

Numbers, limits, and continuity in Gauss

In the eighteenth century, infinitely large and infinitesimal quantities were thought to be entities that could be manipulated in the same way as numbers except for the principle of cancellation

At the beginning of 19th century, Gauss rejected this approach. According to him, a lot of circumspection was required in treating infinite quantities these were to be allowed in analytical reasoning insofar as they can be reduced to the theory of limits (Gauss, *In Disquisitiones generales circa seriem infinitam*, 159]).

More explicitly, in a letter to Schumacher, 12 July 1831, Gauss stated:

"I protest in the first place against the use of infinite magnitude as something completed, which is never allowed in mathematics. The infinite is but a *façon de parler* in that one actually speaks of limits to which certain relations come as close as one desires, while others are allowed to increase without bound." (Gauss [WW, 8:216])

It is clear that Gauss wants to reduce infinite quantities to the theory of limits. But:

What is Gauss's notion of limit?

It is not difficult to realize that there are crucial differences between his notion and the modern one.

For examples, let's consider *De origine proprietatibus que generales numerorum mediorum arithmetico- Geometricorum*, a paper where Gauss gave this definition of the arithmetic-geometric mean (agm)

Given two (positive) numbers a and b , he first considered the sequences alpha and beta

$$a = a_0, \quad a_{n+1} = \frac{a_n + b_n}{2}; \quad \text{and} \quad b = b_0, \quad b_{n+1} = \sqrt{a_n b_n} \quad \text{for } n=1, 2, \dots$$

Then he observed that

- 1) $a_n > b_n$, for $n=1, 2, \dots$,
- 2) a_n were increasing and b_n were decreasing,
- 3) the inequality $a_{n+1} - b_{n+1} = (a_n - b_n)/2$ held.

Then, using equality $(a_n - b_n)(a_n + b_n) = \frac{1}{4}(a_{n+1} - b_{n+1})^2 > 0$, Gauss obtained

[eb'tein+d]

$$\frac{a_{n+1} - b_{n+1}}{a_n - b_n} = \frac{a_n - b_n}{4(a_{n+1} + b_{n+1})} = \frac{a_n - b_n}{2(a_n + b_n) + 4b_{n+1}},$$

Therefore [ˈðee,fχ:]

$$a_{n+1} - b_{n+1} = \frac{(a_n - b_n)^2}{2(a_n + b_n) + 4b_{n+1}} = \frac{a_n - b_n}{2} \cdot \frac{a_n - b_n}{(a_n + b_n) + 2b_{n+1}} < \frac{a_n - b_n}{2},$$

namely

$$a_{n+1} - b_{n+1} < \frac{a_n - b_n}{2}.$$

Property 1 is trivial; properties 2 and 3 involve that both the sequences have limits: manifestum est utramque habere limitem.

Indeed, the first sequence decreases but remains greater the

second, vice versa the second sequence increases but is less than the first.

Gauss denotes the limits by these symbols

(S) a^∞ and b^∞

Another property is this.

Property 4 implies that the sequence of differences $a_n - b_n$ is

decreasing and has 0 as a limit.

Of consequence, the two sequences a_n and b_n have the same limit

$$a^\infty = b^\infty,$$

This limit is the arithmetico-geometric mean of a and b

Gauss denoted it by the symbol $M(a,b)$ (see [WM, 361-362]).

Two remarks are appropriate

First, Gauss did not use the term “limit” in a technical sense, but in a common sense. For him, *limes* means the bound

(In Latin, *limes* means “strip of uncultivated ground marking boundary”).

In Gauss, the concept of limit is not defined but is intuitively given

it is clear that for Gauss limits have property (P):

(P) if c is the limit of c_n then the difference

$$|c_n - c| \quad \text{is less than} \quad \varepsilon$$

for every ε and for large enough n .

In modern mathematics, (P) is the definition of limit and the notion of limit is entirely reduced to this definition, which, in principle, can be changed and so the notion of limit can be changed.

Instead, for Gauss, (P) is not a definition but a property of the intuitive concept of limits which cannot be changed.

The second remark is that Gauss's definition of the arithmetic-geometric means tacitly used this lemma

Lemma.

Given an increasing sequence λ_k

if this sequence has an upper bound, then there is a number L that is the limit of sequence λ_k as k goes to infinity ($k \rightarrow \infty$)

Gauss employed the same lemma at least in another paper, *Disquisitiones generales circa seriem infinitam*, where he used it to prove the convergence test today named after him

Gauss's test stated that if a given sequence M_k , satisfies these conditions:

the ratios (1) M_{k+1}/M_k are of the type

$$(2) \quad \frac{M_{k+1}}{M_k} = \frac{P_\lambda(m+k)}{Q_\lambda(m+k)}, \quad k=0,1,2,\dots,$$

where m is a positive integer, and P and Q are polynomials of the kind

$$(3) \quad P_\lambda(t) = t^\lambda + A_1 t^{\lambda-1} + A_2 t^{\lambda-2} + \dots + A_\lambda$$

$$(4) \quad Q_\lambda(t) = t^\lambda + a_1 t^{\lambda-1} + a_2 t^{\lambda-2} + \dots + a_\lambda,$$

$$(5) \quad A_1 - a_1 < -1.$$

then the series

$$(6) \quad \sum_{k=0}^{\infty} M_k \text{ converges}$$

In one of the steps of his long proof, Gauss showed that under certain condition (if $A_2 - a_2$ is greater than 0)

- the sequence $|M_k|$ taken in absolute value increases and
- there is a decreasing sequence N_k such that

$|M_k|$ is less than $|N_k|$ in absolute value
 $|M_k| < |N_k|$.

Hence - Gauss concluded -

$|M_k|$ is convergent to a finite limit.

De origine proprietatibusque generales numerorum mediorum arithm. Geometricorum, Disquisitiones generales circa seriem infinitam,

In these two papers Gauss did not provide a proof of the lemma,

A question is natural

Is the existence of the supremum of these sequences a hidden assumption in Gauss's reasoning

To answer, it is useful to consider

a discussion of the supremum

which is found in one of Gauss's manuscript
Grundbegriffe der lehre von den Reihen

Here, Gauss considered a series a_n such that all its terms are less than a given number l .

He defined this number l to be the upper bound of the sequence (in Gauss's terms: *limes supra seriem*)

Then Gauss stated that the upper bound has these properties:

1) any number M greater than l is also an upper bound

In the second property Gauss introduces the supremum

2) given a number κ less than a fixed upper bound l , IF

(a) κ is not an upper bound and

(b) κ is allowed to assume the value l by moving through

all the intermediate quantities in a continuous way,

THEN k necessarily reaches the least upper bound L (the supremum of the series).

3) no term of the series can be greater L ,

4) but, for large enough n , there are terms that exceed any number $z < L$

Gauss characterized the supremum correctly; but his reasoning, from a modern point of view, fails

Indeed Gauss again stated the existence of supremum but did not prove it and this is due to the lacking of an adequate construction of real numbers

Let's note that the requirement of this proof is precisely the starting point of Dedekind's theory of real numbers in his *Stetigkeit und irrationale Zahlen*.

Unlike Dedekind, Gauss did not feel the need to construct the set of real numbers

Gauss's approach can be understood if we assume that

his concept of the continuum was essentially different

from the modern one

Indeed, Gauss was linked to a revised notion of the traditional

continuous quantity, which he had inherited from eighteenth-century mathematics

To clarify my thesis I observe that from about 1740s

mathematicians grounded analysis upon the notion of general quantity

General quantity was merely the result of a process of abstraction

practised upon geometric quantities (which, in turns, abstracted [æb'stræktɪd] from physical ['fɪzɪkəl] quantities).

Thus, even if analysis was conceived to be independent of

geometry, general quantities were substantially intuited as segments of a straight line and, therefore, was intrinsically continuous,

in Kantian terms, they derived from an empirical intuition

(Of course [kɪ:s], that made problematic the asserted [e'se:tɪd] independence [ˌɪndɪ'pendəns] of analysis from geometry. I have tried [traɪd] to explain the sense of this claim [kleɪm] of eighteenth-century mathematicians in my [2001].)

Through the 18th century, analysis, which in principle was also independent of arithmetic -the realm of the discrete quantity-, was *de facto* more and more connected with numbers

(rational, irrational, complex numbers)

The status of irrational numbers was ambiguous (they was not true numbers, in the strict sense of word); however any irrational number can be approximate by rational numbers and this (with the aid of the geometrical intuition) was considered sufficient.

Thus, from the end of 18th and the beginnings of 19th century, continuous quantity moved away from its original geometrical model and reduced to the mere intuition of a sequence of numbers, which flows from a value to another assuming all intermediate values -without jumps -

this is precisely the idea of Gauss in his reasoning about the
supremum (This idea was largely [^lɔ:dʒli] shared [^jee+d] in the first [fe:st] decades [^ldekeid+s] of 19th century)

Therefore, Gauss no longer dealt with the 18th century notion of quantity, which was drawn from geometric sources, rather he grounded on the intuitively given numerical continuum

(I use the term 'intuition' in the sense that Gauss gave to it in [WD, 57]: "we can have an idea of quantity in two ways, either by immediate intuition (an immediate [i'mi:djet] idea), or by comparison [kəm'pærisən] with other quantities given by immediate [i'mi:djet] intuition (mediate [mi:di,eit] idea).")

In other terms, Gauss viewed the numerical continuum

as a basic and intuitive object of analysis and considered it

in connection with any sort of an *a priori* intuition (namely, with a

knowledge that does not appeal to any particular experience

Gauss's conception was based upon modified Kantian notions.

Kant believed that space and time were *a priori* intuitions from which the certainty of geometry and arithmetic derived.

It is well known that Gauss did not accept the whole theory of mathematical knowledge of Kant and, in particular, the idea that space is *a priori* intuition

In a letter to Bessel he wrote [rout]: "we cannot establish [i'stæbli] geometry entirely [en'taieli] *a priori*" [January 27, 1829, WW, 8:200]).

However, he shared some crucial points of Kantian philosophy:

- the importance of intuition,

 - the existence of an *a priori* intuition, that guarantees the certainty of mathematical knowledge;
 - the fact that mathematical intuition concerns abstract objects
(universal objects)
- but it can show only by empirical objects
(universal *in concreto*)

For example, in his review of *Commentatio in primum elementorum Euclidis librum*, a book published in 1814 ,

Gauss noted that a great part of the book turned on the contention against Kant that the certainty of geometry was not based on intuition but on definitions and on the *principium identitatis* and the *principium contradictionis*,

According to Gauss, Kant certainly did not wish to deny that use was constantly made in geometry of these logical aids to the presentation and linking of truths.

However, anybody who was acquainted with the essence of geometry knew that they were able to accomplish nothing by themselves, and they put forth only sterile blossoms unless the fertilizing living intuition of the object itself prevailed

everywhere

Intuition as immediate representation of an object seems to be what actually allows one to produce and understand mathematics.

According to Gauss, this intuition was, at least partially *a priori* in particular, he thought that number was *merely* the product of our mind.

But, unlike Kant, he was convinced that the theory of space had a completely different position in our knowledge than did 'the pure theory of quantity'

Following Kant, which had provided a theoretical basis for the numerical continuum by means of the notion of time as a pure intuition, Gauss thought that intuition of the numerical continuum

was originated from a more basic intuition: the *a priori* intuition of time.

In *Zur Metaphysik der Mathematik*, Gauss stated that quantities are time, number¹ and geometrical quantities. He thought that quantity can be represented by numbers by measuring it with regard to a fixed quantity, which is considered as the unit

Gauss did not clarify what the measure of an incommensurable quantity and an irrational number were. He however seems to consider an irrational number as a determination of continuum that can approximate by rationals and individuated by an endless number of steps

Thus, time is crucial to avoid geometrical quantity and to give the impression of analysis as a mere creation of our mind

Nevertheless, in the mathematical practice, time can be easily replaced by its numerical representation and Gauss directly

¹ Here number means whole number or, at most, rational numbers.

operated upon a continuum viewed as *an unbroken flow of numbers*.

By describing the continuum as an unbroken flow of numbers,

I don't give a definition but only a circumlocution to suggest Gauss's idea of the continuum: indeed, Gauss's continuum can not be really defined but only intuited

It is worthwhile noting that this notion allows to separate

continuous quantity from a geometric substratum:

continuity that earlier was referred to an empirical intuition (by means of geometry), is now reduced to the realm of numbers,

even though by the means of the *a priori* intuition of time

This does not mean that numerical continuum was reduced to the

discrete. There was no construction of the continuum: it was

immediately given (with its order structure and the property of completeness).

The relation between numbers and the continuum was inverted with

respect today: numbers did not generate the continuum; rather the

numerical continuum was given and one could cut it and determine any single, specific number.

Apart from Kant's epistemological influence, Gauss's continuum is a slightly revised version of the traditional notion of the continuum, which was at the basis of 18th-century calculus (see Breger)

The traditional continuum did not consist of points, but was given as

a whole. It was an intensional idea characterized by the relation between the whole and its possible parts, unlike in Dedekind-Cantor continuum, which is based on extensional set theory (Breger [1992a, 251])

This conception have remarkable consequences in the calculus: it

is sufficient to think that an interval was always thought of as including its endpoints (in modern terms, it was always a closed interval).

In my opinion, the main difference between Gauss and the 18th-century mathematicians is that the latter thought of a geometric continuum and referred directly to geometric quantities, while the former referred to a numerical continuum, which, in principle,

embodies the flowing of time

To end, I observe that Gauss's conception has important consequences on continuity of functions a point can only break continuity of a function, but continuity is not a property of a point: it has no sense to define continuity of a function at a point;

any reasoning implying continuity of a function did not concern a single point, but a part of the continuum

For instance, Gauss stated: "Continuity of the function $\Gamma(z)$ breaks, every time that the value is infinitely large, i.e. for negative integer value of z " [1812, 147].

The flow that the variable y describes can break for certain [se:ten] exceptional values of z , thus $y=\Gamma(z)$ is discontinuous when

$$z=-1,-2,-3,\dots$$

What is important for Gauss is that the independent variable z could move from a point A of the domain of definition to another B (including the endpoints) in a continuous way (*fluxo continuo*) and that the correspondent variation of the dependent variable y is continuous as well.

One could argue that this idea is unclear, in particular the relationship between the single number and the flow of numbers.

This is true, and I think that some problematic derivations of *Disquisitiones generales* are probably linked to this concept of continuity

For instance,

Gauss stated that it is evident that $F(\alpha,\beta,\gamma+k,1)$ goes to 1 as $k \rightarrow \infty$ [1812, 147]

Limes functiones $F(\alpha, \beta, \gamma+k, 1)$, crescente k in infinitum, manifesto [est] unitas

F is the hypergeometric [haipe+g] function

$$(2) \quad F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} x^3 + \dots$$

In effect, Gauss is calculating this limit:

$$\lim_{k \rightarrow \infty} F(\alpha, \beta, \gamma+k, 1) = \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma+k)_n} = \sum_{n=0}^{\infty} \lim_{k \rightarrow \infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma+k)_n} = 1 + 0 + 0 + \dots,$$

Gauss interchanged the operation of limit: however any justification is lacking as if this interchange was immediate and obvious

and $(x)_n$ is the symbol (3)

$$(3) \quad (x)_n = x(x+1)(x+2)\dots(x+n-1) = \Gamma(x+n)/\Gamma(x) \text{ [Pochhammer symbol]}$$

$$[\text{è } (x)_0 = 1]$$

A similar example is at p.151 of his *Disquisitiones generales*, Gauss interchanged the operation of limits and that of integration.

Indeed, he derived that equality

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_0^n y^{\lambda-1} \left(1 - \frac{y}{n}\right)^n dy \\ &= \int_0^\infty \lim_{n \rightarrow \infty} \left[y^{\lambda-1} \left(1 - \frac{y}{n}\right)^n \right] dy = \int_0^\infty y^{\lambda-1} e^{-y} dy \end{aligned}$$

Gauss performed operations that are legitimate in the specific cases but their legitimacy is not discussed as the legitimacy of these operations was obvious

Today they are justified by means of appropriate theorems, instead Gauss seems to consider them as evident.

It is likely that Gauss approached the matter in a similar manner as Cauchy did in his *Analyse algébrique* some years later, however evidences are lacking for more precise hypotheses He possessed sophisticated knowledge of complex analysis –see, e.g., [WW, 10:368]- though he did not expound them in a systematic way: it is therefore difficult to say of which theorems he was aware

In any event Gauss's concept of the continuum made impossible to distinguish between continuity and uniform continuity, convergence and uniform convergence, which are necessary for the correct formulation of the theorems on the interchange of limits, to which I referred early I end my talk with some observations on continuity of functions .

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In these two examples, Gauss performed some operations that are legitimate in the specific cases but their legitimacy is not discussed

Today they are justified by means of appropriate theorems, instead Gauss seems to consider them as evident.

It is well known that Gauss rejected the infinite extension of finite rules, which were accepted by eighteenth-century mathematicians. Of consequence, it is likely that Gauss approached the matter in a similar manner as Cauchy did in his *Analyse algébrique* some years later, however evidences are lacking for more precise hypotheses

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