathcal{S}^\epsilon$, that will be used in the sequel.

Let $\mathsf{P}^\epsilon(\,\cdot\,,t)$ be the law of $X^\epsilon(t)$ (so *not* the law of $\mathcal{E}^\epsilon(t)$ as it was in the previous subsection). We actually will prove that, for any $t \in [0,1]$, there exists a probability measure $\overline{\mathsf{P}}(\,\cdot\,,t) \in \mathcal{S}^0$, such that $\mathsf{P}^\epsilon(\,\cdot\,,t)$ converges weakly to $\overline{\mathsf{P}}(\,\cdot\,,t)$ as $\epsilon \to 0$, and then that $\overline{\mathsf{P}}(\,\cdot\,,t)$ solves (2.4-2.5). This will imply our result. Indeed, the distribution of $\mathcal{E}^\epsilon(t)$ may be written as $\int_\Phi \mathsf{P}^\epsilon(\mathrm{d}x, t|\mathsf{e})$, and so, as $\epsilon \to 0$, the distribution of $\mathcal{E}^\epsilon(t)$ also will converge to $\overline{\mathsf{P}}(\,\cdot\,,t)$. We will use the following lemma, which proof is deferred to the next section.

Lemma 1. *Let v be a function on $\{\mathsf{e}_1,\dots,\mathsf{e}_{N'}\}^N$ corresponding to a probability measure in $\mathcal{S}^\epsilon$ for every $\epsilon \in]0,\epsilon_0]$. For $\tau > 0$ small enough, and for $0 < \epsilon < \tau^2$,*

$$\mathcal{P}^{*\epsilon,\epsilon^{-1}\tau}\,\mathsf{P} \;=\; \Big(1 - \tau\sum_{k=1}^{N-1}\gamma(e_k,e_{k+1})\Big)\,\mathsf{P} \;+\; \tau\sum_{k=1}^{N-1}\gamma(e_k,e_{k+1})\,\mathsf{P}(\dots,e_{k+1},e_k,\dots) \;+\; \mathcal{O}(\tau^2)\,\mu_{\epsilon,\tau},$$

where $\mathsf{P} \in \mathcal{S}^\epsilon$ is the measure determined by v, and where $\mu_{\epsilon,\tau}$ is a measure on Φ such that $\|\mu_{\epsilon,\tau}\|_1 \leq 1$.

Let now $t \in [0,1]$. Let $\epsilon \in]0,\epsilon_0[$, and let n_ϵ be the largest integer such that $\tau := t/n_\epsilon > \epsilon^{1/2}$. One has $\mathsf{P}^\epsilon(\,\cdot\,,t) = \mathcal{P}^{\epsilon*,\epsilon^{-1}t}\mathsf{P}^\epsilon(\,\cdot\,,0)$. Lemma 1 applies if ϵ is small enough, and one has

$$\mathcal{P}^{\epsilon*,\epsilon^{-1}t}\mathsf{P}(\,\cdot\,,0) \;=\; \mathcal{P}^{\epsilon*,\epsilon^{-1}n_\epsilon\tau}\mathsf{P}(\,\cdot\,,0) \;=\; \mathcal{P}^{\epsilon*,\epsilon^{-1}(n_\epsilon-1)\tau}\big(\mathsf{P}_1 + \mathcal{O}(n_\epsilon^{-2})\,\mu_{1,\epsilon,\tau}\big),$$

where $\mathsf{P}_1 \in \mathcal{S}^\epsilon$ is the measure

$$\mathsf{P}_1 \;:=\; \Big(1-\tau\sum_{k=1}^{N-1}\gamma(e_k,e_{k+1})\Big)\,\mathsf{P}^\epsilon(\,\cdot\,,0) \;+\; \tau\sum_{k=1}^{N-1}\gamma(e_k,e_{k+1})\,\mathsf{P}^\epsilon(\dots,e_{k+1},e_k,\dots,0),$$

and $\mu_{1,\epsilon,\tau}$ a measure such that $\|\mu_{1,\epsilon,\tau}\|_1 \le 1$. Observe that P_1 is a probability measure if τ is small enough, what we assume. Since $\|\mathcal{P}^{\epsilon *,s}\mu\|_1 \le \|\mu\|_1$ for every measure μ and every $s \ge 0$, one may iterate this and get

$$\mathcal{P}^{\epsilon*,\epsilon^{-1}t}\mathsf{P}^\epsilon(\,\cdot\,,0) \;=\; \mathsf{P}_{n_\epsilon} + \mathcal{O}(n_\epsilon^{-2})\sum_{k=1}^{n_\epsilon}\tilde{\mu}_{k,\epsilon,\tau},$$

with P_{n_ϵ} a probability measure in $\mathcal{S}^\epsilon$, and $\tilde{\mu}_{k,\epsilon,\tau}$ measures such that $\|\tilde{\mu}_{k,\epsilon,\tau}\|_1 \le 1$.

Let us first show the existence of a probability measure $\overline{\mathsf{P}}(\,\cdot\,,t) \in \mathcal{S}^0$ in the closure of $(\mathsf{P}_{n_\epsilon})_{\epsilon\le\epsilon_0}$ for the $\|\cdot\|_1$-norm. For this, let v_{n_ϵ} be the function on $\{\mathsf{e}_1,\dots,\mathsf{e}_{N'}\}^N$ determining the measure $\mathsf{P}_{n_\epsilon} \in \mathcal{S}^\epsilon$, and let then $\tilde{\mathsf{P}}_{n_\epsilon}$ be the unique measure in $\mathcal{S}^0$ determined by v_{n_ϵ}. Since $\|\mathsf{P}_{n_\epsilon} - \tilde{\mathsf{P}}_{n_\epsilon}\|_1 \le \mathrm{C}\,\epsilon^2$, a point in the closure of $(\tilde{\mathsf{P}}_{n_\epsilon})_{\epsilon\le\epsilon_0}$ is also in the closure of $(\mathsf{P}_{n_\epsilon})_{\epsilon\le\epsilon_0}$ and is a probability measure. But, since $\mathcal{S}^0$ is a finite-dimensional space, and since $(\tilde{\mathsf{P}}_{n_\epsilon})_{\epsilon\le\epsilon_0}$ is a bounded set for the $\|\cdot\|_1$-norm, the announced measure $\overline{\mathsf{P}}(\,\cdot\,,t)$ indeed exists.

Let us next show that $\overline{\mathsf{P}}(\,\cdot\,,t)$ solves (2.4-2.5). Up to a subsequence, one can write

$$\overline{\mathsf{P}}(\,\cdot\,,t) \;=\; \lim_{\epsilon\to 0}\mathcal{P}^{\epsilon*,\epsilon^{-1}t}\mathsf{P}(\,\cdot\,,0) \tag{4.4}$$

for the $\|\cdot\|_1$-norm, and so

$$\begin{aligned}
\lim_{\tau\to 0}\frac{\overline{\mathsf{P}}(\,\cdot\,,t+\tau)-\overline{\mathsf{P}}(\,\cdot\,,t)}{\tau} \;&=\; \lim_{\tau\to 0}\frac{1}{\tau}\lim_{\epsilon\to 0}\big(\mathcal{P}^{\epsilon*,\epsilon^{-1}\tau}-\mathrm{Id}\big)\mathcal{P}^{\epsilon*,\epsilon^{-1}t}\mathsf{P}(\,\cdot\,,0)\\
&=\; \lim_{\tau\to 0}\frac{1}{\tau}\Big(\lim_{\epsilon\to 0}\big(\mathcal{P}^{\epsilon*,\epsilon^{-1}\tau}-\mathrm{Id}\big)\overline{\mathsf{P}}(\,\cdot\,,t)\Big)\\
&\quad+\lim_{\tau\to 0}\frac{1}{\tau}\Big(\lim_{\epsilon\to 0}\big(\mathcal{P}^{\epsilon*,\epsilon^{-1}\tau}-\mathrm{Id}\big)\big(\mathcal{P}^{\epsilon*,\epsilon^{-1}t}\mathsf{P}(\,\cdot\,,0)-\overline{\mathsf{P}}(\,\cdot\,,t)\big)\Big).
\end{aligned}$$

The second term in the right hand side of this equation is zero since

$$\big\|\big(\mathcal{P}^{\epsilon*,\epsilon^{-1}\tau}-\mathrm{Id}\big)\big(\mathcal{P}^{\epsilon*,\epsilon^{-1}t}\mathsf{P}(\,\cdot\,,0)-\overline{\mathsf{P}}(\,\cdot\,,t)\big)\big\|_1 \le 2\,\big\|\mathcal{P}^{\epsilon*,\epsilon^{-1}t}\mathsf{P}(\,\cdot\,,0)-\overline{\mathsf{P}}(\,\cdot\,,t)\big\|_1 \to 0$$

as $\epsilon\to 0$. By Lemma 1 instead, the first term gives

$$\lim_{\tau\to 0}\frac{1}{\tau}\Big(\lim_{\epsilon\to 0}\big(\mathcal{P}^{\epsilon*,\epsilon^{-1}\tau}-\mathrm{Id}\big)\overline{\mathsf{P}}(\,\cdot\,,t)\Big) = \sum_{k=1}^{N-1}\gamma(e_k,e_{k+1})\Big(\overline{\mathsf{P}}(\dots,e_{k+1},e_k,\dots,t)-\overline{\mathsf{P}}(\dots,e_k,e_{k+1},\dots,t)\Big),$$

which establishes the result. □

4.3 $\mathcal{E}$ is a Markov process

We show that, for any $0=t_0<t_1<\dots<t_n\le 1$ and any $\mathbf{e}_0,\mathbf{e}_1,\dots,\mathbf{e}_n \in \{\mathsf{e}_1,\dots,\mathsf{e}_{N'}\}$, for $n\ge 1$,

$$\mathsf{P}\big(\mathcal{E}_{t_n}=\mathbf{e}_n\big|\mathcal{E}_{t_{n-1}}=\mathbf{e}_{n-1},\dots,\mathcal{E}_0=\mathbf{e}_0\big) \;=\; \mathsf{P}\big(\mathcal{E}_{t_n-t_{n-1}}=\mathbf{e}_n\big|\mathcal{E}_0=\mathbf{e}_{n-1}\big).$$

This is immediate for $n = 1$. We show the case $n = 2$, the generalization being rather straightforward. For $\epsilon > 0$, since X^ϵ is a Markov process, one may write

$$\mathsf{P}\big(\mathcal{E}^\epsilon_{t_2} = \mathbf{e}_2 \big| \mathcal{E}^\epsilon_{t_1} = \mathbf{e}_1, \mathcal{E}^\epsilon_0 = \mathbf{e}_0\big) \;=\; \mathsf{P}_{\mu^\epsilon}\big(\mathcal{E}^\epsilon_{t_2 - t_1} = \mathbf{e}_2\big)$$

where μ^ϵ is an initial measure on Φ given by

$$\mu^\epsilon \;=\; \mathcal{P}^{\epsilon *, \epsilon^{-1} t_1} \mu_0(\,\cdot\,|\mathbf{e} = \mathbf{e}_1) \quad \text{with} \quad \mu_0 \;=\; \mathsf{G}^\epsilon(\,\cdot\,|\mathbf{e} = \mathbf{e}_0).$$

Our claim will now be established if we show that $\|\mu^\epsilon - \mathsf{G}^\epsilon(\,\cdot\,|\mathbf{e} = \mathbf{e}_1)\|_1 \to 0$ as $\epsilon \to 0$. But this follows from the facts that $\|\mathcal{P}^{\epsilon *, \epsilon^{-1} t_1}\mu_0 - \overline{\mathsf{P}}(\,\cdot\,, t_1)\|_1 \to 0$ as $\epsilon \to 0$ by (4.4), that $\overline{\mathsf{P}}(\,\cdot\,, t_1|\mathbf{e} = \mathbf{e}_1) = \mathsf{G}(\,\cdot\,|\mathbf{e} = \mathbf{e}_1)$ since $\overline{\mathsf{P}}(\,\cdot\,, t_1) \in \mathcal{S}^0$, and that $\|\mathsf{G} - \mathsf{G}^\epsilon\|_1 = \mathcal{O}(\epsilon^2)$. $\square$

5 Proof of Lemma 1

The proof of Lemma 1 is conceptually very easy. We start by sketching the method in Subsection 5.1. In Subsection 5.2, some crucial elementary facts are collected. We finally prove Lemma 1 in Subsection 5.3.

5.1 Method

Since $\mathsf{P} \in \mathcal{S}^\epsilon$, this measure is invariant under the uncoupled dynamics, up to an error of order ϵ^2 for the $\|\cdot\|_1$-norm. We recall that $\epsilon < \tau^2$ by hypothesis. Writing the result of Lemma 1 as

$$\mathcal{P}^{\epsilon *, \epsilon^{-1}\tau}\mathsf{P} - \mathcal{P}^{0 *, \epsilon^{-1}\tau}\mathsf{P} = \tau\,(\dots) + \mathcal{O}(\tau^2), \tag{5.1}$$

one sees that one needs to evaluate the difference between the evolution of the coupled and the uncoupled dynamics in the first order in τ. The following identity, known as Duhamel's formula, is suited for this: if S and T are two bounded operators on some space, one has

$$T^n - S^n \;=\; \sum_{k=1}^{n} T^{k-1}(T - S)S^{n-k}. \tag{5.2}$$

Iterating this formula to replace T^{k-1} by S^{k-1}, one finds

$$T^n - S^n \;=\; \sum_{k=1}^{n} S^{k-1}(T - S)S^{n-k} + \sum_{k=1}^{n}\sum_{j=1}^{k-1} T^{j-1}(T - S)S^{k-1-j}(T - S)S^{n-k}. \tag{5.3}$$

Let now η be some small time step (see Subsection 2.4), let $S = \mathcal{P}^{0*,\eta}$, let $T = \mathcal{P}^{\epsilon *,\eta}$, and let n be such that $\epsilon^{-1}\tau = n\eta$. At this point, it could have seemed more natural to work directly in a continuous-time setting. In that case, the sums appearing in (5.2) and (5.3) becomes time integral, whereas $T - S$ should become the part of the generator corresponding to a collision between particles. The problem is that this term is hard to make explicit, since it is only present through the boundary conditions (3.3-3.4), and that is why we decided to divide time in arbitrary small intervals.

All what needs to be done now is to show that the first term in the right hand side of (5.3) coincides with the first term in the right hand side of (5.1) up to an error of order ϵ, and that the second term is $\mathcal{O}(\tau^2)$. The identification of the first terms is not hard since one has a good control on the uncoupled dynamics (see Section 6), i.e. on $S = \mathcal{P}^{0*,\eta}$, and since one can estimate the difference $T - S = \mathcal{P}^{\epsilon*,\eta} - \mathcal{P}^{0*,\eta}$ for η small enough, as explained in Subsection 5.2 below.

The second term represents recollisions. One needs to see that, in a macroscopic time of order τ, the probability of two collisions or more for the same particle is $\mathcal{O}(\tau^2)$, as it should be if they were happening at random. This turns out to be a bit heavy to show, mainly due to the persistence of a deterministic trajectory for microscopic time intervals. The diophantine condition on the velocities has here to be used.

5.2 Crucial elementary estimates

Let $\epsilon \geq 0$ and let $\epsilon > \eta > 0$ be a small time interval. Let us define some subsets of Φ. For this, let

$$\mathsf{p} \;:=\; 2\max\big\{\sqrt{2\mathsf{e}_1}, \dots, \sqrt{2\mathsf{e}_{N'}}\big\}.$$

Then, for $1 \leq k \leq N-1$,

$$\mathrm{Z}_k \;= \{x \in \Phi : |q_{k+1} - q_k + 1 - \epsilon| \leq \mathsf{p}\eta\}, \tag{5.4}$$

$$\mathrm{Coll}_k \;= \{x \in \Phi : \tilde{\mathrm{N}}^{\mathrm{c}}_k(x,\eta) - \tilde{\mathrm{N}}^{\mathrm{c}}_k(x,0) = 1\}. \tag{5.5}$$

Here $\tilde{\mathrm{N}}^{\mathrm{c}}_k(x,s)$ is the value at time s of the process $\mathrm{N}^{\mathrm{c}}_k$ defined in Subsection 3.1 for a deterministic trajectory starting at x. We recall that the use of a tilde always refers to the deterministic dynamics (see Subsection 2.4). One then sets

$$\mathrm{Z} \;=\; \bigcup_{k=1}^{N-1} \mathrm{Z}_k, \qquad \mathrm{Coll} \;=\; \bigcup_{k=1}^{N-1} \mathrm{Coll}_k.$$

First, for $1 \leq k \leq N-1$, the characteristic function of the set Z_k depends only on the variables q_k and q_{k+1}. A computation yields

$$\mathrm{C}\,\epsilon\eta \;\leq\; \int_{[0,1]^2} \chi_{\mathrm{Z}_k}\,\mathrm{d}q_k \mathrm{d}q_{k+1} \;\leq\; \int_{\mathbf{Q}} \chi_{\mathrm{Z}}\,\mathrm{d}\mathbf{q} \;\leq\; \mathrm{C}'\,\epsilon\eta, \tag{5.6}$$

and, for $1 \leq j \neq k \leq N-1$,

$$\int_{\mathbf{Q}} \chi_{\mathrm{Z}_j} \cdot \chi_{\mathrm{Z}_k}\,\mathrm{d}\mathbf{q} \;=\; \mathcal{O}(\epsilon\eta)^2. \tag{5.7}$$

Second, for $1 \leq k \leq N-1$, let us define

$$\mathrm{W}_k \;= \{x \in \Phi : 1 - q_k \leq \mathsf{p}\eta \text{ and } |q_{k+1} - q_k + 1 - \epsilon| \leq \mathsf{p}\eta\},$$

$$\mathrm{W}'_k \;= \{x \in \Phi : q_{k+1} \leq \mathsf{p}\eta \text{ and } |q_{k+1} - q_k + 1 - \epsilon| \leq \mathsf{p}\eta\}.$$

The deterministic dynamics is such that

$$\text{Coll} \subset \{x \in \Phi : p_k > p_{k+1} \text{ and } 0 \leq q_{k+1} - q_k + 1 - \epsilon \leq (p_k - p_{k+1})\eta\} \cup \mathrm{W}_k \cup \mathrm{W}'_k, \tag{5.8}$$

$$\text{Coll} \supset \{x \in \Phi : p_k > p_{k+1} \text{ and } 0 \leq q_{k+1} - q_k + 1 - \epsilon \leq (p_k - p_{k+1})\eta\} - \big(\mathrm{W}_k \cup \mathrm{W}'_k\big). \tag{5.9}$$

So $\text{Coll}_k \subset \mathrm{Z}_k$ and thus $\text{Coll} \subset \mathrm{Z}$, and the characteristic function of Coll_k only depends on the variables x_k and x_{k+1}. A computation based on (5.8-5.9) gives

$$\int_{\mathbf{Q}} \chi_{\text{Coll}_k} \, \mathrm{d}q_k \mathrm{d}q_{k+1} \;=\; \chi_{\mathbb{R}_+}(p_k - p_{k+1}) \,.\, (p_k - p_{k+1})\eta\epsilon + \mathcal{O}(\eta^2), \tag{5.10}$$

for any values of the impulsions. It follows moreover from (5.7) that, for $1 \leq j \neq k \leq N-1$,

$$\int \chi_{\text{Coll}_j} \cdot \chi_{\text{Coll}_k} \, \mathrm{d}\mathbf{q} \;=\; \mathcal{O}(\epsilon\eta)^2 \tag{5.11}$$

for any values of the impulsions.

Let us finally estimate the difference between the evolution of the coupled and uncoupled dynamics over a time interval η. Let us define the operator

$$L \;:=\; \frac{1}{\eta}(\mathcal{P}^{\epsilon,\eta} - \mathcal{P}^{0,\eta}). \tag{5.12}$$

One has

$$\tilde{L} \;:=\; \frac{1}{\eta}(\tilde{\mathcal{P}}^{\epsilon,\eta} - \tilde{\mathcal{P}}^{0,\eta}) \;=\; \chi_{\text{Coll}} \cdot \frac{1}{\eta}(\tilde{\mathcal{P}}^{\epsilon,\eta} - \tilde{\mathcal{P}}^{0,\eta}) \tag{5.13}$$

and

Lemma 2. *There exists a constant* $\mathrm{C} < +\infty$ *such that, for all* $\eta > 0$ *small enough, and for all* $u \in \mathrm{L}^\infty(\Phi)$*,*

$$Lu = \tilde{L}u + \chi_{\mathrm{Z}}\, u'$$

where $u' \in \mathrm{L}^\infty(\Phi)$ *is such that* $\|u'\|_\infty \leq \mathrm{C}\|u\|_\infty$.

Proof. Let $x \in \Phi$. If $x \notin \mathrm{Z}$, then $X^\epsilon(x,\eta) = X^0(x,\eta)$ for every realization of the Poisson processes. Therefore

$$Lu = \chi_{\mathrm{Z}} \,.\, Lu.$$

Next, the probability that one of the Poisson processes have a jump in a time interval of length η is itself $\mathcal{O}(\eta)$ so that

$$\mathcal{P}^{\epsilon,\eta}u(x) = \mathsf{E}(u \circ X^\epsilon(x,\eta)) = u \circ \tilde{X}^\epsilon(x,\eta) + \mathcal{O}(\eta\|u\|_\infty) = \tilde{\mathcal{P}}^{\epsilon,\eta}u(x) + \mathcal{O}(\eta\|u\|_\infty).$$

This last formula still holds also in particular for $\epsilon = 0$, and therefore $Lu = \frac{1}{\eta}\tilde{L}u + \chi_{\mathrm{Z}}u'$ with $\|u'\|_\infty \leq \mathrm{C}\,\|u\|_\infty$ for some $\mathrm{C} < +\infty$. □

5.3 Proof of Lemma 1

1. Preliminary simplifications. We will give the proof in the case $N' = N$, indicating at the very end the small needed adaptations to bring for dealing with the case $N' < N$. The space $\mathcal{S}^\epsilon$ is generated by functions of the form

$$\chi_{\mathsf{e}_{k_1},\dots,\mathsf{e}_{k_N}}(e_1,\dots,e_N)\,.\,\chi_{\mathbf{Q}^\epsilon}(\mathbf{q}),$$

where $k_1,\dots,k_N$ is a permutation of $1,\dots,N$. By linearity, it suffices to consider the case where P is proportional to a function of that form. Moreover, to simplify notations, we will assume that $(k_1,\dots,k_N) = (1,\dots,N)$. Let then

$$\mathsf{P}' \;:=\; \frac{1}{2^N}\cdot\chi_{\mathsf{e}_1,\dots,\mathsf{e}_N}(e_1,\dots,e_N)\,.\,\chi_{\mathbf{Q}}(\mathbf{q}) \;=\; \frac{1}{2^N}\cdot\chi_{\mathsf{e}_1,\dots,\mathsf{e}_N}(e_1,\dots,e_N). \tag{5.14}$$

This measure is close to P when $\epsilon \to 0$:

$$\|\mathsf{P}-\mathsf{P}'\|_1 \;=\; \mathcal{O}(\epsilon^2).$$

Because $\|\mathcal{P}^{\epsilon *,t}\mu\|_1 \le \|\mu\|_1$ for every μ and every $t \ge 0$, one may therefore show the result with P' instead of P. This will be advantageous since P' is invariant under the uncoupled dynamics.

2. Applying Duhamel's formula. Let $\tau > 0$, let $\epsilon \in]0,\tau^2[$, and let $\eta \in]\epsilon^4, 2\epsilon^4[$ be such that, for some integer $n \ge 1$, one has

$$\epsilon^{-1}\tau = \eta n. \tag{5.15}$$

Applying Duhamel's formula (5.2) and noting that $\mathsf{P}^{0*,t}\mathsf{P}' = \mathsf{P}'$ for every $t \ge 0$, one gets

$$\mathcal{P}^{\epsilon *,\epsilon^{-1}\tau}\mathsf{P}' = \mathsf{P}' + \eta\sum_{k=1}^{n}\mathcal{P}^{\epsilon *,(k-1)\eta}L^*\mathsf{P}',$$

with L the operator defined in (5.12). One would like to replace L^* by $\tilde{L}^*$, with $\tilde{L}$ defined in (5.13). One has

$$\begin{aligned}\|\mathsf{P}^{\epsilon *,(k-1)\eta}(L^*-\tilde{L}^*)\mathsf{P}'\|_1 \le \|(L^*-\tilde{L}^*)\mathsf{P}'\|_1 &= \sup_{u:\|u\|_\infty\le 1}\mathsf{P}'\big((L-\tilde{L})u\big)\\ &\le \sup_{u:\|u\|_\infty\le 1}\max_{\mathbf{p}\in\mathbf{P}^N}\int_{\mathbf{Q}}\big|(L-\tilde{L})u(\mathbf{q},\mathbf{p})\big|\,\mathrm{d}\mathbf{q} \;=\; \mathcal{O}(\epsilon\eta).\end{aligned}$$

Here the inequality follows from the fact that P' is a probability measure independent of $\mathbf{q}$, whereas the last equality is a consequence of Lemma 2 and (5.6). Therefore

$$\mathcal{P}^{\epsilon *,\epsilon^{-1}\tau}\mathsf{P}' = \mathsf{P}' + \eta\sum_{k=1}^{n}\mathcal{P}^{\epsilon *,(k-1)\eta}\tilde{L}^*\mathsf{P}' + \mu_0 \tag{5.16}$$

where μ_0 is a measure such that

$$\|\mu_0\|_1 = \mathcal{O}(\eta n\,.\,\epsilon\eta) = \mathcal{O}(\tau^2). \tag{5.17}$$

One then uses Duhamel's formula once more to replace $\mathcal{P}^{\epsilon*,(k-1)\eta}$ by $\mathcal{P}^{0*,(k-1)\eta}$ in (5.16):

$$\begin{aligned}\mathcal{P}^{\epsilon*,\epsilon^{-1}\tau}\mathsf{P}' = &\ \mathsf{P}' + \eta\sum_{k=1}^{n}\mathcal{P}^{0*,(k-1)\eta}\tilde{L}^*\mathsf{P}' + \eta^2\sum_{k=1}^{n}\sum_{j=1}^{k-1}\mathcal{P}^{\epsilon*,(j-1)\eta}L^*\mathcal{P}^{0*,(k-1-j)\eta}\tilde{L}^*\mathsf{P}' + \mu_0\\ := &\ \mathsf{P}' + \mu_1 + \mu_2 + \mu_0.\end{aligned}$$

It is not clear at this point that we can replace the remaining operator L^* by $\tilde{L}^*$ up to a negligible error, and so we directly proceed now to the analysis of μ_1 and μ_2. In a first step we will show that μ_1 is given by (5.23) below, whereas in a second step, one will prove that $\|\mu_2\|_1 = \mathcal{O}(\tau^2)$. Thanks to the bound (5.17) on $\|\mu_0\|_1$, this will finish the proof.

3. Analyzing μ_1. We here show that μ_1 can be expressed as in (5.23) below. Given a measure μ on Φ, we write $\mu(1|\mathbf{e})$ the measure in $\mathcal{S}^0$ explicitly given by

$$\mu(1|\mathbf{e}) \ = \ \frac{1}{2^N}\sum_{p_1=\pm\sqrt{2e_1}}\cdots\sum_{p_N=\pm\sqrt{2e_n}}\int_{\mathbf{Q}}\mu(\mathrm{d}\mathbf{q},\mathbf{p}), \tag{5.18}$$

where the factor $1/2^N$ appears since we want to see $\mu(1|\mathbf{e})$ as a measure on Φ, and not as a measure on $\{\mathsf{e}_1,\dots,\mathsf{e}_N\}^N$. Because $\mathcal{P}^{0*,t}\mu(1|\mathsf{e}) = \mu(1|\mathsf{e})$ for any μ on Φ and $t \geq 0$, and since $\eta n = \epsilon^{-1}\tau$, one has

$$\mu_1 = \epsilon^{-1}\tau\,.\,\tilde{L}^*\mathsf{P}'(1|\mathbf{e}) + \eta\sum_{k=1}^{n}\mathcal{P}^{0*,(k-1)\eta}\big(\tilde{L}^*\mathsf{P}' - \tilde{L}^*\mathsf{P}'(1|\mathbf{e})\big). \tag{5.19}$$

Let us first compute $\tilde{L}^*\mathsf{P}'(1|\mathbf{e})$. It follows from (5.18) that

$$\tilde{L}^*\mathsf{P}'(1|\mathsf{e}) \ = \ \frac{1}{2^N}\tilde{L}^*\mathsf{P}'(\chi_{\mathbf{e}}) \ = \ \frac{1}{2^N}\mathsf{P}'(\tilde{L}\chi_{\mathbf{e}}).$$

Now, from (5.13) and then (5.11),

$$\begin{aligned}\mathsf{P}'(\tilde{L}\chi_{\mathbf{e}}) \ &= \frac{1}{\eta}\Big(\mathsf{P}'(\chi_{\mathrm{Coll}}\,.\,\tilde{\mathcal{P}}^{\epsilon,\eta}\chi_{\mathbf{e}}) - \mathsf{P}'(\chi_{\mathrm{Coll}}\,.\,\tilde{\mathcal{P}}^{0,\eta}\chi_{\mathbf{e}})\Big)\\ &= \frac{1}{\eta}\sum_{k=1}^{N-1}\Big(\mathsf{P}'(\chi_{\mathrm{Coll}_k}\,.\,\tilde{\mathcal{P}}^{\epsilon,\eta}\chi_{\mathbf{e}}) - \mathsf{P}'(\chi_{\mathrm{Coll}_k}\,.\,\tilde{\mathcal{P}}^{0,\eta}\chi_{\mathbf{e}})\Big) \ + \ \mathcal{O}(\eta).\end{aligned}$$

On the one hand, from the definition (5.5), one has

$$\tilde{\mathcal{P}}^{\epsilon,\eta}\chi_{\mathbf{e}}(x) \ = \ \chi_{\mathbf{e}}\big(\tilde{X}^{\epsilon}(x,\eta)\big) \ = \ \chi_{\sigma_k\mathbf{e}}(x) \quad \text{for} \quad x \in \mathrm{Coll}_k - \bigcup_{j\neq k}\mathrm{Coll}_j,$$

with

$$\sigma_k\mathbf{e} \ = \ (e_1,\dots,e_{k+1},e_k,\dots,e_N).$$

On the other hand, $\tilde{\mathcal{P}}^{0,\eta}\chi_{\mathbf{e}} = \chi_{\mathbf{e}}$ since the uncoupled dynamics preserves the energy of individual particles. Therefore, using (5.11) once more,

$$\mathsf{P}'(\tilde{L}\chi_{\mathbf{e}}) \ = \ \frac{1}{\eta}\sum_{k=1}^{N-1}\Big(\mathsf{P}'(\chi_{\mathrm{Coll}_k}\cdot\chi_{\sigma_k\mathbf{e}}) - \mathsf{P}'(\chi_{\mathrm{Coll}_k}\cdot\chi_{\mathbf{e}})\Big) \ + \ \mathcal{O}(\eta).$$

From the particular form (5.14) of P' and from (5.10), one obtains

$$\begin{aligned}
\mathsf{P}'(\chi_{\mathrm{Coll}_k}\,.\,\chi_{\sigma_k\mathbf{e}}) &= \frac{1}{2^N}\sum_{\mathbf{p}'\in\mathbf{P}^N}\chi_{(\mathsf{e}_1,\dots,\mathsf{e}_N)}(\mathbf{p}')\,.\,\chi_{\sigma_k\mathbf{e}}(\mathbf{p}')\,.\,\int_{\mathbf{Q}}\chi_{\mathrm{Coll}_k}(\mathbf{q}')\,\mathrm{d}\mathbf{q}'\\
&= \frac{\eta\epsilon}{2^N}\sum_{\mathbf{p}'\in\mathbf{P}^N}\chi_{(\mathsf{e}_1,\dots,\mathsf{e}_N)}(\mathbf{p}')\,.\,\chi_{\sigma_k\mathbf{e}}(\mathbf{p}')\,.\,\chi_{\mathbb{R}_+}(p'_k-p'_{k+1})\,.\,(p'_k-p'_{k+1})\;+\;\mathcal{O}(\eta^2)\\
&= \frac{\eta\epsilon}{4}\,.\,\chi_{(\mathsf{e}_1,\dots,\mathsf{e}_N)}(\sigma_k\mathbf{e})\sum_{\substack{p'_k=\pm\sqrt{2e_{k+1}}\\ p'_{k+1}=\pm\sqrt{2e_k}}}\chi_{\mathbb{R}_+}(p'_k-p'_{k+1})\,.\,(p'_k-p'_{k+1})\;+\;\mathcal{O}(\eta^2)\\
&= \eta\epsilon\,.\,\chi_{(\mathsf{e}_1,\dots,\mathsf{e}_N)}(\sigma_k\mathbf{e})\,.\,\gamma(e_k,e_{k+1})\;+\;\mathcal{O}(\eta^2),
\end{aligned}$$

where γ is the function defined by (2.1). A similar computation shows that

$$\mathsf{P}'(\chi_{\mathrm{Coll}_k}\,.\,\chi_{\mathbf{e}}) \;=\; \eta\epsilon\,.\,\chi_{(\mathsf{e}_1,\dots,\mathsf{e}_N)}(\mathbf{e})\,.\,\gamma(e_k,e_{k+1})\;+\;\mathcal{O}(\eta^2),$$

so that one concludes that

$$\begin{aligned}
\tilde{L}^*\mathsf{P}'(1|\mathbf{e}) &= \frac{\epsilon}{2^N}\sum_{k=1}^{N-1}\gamma(e_k,e_{k+1})\,.\,\big(\chi_{\mathsf{e}_1,\dots,\mathsf{e}_N}(\sigma_k\mathbf{e})-\chi_{\mathsf{e}_1,\dots,\mathsf{e}_N}(\mathbf{e})\big)+\mathcal{O}(\eta)\\
&= \epsilon\sum_{k=1}^{N}\gamma(e_k,e_{k+1})\,.\,\big(\mathsf{P}'(\sigma_k\mathbf{e})-\mathsf{P}'(\mathbf{e})\big)+\mathcal{O}(\eta). \qquad (5.20)
\end{aligned}$$

We then need to bound the second term in the right hand side of (5.19). Let us first observe that, by the particular form (5.14) of P', by the definition (5.13) of $\tilde{L}$, by the fact that Coll $\subset$ Z, and finally by (5.6),

$$\|\tilde{L}^*\mathsf{P}'\|_1 \;\le\; \sup_{u:\|u\|_\infty\le 1}\;\max_{\mathbf{p}\in\mathbf{P}^N}\int_{\mathbf{Q}}|\tilde{L}u(\mathbf{q},\mathbf{p})|\,\mathrm{d}\mathbf{q}\;\le\;\max_{\mathbf{p}\in\mathbf{P}^N}\frac{2}{\eta}\int_{\mathbf{Q}}\chi_{\mathrm{Z}}(\mathbf{q})\,\mathrm{d}\mathbf{q}\;=\;\mathcal{O}(\epsilon). \qquad (5.21)$$

The uncoupled dynamics is studied in more details in the next section ; it follows from (5.21) and (6.8) there that

$$\eta\sum_{k=1}^{n}\big\|\mathcal{P}^{0*,(k-1)\eta}\big(\tilde{L}^*\mathsf{P}'-\tilde{L}^*\mathsf{P}'(1|\mathbf{e})\big)\big\|_1 \;=\; \mathcal{O}\Big(\eta\sum_{k=1}^{n}\mathrm{e}^{-c(k-1)\eta}\epsilon\Big)\;=\;\mathcal{O}(\epsilon), \qquad (5.22)$$

where c denotes some strictly positive constant. Putting (5.20) and (5.22) in (5.19), one ends up with

$$\mu_1 \;=\; \tau\sum_{k=1}^{N}\gamma(e_k,e_{k+1})\,.\,\big(\mathsf{P}'(\sigma_k\mathbf{e})-\mathsf{P}'(\mathbf{e})\big)\;+\;\mu'_1, \qquad (5.23)$$

with $\|\mu'_1\|_1=\mathcal{O}(\tau^2)$, since $\eta<\epsilon<\tau^2$.

4. Bounding μ_2. One here establishes that $\|\mu_2\|_1=\mathcal{O}(\tau^2)$. Since $\|\mathcal{P}^{\epsilon*,s}\mu\|_1\le\|\mu\|_1$ for every $s\ge 0$ and every measure μ, one has

$$\|\mu_2\|_1\le(\epsilon^{-1}\tau)\eta\sum_{j=1}^{n}\|L^*\mathcal{P}^{0*,(j-1)\eta}\tilde{L}^*\mathsf{P}'\|_1, \qquad (5.24)$$

and, because of the particular form (5.14) of P',

$$\|L^*\mathcal{P}^{0*,(j-1)\eta}\tilde{L}^*\mathsf{P}'\|_1 \;\le\; \sup_{u:\|u\|_\infty\le 1}\; \max_{\substack{\mathbf{p}\,:\,p_1^2=2\mathsf{e}_1\\ \cdots\\ p_N^2=2\mathsf{e}_N}} \int_{\mathbf{Q}} |\tilde{L}\mathcal{P}^{0,(j-1)\eta}Lu(\mathbf{q},\mathbf{p})|\,\mathrm{d}\mathbf{q}. \tag{5.25}$$

So let us take $u \in \mathrm{L}^\infty(\Phi)$ such that $\|u\|_\infty \le 1$, and $\mathbf{p} \in \mathbf{P}^N$ such that $p_k^2 = 2\mathsf{e}_k$ for $1 \le k \le N$. To lighten some further notations, let us also define the time

$$t = (j-1)\eta. \tag{5.26}$$

By Lemma 2,

$$\tilde{L}\mathcal{P}^{0,t}Lu \;=\; \tilde{L}\mathcal{P}^{0,t}\tilde{L}u \;+\; \tilde{L}\mathcal{P}^{0,t}\chi_{\mathrm{Z}}u',$$

with $\|u'\|_\infty = \mathcal{O}(1)$. To avoid to deal with too many constants, one will assume that actually $\|u'\|_\infty \le 1$. Then, from the definition (5.13) of $\tilde{L}$,

$$\begin{aligned} |\tilde{L}\mathcal{P}^{0,t}\tilde{L}u| \;&\le \frac{1}{\eta^2}\Big|\chi_{\mathrm{Coll}}\,.\,(\tilde{\mathcal{P}}^{\epsilon,\eta}-\tilde{\mathcal{P}}^{0,\eta})\mathcal{P}^{0,t}\big(\chi_{\mathrm{Coll}}\,.\,(\tilde{\mathcal{P}}^{\epsilon,\eta}-\tilde{\mathcal{P}}^{0,\eta})u\big)\Big| \\ &\le \frac{2}{\eta^2}\Big(\chi_{\mathrm{Coll}}\,.\,\tilde{\mathcal{P}}^{\epsilon,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Coll}} \;+\; \chi_{\mathrm{Coll}}\,.\,\tilde{\mathcal{P}}^{0,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Coll}}\Big) \end{aligned}$$

and, similarly,

$$|\tilde{L}\mathcal{P}^{0,t}\chi_{\mathrm{Z}}u'| \;\le\; \frac{2}{\eta}\Big(\chi_{\mathrm{Coll}}\,.\,\tilde{\mathcal{P}}^{\epsilon,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Z}} \;+\; \chi_{\mathrm{Coll}}\,.\,\tilde{\mathcal{P}}^{0,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Z}}\Big).$$

Therefore

$$\int_{\mathbf{Q}} |\tilde{L}\mathcal{P}^{0,t}Lu(\mathbf{q},\mathbf{p})|\,\mathrm{d}\mathbf{q} \;\le\; \frac{2}{\eta^2}\int_{\mathbf{Q}} \chi_{\mathrm{Coll}}(\mathbf{q},\mathbf{p})\,.\,\tilde{\mathcal{P}}^{\epsilon,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Coll}}(\mathbf{q},\mathbf{p})\,\mathrm{d}\mathbf{q} \tag{5.27}$$

$$+\;\frac{2}{\eta^2}\int_{\mathbf{Q}} \chi_{\mathrm{Coll}}(\mathbf{q},\mathbf{p})\,.\,\tilde{\mathcal{P}}^{0,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Coll}}(\mathbf{q},\mathbf{p})\,\mathrm{d}\mathbf{q} \tag{5.28}$$

$$+\;\frac{2}{\eta}\int_{\mathbf{Q}} \chi_{\mathrm{Coll}}(\mathbf{q},\mathbf{p})\,.\,\tilde{\mathcal{P}}^{\epsilon,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Z}}(\mathbf{q},\mathbf{p})\,\mathrm{d}\mathbf{q} \tag{5.29}$$

$$+\;\frac{2}{\eta}\int_{\mathbf{Q}} \chi_{\mathrm{Coll}}(\mathbf{q},\mathbf{p})\,.\,\tilde{\mathcal{P}}^{0,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Z}}(\mathbf{q},\mathbf{p})\,\mathrm{d}\mathbf{q}. \tag{5.30}$$

The four terms appearing in the right hand side of this inequality are bounded in a very similar way, and we will only deal in detail with the first one, indicating at the end the minor needed adaptations to bound the three others. We write

$$\int_{\mathbf{Q}} \chi_{\mathrm{Coll}}\,.\,\tilde{\mathcal{P}}^{\epsilon,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Coll}}\,\mathrm{d}\mathbf{q} \;\le \sum_{1\le k,l\le N-1}\int_{\mathbf{Q}} \chi_{\mathrm{Coll}_k}\,.\,\tilde{\mathcal{P}}^{\epsilon,\eta}\mathcal{P}^{0,t}\chi_{\mathrm{Coll}_l}\,\mathrm{d}\mathbf{q}. \tag{5.31}$$

It follows from (6.6-6.7) and then (6.3) in the next section, that

$$\begin{aligned} \mathcal{P}^{0,t}\chi_{\mathrm{Coll}_l}(x_l,x_{l+1}) \;=\;\; & \mathrm{e}^{-2\lambda t}\,.\,\chi_{\mathrm{Coll}_l}(\tilde{X}_l^0(x_l,t),\tilde{X}_{l+1}^0(x_{l+1},t)) \\ &+\;\mathrm{e}^{-\lambda t}\int g(x_{l+1},x'_{l+1},t)\,.\,\chi_{\mathrm{Coll}_l}(\tilde{X}_l^0(x_l,t),x'_{l+1})\,\mathrm{d}x'_{l+1} \\ &+\;\mathrm{e}^{-\lambda t}\int g(x_l,x'_l,t)\,.\,\chi_{\mathrm{Coll}_l}(x'_l,\tilde{X}_{l+1}^0(x_{l+1},t))\,\mathrm{d}x'_l \\ &+\;\int g(x_l,x'_l,t)\,.\,g(x_{l+1},x'_{l+1},t)\,.\,\chi_{\mathrm{Coll}_l}(x')\,\mathrm{d}x'_1\,\mathrm{d}x'_{l+1}, \end{aligned}$$

where we have used the notation

$$\int u \, \mathrm{d}x_m \;=\; \sum_{p_m=\pm|p_m|} \int_{[0,1]} u \, \mathrm{d}q_m, \qquad 1 \le m \le N.$$

The first term in the right hand side of this expression cannot be simplified further. As far as the second is concerned, by the bound $\mathrm{Coll}_l \subset \mathrm{Z}_l$ and the definition (5.4) of Z_l, one obtains

$$\chi_{\mathrm{Coll}_l}(\tilde{X}_l^0(x_l,t), x'_{l+1}) \;\le\; \chi_{[1-\epsilon,1]}\big(\tilde{q}_l^0(x_l,t)\big) \,.\, \chi_{[0,\mathrm{p}\eta]}\big(\big|q'_{l+1} - \tilde{q}_l^0(x_l,t) + 1 - \epsilon\big|\big).$$

Now, g is uniformly bounded according to Lemma 3 in the next section, and so

$$\int g(x_{l+1}, x'_{l+1}, t) \,.\, \chi_{\mathrm{Coll}_l}(\tilde{X}_l^0(x_l,t), x'_{l+1}) \, \mathrm{d}x'_{l+1} \;=\; \mathcal{O}(\eta) \,.\, \chi_{[1-\epsilon,1]}\big(\tilde{q}_l^0(x_l,t)\big).$$

By a similar computation for the third term, and using (5.6) for the fourth one, one concludes that

$$\begin{aligned} \mathcal{P}^{0,t}\chi_{\mathrm{Coll}_l}(x_l, x_{l+1}) \;&\le \mathrm{e}^{-2\lambda t} \,.\, \chi_{\mathrm{Coll}_l}(\tilde{X}_l^0(x_l,t), \tilde{X}_{l+1}^0(x_{l+1},t)) \\ &\quad + \; \mathcal{O}(\eta) \,.\, \chi_{[1-\epsilon,1]}\big(\tilde{q}_l^0(x_l,t)\big) \;+\; \mathcal{O}(\eta) \,.\, \chi_{[0,\epsilon]}\big(\tilde{q}_{l+1}^0(x_{l+1},t)\big) \;+\; \mathcal{O}(\eta\epsilon) \\ &:= \sum_{i=1}^{4} \mathrm{A}_{i,j,l}(x_l, x_{l+1}) \end{aligned} \tag{5.32}$$

where the index j comes from the fact that $t = (j-1)\eta$ by (5.26).

Defining then

$$\mathrm{B}_{i,j,k,l}(p_k, p_{k+1}) \;=\; \frac{1}{\eta^2} \int_{\mathbf{Q}} \chi_{\mathrm{Coll}_k} \,.\, \mathcal{P}^{\epsilon,\eta} \mathrm{A}_{i,j,l} \, \mathrm{d}\mathbf{q},$$

and going backwards in the expressions (5.24-5.32), one concludes that it will be enough to show that, for $i = 1,2,3,4$, and for $1 \le k,l \le N-1$, one has

$$\eta \sum_{j=1}^{n} \mathrm{B}_{i,j,k,l}(p_k, p_{k+1}) \;=\; \mathcal{O}(\tau\epsilon), \tag{5.33}$$

which in particular will be the case if one establishes that

$$\mathrm{B}_{i,j,k,l}(p_k, p_{k+1}) = \mathcal{O}(\epsilon^2).$$

Five cases will be analyzed separately: $(i = 1, l = k)$, $(i = 1, l = k \pm 1)$, $(i = 1, |l-k| > 1)$, $(i = 2,3)$, $(i = 4)$. The diophantine condition on the velocities only needs to be used in the case $(i = 1, l = k)$.

1. Bounding $\mathrm{B}_{1,j,k,k}$. One has

$$\mathrm{B}_{1,j,k,k} \;\le\; \frac{\mathrm{e}^{-2\lambda t}}{\eta^2} \int_{\mathbf{Q}} \chi_{\mathrm{Coll}_k}(x) \,.\, \chi_{\mathrm{Coll}_k}\big(\tilde{X}_k^0(\tilde{X}_k^\epsilon(x,\eta),t), \tilde{X}_{k+1}^0(\tilde{X}_{k+1}^\epsilon(x,\eta),t)\big) \, \mathrm{d}\mathbf{q}. \tag{5.34}$$

Let us take some $x \in \mathrm{Coll}_k$. The first point to observe is that, for the deterministic realization $\tilde{X}^\epsilon$ of the process X^ϵ, collisions involving a given particle are spaced by time intervals of order 1 at least. There exists therefore a constant $c > 0$ such that, for $t = (j-1)\eta \le c$,

$$\big(\tilde{X}_k^0(\tilde{X}_k^\epsilon(x,\eta),t), \tilde{X}_{k+1}^0(\tilde{X}_{k+1}^\epsilon(x,\eta),t)\big) \;\notin\; \mathrm{Coll}_k,$$

so that $B_{1,j,k,k} = 0$ in that case. Let then suppose $t \geq c$, and let us just bound the second factor in the integrand of (5.34) by

$$\chi_{\mathrm{Coll}_k}(x_k, x_{k+1}) \;\leq\; \chi_{[1-2\epsilon,1]\times[0,2\epsilon]}(q_k, q_{k+1}),$$

valid since $\mathrm{Coll}_k \subset \mathrm{Z}_k$, and by the definition (5.4) of Z_k. Let then $t^* > c$ be the smallest time such that

$$\big(\tilde{q}^0_k(\tilde{X}^\epsilon_k(x,\eta),t^*), \tilde{q}^0_{k+1}(\tilde{X}^\epsilon_{k+1}(x,\eta),t^*)\big) \in [1-2\epsilon,1]\times[0,2\epsilon].$$

There has to be two integers $r, r' \neq 0$ such that

$$t^* = \frac{2r}{p_k} + \mathcal{O}(\epsilon) = \frac{2r'}{p_{k+1}} + \mathcal{O}(\epsilon),$$

and so

$$\Big|\frac{p_k}{p_{k+1}} - \frac{r}{r'}\Big| \;=\; \mathcal{O}(\epsilon) \quad \text{and} \quad \Big|\frac{p_{k+1}}{p_k} - \frac{r'}{r}\Big| \;=\; \mathcal{O}(\epsilon).$$

As said after (5.25), $p_k^2 = 2\mathsf{e}_k$ and $p_{k+1}^2 = 2\mathsf{e}_{k+1}$, and, by hypothesis, the quotients p_k/p_{k+1} and p_{k+1}/p_k are thus diophantine by hypothesis. This condition guaranties the existence of a constant $c > 0$ such that $r, r' \;\geq\; c\,\epsilon^{-1/\beta}$, and so one has also $t^* > c'\,\epsilon^{-1/\beta}$ for some $c' > 0$. Since $B_{1,j,k,k} = 0$ as long as $t = (j-1)\eta \leq t^*$, and since $\eta > \epsilon^4$, one concludes that

$$B_{1,j,k,k} \;\leq\; \frac{\mathrm{e}^{-2\lambda\, c'\,\epsilon^{-1/\beta}}}{\eta^2} \;=\; \mathcal{O}(\epsilon^2). \tag{5.35}$$

2. Bounding $\eta\sum_{j=1}^n B_{1,j,k,k\pm1}$. The two cases, $l = k+1$ and $l = k-1$, are analogous, and one will treat the first one only. One has

$$B_{1,j,k,k+1} \;\leq\; \frac{C}{\eta^2}\int_{\mathbf{Q}} \chi_{\mathrm{Coll}_k}(x)\,.\,\chi_{\mathrm{Coll}_{k+1}}\big(\tilde{X}^0_{k+1}(\tilde{X}^\epsilon_{k+1}(x,\eta),t), \tilde{X}^0_{k+2}(\tilde{X}^\epsilon_{k+2}(x,\eta),t)\big)\,\mathrm{d}\mathbf{q}.$$

Integrating over q_k and q_{k+1}, one gets

$$B_{1,j,k,k+1} \;\leq\; \frac{C\,\epsilon}{\eta} \sup_{\substack{q_k\in[1-2\epsilon,1]\\ q_{k+1}\in[0,2\epsilon]}} \int \chi_{\mathrm{Z}_{k+1}}\big(\tilde{X}^0_{k+1}(\tilde{X}^\epsilon_{k+1}(x,\eta),t), \tilde{X}^0_{k+2}(\tilde{X}^\epsilon_{k+2}(x,\eta),t)\big)\,\mathrm{d}q_{k+2}\mathrm{d}q_{k+3}$$

One may obviously insert the factor $\chi_{\mathrm{Z}_{k+2}} + (1-\chi_{\mathrm{Z}_{k+2}})$ in this integral, and split it accordingly. In the term containing the factor $\chi_{\mathrm{Z}_{k+2}}$, one may just use the bound $\chi_{\mathrm{Z}_{k+1}}(\dots) \leq 1$ so that this term is $\mathcal{O}(\eta\epsilon)$, whereas, in the term containing the factor $(1-\chi_{\mathrm{Z}_{k+2}})$, one may replace $\tilde{X}^\epsilon_{k+2}$ by $\tilde{X}^0_{k+2}$. At the end,

$$B_{1,j,k,k+1} \;\leq\; C\epsilon^2 \;+\; \frac{C\epsilon}{\eta} \sup_{\substack{q_k\in[1-2\epsilon,1]\\ q_{k+1}\in[0,2\epsilon]}} \int \chi_{\mathrm{Z}_{k+1}}\big(\tilde{X}^0_{k+1}(\tilde{X}^\epsilon_{k+1}(x,\eta),t), \tilde{X}^0_{k+2}(x,t+\eta)\big)\,\mathrm{d}q_{k+2}.$$

For the integrand to not be 0, the two following conditions need to be fulfilled:

$$\tilde{q}^0_{k+1}(\tilde{X}^\epsilon_{k+1}(x_k,x_{k+1},\eta),t) \;\in\; [1-2\epsilon,1],$$
$$\big|\tilde{q}^0_{k+2}(x_{k+2},t+\eta) - \tilde{X}^0_{k+1}(\tilde{q}^\epsilon_{k+1}(x_k,x_{k+1},\eta),t)\big| \;=\; \mathcal{O}(\eta).$$

The uncoupled dynamics is such that, for any $a \in [0,1]$, any $s \geq 0$,

$$\mathrm{Leb}\big(|\tilde{q}^0_{k+2}(x_{k+2}, s) - a| = \mathcal{O}(\eta)\big) \;=\; \mathcal{O}(\eta).$$

So, integrating over q_{k+2},

$$\mathrm{B}_{1,j,k,k+1} \;\leq\; \mathrm{C}\epsilon^2 \;+\; \mathrm{C}\epsilon \sup_{\substack{q_k \in [1-2\epsilon,1] \\ q_{k+1} \in [0,2\epsilon]}} \chi_{[1-2\epsilon,1]}\big(\tilde{q}^0_{k+1}(\tilde{X}^\epsilon_{k+1}(x_k, x_{k+1}, \eta), (\eta-1)j)\big),$$

where one has used the definition (5.26) of t. Let us thus fix some $q_k \in [1-2\epsilon, 1]$ and some $q_{k+1} \in [0, 2\epsilon]$. Summing $\mathrm{B}_{1,j,k,k+1} - \mathrm{C}\epsilon^2$ over j, one sees that only a proportion ϵ of the terms are non zero, and so

$$\eta \sum_{j=1}^{n} \mathrm{B}_{1,j,k,k+1} \;=\; \mathcal{O}(\epsilon\tau). \tag{5.36}$$

3. Bounding $\mathrm{B}_{1,j,k,l}$, $|l-k| \geq 1$. One has

$$\mathrm{B}_{1,j,k,l} \;\leq\; \frac{1}{\eta^2} \int_{\mathbf{Q}} \chi_{\mathrm{Coll}_k}(x) \,.\, \chi_{\mathrm{Coll}_l}\big(\tilde{X}^0_l(\tilde{X}^\epsilon_l(x,\eta),t), \tilde{X}^0_{l+1}(\tilde{X}^\epsilon_{l+1}(x,\eta),t)\big)\, \mathrm{d}\mathbf{q}.$$

Integrating over the variables q_k, q_{k+1}, one gets,

$$\mathrm{B}_{1,j,k,l} \;\leq\; \frac{\mathrm{C}\,\epsilon}{\eta} \sup_{\substack{q_k \in [1-2\epsilon,1] \\ q_{k+1} \in [0,2\epsilon]}} \int \chi_{\mathrm{Coll}_l}\big(\tilde{X}^0_l(\tilde{X}^\epsilon_l(x,\eta),t), \tilde{X}^0_{l+1}(\tilde{X}^\epsilon_{l+1}(x,\eta),t)\big)\, \mathrm{d}\mathbf{q}_{k,k+1}$$

where $\mathrm{d}\mathbf{q}_{k,k+1} = \mathrm{d}q_1 \dots \mathrm{d}q_{k-1} \mathrm{d}q_{k+2} \dots \mathrm{d}q_N$. One may obviously insert the factor

$$\chi_{\mathrm{Z}_{l-1} \cup \mathrm{Z}_l \cup \mathrm{Z}_{l+1}}(x) \;+\; \big(1 - \chi_{\mathrm{Z}_{l-1} \cup \mathrm{Z}_l \cup \mathrm{Z}_{l+1}}(x)\big)$$

into this last integral, and split it accordingly. In the term involving the factor $\chi_{\mathrm{Z}_{l-1} \cup \mathrm{Z}_l \cup \mathrm{Z}_{l+1}}$, one just uses the bound $\chi_{\mathrm{Coll}_l}(\dots) \leq 1$ to obtain that this term is $\mathcal{O}(\eta\epsilon)$, whereas in the other one, one may replace $\tilde{X}^\epsilon$ by $\tilde{X}^0$. At the end,

$$\mathrm{B}_{1,j,k,l} \;\leq\; \mathrm{C}\,\epsilon^2 \;+\; \frac{\mathrm{C}\,\epsilon}{\eta} \int \chi_{\mathrm{Coll}_{k+1}}\big(\tilde{X}^0_l(x, t+\eta), \tilde{X}^0_{l+1}(x, t+\eta)\big)\, \mathrm{d}\mathbf{q}_{k,k+1},$$

and this last integral is $\mathcal{O}(\eta\epsilon)$. So

$$\mathrm{B}_{1,j,k,l} \;=\; \mathcal{O}(\epsilon^2). \tag{5.37}$$

4. Bounding $\eta \sum_{j=1}^{n} \mathrm{B}_{2,j,k,l}$ *and* $\eta \sum_{j=1}^{n} \mathrm{B}_{3,j,k,l}$. These two cases are similar, and we treat only the first one:

$$\begin{aligned} \eta \sum_{j=1}^{n} \mathrm{B}_{2,j,k,l} \;&\leq \frac{\mathrm{C}}{\eta} \int_{\mathbf{Q}} \chi_{\mathrm{Coll}_k}(x) \,.\, \tilde{\mathsf{P}}^{\epsilon,\eta} \eta \sum_{j=1}^{n} \chi_{[1-\epsilon,1]}\big(\tilde{X}_l(x_l, (j-1)\eta)\big)\, \mathrm{d}\mathbf{q} \\ &\leq \mathrm{C}\,\epsilon \,.\, \sup_{x \in \Phi} \eta \sum_{j=1}^{n} \chi_{[1-\epsilon,1]}\big(\tilde{X}_l(x_l, (j-1)\eta)\big). \end{aligned}$$

Since, for any $x \in \Phi$, the number of non zero terms in the sum is $\mathcal{O}(n\epsilon)$,

$$\eta \sum_{j=1}^{n} \mathrm{B}_{2,j,k,l} \;=\; \mathcal{O}(\epsilon\tau). \tag{5.38}$$

5. Bounding $\mathrm{B}_{4,j,k,l}$.

$$\mathrm{B}_{4,j,k,l} \;\leq\; \frac{\mathrm{C}}{\eta^2} \int_Q \chi_{\mathrm{Coll}_k}(x) \,.\, \tilde{\mathcal{P}}^{\epsilon,\eta} \epsilon\eta \,\mathrm{d}\mathbf{q} \;=\; \mathcal{O}(\epsilon^2) \tag{5.39}$$

6. Concluding the bound on μ_2. The bounds (5.35-5.39) show that (5.33) holds for $i = 1, 2, 3, 4$ and $1 \leq k, l \leq N-1$. However, as said after (5.27-5.30), one still needs to explain how to adapt the proof when one starts with one of the terms (5.28-5.30), instead of (5.27). If one starts from (5.28), all the previous arguments actually still hold. If instead one starts from (5.29), the only place where some non-immediate adaptation is required, concerns the bound on $\mathrm{B}_{1,j,k,k}$ for $t = (j-1)\eta < c$, where c is the constant introduced just after (5.34). Indeed, for $x \in \mathrm{Coll}$, it can be that

$$\big(\tilde{X}^0_k(\tilde{X}^\epsilon_k(x,\eta),t), \tilde{X}^0_{k+1}(\tilde{X}^\epsilon_{k+1}(x,\eta),t)\big) \;\in\; \mathrm{Z}_k$$

for a finite and constant number of times $t = (j-1)\eta < c$. Since now the factor $1/\eta^2$ in the beginning of right hand side of (5.34) may be replaced by a factor $1/\eta$, one have, for such times, only the bound

$$\mathrm{B}_{1,j,k,k} \;\leq\; \frac{1}{\eta} \int_{\mathbf{Q}} \chi_{\mathrm{Coll}_k}(x) \,\mathrm{d}\mathbf{q}. \;=\; \mathcal{O}(\epsilon)$$

But then, inserting this in (5.33), one gets $\eta \sum_j \mathrm{B}_{1,j,k,k} = \mathcal{O}(\eta\epsilon) = \mathcal{O}(\epsilon\tau)$, since the sum contains only a finite and constant number of non-zero terms. The case of (5.30) is analogous. □

5. The case $N' < N$. If $N' < N$, the arguments leading from (5.34) to (5.35) do not hold anymore, and this is the only place where a problem occurs. Indeed one assumed there that all the energies were different in order to make use of the diophantine condition. This problem is clearly superficial, since we do not care about collisions between particles having the same energy, and it can indeed be fixed by a slight modification of the proof.

Let us go back to the general strategy explained in Subsection 5.1, and let us again start from a measure P assigning to each particle a given energy. When Duhamel's formula is used for the first time, in (5.2), instead of taking S to be the operator $\mathcal{P}^{0*,\eta}$ decoupling all the particles, one can take an operator which decouples only blocs of particles having a given energy. Up to an error of order ϵ^2, this measure is still invariant under this dynamics, and this is in fact all what we need to proceed. When Duhamel's formula is used for the second time, in (5.3), one still takes S to be $\mathcal{P}^{0*,\eta}$ however, since otherwise one could not make use of the properties obtained in the Section 6.

It can then be checked that the proof essentially remains unchanged but that, when bounding μ_2, one may now assume that recollisions only occur between particles of different energies, allowing thus the use of the diophanitine condition. We decided however to not implement these changes explicitly to avoid even more heavy notations.

6 Uncoupled dynamics

We analyze here the dynamics when $\epsilon = 0$, so that all the particles evolve independently.

6.1 $N = 1$ particle

Take $N = 1$, and let e_1 be the energy of the only particle. Let $\overline{X} = (\overline{q}, \overline{p})$ be the chain on $\mathbb{R} \times \mathbf{P} = \mathbb{R} \times \{-\mathsf{e}_1, \mathsf{e}_1\}$ solving the equations

$$\mathrm{d}\overline{q} \;=\; \overline{p}\,\mathrm{d}t, \qquad \mathrm{d}\overline{p} \;=\; -2\overline{q}\,\mathrm{dN},$$

where N is a Poisson process of parameter $\lambda > 0$. For a given smooth function v on $\mathbb{R} \times \mathbf{P}$, let

$$u(x,t) = \mathsf{E}\big(v \circ \overline{X}(x,t)\big),$$

which, for $t \geq 0$, satisfies the equation

$$\partial_t u(q,p,t) = p\,\partial_q u(q,p,t) + \lambda\big(u(q,-p,t) - u(q,p,t)\big).$$

It is lengthy but straightforward to check that u may be written as

$$u(q,p,t) = \sum_{p'=\pm p} \int_{-\infty}^{+\infty} \rho(q,p,q',p',t)\, v(q',p')\,\mathrm{d}q'$$

where the probability measure ρ is given by

$$\rho(q,p,q',p',t) = \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \rho_n(q,p,q',p',t) \tag{6.1}$$

with

$$\begin{aligned}
\rho_0(q,p,q',p',t) &= \chi_{\{0\}}(p-p')\delta_0(pt+q-q'),\\
\rho_n(q,p,q',p',t) &= \chi_{[-|p|t,|p|t]}(q'-q)\chi_{\{0\}}(p-p')\frac{n!\left(1+\frac{q'-q}{pt}\right)^{\frac{n}{2}}\left(1-\frac{q'-q}{pt}\right)^{\frac{n}{2}-1}}{|p|t\,2^n\left(\frac{n}{2}\right)!\left(\frac{n}{2}-1\right)!}, \quad n \geq 2 \text{ even},\\
\rho_n(q,p,q',p',t) &= \chi_{[-|p|t,|p|t]}(q'-q)\chi_{\{0\}}(p+p')\frac{n!\left(1+\frac{q'-q}{pt}\right)^{\frac{n-1}{2}}\left(1-\frac{q'-q}{pt}\right)^{\frac{n-1}{2}}}{|p|t\,2^n\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}, \quad n \geq 1 \text{ uneven}.
\end{aligned}$$

This allows us to give an expression for the kernel of the evolution operator $\mathcal{P}^{0,t}$ of the chain X on $\Phi = [0,1] \times \{-\mathsf{e}_1, \mathsf{e}_1\}$ defined by (3.1-3.2). Given a function u on Φ, and given $t \geq 0$, let us write

$$\mathcal{P}^{0,t}u(x) = \sum_{p'=\pm p} \int_0^1 f(q,p,q',p',t)\, u(q',p')\,\mathrm{d}q'.$$

The trajectories of X are obtained from these of $\overline{X}$ by reflecting them against the boundaries at $q = 0$ and $q = 1$, and therefore

$$f(q,p,q',p',t) = \sum_{k\in\mathbb{Z},\text{even}} \rho(q,p,q'+k,p',t) + \sum_{k\in\mathbb{Z},\text{uneven}} \rho(q,p,1-q'+k,-p',t). \tag{6.2}$$

Lemma 3. *One has this.*

1. *(Doeblin's condition) There exists* $\alpha > 0$ *and* $t_0 \geq 0$ *such that* $f(x, x', t_0) \geq \frac{\alpha}{2}$ *for every* $x, x' \in \Phi$.
2. *There exists a function* $g \in \mathrm{L}^\infty(\Phi^2 \times \mathbb{R}_+)$ *such that*

$$f(x, x', t) = \mathrm{e}^{-\lambda t} \chi_{\{\tilde{p}(x,t)\}}(p')\, \delta_{\tilde{q}(x,t)}(q') + g(x, x', t). \tag{6.3}$$

Proof. The first part is directly established using (6.1) and (6.2). Let us show the second one. By (6.2), the first term in the right hand side of (6.3) directly comes from ρ_0 in (6.1). Then, inserting (6.1) in (6.2), and putting the sums over k inside the sums over n, one sees that it suffices to find a constant $\mathrm{C} < +\infty$ such that, for every $p \in \mathbf{P}$, for every $z \in \mathbb{R}$ and for every $n \geq 1$, one has

$$\frac{n!}{2^n \left(\frac{n}{2}\right)! \left(\frac{n}{2} - 1\right)!} \frac{1}{|p|t} \sum_{\substack{k \in \mathbb{Z} \\ |z+k| \leq |p|t}} \left(1 + \frac{z+k}{pt}\right)^{\frac{n}{2}} \left(1 - \frac{z+k}{pt}\right)^{\frac{n}{2}-1} \leq \mathrm{C}, \tag{6.4}$$

if n is even, and

$$\frac{n!}{2^n \left(\frac{n-1}{2}\right)! \left(\frac{n-1}{2}\right)!} \frac{1}{|p|t} \sum_{\substack{k \in \mathbb{Z} \\ |z+k| \leq |p|t}} \left(1 + \frac{z+k}{pt}\right)^{\frac{n-1}{2}} \left(1 - \frac{z+k}{pt}\right)^{\frac{n-1}{2}} \leq \mathrm{C}, \tag{6.5}$$

if n is uneven. Let us show (6.4) ; the proof of (6.5) is analogous. One finds $c > 0$ such that, for every $x \in [-1, 1]$,

$$(1+x)^{n/2}(1-x)^{n/2-1} \;\leq\; 2(1-x^2)^{n/2-1} \;=\; \mathcal{O}\big(\mathrm{e}^{-cnx^2}\big).$$

But

$$\frac{1}{|p|t} \sum_{\substack{k \in \mathbb{Z} \\ |z+k| \leq |p|t}} \mathrm{e}^{-cn\left(\frac{z+k}{pt}\right)^2} \;=\; \mathcal{O}(n^{-1/2}).$$

On the other hand, one establishes by means of Stirling's formula that the combinatoric factor in the left hand side of (6.4) is $\mathcal{O}(n^{1/2})$. This completes the proof. □

6.2 $N \geq 1$ particles

Lemma 3 has direct implications to the case $N \geq N' \geq 1$. Since the particles evolve independently, the operator $\mathcal{P}^{0,t}$ can be written as the tensor product

$$\mathcal{P}^{0,t} = \mathcal{P}_1^t \ldots \mathcal{P}_N^t, \tag{6.6}$$

where, for $1 \leq k \leq N$, $\mathcal{P}_k^t$ is the evolution operator of a chain which moves only particle k and let fixed the other ones:

$$\mathcal{P}_k^t u(x) = \sum_{p'_k = \pm p_k} \int_0^1 f(q_k, p_k, q'_k, p'_k, t)\, u(q_1, \ldots, q'_k, \ldots, q_N, p_1, \ldots, p'_k, \ldots, p_N)\, \mathrm{d}q'_k, \tag{6.7}$$

with f defined by (6.2). Now, Doeblin's condition (the first part of Lemma 3) means that the operators $\mathcal{P}_k^{0*,t_0}$ have a spectral gap for the $\|\,.\,\|_1$-norm. By the tensor product decomposition (6.6), one concludes thus that there exist constants $\mathrm{C} < +\infty$ and $c > 0$ such that, for every measure μ on Φ,

$$\left\|\mathcal{P}^{0*,t}\mu - \mu(1|\mathbf{e})\right\|_1 \;\leq\; \mathrm{C}\,\mathrm{e}^{-ct}\|\mu\|_1, \tag{6.8}$$

with $\mu(1|\mathbf{e})$ defined by (5.18), $\mathbf{e} = (e_1, \ldots, e_N)$, and e_k the energy of particle k.

7 Figures

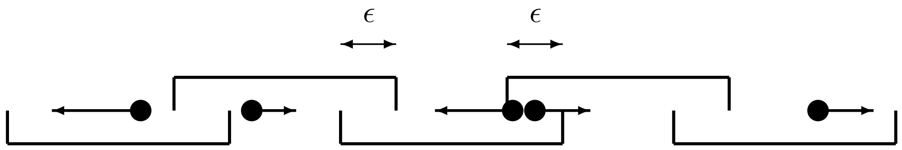

Figure 1: The dynamical system for $N = 5$. Particles may collide elastically with their neighbor when they enter in the overlap region of size ϵ.

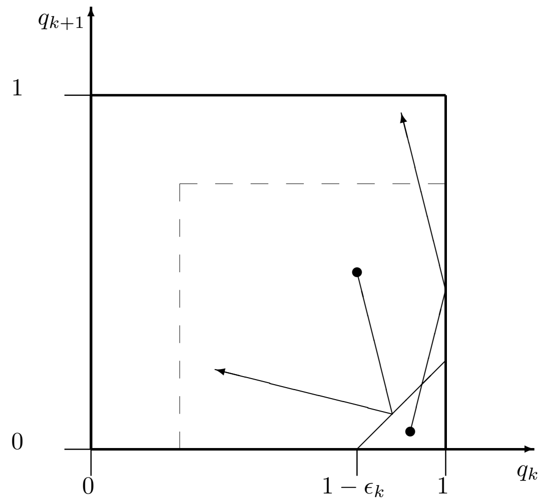

Figure 2: Projection of Φ on the $(q_k, q_{k+1})-$plane: the two circles represent projections of two possible states of the N-particle system. Points in the triangle in the lower right corner belong to $\Phi - \Phi^\epsilon$, since $q_{k+1} < q_k - 1 + \epsilon$ there. In the region on the left of the dotted line, particle k may collide with particle $k-1$, whereas in the region above the dotted line, particle $k+1$ may collide with particle $k+2$.

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