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(Co)homologies and K-theory of Bianchi groups using computational geometric models

Alexander Rahm

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UNIVERSITÉ DE GRENOBLE

UNIVERSITÄT GÖTTINGEN

THÈSE

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Alexander Daniel RAHM

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(Co)homologies et K-théorie de groupes de Bianchi

par des modèles géométriques calculatoires

(Co)homologies and K-theory of Bianchi groups using computational geometric models

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Thèse dirigée par Philippe Elbaz-Vincent et codirigée par Thomas Schick

JURY

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This thesis is dedicated to Fritz Grunewald (1949-2010), for his life-long contributions to the understanding of the groups acting on hyperbolic space, and many other areas of mathematics.

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0.1 Introduction en français

Cette thèse consiste en une étude de la géométrie d'une certaine classe de groupes arithmétiques, à travers une action propre sur un espace contractile. Nous allons calculer explicitement leur homologie de groupe, et leur K-homologie équivariante.

Plus précisément, considérons un corps de nombres quadratique imaginaire $\mathbb{Q}(\sqrt{-m})$, où m est un entier positif ne contenant pas de carré, et son anneau d'entiers $\mathcal{O}$. Les *groupes de Bianchi* sont les groupes $\mathrm{SL}_2(\mathcal{O})$ et $\mathrm{PSL}_2(\mathcal{O})$. Ces groupes agissent d'une manière naturelle sur l'espace hyperbolique à trois dimensions. En 1892, Luigi Bianchi [8] a calculé des domaines fondamentaux pour cette action, pour les valeurs $m = 1$, 2, 3, 5, 6, 7, 10, 11, 13, 15 et 19. Un tel domaine fondamental a la forme d'un polyèdre hyperbolique (à quelques sommets manquants près), et nous l'appelerons le *polyèdre fondamental de Bianchi*.

Les groupes de Bianchi peuvent être considérés comme un élement clé pour l'étude d'une classe plus large de groupes, les groupes *Kleiniens*, qui ont déjà été étudiés par Henri Poincaré [29]. En fait, chaque groupe Kleinien arithmétique non-cocompact est commensurable avec un des groupes de Bianchi [26]. Un éventail d'informations sur les groupes de Bianchi peut être trouvé dans les monographies [17, 19, 26].

Nous utilisons une implémentation sur ordinateur [31] écrite dans le language Pari/GP [1] pour calculer le polyèdre fondamental de Bianchi, et décrivons les algorithmes sous-jacents dans le chapitre 1.

Deux programmes pour des tâches très similaires ont été réalisés : l'un par Karen Vogtmann (publication en 1985), et l'autre par Robert Riley (publication en 1983). Concernant le programme de Riley [33], une bibliothèque de routines en FORTRAN est toujours disponible [34]. Mais comme Riley est décédé, ces routines qui sont issues d'une documentation minimale ne peuvent pas facilement être utilisées pour reconstruire un programme pouvant traiter les groupes de Bianchi. Le point commun du programme de Riley avec celui mis en oeuvre dans le cadre de cette thèse, est le calcul du polyèdre fondamental de Bianchi. Ensuite, le programme de Riley a produit des présentations pour les groupes de Bianchi, et il a calculé à la main quelques informations sur leurs abélianisations.

Le code source du programme de Vogtmann n'a pas été conservé, mais une description et les résultats qu'il a produit peuvent être trouvés dans [40]. À la place du polyèdre fondamental de Bianchi, le programme de Vogtmann calculait un domaine fontamental pour le rétracte équivariant de l'espace hyperbolique à trois dimensions décrit par Eduardo Mendoza [27]. Précisément dans les cas où l'anneau $\mathcal{O}$ est un anneau principal, ce rétracte coïncide avec le rétracte décrit par Dieter Flöge [20] et utilisé dans [30] et le présent mémoire pour des calculs d'homologie de groupe. Le programme de Vogtmann calculait alors l'homologie de groupe à coefficients rationnels.

Un modèle complètement différent a été choisi par Dan Yasaki [43], qui a implémenté un algorithme de Paul Gunnells [22] pour calculer les formes parfaites modulo l'action de $\mathrm{GL}_2(\mathcal{O})$ et obtenir les facettes du polyèdre de Voronoï qui proviennent d'une construction d' Avner Ash [2]. Yasaki pourrait en déduire des présentations pour ces groupes.

Nous décrivons dans le chapitre 1.3 notre manière de calculer les stabilisateurs et identifications dans le bord du polyèdre fondamental de Bianchi. Grâce à ces informations, nous obtenons une description explicite des orbi-espaces donnés par l'action des groupes de Bianchi sur l'espace hyperbolique réel à trois dimensions. Ces orbi-espaces ont d'ailleurs des applications en Physique [3]. Nous allons exploiter nos informations sur ceux-ci de différentes manières, afin d' observer des aspects différents de la géométrie de nos groupes.

- Nous calculons l'homologie de groupe comme définie dans [11]. Pour ceci, nous nous servons de la suite spectrale équivariante de Leray/Serre de l'homologie de groupe des stabilisateurs de cellules dans l'espace quotient, convergente vers l'homologie des groupes de Bianchi. En degrés supérieurs à la dimension cohomologique virtuelle de nos groupes, nous pouvons décomposer la différentielle d^1 de cette suite spectrale dans ses parties 2–primaire et 3–primaire. Nous étudions ces différentielles dans la partie 2.2. Ceci nous permet de calculer l'aboutissement et le dévissage de la suite spectrale, ce que nous faisons dans le chapitre 3 pour tous les groupes de Bianchi définis sur des anneaux principaux.

- Dans le chapitre 4, nous calculons l'homologie de Bredon des groupes de Bianchi, de laquelle nous déduisons leur K-homologie équivariante. Par la conjecture de Baum/Connes, qui est vérifiée par les groupes de Bianchi, nous obtenons la K-théorie des C^*-algèbres réduites des groupes de Bianchi, comme des images isomorphes.

- Dans le chapitre 5, nous allons complexifier nos orbi-espaces, en complexifiant l'espace hyperbolique à trois dimensions. Nous obtenons des orbi-espaces donnés par l'action induite des groupes de Bianchi sur l'espace hyperbolique à trois dimensions complexes. Ensuite, nous calculons la cohomologie d'orbi-espace de Chen/Ruan pour ces orbi-espaces complexes. Ceci est l'un des deux côtés de l'isomorphisme présumé dans la conjecture sur la résolution cohomologique crépante de Ruan [12]. Nous pouvons déterminer explicitement la structure d'espace vectoriel de la cohomologie de Chen/Ruan en utilisant [31]; et nous déterminons sa structure de produit cohomologique dans un énoncé général, théorème 86.

Nos resultats dans la K-homologie équivariante sont les suivants. Soit $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$. Alors, pour $\mathcal{O}_{-m}$ principal, la K-homologie équivariante de Γ est de type d'isomorphie

	$m=1$	$m=2$	$m=3$	$m=7$	$m=11$	$m\in\{19,43,67,163\}$
$K_0^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}^6$	$\mathbb{Z}^5\oplus\mathbb{Z}/2$	$\mathbb{Z}^5\oplus\mathbb{Z}/2$	$\mathbb{Z}^3$	$\mathbb{Z}^4\oplus\mathbb{Z}/2$	$\mathbb{Z}^{\beta_2}\oplus\mathbb{Z}^3\oplus\mathbb{Z}/2$
$K_1^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}$	$\mathbb{Z}^3$	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}\oplus\mathbb{Z}^{\beta_1}$,

où les nombres de Betti sont

m	19	43	67	163
β_1	1	2	3	7
β_2	0	1	2	6,

et la continuation de la K-homologie équivariante de Γ est donnée par la 2-périodicité de Bott.

Nos calculs d'homologie de groupes nous mènent à la découverte de quelques règles générales très restrictives pour l'action des sous-groupes finis des groupes de Bianchi sur l'espace tangent à leurs points fixes. Nous allons constater ceci en détail dans la partie 2.1. Il est utile d'avoir des connaissances sur ces situations géométriques locales pour toutes les études homologiques mentionnées. En fait, nous allons dévélopper un traitement généralisé du sous-complexe dans l'espace quotient, qui consiste en des cellules fixées par la ℓ-torsion, pour un nombre premier ℓ. Ceci nous permet de démontrer, dans la partie 2.3, que le type d'homéomorphie de ce sous-complexe détermine complètement l'homologie à coefficients $\mathbb{Z}/\ell$ des groupes de Bianchi en degrés supérieurs à leur dimension cohomologique virtuelle.

Précisons la méthode pour les calculs d'homologie de groupe. Poincaré a donné une formule explicite pour l'action sur l'espace hyperbolique à trois dimensions, $\mathcal{H}$. Pourtant, les groupes arithmétiques qui sont des réseaux dans $\mathrm{SL}_2(\mathbb{C})$ sont de dimension cohomologique virtuelle 2, donc il est désirable de restreindre cette action propre sur $\mathcal{H}$ à un espace cellulaire contractile à deux dimensions. De plus, on aimerait se trouver dans la situation où cet espace soit cocompact. En principe, ceci a été mis en place par Mendoza [27] et aussi par Flöge [20], en utilisant la théorie de réduction de Minkowski, Humbert, Harder et autres. Le point commun des deux approches est qu'ils considèrent des rétractes Γ-équivariants à deux dimensions qui sont cocompacts et portent une structure naturelle de complexe cellulaire telle que l'action de Γ est cellulaire et que le quotient est un complexe cellulaire fini.

Dans les cas de groupe de classes d'idéaux trivial, les deux approches coïncident. Schwermer et Vogtmann [37] ont exposé la suite spectrale équivariante de Leray/Serre associée, convergeante vers l'homologie à coefficients entiers de Γ, et s'en sont servis pour calculer cette dernière dans les cas d'anneau Euclidien, $m = 1, 2, 3, 7, 11$.

Dans les cas de groupe des classes d'idéaux non-trivial, il y a une différence entre l'approche de Mendoza et celle de Flöge. Nous utilisons le modèle du demi-espace supérieur pour $\mathcal{H}$ et identifions son bord avec $\mathbb{C} \cup \infty \cong \mathbb{C}P^1$. Nous définissons les *pointes* comme ∞ et les éléments du corps de nombres $\mathbb{Q}\left(\sqrt{-m}\,\right)$, vus comme éléments du bord canonique $\mathbb{C}P^1$. Les éléments du groupe des classes d'idéaux sont en bijection avec les Γ-orbites de pointes.

Les pointes qui représentent un élément non-trivial du groupe des classes d'idéaux sont traditionellement appelées les pointes *singulières*. Tandis que Mendoza contracte en s'éloignant de toutes les pointes, Flöge contracte en s'éloignant seulement des pointes non-singulières. À la place de l'espace $\mathcal{H}$ lui-même, Flöge considère l'espace $\widehat{\mathcal{H}}$ obtenu

de $\mathcal{H}$ en ajoutant les Γ-orbites des pointes singulières. Ainsi, une rétraction géodésique de $\widehat{\mathcal{H}}$ peut être définie sur le polyèdre fondamental de Bianchi, en incluant les pointes singulières dans le rétracte X de $\widehat{\mathcal{H}}$. Il s'avère alors que l'espace quotient de X par Γ est compact, et X est un Γ-complexe contractile convenable à deux dimensions également dans les cas de groupe des classes d'idéaux non-trivial, comme l'ont vérifié Mathias Fuchs et l'auteur [30].

L'avantage du complexe cellulaire de Mendoza, qui est contenu dans $\mathcal{H}$ et exclut les pointes, consiste dans le fait que ses stabilisateurs de cellules sont tous finis. En conséquence, l'homologie de Γ à coefficients rationnels est isomorphe à l'homologie de l'espace quotient. En se servant de cette propriété du modèle de Mendoza, Karen Vogtmann [40] a calculé l'homologie de groupe à coefficients rationnels d'une manière efficace, dans tous les cas où la valeur absolue du discriminant du corps de nombres est inférieure à 100.

Comparé avec ceci, le complexe cellulaire de Flöge admet l'inconvénient que le calcul de l'homologie de Γ en petits degrés — jusqu'à la dimension cohomologique virtuelle de Γ — nécéssite un travail beaucoup plus dur avec la suite spectrale associée, parce que celle-ci dégénère une page plus tard. Pour calculer la différentielle de degré 2 non-triviale qui provoque cette situation, on a besoin de certaines informations topologiques explicites sur les cellules touchant les pointes. Ce calcul a été achevé dans cinq cas de groupe de classe d'idéaux non-trivial [30].

Mais en degrés supérieurs à la dimension cohomologique virtuelle, l'avantage du modèle de Flöge — de préserver la géométrie du polyèdre fondamental de Bianchi — montre sa valeur. Nous faisons quelques découvertes sur l'opération des stabilisateurs et nous pouvons en faire usage pour le calcul des différentielles sur la première page de notre suite spectrale. Ceci nous rend possible le calcul de l'homologie à coefficients entiers des groupes de Bianchi en tous ces degrés supérieurs.

D'ailleurs l'approche de Flöge lui a permis de donner une présentation de quelques groupes de Bianchi en termes de générateurs et relations, de laquelle Berkove [6] a calculé l'anneau de cohomologie de groupe à coefficients entiers dans les cas $m = 2$, 3, 5, 6, 7, 10 et 11. Également, on ne peut pas obtenir la structure d'orbi-espace complexe que nous allons étudier dans le chapitre 5 par le modèle de Mendoza, qui préserve seulement la topologie Γ-équivariante et non la géométrie. Pour ces raisons, et pour obtenir des informations complémentaires aux résultats de Vogtmann, nous allons choisir le complexe cellulaire de Flöge.

Concernant notre suite spectrale équivariante de Leray/Serre, nous pourrions nous servir de son analogue en cohomologie de groupes, et calculer l'homologie de $\mathrm{SL}_2(\mathcal{O})$ à coefficients dans le module de Steinberg – via sa cohomologie de Farrell/Tate, comme ceci a été effectué par Schwermer et Vogtmann dans les cas Euclidiens. Ceci nous offrirait la possibilité d'obtenir le K-groupe algébrique $K_2\mathcal{O}$. Ce dernier a déjà été calculé pour une grande gamme d'anneaux d'entiers de corps de nombres quadratiques imaginaires par Karim Belabas et Herbert Gangl [5], en utilisant une méthode complètement différente.

Il convient de formuler deux remarques supplémentaires. Si l'espace quotient $_\Gamma\backslash\mathcal{H}$ était une variété différentielle, il serait possible de la traiter avec le programme Snap-

Pea [42] de Jeffrey Weeks, qui construirait une triangulation afin de calculer des informations utiles de plusieurs types : par exemple le groupe fondamental, le volume, le revêtement d'orientation et le groupe de symétrie de cette variété. Mais selon Nathan Dunfield, qui a écrit une interface pour SnapPea ensemble avec Marc Culler, la méthode de triangulation utilisée ne peut pas traiter des orbi-espaces pour lesquels le revêtement $\mathcal{H} \to {}_{\Gamma}\backslash\mathcal{H}$ est ramifié. Ceci est le cas pour tous les groupes de Bianchi car ils admettent des points à stabilisateurs non-triviaux dans $\mathcal{H}$.

Un champ d'études intéressant, qui par ailleurs devient accessible grâce au savoir explicite sur l'espace quotient, sont les sous-groupes d'indice fini dans les groupes de Bianchi ; voir par exemple [15, 16] pour les cas Euclidiens.

0.2 Introduction in English

This thesis consists of the study of the geometry of a certain class of arithmetic groups by means of a proper action on a contractible space. We will explicitly compute their group homology, and their equivariant K-homology.

More precisely, denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by $\mathcal{O}$ its ring of integers. The *Bianchi groups* are the groups $\mathrm{SL}_2(\mathcal{O})$ and $\mathrm{PSL}_2(\mathcal{O})$. These groups act in a natural way on hyperbolic three-space. In 1892, Luigi Bianchi [8] computed fundamental domains for this action for the values $m = 1, 2, 3, 5, 6, 7, 10, 11, 13, 15$ and 19. Such a fundamental domain has the shape of a hyperbolic polyhedron (up to some missing vertices), so we will call it the *Bianchi fundamental polyhedron.*

The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian* groups, which dates back to Henri Poincaré's works [29]. In fact, each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [26]. A wealth of information on the Bianchi groups can be found in the monographs [17, 19, 26].

We use a computer implementation [31] in the language Pari/GP [1] to compute the Bianchi fundamental polyhedron, and describe the relevant algorithms in chapter 1.

Two computer programs have been written that do closely related computations: one by Karen Vogtmann (published in 1985), and one by Robert Riley (published in 1983). Concerning Riley's program [33], a library of FORTRAN routines is still available [34]. But since Riley has passed away, these sparsely documented routines cannot easily be used to reconstruct a program that could treat the Bianchi groups. The common point of Riley's program with the one written for this thesis is the computation of the Bianchi fundamental polyhedron. Afterwards, Riley's program produced presentations for the Bianchi groups, and he computed by hand some information on their abelianizations.

The source code of Vogtmann's program has not been preserved, but both a description and the obtained results can be found in [40]. Instead of the Bianchi fundamental polyhedron, Vogtmann's program computed a fundamental domain for the equivariant retract of hyperbolic three-space described by Eduardo Mendoza [27]. Precisely in the cases where the ring $\mathcal{O}$ is a principal ideal domain, this retract coincides with the retract described by Dieter Flöge [20] and used in [30] and the present thesis for group homology computations. Vogtmann's program further computed the rational group homology.

A completely different model has been chosen by Dan Yasaki [43], who has implemented an algorithm of Paul Gunnells [22] to compute the perfect forms modulo the action of $\mathrm{GL}_2(\mathcal{O})$ and obtain the facets of the Voronoï polyhedron arising from a construction of Avner Ash [2]. Yasaki can possibly deduce presentations of $\mathrm{GL}_2(\mathcal{O})$ from this.

We describe in section 1.3 how we compute the stabilisers and identifications on the Bianchi fundamental polyhedron. This information provide an explicit knowledge of the orbifolds given by the action of the Bianchi groups on real hyperbolic three-space. These orbifolds also have applications in physics [3]. We will exploit our computations of them in different ways, in order to view different aspects of the geometry of our groups:

- We compute group homology as defined in [11]. To this end, we make use of the equivariant Leray/Serre spectral sequence from the group homology of the stabilisers of representatives of the cells in the orbit space, converging to the group homology of the Bianchi groups. In degrees greater than the virtual cohomological dimension of our groups, we can decompose the d^1-differentials of this spectral sequence into their 2–primary and 3–primary parts. We study these differentials in section 2.2. This allows us to calculate the abutment and the dévissage of the spectral sequence, which we do for all Bianchi groups over principal ideal domains in chapter 3.

- In chapter 4, we compute the Bredon homology of the Bianchi groups, from which we deduce their equivariant K-homology. By the Baum/Connes conjecture, which is verified by the Bianchi groups, we obtain the K-theory of the reduced C^*-algebras of the Bianchi groups as isomorphic images.

- In chapter 5, we complexify our orbifolds by complexifying the real hyperbolic three-space. We obtain orbifolds given by the induced action of the Bianchi groups on complex hyperbolic three-space. Then we compute the Chen/Ruan orbifold cohomology for these complex orbifolds. This is one side of Ruan's cohomological crepant resolution conjecture [12]. We can determine the vector space structure of this cohomology theory using [31]; and we determine its cohomological product structure in a general statement, theorem 86.

Our results in equivariant K-homology are the following. Let $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$. Then, for $\mathcal{O}_{-m}$ principal, the equivariant K-homology of Γ has isomorphy types

	$m=1$	$m=2$	$m=3$	$m=7$	$m=11$	$m\in\{19,43,67,163\}$
$K_0^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}^6$	$\mathbb{Z}^5\oplus\mathbb{Z}/2$	$\mathbb{Z}^5\oplus\mathbb{Z}/2$	$\mathbb{Z}^3$	$\mathbb{Z}^4\oplus\mathbb{Z}/2$	$\mathbb{Z}^{\beta_2}\oplus\mathbb{Z}^3\oplus\mathbb{Z}/2$
$K_1^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}$	$\mathbb{Z}^3$	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}\oplus\mathbb{Z}^{\beta_1}$,

where the Betti numbers are

m	19	43	67	163
β_1	1	2	3	7
β_2	0	1	2	6,

and the remainder of the equivariant K-homology of Γ is given by Bott 2-periodicity.

Our group homology calculations make us discover that the action of the finite subgroups in the Bianchi groups on the tangent space at their fixed points follows very restrictive general rules. We shall state them precisely and in detail in section 2.1. The knowledge about these local geometric situations is in turn helpful for all these three homological studies. Indeed, we are going to develop a general treatment of the subcomplex in the orbit space, which consists of the cells fixed by ℓ-torsion, for a prime ℓ.

This allows us to prove in section 2.3 that the homeomorphy type of this sub-complex completely determines the homology with $\mathbb{Z}/\ell$–coefficients of the Bianchi groups in degrees greater than their virtual cohomological dimension.

Let us have a closer look at the group homology computations. Poincaré gave an explicit formula for the action on hyperbolic three-space $\mathcal{H}$. However, the virtual cohomological dimension of arithmetic groups that are lattices in $\mathrm{SL}_2(\mathbb{C})$ is two, so it is desirable to restrict this proper action on $\mathcal{H}$ to one on a contractible cellular two-dimensional subspace. Moreover, this space should be cocompact. In principle, this has been achieved by Mendoza [27] and by Flöge [20], using reduction theory of Minkowski, Humbert, Harder and others. These two approaches have in common that they consider two-dimensional Γ-equivariant retracts which are cocompact and are endowed with a natural CW-structure such that the action of Γ is cellular and the quotient is a finite CW-complex.

In the cases of trivial class group, these two approaches coincide. Schwermer and Vogtmann [37] have exposed the associated equivariant Leray/Serre spectral sequence converging to the integral homology of Γ, and used this to calculate the latter in the Euclidean principal ideal domain cases, $m = 1$, 2, 3, 7 and 11.

In the cases of non-trivial ideal class group, there is a difference between the models of Mendoza and Flöge. We use the upper-half-space model of $\mathcal{H}$ and identify its boundary with $\mathbb{C} \cup \infty \cong \mathbb{C}P^1$. The elements of the class group of the number field are in bijection with the Γ-orbits of the cusps, where the cusps are ∞ and the elements of the number field $\mathbb{Q}\left(\sqrt{-m}\,\right)$, thought of as elements of the canonical boundary $\mathbb{C}P^1$. The cusps that represent a non-trivial element of the class group are commonly called *singular* points. While Mendoza retracts away from all cusps, Flöge retracts away only from the non-singular ones. Rather than the space $\mathcal{H}$ itself, he considers the space $\widehat{\mathcal{H}}$ obtained from $\mathcal{H}$ by adjoining the Γ-orbits of the singular points. Then, a geodesic retraction of $\widehat{\mathcal{H}}$ can be defined on the Bianchi fundamental polyhedron, including the singular cusps into the retract X of $\widehat{\mathcal{H}}$. Now it turns out that the quotient space of X by Γ is compact, and X is a suitable contractible 2-dimensional Γ-complex also in the case of non-trivial class group, as has been verified by Mathias Fuchs and the author [30].

The advantage of Mendoza's cellular complex, being contained in $\mathcal{H}$ and excluding the cusps, is that its cell stabilisers are all finite. As a consequence, the rational homology of Γ is isomorphic to the rational homology of the quotient space. Using this property of Mendoza's model, Karen Vogtmann [40] has computed the rational group homology in an efficient way, in all cases where the absolute value of the discriminant of the number field is inferior to 100.

Compared to this, Flöge's cellular complex has the inconvenient that for computing the homology of Γ in low degrees — up to the virtual cohomological dimension of Γ — there is much harder work with the associated spectral sequence, as it degenerates one page later. The computation of the non-trivial differential of degree 2 that provokes this situation requires some explicit topological knowledge about the cells adjacent to the cusps. However, this calculation has been achieved in five cases of non-trivial class group [30].

But in degrees superior to the virtual cohomological dimension, the advantage of Flöge's model — to preserve the geometry of the Bianchi fundamental polyhedron — pays off. We can use some discoveries that we make about the action of the cell stabilisers for the computation of the differentials on the first page of our spectral sequence. This enables us to calculate the integral homology of the Bianchi groups in all these higher degrees.

A further fruit of Flöge's approach is that it has allowed him to give a presentation of some Bianchi groups with generators and relations, from which Berkove [6] has calculated the integral group cohomology ring in the cases $m = 2$, 3, 5, 6, 7, 10 and 11. Also, the complex orbifold structure that we study in chapter 5 cannot be obtained from Mendoza's model, which preserves only the Γ-equivariant topology and not the geometry. Therefore, and to obtain information complementary to the results Vogtmann has obtained with Mendoza's model already in many cases, we shall choose Flöge's cellular complex.

Concerning our equivariant Leray/Serre spectral sequence, we could use its analogon in group cohomology, and compute the homology of $\mathrm{SL}_2(\mathcal{O})$ with coefficients in the Steinberg module – via its Farrell/Tate cohomology, as it has been done by Schwermer and Vogtmann in the Euclidean cases. This would enable us to obtain the algebraic K-group $K_2(\mathcal{O})$. The latter has already been computed for a large collection of rings of integers in imaginary quadratic number fields by Karim Belabas and Herbert Gangl [5], using a completely different method.

Let us make two further remarks. If the quotient space ${}_\Gamma\backslash\mathcal{H}$ were a smooth manifold, it would be possible to enter it into the program SnapPea [42] by Jeffrey Weeks, which would construct a triangulation for it and compute useful information of several types: for example the fundamental group, the volume, the orientation cover and the symmetry group of such a manifold. But according to Nathan Dunfield, who has written an interface for SnapPea with Marc Culler, the triangulation method in use cannot deal with orbifolds for which the covering map $\mathcal{H} \to {}_\Gamma\backslash\mathcal{H}$ is ramified. The latter is the case for all the Bianchi groups because they admit non-trivially stabilized points in $\mathcal{H}$.

An interesting field of study, which further becomes accessible with the explicit knowledge of the quotient space, are the subgroups of finite index in the Bianchi groups; see for instance [15, 16] in the Euclidean cases.

Chapter 1

Algorithms to compute the quotient space

The aim of this chapter is to describe an algorithm that, given any Bianchi group, computes a fundamental domain for its action on hyperbolic three-space, and also computes the associated quotient space.

1.1 Swan's concept

This section recalls Richard G. Swan's work [38], which describes — from the theoretical viewpoint — an algorithm to compute the Bianchi fundamental polyhedron. To the author's as well as Swan's knowledge, this algorithm has never been put into practice until the realization described in section 1.2.

1.1.1 Defining the Bianchi fundamental polyhedron

Let m be a squarefree positive integer and $K = \mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic number field with ring of integers $\mathcal{O}_{-m}$, which we also just denote by $\mathcal{O}$. Consider the familiar action (we give an explicit formula for it in lemma 22) of the group $\Gamma := \mathrm{SL}_2(\mathcal{O}) \subset \mathrm{GL}_2(\mathbb{C})$ on hyperbolic three-space, for which we will use the upper-half space model $\mathcal{H}$. As a set,

$$\mathcal{H} = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}.$$

We will call the coordinate ζ the *height*.

The Bianchi/Humbert theory [8, 24] gives a fundamental domain for this action. We will start by giving a geometric description of it, and the arguments why it is a fundamental domain.

Definition 1. *A pair of elements* $(\mu, \lambda) \in \mathcal{O}^2$ *is called* unimodular *if the ideal sum* $\mu\mathcal{O} + \lambda\mathcal{O}$ *equals* $\mathcal{O}$.

The boundary of $\mathcal{H}$ is the Riemann sphere $\partial\mathcal{H} = \mathbb{C} \cup \{\infty\}$ (as a set), which contains the complex plane $\mathbb{C}$. The totally geodesic surfaces in $\mathcal{H}$ are the Euclidean vertical planes (we define *vertical* as orthogonal to the complex plane) and the Euclidean hemispheres centred on the complex plane.

***Notation* 2.** Given a unimodular pair $(\mu, \lambda) \in \mathcal{O}^2$ with $\mu \neq 0$, let $S_{\mu,\lambda} \subset \mathcal{H}$ denote the hemisphere given by the equation $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1$.

This hemisphere has centre λ/μ on the complex plane $\mathbb{C}$, and radius $1/|\mu|$. Let

$$B := \big\{(z,\zeta) \in \mathcal{H}\colon \text{The inequality } |\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1 \text{ is fulfilled for all unimodular pairs } (\mu, \lambda) \in \mathcal{O}^2 \text{ with } \mu \neq 0 \big\}.$$

Then B is the set of points in $\mathcal{H}$ which lie above or on all hemispheres $S_{\mu,\lambda}$.

Lemma 3 ([38])**.** *The set B contains representatives for all the orbits of points under the action of $\mathrm{SL}_2(\mathcal{O})$ on $\mathcal{H}$.*

Proof. Consider hyperbolic three-space as the set of positive definite Hermitian forms f in two complex variables, modulo homotheties. The action of $\mathrm{GL}_2(\mathbb{C})$ on the variables by linear automorphisms of $\mathbb{C}^2$ induces an action on this set by the formula $\gamma \cdot f(z) := f(\gamma^{-1}z)$ for $\gamma \in \mathrm{GL}_2(\mathbb{C})$, $z \in \mathbb{C}^2$. The latter action corresponds to the familiar action on $\mathcal{H}$, which Swan even defines this way. Now the set B corresponds to the forms which take their "proper minimum" at the argument $(1, 0)$. From Humbert [24], it follows that for any binary Hermitian form f, there exists an element $\gamma \in \mathrm{SL}_2(\mathcal{O})$ such that $\gamma \cdot f$ takes its proper minimum at $(1, 0)$. □

The action extends continuously to the boundary $\partial\mathcal{H}$, which is a Riemann sphere. In $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$, consider the stabiliser subgroup Γ_∞ of the point $\infty \in \partial\mathcal{H}$. In the cases $m = 1$ and $m = 3$, the latter group contains some rotation matrices like $\left(\begin{smallmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{smallmatrix}\right)$, which we want to exclude. These two cases have been treated in [27, 37] and others, and we assume $m \neq 1$, $m \neq 3$ throughout this chapter. Then,

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathcal{O} \right\},$$

which performs translations by the elements of $\mathcal{O}$ with respect to the Euclidean geometry of the upper-half space $\mathcal{H}$.

***Notation* 4.** A fundamental domain for Γ_∞ in the complex plane (as a subset of $\partial\mathcal{H}$) is given by the rectangle

$$D_0 := \begin{cases} \{x + y\sqrt{-m} \in \mathbb{C} \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}, & m \equiv 1 \text{ or } 2 \mod 4, \\ \{x + y\sqrt{-m} \in \mathbb{C} \mid \frac{-1}{2} \leqslant x \leqslant \frac{1}{2},\ 0 \leqslant y \leqslant \frac{1}{2}\}, & m \equiv 3 \mod 4. \end{cases}$$

And a fundamental domain for Γ_∞ in $\mathcal{H}$ is given by

$$D_\infty := \{(z,\zeta) \in \mathcal{H} \mid z \in D_0\}.$$

Definition 5. *We define the* Bianchi fundamental polyhedron *as*

$$D := D_\infty \cap B.$$

It is a polyhedron in hyperbolic space up to the missing vertex ∞, and up to missing vertices at the singular points if $\mathcal{O}$ is not a principal ideal domain (see subsection 1.1.3). As Lemma 3 states $\Gamma \cdot B = \mathcal{H}$, and as $\Gamma_\infty \cdot D_\infty = \mathcal{H}$ yields $\Gamma_\infty \cdot D = B$, we have $\Gamma \cdot D = \mathcal{H}$. We observe the following notion of strictness of the fundamental domain: the interior of the Bianchi fundamental polyhedron contains no two points which are identified by Γ. Swan proves the following theorem, which implies that the boundary of the Bianchi fundamental polyhedron consists of finitely many cells.

Theorem 6 ([38])**.** *There is only a finite number of unimodular pairs (λ, μ) such that the intersection of $S_{\mu,\lambda}$ with the Bianchi fundamental polyhedron is non-empty.*

He also proves a corollary, from which it can be deduced that the action of Γ on $\mathcal{H}$ is properly discontinuous.

Corollary 7 ([38])**.** *There are only finitely many matrices $\gamma \in \mathrm{SL}_2(\mathcal{O})$ such that $D \cap \gamma \cdot D \neq \emptyset$.*

1.1.2 Determining the Bianchi fundamental polyhedron

The set B which determines the Bianchi fundamental polyhedron has been defined using infinitely many hemispheres. But we will see that only a finite number of them are significant for this purpose and need to be computed. We will state a criterion for what is an appropriate choice that gives us precisely the set B. This criterion is easy to verify in practice.
Suppose we have made a finite selection of n hemispheres. The index i running from 1 through n, we denote the i-th hemisphere by $S(\alpha_i)$, where α_i is its centre and given by a fraction $\alpha_i = \frac{\lambda_i}{\mu_i}$ in the number field K. Here, we require the ideal (λ_i, μ_i) to be the whole ring of integers $\mathcal{O}$. This requirement is just the one already made for all the hemispheres in the definition of B. Now, we can do an approximation of notation 2, using, modulo the translation group Γ_∞, a finite number of hemispheres.
***Notation* 8.** Let $B(\alpha_1, \ldots, \alpha_n) := \big\{(z, \zeta) \in \mathcal{H}$:

The inequality $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 \geqslant 1$ is fulfilled for all unimodular pairs $(\mu,\, \lambda) \in \mathcal{O}^2$ with $\frac{\lambda}{\mu} = \alpha_i + \gamma$, for some $i \in \{1, \ldots, n\}$ and some $\gamma \in \mathcal{O}$ $\big\}$.

Then $B(\alpha_1, \ldots, \alpha_n)$ is the set of all points in $\mathcal{H}$ lying above or on all hemispheres $S(\alpha_i + \gamma)$, $i = 1, \ldots, n$; for any $\gamma \in \mathcal{O}$.

The intersection $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ with the fundamental domain D_∞ for the translation group Γ_∞, is our candidate to equal the Bianchi fundamental polyhedron.

Convergence of the approximation. We will give a method to decide when $B(\alpha_1, \ldots, \alpha_n) = B$. This gives us an effective way to find B by adding more and more elements to the set $\{\alpha_1, \ldots, \alpha_n\}$ until we find $B(\alpha_1, \ldots, \alpha_n) = B$.

We consider the boundary $\partial B(\alpha_1, \ldots, \alpha_n)$ of $B(\alpha_1, \ldots, \alpha_n)$ in $\mathcal{H} \cup \mathbb{C}$. It consists of the points $(z, \zeta) \in \mathcal{H} \cup \mathbb{C}$ satisfying all the non-strict inequalities $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$ that we have used to define $B(\alpha_1, \ldots, \alpha_n)$, and satisfy the additional condition that at least one of these non-strict inequalities is an equality.

We will see below that $\partial B(\alpha_1, \ldots, \alpha_n)$ carries a natural cell structure. This, together with the following definitions, makes it possible to state the criterion which tells us when we have found all the hemispheres relevant for the Bianchi fundamental polyhedron.

Definition 9. *We shall say that the hemisphere $S_{\mu,\lambda}$ is* strictly below *the hemisphere $S_{\beta,\alpha}$ at a point $z \in \mathbb{C}$ if the following inequality is satisfied:*

$$\left|z - \frac{\alpha}{\beta}\right|^2 - \frac{1}{|\beta|^2} < \left|z - \frac{\lambda}{\mu}\right|^2 - \frac{1}{|\mu|^2}.$$

This is, of course, an abuse of language because there may not be any points on $S_{\beta,\alpha}$ or $S_{\mu,\lambda}$ with coordinate z. However, if there is a point (z, ζ) on $S_{\mu,\lambda}$, the right hand side of the inequality is just $-\zeta^2$. Thus the left hand side is negative and so of the form $-(\zeta')^2$. Clearly, $(z, \zeta') \in S_{\beta,\alpha}$ and $\zeta' > \zeta$.
We will further say that a point $(z, \zeta) \in \mathcal{H} \cup \mathbb{C}$ is *strictly below* a hemisphere $S_{\mu,\lambda}$, if there is a point $(z, \zeta') \in S_{\mu,\lambda}$ with $\zeta' > \zeta$.

1.1.3 Singular points

We call *cusps* the elements of the number field considered as points in the boundary of hyperbolic space, via the inclusion $K \subset \mathbb{C} \cup \{\infty\} \cong \partial\mathcal{H}$. We write $\infty = \frac{1}{0}$, which we also consider as a cusp. It is well-known that the set of cusps is closed under the action of $\mathrm{SL}_2(\mathcal{O})$ on $\partial\mathcal{H}$; and that we have the following bijective correspondence between the $\mathrm{SL}_2(\mathcal{O})$-orbits of cusps and the ideal classes in $\mathcal{O}$. A cusp $\frac{\lambda}{\mu}$ is in the $\mathrm{SL}_2(\mathcal{O})$-orbit of the cusp $\frac{\lambda'}{\mu'}$, if and only if the ideals (λ', μ') and (λ, μ) are in the same ideal class. It immediately follows that the orbit of the cusp $\infty = \frac{1}{0}$ corresponds to the principal ideals. Let us call *singular* the cusps $\frac{\lambda}{\mu}$ such that (λ, μ) is not principal. And let us call *singular points* the singular cusps which lie in ∂B. It follows from the characterisation of the singular points by Bianchi that they are precisely the points in $\mathbb{C} \subset \partial\mathcal{H}$ which cannot be strictly below any hemisphere. In the cases where $\mathcal{O}$ is a principal ideal domain, $K \cup \{\infty\}$ consists of only one $\mathrm{SL}_2(\mathcal{O})$-orbit, so there are no singular points. We use the following formulae derived by Swan, to compute representatives modulo the translations by Γ_∞, of the singular points.

Lemma 10 ([38])**.** *The singular points of K, $\bmod \mathcal{O}$, are given by $\frac{p(r+\sqrt{-m})}{s}$, where $p, r, s \in \mathbb{Z}$, $s > 0$, $\quad \frac{-s}{2} < r \leqslant \frac{s}{2}$, $\quad s^2 \leqslant r^2 + m$, and*

- *if* $m \equiv 1$ *or* $2 \mod 4$,
  $s \neq 1$, $s \mid r^2 + m$, *the numbers* p *and* s *are coprime, and* p *is taken* $\mod s$;
- *if* $m \equiv 3 \mod 4$,
  s *is even,* $s \neq 2$, $2s \mid r^2 + m$, *the numbers* p *and* $\frac{s}{2}$ *are coprime;* p *is taken* $\mod \frac{s}{2}$.

The singular points need not be considered in Swan's termination criterion, because they cannot be strictly below any hemisphere $S_{\mu,\lambda}$.

1.1.4 Swan's termination criterion

We observe that the set of $z \in \mathbb{C}$ over which some hemisphere is strictly below another is $\mathbb{C}$ or an open half-plane. In the latter case, the boundary of this is a line.

***Notation* 11.** Denote by $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ the set of $z \in \mathbb{C}$ over which neither $S_{\beta,\alpha}$ is strictly below $S_{\mu,\lambda}$ nor vice versa.

This line is computed by turning the inequality in definition 9 into an equation. Swan calls it the line over which the two hemispheres *agree*, and we will see later that the most important edges of the Bianchi fundamental polyhedron lie on the preimages of such lines.

We now restrict our attention to a set of hemispheres which is finite modulo the translations in Γ_∞.

Consider a set of hemispheres $S(\alpha_i + \gamma)$, where the index i runs from 1 through n, and γ runs through $\mathcal{O}$. We call this set of hemispheres a *collection*, if every non-singular point $z \in \mathbb{C} \subset \partial\mathcal{H}$ is strictly below some hemisphere in our set.

Now consider a set $B(\alpha_1, \ldots, \alpha_n)$ which is determined by such a collection of hemispheres.

Theorem 12 (Swan's termination criterion [38])**.** *We have* $B(\alpha_1, \ldots, \alpha_n) = B$ *if and only if no vertex of* $\partial B(\alpha_1, \ldots, \alpha_n)$ *can be strictly below any hemisphere* $S_{\mu,\lambda}$.

In other words, no vertex v of $\partial B(\alpha_1, \ldots, \alpha_n)$ can lie strictly below any hemisphere $S_{\mu,\lambda}$.
With this criterion, it suffices to compute the cell structure of $\partial B(\alpha_1, \ldots, \alpha_n)$ to see if our choice of hemispheres gives us the Bianchi fundamental polyhedron. This has only to be done modulo the translations of Γ_∞, which preserve the height and hence the situations of being strictly below. Thus our computations only need to be carried out on a finite set of hemispheres.

1.1.5 Computing the cell structure in the complex plane

We will in a first step compute the image of the cell structure under the homeomorphism from $\partial B(\alpha_1, \ldots, \alpha_n)$ to $\mathbb{C}$ given by the vertical projection. For each 2-cell of this structure, there is an associated hemisphere $S_{\mu,\lambda}$. The interior of this 2-cell consists of the points $z \in \mathbb{C}$ where all other hemispheres in our collection are strictly below $S_{\mu,\lambda}$. Swan shows that this is the interior of a convex polygon.
The edges of these polygons lie on real lines in $\mathbb{C}$ specified in notation 11.
A vertex is an intersection point z of any two of these lines involving the same hemisphere $S_{\mu,\lambda}$, if all other hemispheres in our collection are strictly below, or agree with, $S_{\mu,\lambda}$ at z.

Lifting the cell structure back to hyperbolic space

Now we can lift the cell structure back to $\partial B(\alpha_1, \ldots, \alpha_n)$, using the projection homeomorphism onto $\mathbb{C}$. The preimages of the convex polygons of the cell structure on $\mathbb{C}$, are totally geodesic hyperbolic polygons each lying on one of the hemispheres in our collection. These are the 2-cells of $\partial B(\alpha_1, \ldots, \alpha_n)$.
The edges of these hyperbolic polygons lie on the intersection arcs of pairs of hemispheres in our collection. As two Euclidean 2-spheres intersect, if they do so non-trivially, in a circle centred on the straight line which connects the two 2-sphere centres, such an intersection arc lies on a semicircle centred in the complex plane. The plane which contains this semicircle must be orthogonal to the connecting line, hence a vertical plane in $\mathcal{H}$. We can alternatively conclude the latter facts observing that an edge which two totally geodesic polygons have in common must be a geodesic segment.
Lifting the vertices becomes now obvious from their definition. This enables us to check Swan's termination criterion.

We will now sketch Swan's proof of this criterion. Let P be one of the convex polygons of the cell structure on $\mathbb{C}$. The preimage of P lies on one hemisphere $S(\alpha_i)$ of our collection. Now the condition stated in theorem 12 says that at the vertices of P, the hemisphere $S(\alpha_i)$ cannot be strictly below any other hemisphere. The points where $S(\alpha_i)$ can be strictly below some hemisphere constitute an open half-plane in $\mathbb{C}$, and hence cannot lie in the convex hull of the vertices of P, which is P. Theorem 12 now follows because $\mathbb{C}$ is tessellated by these convex polygons.

1.2 Realization of Swan's algorithm

From now on, we will work on putting Swan's concept into practice.

We can reduce the set of hemispheres on which we carry out our computations, with the help of the following notion.

Definition 13. *A hemisphere $S_{\mu,\lambda}$ is said to be* everywhere below *a hemisphere $S_{\beta,\alpha}$ when:*

$$\left|\frac{\lambda}{\mu} - \frac{\alpha}{\beta}\right| \leqslant \frac{1}{|\beta|} - \frac{1}{|\mu|}.$$

Note that this is also the case when $S_{\mu,\lambda} = S_{\beta,\alpha}$. Any hemisphere which is everywhere below another one, does not contribute to the Bianchi fundamental polyhedron, in the following sense.

Proposition 14. *Let $S(\alpha_n)$ be a hemisphere everywhere below some other hemisphere $S(\alpha_i)$, where $i \in \{1, \ldots, n-1\}$.*
Then $B(\alpha_1, \ldots, \alpha_n) = B(\alpha_1, \ldots, \alpha_{n-1})$.

Proof. Write $\alpha_n = \frac{\lambda}{\mu}$ and $\alpha_i = \frac{\theta}{\tau}$ with $\lambda, \mu, \theta, \tau \in \mathcal{O}$. We take any point (z, ζ) strictly below $S_{\mu,\lambda}$ and show that it is also strictly below $S_{\tau,\theta}$. In terms of notation 8, this problem looks as follows: we assume that the inequality $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 < 1$ is satisfied, and show that this implies the inequality $|\tau z - \theta|^2 + |\tau|^2\zeta^2 < 1$. The first inequality can be transformed into $\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2 < \frac{1}{|\mu|^2}$. Hence, $\sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2} < \frac{1}{|\mu|}$. We will insert this into the triangle inequality for the Euclidean distance in $\mathbb{C} \times \mathbb{R}$ applied to the three points (z, ζ), $(\frac{\lambda}{\mu}, 0)$ and $(\frac{\theta}{\tau}, 0)$, which is

$$\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2}.$$

So we obtain $\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \frac{1}{|\mu|}$. By definition 13, the expression on the right hand side is smaller than or equal to $\frac{1}{|\tau|}$. Therefore, we take the square and obtain $\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2 < \frac{1}{|\tau|^2}$, which is equivalent to the claimed inequality. □

Another notion that will be useful for our algorithm, is the following.

Definition 15. *Let $z \in \mathbb{C}$ be a point lying within the vertical projection of $S_{\mu,\lambda}$. Define the* lift on the hemisphere $S_{\mu,\lambda}$ *of z as the point on $S_{\mu,\lambda}$ the vertical projection of which is z.*

***Notation* 16.** Denote by the *hemisphere list* a list into which we will record a finite number of hemispheres $S(\alpha_1)$,...,$S(\alpha_n)$. Its purpose is to determine a set $B(\alpha_1, \ldots, \alpha_n)$ in order to approximate, and finally obtain, the Bianchi fundamental polyhedron.

The algorithm computing the Bianchi fundamental polyhedron

We now describe the algorithm that we have realized using Swan's description; it is decomposed into algorithms 1 through 3 below.

Initial step. We begin with the smallest value which the norm of a non-zero element $\mu \in \mathcal{O}$ can take, namely 1. Then μ is a unit in $\mathcal{O}$, and for any $\lambda \in \mathcal{O}$, the pair (μ, λ) is unimodular. And we can rewrite the fraction $\frac{\lambda}{\mu}$ such that $\mu = 1$. We obtain the unit hemispheres (of radius 1), centred at the imaginary quadratic integers $\lambda \in \mathcal{O}$. We record into the hemisphere list the ones which touch the Bianchi fundamental polyhedron, i.e. the ones the centre of which lies in the fundamental rectangle D_0 (of notation 4) for the action of Γ_∞ on the complex plane.

Step A. Increase $|\mu|$ to the next higher value which the norm takes on elements of $\mathcal{O}$. Run through all the finitely many μ which have this norm. For each of these μ, run through all the finitely many λ with $\frac{\lambda}{\mu}$ in the fundamental rectangle D_0. Check that $(\mu, \lambda) = \mathcal{O}$ and that the hemisphere $S_{\mu,\lambda}$ is not everywhere below a hemisphere $S_{\beta,\alpha}$ in the hemisphere list. If these two checks are passed, record (μ, λ) into the hemisphere list.

We repeat step **A** until $|\mu|$ has reached an expected value. Then we check if we have found all the hemispheres which touch the Bianchi fundamental polyhedron, as follows.

Step B. We compute the lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ of definition 11, over which two hemispheres agree, for all pairs $S_{\beta,\alpha}$, $S_{\mu,\lambda}$ in the hemisphere list which touch one another.
Then, for each hemisphere $S_{\beta,\alpha}$, we compute the intersection points of each two lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$ referring to $\frac{\alpha}{\beta}$.
We drop the intersection points at which $S_{\beta,\alpha}$ is strictly below some hemisphere in the list.
We erase the hemispheres from our list, for which less than three intersection points remain. We can do this because a hemisphere which touches the Bianchi fundamental polyhedron only in two vertices shares only an edge with it and no 2-cell.
Now, the vertices of $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ are the lifts of the remaining intersection points.
Thus we can check Swan's termination criterion (theorem 12), which we do as follows.
We pick the lowest value $\zeta > 0$ for which $(z, \zeta) \in \mathcal{H}$ is the lift inside Hyperbolic Space of a remaining intersection point z.
If $\zeta \geqslant \frac{1}{|\mu|}$, then all (infinitely many) remaining hemispheres have radius equal or smaller than ζ, so (z, ζ) cannot be strictly below them. So Swan's termination criterion is fulfilled, we have found the Bianchi fundamental polyhedron, and can proceed by determining its cell structure.
Else, ζ becomes the new expected value for $\frac{1}{|\mu|}$. We repeat step **A** until $|\mu|$ reaches $\frac{1}{\zeta}$ and then proceed again with step **B**.

Proposition 17. *The hemisphere list, as computed by algorithm* 1*, determines the Bianchi fundamental polyhedron. This algorithm terminates within finite time.*

Algorithm 1 Computation of the Bianchi fundamental polyhedron

Input: A square-free positive integer m.
Output: The hemisphere list, containing entries $S(\alpha_1)$,...,$S(\alpha_n)$ such that $B(\alpha_1, \ldots, \alpha_n) = B$.

Let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{-m})$.
Let $h_{\mathcal{O}}$ be the class number of $\mathcal{O}$. Compute $h_{\mathcal{O}}$.
Estimate the highest value for $|\mu|$ which will occur in notation 8 by
the formula $E := \begin{cases} \frac{5m}{2} h_{\mathcal{O}} - 2m + \frac{1}{2}, & m \equiv 3 \mod 4, \\ 21 m h_{\mathcal{O}} - 19m, & \text{else.} \end{cases}$
$\mathcal{N} := 1$.
Swan's_cancel_criterion_fulfilled := false.

while Swan's_cancel_criterion_fulfilled = false, **do**
  while $\mathcal{N} \leqslant E$ **do**
    Execute algorithm 2 with argument $\mathcal{N}$.
    Increase $\mathcal{N}$ to the next greater value in
    the set $\{\sqrt{n^2 m + j^2} \mid n, j \in \mathbb{N}\}$ of values of the norm on $\mathcal{O}$.
  end while
  Compute ζ with algorithm 3.
  if $\zeta \geqslant \frac{1}{\mathcal{N}}$, **then**
    `All (infinitely many) remaining hemispheres have radius smaller than` ζ`, so` (z, ζ) `cannot be strictly below any of them.`
    Swan's_cancel_criterion_fulfilled := true.
  else
    ζ `becomes the new expected lowest value for` $\frac{1}{\mathcal{N}}$`:`
    $E := \frac{1}{\zeta}$.
  end if
end while

Proof.

- The value ζ is the minimal height of the non-singular vertices of the cell complex $\partial B(\alpha_1, \ldots, \alpha_n)$ determined by the hemisphere list $\{S(\alpha_1), \ldots, S(\alpha_n)\}$. All the hemispheres which are not in the list, have radius smaller than $\frac{1}{\mathcal{N}}$. By remark 19, the inequality $\zeta \geqslant \frac{1}{\mathcal{N}}$ will become satisfied; and then no non-singular vertex of $\partial B(\alpha_1, \ldots, \alpha_n)$ can be strictly below any of them.
  Hence by theorem 12, $B(\alpha_1, \ldots, \alpha_n) = B$; and we obtain the Bianchi fundamental polyhedron as $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$.

- We now consider the run-time. By theorem 6, the set of hemispheres

  $$\{S_{\mu,\lambda} \mid S_{\mu,\lambda} \text{ touches the Bianchi Fundamental Polyhedron}\}$$

  is finite. So, there exists an $S_{\mu,\lambda}$ for which the norm of μ takes its maximum on this finite set. The variable $\mathcal{N}$ reaches this maximum for $|\mu|$ after a finite number of steps; and then Swan's termination criterion is fulfilled. The latter steps require a finite run-time because of propositions 20 and 21.

□

Swan explains furthermore how to obtain an a priori bound for the norm of the $\mu \in \mathcal{O}$ occurring for such hemispheres $S_{\mu,\lambda}$. But he states that this upper bound for $|\mu|$ is much too large. So instead of the theory behind theorem 6, we use Swan's termination criterion (theorem 12 above) to limit the number of steps in our computations. We then get the following.

***Observation* 18.** We can give bounds for $|\mu|$ in the cases where K is of class number 1 or 2 (there are nine cases of class number 1 and eighteen cases of class number 2, and we have done the computation for all of them). They are the following:

$$\begin{cases} K \text{ of class number 1:} & |\mu| \leqslant \frac{|\Delta|+1}{2}, \\ K \text{ of class number 2:} & \begin{cases} |\mu| \leqslant 3|\Delta|, & m \equiv 3 \mod 4, \\ |\mu| \leqslant (5 + \frac{61}{116})|\Delta|, & \text{else}, \end{cases} \end{cases}$$

where Δ is the discriminant of $K = \mathbb{Q}(\sqrt{-m})$,
i.e., $|\Delta| = \begin{cases} m, & m \equiv 3 \mod 4, \\ 4m, & \text{else}. \end{cases}$

***Remark* 19.** In algorithm 1, we have chosen the value E by an extrapolation formula for observation 18. If this is greater than the exact bound for $|\mu|$, the algorithm computes additional hemispheres which do not contribute to the Bianchi fundamental polyhedron. On the other hand, if E is smaller than the exact bound for $|\mu|$, it will be increased in the outer while loop of the algorithm, until it is sufficiently large. But then, the algorithm performs some preliminary computations of the intersection lines and vertices, which cost additional run-time. Thus our extrapolation formula is aimed at choosing E slightly greater than the exact bound for $|\mu|$ we expect.

Proposition 20. *Algorithm 2 finds all the hemispheres of radius $\frac{1}{N}$, on which a 2-cell of the Bianchi fundamental polyhedron can lie. This algorithm terminates within finite time.*

Proof.

- Directly from the definition of the hemispheres $S_{\mu,\lambda}$, it follows that the radius is given by $\frac{1}{|\mu|}$. So our algorithm runs through all μ in question. By construction of the Bianchi fundamental polyhedron D, the hemispheres on which a 2-cell of D lies must have their centre in the fundamental rectangle D_0. By proposition 14, such hemispheres cannot be everywhere below some other hemisphere in the list.
- Now we consider the run-time of the algorithm. There are finitely many $\mu \in \mathcal{O}$ the norm of which takes a given value. And for a given μ, there are finitely many $\lambda \in \mathcal{O}$ such that $\frac{\lambda}{\mu}$ is in the fundamental rectangle D_0. Therefore, this algorithm consists of finite loops and terminates within finite time.

□

Proposition 21. *Algorithm 3 finds the minimal height occurring amongst the non-singular vertices of $\partial B(\alpha_1, \ldots, \alpha_n)$. This algorithm erases only such hemispheres from the list, which do not change $\partial B(\alpha_1, \ldots, \alpha_n)$. It terminates within finite time.*

Proof.

- The heights of the points in $\mathcal{H}$ are preserved by the action of the translation group Γ_∞, so we only need to consider representatives in the fundamental domain D_∞ for this action. Our algorithm computes the entire cell structure of $\partial B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$, as described in subsection 1.1.5. The number of lines to intersect is smaller than the square of the length of the hemisphere list, and thus finite. As a consequence, the minimum of the height has to be taken only on a finite set of intersection points, whence the first claim.
- If a cell of $\partial B(\alpha_1, \ldots, \alpha_n)$ lies on a hemisphere, then its vertices are lifts of intersection points. So we can erase the hemispheres which are strictly below some other hemispheres at all the intersection points, without changing $\partial B(\alpha_1, \ldots, \alpha_n)$.

Algorithm 2 Recording the hemispheres of radius $\frac{1}{\mathcal{N}}$

Input: The value $\mathcal{N}$, and the hemisphere list (empty by default).
Output: The hemisphere list with some hemispheres of radius $\frac{1}{\mathcal{N}}$ added.

```
for a running from 0 through N within Z,  do
    for b in Z such that |a + bω| = N,  do
        Let μ := a + bω.
        for all the λ ∈ O with λ/μ in the fundamental rectangle D_0, do
            if  the pair (μ, λ) is unimodular, then
                Let L be the length of the hemisphere list.
                everywhere_below := false,        j := 1.
                while  everywhere_below = false and j ⩽ L, do
                    Let S_{β,α} be the j'th entry in the hemisphere list;
                    if  S_{μ,λ} is everywhere below S_{β,α}, then
                        everywhere_below := true.
                    end if
                    Increase j by 1.
                end while
                if  everywhere_below = false, then
                    Record S_{μ,λ} into the hemisphere list.
                end if
            end if
        end for
    end for
end for
```

We recall that the notion ``everywhere below'' has been made precise in definition 13; and that the fundamental rectangle D_0 has been specified in notation 4.

- Now we consider the run-time. This algorithm consists of loops running through the hemisphere list, which has finite length. Within one of these loops, there is a loop running through the set of pairs of lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$. A (far too large) bound for the cardinality of this set is given by the fourth power of the length of the hemisphere list.
  The steps performed within these loops are very delimited and easily seen to be of finite run-time.

□

Algorithm 3 Computing the minimal proper vertex height

Input: The hemisphere list $\{S(\alpha_1), \ldots, S(\alpha_n)\}$.
Output: The lowest height ζ of a non-singular vertex of $\partial B(\alpha_1, \ldots, \alpha_n)$. And the hemisphere list with some hemispheres removed which do not touch the Bianchi fundamental polyhedron.

for all pairs $S_{\beta,\alpha}$, $S_{\mu,\lambda}$ in the hemisphere list which touch one another, **do**
  compute the line $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ of notation 11.
end for

for each hemisphere $S_{\beta,\alpha}$ in the hemisphere list, **do**
  for each two lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$ referring to $\frac{\alpha}{\beta}$, **do**
    Compute the intersection point of $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$, if it exists.
  end for
end for
Drop the intersection points at which $S_{\beta,\alpha}$ is strictly below some hemisphere in the list.
Erase the hemispheres from our list, for which no intersection points remain.
`Now the vertices of` $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ `are the lifts (specified in definition 15) on the appropriate hemispheres of the remaining intersection points.`
Pick the lowest value $\zeta > 0$ for which $(z, \zeta) \in \mathcal{H}$ is the lift on some hemisphere of a remaining intersection point z.
Return ζ.

1.3 The cell complex and its orbit space

With the method described in subsection 1.1.5, we obtain a cell structure on the boundary of the Bianchi fundamental polyhedron. The cells in this structure which touch the cusp ∞ are easily determined: they are four 2-cells each lying on one of the Euclidean vertical planes bounding the fundamental domain D_∞ for Γ_∞ specified in notation 4; and four 1-cells each lying on one of the intersection lines of these planes. The other 2-cells in this structure lie each on one of the hemispheres determined with our realization of Swan's algorithm.
As the Bianchi fundamental polyhedron is a hyperbolic polyhedron up to some missing cusps, its boundary cells can be oriented as its facets. Once the cell structure is subdivided until the cells are fixed pointwise by their stabilisers, this cell structure with orientation is transported onto the whole hyperbolic space by the action of Γ.

1.3.1 Computing the vertex stabilisers and identifications

Let us state explicitly the Γ-action on the upper-half space model $\mathcal{H}$, in the form in which we will use it rather than in its historical form.

Lemma 22 (Poincaré)**.** *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$*, the action of* γ *on* $\mathcal{H}$ *is given by* $\gamma \cdot (z, \zeta) = (z', \zeta')$*, where*

$$\zeta' = \frac{|\det \gamma| \zeta}{|cz - d|^2 + \zeta^2 |c|^2},$$

$$z' = \frac{\left(\overline{d - cz}\right)(az - b) - \zeta^2 \bar{c} a}{|cz - d|^2 + \zeta^2 |c|^2}.$$

From this operation formula, we establish equations and inequalities on the entries of a matrix sending a given point (z, ζ) to another given point (z', ζ') in $\mathcal{H}$. We will use them in algorithm 4 to compute such matrices. For the computation of the vertex stabilisers, we have $(z, \zeta) = (z', \zeta')$ which simplifies the below equations and inequalities as well as the pertinent algorithm.

First, we fix a basis for $\mathcal{O}$ as the elements 1 and

$$\omega := \begin{cases} \sqrt{-m}, & m \equiv 1 \text{ or } 2 \mod 4, \\ -\frac{1}{2} + \frac{1}{2}\sqrt{-m}, & m \equiv 3 \mod 4. \end{cases}$$

As we have put $m \neq 1$ and $m \neq 3$, the only units in the ring $\mathcal{O}$ are ± 1. We will use the notations $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geqslant x\}$ and $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leqslant x\}$ for $x \in \mathbb{R}$.

Lemma 23. *Let* $m \equiv 3 \mod 4$*. Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ *be a matrix sending* (z, r) *to* $(\zeta, \rho) \in \mathcal{H}$*. Write* c *in the basis as* $j + k\omega$ *, where* $j, k \in \mathbb{Z}$*. Then* $|c|^2 \leqslant \frac{1}{r\rho}$*,* $|j| \leqslant \sqrt{\frac{1+\frac{1}{m}}{r\rho}}$ *and*

$$\frac{2j}{m+1} - 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m+1} \leqslant k \leqslant \frac{2j}{m+1} + 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m+1}.$$

Algorithm 4 Computation of the matrices identifying two points in $\mathcal{H}$.

Input: The points (z, r), (ζ, ρ) in the interior of $\mathcal{H}$, where z, $\zeta \in K$ and r^2, $\rho^2 \in \mathbb{Q}$.
Output: The set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_{-m})$, $m \equiv 3 \mod 4$, with nonzero entry c, sending the first of the input points to the second one.

c `will run through` $\mathcal{O}$ `with` $0 < |c|^2 \leqslant \frac{1}{r\rho}$.
Write c in the basis as $j + k\omega$, where $j, k \in \mathbb{Z}$.

for j running from $-\left\lceil \sqrt{\frac{1+\frac{1}{m}}{r\rho}} \right\rceil$ through $\left\lceil \sqrt{\frac{1+\frac{1}{m}}{r\rho}} \right\rceil$ **do**
  $k^{\pm}_{\text{limit}} := 2\frac{j}{m+1} \pm 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m+1}$.

  for k running from $\lfloor k^{-}_{\text{limit}} \rfloor$ through $\lceil k^{+}_{\text{limit}} \rceil$ **do**
    $c := j + k\omega$;
    if $|c|^2 \leqslant \frac{1}{r\rho}$ and c nonzero, **then**
      Write cz in the basis as $R(cz) + W(cz)\omega$ with $R(cz), W(cz) \in \mathbb{Q}$.
      d `will run through` $\mathcal{O}$ `with` $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$.
      Write d in the basis as $q + s\omega$, where $q, s \in \mathbb{Z}$.
      $s^{\pm}_{\text{limit}} := W(cz) \pm 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}}$.
      for s running from $\lfloor s^{-}_{\text{limit}} \rfloor$ through $\lceil s^{+}_{\text{limit}} \rceil$ **do**

        $\Delta := \frac{r}{\rho} - r^2|c|^2 - m\left(\frac{W(cz)}{2} - \frac{s}{2}\right)^2$;
        if Δ is a rational square, **then**
          $q_{\pm} := R(cz) - \frac{W(cz)}{2} + \frac{s}{2} \pm \sqrt{\Delta}$.
          Do the following for both $q_{\pm} = q_{+}$ and $q_{\pm} = q_{-}$ if $\Delta \neq 0$.
          if $q_{\pm} \in \mathbb{Z}$, **then**
            $d := q_{\pm} + s\omega$;
            if $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$ and (c, d) unimodular, **then**
              $a := \frac{\rho}{r}\overline{d} - \frac{\rho}{r}\overline{cz} - c\zeta$.
              if a is in the ring of integers, **then**
                b `is determined by the determinant` 1:
                $b := \frac{ad-1}{c}$.
                if b is in the ring of integers, **then**
                Check that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = (\zeta, \rho)$.
                Return $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
                end if
              end if
            end if
          end if
        end if
      end for
    end if
  end for
end for

Proof. From the operation equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = (\zeta, \rho)$, we deduce $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$ and conclude $r^2|c|^2 \leqslant \frac{r}{\rho}$, whence the first inequality. We insert $|c|^2 = \left(j - \frac{k}{2}\right)^2 + m\left(\frac{k}{2}\right)^2$ $= j^2 + \frac{m+1}{4}k^2 - jk$ into it, and obtain

$$0 \geqslant k^2 - \frac{4j}{m+1}k + \frac{4}{m+1}\left(j^2 - \frac{1}{r\rho}\right) =: f(k).$$

We observe that $f(k)$ is a quadratic function in $k \in \mathbb{Z} \subset \mathbb{R}$, taking its values exclusively in $\mathbb{R}$. Hence its graph has the shape of a parabola, and the negative values of $f(k)$ appear exactly on the interval where k is between its two zeroes,

$$k_\pm = \tfrac{2j}{m+1} \pm 2\tfrac{\sqrt{\Delta}}{m+1}, \qquad \text{where } \Delta = \tfrac{m+1}{r\rho} - j^2 m.$$

This implies the third and fourth claimed inequalities. As k is a real number, Δ must be non-negative in order that $f(k)$ be non-positive. Hence $j^2 \leqslant \frac{1+\frac{1}{m}}{r\rho}$, which gives the second claimed inequality. □

Lemma 24. *Under the assumptions of lemma 23, write d in the basis as $q + s\omega$, where $q, s \in \mathbb{Z}$. Write cz in the basis as $R(cz) + W(cz)\omega$, where $R(cz), W(cz) \in \mathbb{Q}$. Then $W(cz) - 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}} \leqslant s \leqslant W(cz) + 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}}$, and*

$$q = R(cz) - \frac{W(cz)}{2} + \frac{s}{2} \pm \sqrt{\frac{r}{\rho} - r^2|c|^2 - m\left(\frac{W(cz)}{2} - \frac{s}{2}\right)^2}.$$

Proof. Recall that $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-m}$, so $\overline{q + s\omega} = q - \frac{s}{2} - \frac{s}{2}\sqrt{-m}$. The operation equation yields $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$. From this, we derive

$$\begin{aligned} \tfrac{r}{\rho} - r^2|c|^2 &= (cz - (q + s\omega))\left(\overline{cz} - (q - \tfrac{s}{2} - \tfrac{s}{2}\sqrt{-m})\right) \\ &= \left(\mathrm{Re}(cz) - q + \tfrac{s}{2}\right)^2 + \left(\mathrm{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2 \\ &= \mathrm{Re}(cz)^2 + q^2 - qs + \tfrac{s^2}{4} - 2\mathrm{Re}(cz)q + \mathrm{Re}(cz)s + \left(\mathrm{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2. \end{aligned}$$

We solve for q,

$$q^2 + (-2\mathrm{Re}(cz) - s)\,q + \left(\mathrm{Re}(cz) + \frac{s}{2}\right)^2 + \left(\mathrm{Im}(cz) - \frac{s}{2}\sqrt{m}\right)^2 - \frac{r}{\rho} + r^2|c|^2 = 0$$

and find

$$q_\pm = \mathrm{Re}(cz) + \tfrac{s}{2} \pm \sqrt{\Delta}, \qquad \text{where } \Delta = \tfrac{r}{\rho} - r^2|c|^2 - \left(\mathrm{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2.$$

We express this as

$$q_\pm = R(cz) - \tfrac{W(cz)}{2} + \tfrac{s}{2} \pm \sqrt{\Delta}, \qquad \text{where } \Delta = \tfrac{r}{\rho} - r^2|c|^2 - m\left(\tfrac{W(cz)}{2} - \tfrac{s}{2}\right)^2,$$

which is the claimed equation. The condition that q must be a rational integer implies $\Delta \geqslant 0$, which can be rewritten in the claimed inequalities. □

We further state a simple inequality in order to prove that algorithm 4 terminates in finite time.

Lemma 25. *Let $K = \mathbb{Q}(\sqrt{-m})$ with $m \neq 3$. Let $c, z \in K$. Write their product cz in the $\mathbb{Q}$-basis $\{1, \omega\}$ for K as $R(cz) + W(cz)\omega$.*
Then $|W(cz)| \leqslant |c| \cdot |z|$.

Proof. Let $x + y\omega \in K$ with $x, y \in \mathbb{Q}$. Our first step is to show that $|y| \leqslant |x + y\omega|$. Consider the case $m \equiv 1$ or $2 \mod 4$. Then

$$|x + y\omega| = \sqrt{x^2 + my^2} \geqslant \sqrt{m}|y| \geqslant |y|,$$

and we have shown our claim. Else consider the case $m \equiv 3 \mod 4$. Then,

$$|x + y\omega| = \sqrt{(x + \omega y)(x + \overline{\omega} y)} = \sqrt{\left(x^2 - 2x\frac{y}{2} + \frac{y^2}{4}\right) + \frac{m}{4}y^2} \geqslant \frac{\sqrt{m}}{2}|y|,$$

and our claim follows for $m > 3$. Now we have shown that $|W(cz)| \leqslant |cz|$; and we use some embedding of K into $\mathbb{C}$ to verify the equation $|cz| = |c| \cdot |z|$. □

Proposition 26. *Let $m \equiv 3 \mod 4$. Then algorithm 4 gives all the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ with $c \neq 0$, sending (z, r) to $(\zeta, \rho) \in \mathcal{H}$. It terminates in finite time.*

Proof.
- The first claim is easily established using the bounds and formulae stated in lemmata 23 and 24.
- Now we consider the run-time. This algorithm consists of three loops the limits of which are at most linear expressions in $\frac{1}{\sqrt{r\rho}}$. For $s^{\pm}_{\text{limit}}$, we use lemma 25 and $r^2|c|^2 \leqslant \frac{r}{\rho}$ to see this (we get a factor $|z|$ here, which we can neglect).

□

1.3.2 Pointwise stabilisation

We subdivide our cell structure in order to make the stabilisers *fix the cells pointwise.* By the latter property, we mean that if an element $\gamma \in \Gamma$ fixes a cell σ, then it fixes all the points $x \in \sigma$. We call the resulting object the *refined cellular complex.* We perform a check if all the cells in the cellular complex are fixed pointwise by their stabiliser. This is obviously true for vertices.

Cell subdivision

There is an obstacle to having pointwise-fixed 2-cells. For instance, consider a unit hemisphere $S_{1,\lambda}$. The matrix $\begin{pmatrix} -\lambda & -\lambda^2 - 1 \\ 1 & \lambda \end{pmatrix}$ of order 2 in $\mathrm{PSL}_2(\mathcal{O})$ sends ∞ to the hemisphere centre $\lambda \in \mathcal{O} \subset \partial\mathcal{H}$ and vice versa. It sends the unit hemisphere $S_{1,\lambda}$ onto itself, rotating around the geodesic arc the projection of which onto $\mathbb{C}$ is parallel to the

imaginary axis and passes through the hemisphere centre $\frac{\lambda}{1}$. We have a 2-cell in our cellular complex lying on $S_{1,\lambda}$, with this rotation axis passing through its interior. It is sent onto itself by our matrix, reverting its orientation.
More generally, let (μ, λ) be any unimodular pair defining one of the hemispheres $S_{\mu,\lambda}$ on which a 2-cell of the Bianchi fundamental polyhedron lies. Flöge [21] has given a method to find a matrix in Γ sending this 2-cell onto some 2-cell again in the Bianchi fundamental polyhedron, reverting its orientation if it is sent onto itself. As $(\mu, \lambda) = \mathcal{O}$, there exist u and v in $\mathcal{O}$ such that $\begin{pmatrix} u & v \\ \mu & \lambda \end{pmatrix} \in \Gamma$. For any point $(z, \zeta) \in S_{\mu,\lambda}$, we have $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1$ by notation 2; and hence by the operation formula in lemma 22, our matrix preserves the height ζ when applied to (z, ζ). Then, Flöge shows that if $(z, \zeta) \in \partial B$, its image by our matrix is again in ∂B. Composing with a translation from Γ_∞, we obtain an element of Γ which sends our 2-cell into the boundary of the Bianchi fundamental polyhedron.

The following cell subdivision has been introduced by Flöge. In order to obtain pointwise-stabilised 2-cells, we subdivide our 2-cells along the possible symmetry axes of the described matrices. Into the 2-cell lying on $S_{\mu,\lambda}$, we insert an edge the projection of which onto $\mathbb{C}$ is parallel to the imaginary axis and passes through the hemisphere centre $\frac{\lambda}{\mu}$. If this edge lies in the interior of our 2-cell, we replace our 2-cell by two 2-cells such that their intersection is this edge, and their union is the former 2-cell.

Checking the pointwise stabilisation of the edges

We want to check if some edge can be sent onto itself reverting its orientation. This is only possible when origin and end of the edge are identified by some element of Γ. As mentioned above, we have computed all such equivalences of vertices. Thus we can check if this applies to some edge. In this case, we produce an error message calling for a cellular subdivision of the edge.

Checking the pointwise stabilisation of the 2-cells

We will now make use of the fact that real hyperbolic space is *non-positively curved*, which we call the CAT(0) property (after Mikhail Gromov's acronym for Élie Cartan, Aleksandr D. Aleksandrov and Victor A. Toponogov). For the definition of this property, as well as for the claim that it applies to hyperbolic space, we refer to [10]. This allows us to formulate the following statement with quite mild hypotheses.

Lemma 27. *Let σ be a polygon, and G be a group of isometries of an ambient* CAT(0) *space. Suppose there are at least three vertices of σ which are the unique representative amongst the vertices of σ, of their respective G-orbit. Then the stabiliser of σ in G must fix σ pointwise.*

Proof. Consider an element g of the stabiliser of σ. The isometry g must preserve the set of the vertices of σ up to a permutation. Furthermore, a vertex which is not G-equivalent to any other in this set, must be fixed by g. Under the hypothesis of our lemma, g must

hence fix three vertices of σ. As g is a CAT(0) isometry, it must fix pointwise the whole triangle with these three vertices as corners. This triangle is contained in the polygon σ and determines the isometric automorphisms of σ. Thus g must fix σ pointwise. $\square$

Hence the check on our cell structure consists of making sure that each 2-cell has at least three vertices which are unique as representative of their respective Γ-orbit, amongst the vertices of the 2-cell. Again, we can do this because we already have computed the Γ-equivalences of vertices.

The guarantee that all cells are fixed pointwise, allows us to obtain the stabilisers of the higher dimensional cells simply by intersection of their vertex stabilisers. Even more, in order to check the equivalence of two cells σ and σ', we only need to intersect the sets of elements of Γ which identify the vertices of σ with the ones of σ'.

1.3.3 Isomorphy types of stabilisers

Throughout this thesis, we will use the "number theorist's notation" $\mathbb{Z}/n$ for the cyclic group of order n.

Lemma 28 (Schwermer/Vogtmann [37])**.** *The finite subgroups in* $\mathrm{PSL}_2(\mathcal{O})$ *are exclusively of isomorphism types the cyclic groups of orders two and three, the trivial group, the Klein four-group* $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$*, the symmetric group* $\mathcal{S}_3$ *and the alternating group* $\mathcal{A}_4$*. The homology with trivial* $\mathbb{Z}$*-coefficients, of these groups is*

$$\mathrm{H}_q(\mathbb{Z}/n;\,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/n, & q \text{ odd}, \\ 0, & q \text{ even},\ q > 0; \end{cases}$$

$$\mathrm{H}_q(\mathcal{D}_2;\,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0, \\ (\mathbb{Z}/2)^{\frac{q+3}{2}}, & q \text{ odd}, \\ (\mathbb{Z}/2)^{\frac{q}{2}}, & q \text{ even},\ q > 0; \end{cases}$$

$$\mathrm{H}_q(\mathcal{S}_3;\,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/2, & q \equiv 1 \mod 4, \\ 0, & q \equiv 2 \mod 4, \\ \mathbb{Z}/6, & q \equiv 3 \mod 4, \\ 0, & q \equiv 0 \mod 4,\ q > 0; \end{cases}$$

$$\mathrm{H}_q(\mathcal{A}_4;\,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q = 0, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/3, & q = 6k+1, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2, & q = 6k+2, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/6, & q = 6k+3, \\ (\mathbb{Z}/2)^k, & q = 6k+4, \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6, & q = 6k+5, \\ (\mathbb{Z}/2)^{k+1}, & q = 6k+6. \end{cases}$$

The stabilisers of the points inside $\mathcal{H}$ are finite and hence of the above-listed types. Using the Universal Coefficient Theorem, we obtain the following corollary.

Corollary 29. *Consider group homology with trivial $\mathbb{Z}/n$-coefficients.*

- $\mathrm{H}_q(\mathbb{Z}/n;\,\mathbb{Z}/n) \cong \mathbb{Z}/n$ *for all* $q \in \mathbb{N} \cup \{0\}$*;*
- $\mathrm{H}_q(\mathcal{D}_2;\,\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{q+1}$*;*
- $\mathrm{H}_q(\mathcal{S}_3;\,\mathbb{Z}/2) \cong \mathbb{Z}/2$ *for all* $q \in \mathbb{N} \cup \{0\}$*;*
- $\mathrm{H}_q(\mathcal{A}_4;\,\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & q = 0, \\ (\mathbb{Z}/2)^{2k}, & q = 6k+1, \\ (\mathbb{Z}/2)^{2k+1}, & q = 6k+2, \\ (\mathbb{Z}/2)^{2k+2}, & q = 6k+3, \\ (\mathbb{Z}/2)^{2k+1}, & q = 6k+4, \\ (\mathbb{Z}/2)^{2k+2}, & q = 6k+5, \\ (\mathbb{Z}/2)^{2k+3}, & q = 6k+6. \end{cases}$

Lemma 30. *The stabilisers in* $\mathrm{PSL}_2(\mathcal{O})$ *of pointwise-fixed*

- *edges in* $\mathcal{H}$*, are trivial or of type* $\mathbb{Z}/2$ *or* $\mathbb{Z}/3$*;*
- *2-cells and 3-cells in* $\mathcal{H}$*, are trivial.*

Proof. As $\mathrm{SL}_2(\mathbb{C})$ acts as orientation-preserving isometries on hyperbolic three-space, the stabiliser of a pointwise-fixed edge can only perform a rotation, with this edge lying on the rotation axis. This is possible because the edges in $\mathcal{H}$ are geodesic segments. The group of rotations around one given axis must be abelian; and it is easy to see that it cannot be of Klein four-group type. Thus amongst the above-listed types of subgroups of $\mathrm{PSL}_2(\mathcal{O})$, which fix points in $\mathcal{H}$, the only non-trivial ones which can fix edges pointwise, are $\mathbb{Z}/2$ and $\mathbb{Z}/3$.
In a pointwise-fixed 2-cell or 3-cell, we can choose two non-aligned pointwise-fixed edges, a rotation around one of which only fixes the other edge pointwise if it is the trivial rotation. □

Chapter 2

Torsion in the Bianchi groups

2.1 Geometric rigidity

We will now study closely, and from a geometrical viewpoint, the action of our cell stabilisers. This will allow us to reduce the efforts needed in our homological computations. Furthermore, this study will help us understand the results stated in chapter 3. The statements in this section are aimed to prove theorem 39, which will be of practical use in section 2.2.

Let us begin with Felix Klein's classification of the elements in $\mathrm{SL}_2(\mathbb{C})$, which passes to $\mathrm{PSL}_2(\mathbb{C})$.

Definition 31. *An element $\gamma \in \mathrm{SL}_2(\mathbb{C})$, $\gamma \neq \pm 1$, is called* loxodromic *if its trace is not a real number. Else it is called* $\begin{cases} \text{parabolic}, & |\mathrm{tr}(\gamma)| = 2, \\ \text{hyperbolic}, & |\mathrm{tr}(\gamma)| > 2 \\ \text{elliptic}, & |\mathrm{tr}(\gamma)| < 2 \end{cases}$.

We find the geometric meaning of this classification in the following proposition, which is mostly known from Felix Klein's lectures.

Proposition 32 ([17])**.** *Let γ be a non-trivial element of $\mathrm{PSL}_2(\mathbb{C})$. Then the following holds:*

- *γ is parabolic if and only if γ has exactly one fixed point in $\partial\mathcal{H}$.*
- *γ is elliptic if and only if it has two fixed points in $\partial\mathcal{H}$ and if the points on the geodesic line in $\mathcal{H}$ joining these two points are also left fixed. The action of γ is then a rotation around this line.*
- *γ is hyperbolic if and only if it has two fixed points in $\partial\mathcal{H}$ and if any circle in $\partial\mathcal{H}$ through these points together with its interior is left invariant. The line in $\mathcal{H}$ joining these two fixed points is then left invariant, but γ has no fixed point in $\mathcal{H}$.*

- *γ is loxodromic in all other cases. The action of γ has then two fixed points in $\partial\mathcal{H}$ and no fixed point in $\mathcal{H}$. The geodesic joining the two fixed points is the only geodesic in $\mathcal{H}$ which is left invariant.*

For instance in [32], it is stated that the parabolic elements do not have a fixed point in the interior of $\mathcal{H}$. So by excluding the parabolic, hyperbolic and loxodromic cases, we obtain the following corollary.

Corollary 33. *Let γ be a non-trivial element of $\mathrm{PSL}_2(\mathbb{C})$, admitting a fixed point $v \in \mathcal{H}$. Then γ fixes pointwise a geodesic line through v, and performs a rotation around this line.*

We now introduce a type of objects which is crucial for the geometry of the stabilisers' action. Its meaning is illustrated by the following four statements.

Definition 34. *We will call a geodesic line passing through the point $v \in \mathcal{H}$ a Γ_v-axis, if there exists a non-trivial element of Γ fixing this line pointwise.*

Theorem 35. *For any vertex $v \in \mathcal{H}$, there is a bijection between the Γ_v-axes and the non-trivial cyclic subgroups of the stabiliser Γ_v. It is given by associating to a Γ_v-axis the subgroup in Γ of rotations around this axis.*

Proof. First we show that any Γ_v-axis is attributed to some non-trivial cyclic subgroup of Γ_v.
Let l be a Γ_v-axis, and let $\dot{\Gamma}_l$ be the subgroup in Γ fixing l pointwise. By the definition of the Γ_v-axes, this subgroup is non-trivial. It is a subgroup of Γ_v, because $\dot{\Gamma}_l$ fixes l pointwise and thus fixes v. And it is cyclic because by corollary 33, $\dot{\Gamma}_l$ consists only of rotations around l.
Now we show that any non-trivial cyclic subgroup of Γ_v is attributed to some Γ_v-axis. Let γ be the generator of a non-trivial cyclic subgroup of Γ_v. By corollary 33, there is a geodesic line containing v, around which γ performs a rotation. □

Corollary 36. *For any vertex $v \in \mathcal{H}$, the action of the stabiliser Γ_v on the set of Γ_v-axes, induced by the action of Γ on $\mathcal{H}$, is given by conjugation of its non-trivial cyclic subgroups.*

Proof. Let l be a Γ_v-axis, and $\gamma \in \Gamma_v$. Let $\dot{\Gamma}_l$ be the subgroup of Γ fixing l pointwise. Then $\gamma \cdot l$ is again a Γ_v-axis; and the subgroup of Γ fixing $\gamma \cdot l$ pointwise is $\gamma\dot{\Gamma}_l\gamma^{-1}$. Hence by theorem 35, we can transfer the action to Γ_v-conjugation of the nontrivial cyclic subgroups. □

Lemma 37. *Each Γ_v-axis contains two edges of our refined cell complex that are adjacent to v.*

Proof. Let l be a Γ_v-axis for which this is not the case. Then l passes through the interior of a 2- or 3-cell σ adjacent to v. Let γ be a non-trivial element of Γ fixing l. As the Γ-action preserves our cell structure, γ must send σ to another cell of its dimension.

Since γ fixes the points in the non-empty intersection of l with the interior of σ, and since the interior of σ intersects trivially with the interior of any other cell, γ must fix σ. Hence γ is in the stabiliser of σ, and trivial by lemma 30. But γ has been chosen non-trivial, contradiction. □

Lemma 38. *Let e be an edge fixed pointwise by a non-trivial element $\gamma \in \Gamma$. Let v be a vertex adjacent to e. Then e lies on a Γ_v-axis.*

Proof. By corollary 33, γ must perform a rotation around an axis passing through v and all the points of e. This is the Γ_v-axis containing e. □

The above study allows us to prove the announced geometric rigidity theorem, which we will use in sections 2.2 and 2.3.

Theorem 39. *Let v be a non-singular vertex in our refined cell complex. Then the number n of orbits of edges in our refined cell complex adjacent to v, with stabiliser in Γ isomorphic to $\mathbb{Z}/\ell$, is given as follows for $\ell = 2$ and $\ell = 3$.*

Isomorphy type of Γ_v	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
n for $\ell = 2$	0	2	0	3	2	1
n for $\ell = 3$	0	0	2	0	1	2.

Proof. Due to lemma 38, any edge with the requested properties must lie on some Γ_v-axis. Thus the cases where $n = 0$ follow directly from theorem 35. It remains to distinguish the following cases.

- Let $\ell = 2$ and $\Gamma_v \cong \mathcal{A}_4$.
  There is only one conjugacy class of order-2-elements in $\mathcal{A}_4$, so by corollary 36, there is just one Γ_v-orbit of Γ_v-axes such that an element of order 2 fixes this axis pointwise. We will see that such an axis is subject to a rotation of angle π around an equivalent Γ_v-axis:
  We pick two order-2-elements in Γ_v and denote them by α and β. Denote by l the axis stabilised pointwise by β. The order-2-elements in $\mathcal{A}_4$ commute, so α sends l to itself. But this cannot be carried out by the identity map, because in that case α would fix two axes and thus also the hyperbolic plane spanned by them, in contradiction to corollary 33.
  Hence, l is non-trivially rotated onto itself by α, with a rotation axis passing through v. As α preserves our cell structure, it must permute the two edges adjacent to v, lying on l due to lemma 37.
  So, just one $\mathbb{Z}/2$-stabilised edge adjacent to v is in the quotient by Γ_v.

- Let $\ell = 2$ and $\Gamma_v \cong \mathcal{D}_2$.
  There are exactly three conjugacy classes of order-2-elements in $\mathcal{D}_2$, so by corollary 36, there are three Γ_v-orbits of Γ_v-axes. We will see that such an axis is

subject to rotations of angle π around the other two Γ_v-axes:
We pick two order-2-elements in Γ_v and denote them by α and β. Denote by l the axis stabilised pointwise by β. All elements in $\mathcal{D}_2$ commute, so α sends l to itself. But this cannot be carried out by the identity map, because in that case α would fix two axes and thus also the hyperbolic plane spanned by them, in contradiction to corollary 33.
Hence, l is non-trivially rotated onto itself by α, with a rotation axis passing through v. As α preserves our cell structure, it must permute the two edges adjacent to v, lying on l due to lemma 37. So, there are exactly three representatives of non-trivially stabilised edges adjacent to v, one on each Γ_v-axis. Their stabilisers are the order-2-subgroups of Γ_v, as we see from theorem 35.

- Let $\ell = 2$ and $\Gamma_v \cong \mathcal{S}_3$.
  There is only one conjugacy class of order-2-elements in $\mathcal{S}_3$, so by corollary 36, there is just one Γ_v-orbit of Γ_v-axes such that an element of order 2 fixes this axis pointwise. Denote its representative by l. Denote by γ the generator of the stabiliser of l. As γ is of order 2, the only elements in $\Gamma_v \cong \mathcal{S}_3$ which commute with γ, are γ itself and 1. So by corollary 36, the only elements of Γ_v sending l to itself are γ and 1. Denote by (a, v) and (v, b), the two edges which lie on l by lemma 37. Let $\delta \in \Gamma$ be an isometry identifying these two edges. As γ and 1 both fix l pointwise, by the last conclusion δ cannot be an element of Γ_v. Thus δ must send a to v and v to b. Therefore all three vertices a, b and v are Γ-equivalent, and in particular origin and end of such an edge are identified, which we have excluded for our refined cell complex in the first check of section 1.3.2. Hence δ does not exist, and the two edges lie on different orbits.

- Let Γ_v be a non-trivial cyclic group.
  As we see from corollary 33, any vertex with stabiliser a cyclic group lies on a single rotation axis, around which its stabiliser performs rotations. Hence by lemma 37, there are two edges in our cell complex adjacent to v which have the same stabiliser as v; and any other edge adjacent to v has the trivial stabiliser. It remains to see that these two edges are not identified when passing to the quotient space.
  Let $\gamma \in \Gamma$ be an isometry identifying these two edges. Denote these edges by (a, v) and (v, b). As the only elements of Γ fixing v, fix the whole rotation axis, γ cannot send a to b and fix v. Thus γ must send a to v and v to b. Therefore all three vertices a, b and v are Γ-equivalent, and in particular origin and end of such an edge are identified, which we have excluded for our refined cell complex in the first check of section 1.3.2.

- Let $\ell = 3$ and $\Gamma_v \cong \mathcal{S}_3$.
  There is only one cyclic subgroup of order 3 in $\mathcal{S}_3$. So by theorem 35, there is just one such Γ_v-axis. Denote this axis by l. Then by corollary 36, l is invariant under the action of the elements of order 2 in Γ_v, because the unique order three subgroup must be preserved under conjugation. But then l must be rotated by the angle π onto itself by the order two elements, which else would fix l and their own

axis, hence a hyperbolic plane. So the two edges which lie on l by lemma 37 must be equivalent.

- Let $\ell = 3$ and $\Gamma_v \cong \mathcal{A}_4$.
  There are four cyclic subgroups of order 3 in $\mathcal{A}_4$. They are all conjugate, so just one Γ_v-axis is in the quotient by Γ_v. Denote its representative by l. Denote by γ the generator of the stabiliser of l. Conjugation with elements of order 2 in $\mathcal{A}_4$ permutes the four order-3-subgroups. The only elements in $\Gamma_v \cong \mathcal{A}_4$, the conjugation with which fixes the cyclic subgroup generated by γ, are γ itself, γ^2 and 1. So by corollary 36, the only elements of Γ_v sending l to itself are γ, γ^2 and 1. Denote by (a, v) and (v, b), the two edges which lie on l by lemma 37. Let $\delta \in \Gamma$ be an isometry identifying these two edges. As γ, γ^2 and 1 all fix l pointwise, by the last conclusion δ cannot be an element of Γ_v. Thus δ must send a to v and v to b. Therefore all three vertices a, b and v are Γ-equivalent, and in particular origin and end of such an edge are identified, which we have excluded for our refined cell complex in the first check of section 1.3.2. Hence δ does not exist, and the two edges lie on different orbits.

□

The proofs of the following two corollaries are included in the above proof.

Corollary 40. *Consider the case $\Gamma_v \cong \mathcal{D}_2$ of theorem 39. Then the three stabilisers of edge representatives adjacent to v which are not trivial, are precisely the three order-2-subgroups of Γ_v.*

Corollary 41. *Consider the cases of theorem 39 where Γ_v is a non-trivial cyclic group. Then the two edges adjacent to v, which have a non-trivial stabiliser, must have the same stabiliser as v.*

2.2 The equivariant spectral sequence to group homology

Let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$. In order to obtain a compact quotient space, we retract our refined cell complex Γ-equivariantly in the way described by Flöge [20]. On the Bianchi fundamental polyhedron, this retraction is given by the vertical projection onto the surface ∂B described in subsection 1.1.2 on page 20. So in the principal ideal domain cases, the Bianchi fundamental polyhedron is contracted onto its cells which do not touch the boundary of $\mathcal{H}$, and by means of the Γ-action, we extend this retraction to the whole space $\mathcal{H}$. The equivariantly retracted cellular complex allows us to compute the group homology of Γ with the method of [37]. In the cases where the class group of $\mathcal{O}_{-m}$ is non-trivial, an obstacle to the retraction on ∂B appears: the intersection of ∂B with the boundary of $\mathcal{H}$ is non-empty and consists of the singular points. Flöge then passes to an extended space which consists of the union of $\mathcal{H}$ with the Γ-orbits of the singular points. This space can be retracted in the intended way. We call the Γ-invariant retraction of this space the *Flöge cellular complex* and denote it by X. In order to have no torsion in

Γ_∞ (see the discussion preceding notation 4), let us assume that $m \neq 1$ and $m \neq 3$ for the remainder of this chapter.

The spectral sequence. We will use the Flöge cellular complex X to compute the group homology of Γ with trivial $\mathbb{Z}$–coefficients, as defined in [11]. We proceed following [11, VII] and [37]. Let us consider the homology $\mathrm{H}_*(\Gamma; C_\bullet(X))$ of Γ with coefficients in the cellular chain complex $C_\bullet(X)$ associated to X; and call it the Γ-equivariant homology of X. As X is contractible, the map $X \to pt.$ to the point $pt.$ induces an isomorphism

$$\mathrm{H}_*(\Gamma;\, C_\bullet(X)) \to \mathrm{H}_*(\Gamma;\, C_\bullet(pt.)) \cong \mathrm{H}_*(\Gamma;\, \mathbb{Z}).$$

Denote by X^p the set of p-cells of X, and make use of that the stabiliser Γ_σ in Γ of any p-cell σ of X fixes σ pointwise. Then from

$$C_p(X) = \bigoplus_{\sigma \in X^p} \mathbb{Z} \cong \bigoplus_{\sigma \in \Gamma \backslash X^p} \mathrm{Ind}_{\Gamma_\sigma}^{\Gamma} \mathbb{Z},$$

Shapiro's lemma yields

$$\mathrm{H}_q(\Gamma;\, C_p(X)) \cong \bigoplus_{\sigma \in \Gamma \backslash X^p} \mathrm{H}_q(\Gamma_\sigma;\, \mathbb{Z});$$

and the equivariant Leray/Serre spectral sequence takes the form

$$E^1_{p,q} = \bigoplus_{\sigma \in \Gamma \backslash X^p} \mathrm{H}_q(\Gamma_\sigma;\, \mathbb{Z}) \implies \mathrm{H}_{p+q}(\Gamma;\, C_\bullet(X)),$$

converging to the Γ-equivariant homology of X, which is, as we have already seen, isomorphic to $\mathrm{H}_{p+q}(\Gamma;\, \mathbb{Z})$ with the trivial action on the coefficients $\mathbb{Z}$.

We shall also make extensive use of the description given in [37], of the d^1-differential in this spectral sequence. The technical difference to the cases of trivial class group, treated by [37], is that the stabilisers of the singular points are free abelian groups of rank two. In particular, the Γ-action on our complex $X^\bullet$ is not a *proper action* (in the sense that all stabilisers are finite). As a consequence, the resulting spectral sequence does not degenerate on the E^2-level as it does in Schwermer and Vogtmann's cases. It is explained in [30] how to handle the non-trivial d^2-differentials in this spectral sequence.

Let us now describe how to compute explicitly the d^1-differentials, making use of the knowledge from lemma 28 about the isomorphy types of the stabilisers, and lemma 45 about their inclusions. The bottom row of the E^1-term, more precisely the chain complex given by the $E^1_{p,0}$-modules and the $d^1_{p,0}$-maps, is equivalent to the $\mathbb{Z}$-chain complex giving the homology of the quotient space of our cell complex by the Γ-action.

From lemma 30, we see that for $q > 0$, the $E^1_{p,q}$-terms are concentrated in the two columns $p = 0$ and $p = 1$. So for $q > 0$, we only need to compute the differentials

$$\bigoplus_{\sigma \in \Gamma \backslash X^0} \mathrm{H}_q(\Gamma_\sigma; \mathbb{Z}) \xleftarrow{d^1_{1,q}} \bigoplus_{\sigma \in \Gamma \backslash X^1} \mathrm{H}_q(\Gamma_\sigma; \mathbb{Z}).$$

These differentials arise from the following cell stabiliser inclusions. For any edge in ${}_\Gamma\backslash X^1$, we have, because it is fixed pointwise, an inclusion ι of its stabiliser into the stabiliser of its origin vertex. Choose any matrix g which sends the origin vertex of this edge to its vertex representative in ${}_\Gamma\backslash X^0$.

***Notation* 42.** The *cell stabiliser inclusion associated to the origin* of our edge is the composition of the conjugation by g after the inclusion ι.

Up to inner automorphisms of the origin vertex stabiliser, this conjugation map does not depend on the choice of g, because g is determined up to multiplication with elements of the origin vertex stabiliser.
We denote the cell stabiliser inclusion associated to the end of an edge analogously. We see in [11, VII.8] that these cell stabiliser inclusions induce the differential d^1 of the equivariant spectral sequence.

The low terms of a free resolution for $\mathcal{A}_4$. We will use Wall's Lemma to construct a free resolution for $\mathcal{A}_4$, and compute its three differentials of lowest degrees explicitly. This resolution will help us determine the maps induced on homology by inclusions into vertex stabilisers of type $\mathcal{A}_4$.

So let us recall Wall's lemma. Given a group extension $1 \to K \to G \to H \to 1$, and free resolutions B for K, and C for H, we construct a free resolution for G in terms of the following double chain complex. For any $s \in \mathbb{N} \cup \{0\}$, let C_s be free on α_s generators. We define D_s as the direct sum of α_s copies of $\mathbb{Z}[G] \otimes_K B$. Then we have an augmentation of D_s onto the direct sum of α_s copies of $\mathbb{Z}[H]$, which we will identify with C_s, and write $\varepsilon_s : D_s \to C_s$. If $A_{r,s}$ is the submodule of D_s which is the direct sum of α_s copies of $\mathbb{Z}[G] \otimes_K B_r$, then $A_{r,s}$ is a free G-module, and D_s is the direct sum of the $A_{r,s}$.

Lemma 43 (C.T.C. Wall [41])**.** *There exist G-maps $d^k_{r,s} : A_{r,s} \to A_{r+k-1,s-k}$ for $k \geqslant 1$, $s \geqslant k$ such that*

- $\varepsilon_{s-1} \circ d^1_{0,s} = d^C_s \circ \varepsilon_s : A_{0,s} \to C_{s-1}$ *where d^C denotes the differential in C,*
- $\sum_{i=0}^{k} d^{k-i} \circ d^i = 0,$ *for each k, where $d^k_{r,s}$ is interpreted as zero if $r = k = 0$, or if $s < k$.*

Finally, we let A denote the direct sum of the $A_{r,s}$ graded by $\dim A_{r,s} = r + s$. And let $d := \sum_k d^k$.

Theorem 44 (C.T.C. Wall [41])**.** *(A, d) is acyclic, and so yields a free resolution for G.*

Let t be a generator of $\mathbb{Z}/n$. Let $F(n)$ be the periodic resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/n]$ given by

$$\cdots \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{t^n+\ldots+t+1} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{\text{augmentation}} \mathbb{Z}.$$

We consider the group extension $1 \to \mathcal{D}_2 \to \mathcal{A}_4 \to \mathbb{Z}/3 \to 1$, the resolution $F(3)$ for $\mathbb{Z}/3$ and the resolution $F(2) \otimes F(2)$ for $\mathcal{D}_2$. Then, $A_{r,s} = \mathbb{Z}[\mathcal{A}_4] \otimes_{\mathbb{Z}[\mathcal{D}_2]} (\mathbb{Z}[\mathcal{D}_2])^r \cong (\mathbb{Z}[\mathcal{A}_4])^r$. Let us use the cycle notation for the elements in the alternating group on four letters. Then in low degrees, the differential of $F(2) \otimes F(2)$ becomes

$$d^0_{1,s} = ((12)(34) - 1, \quad (14)(23) - 1)\,,$$

$$d^0_{2,s} = \begin{pmatrix} (12)(34)+1 & 1-(14)(23) & 0 \\ 0 & (12)(34)-1 & (14)(23)+1 \end{pmatrix},$$

$$d^0_{3,s} = \begin{pmatrix} (12)(34)-1 & (14)(23)-1 & 0 & 0 \\ 0 & -(12)(34)-1 & (14)(23)+1 & 0 \\ 0 & 0 & (12)(34)-1 & (14)(23)-1 \end{pmatrix},$$

for all $s \in \mathbb{N}$. At the same time, we can set $d^1_{0,2k} = ((132) + (123) + 1)$ and $d^1_{0,2k+1} = ((123) - 1)$ for all $k \in \mathbb{N}$, which satisfies the first condition in Wall's Lemma. Further, we set

$$d^1_{1,1} = \begin{pmatrix} 1 & (142) \\ -(123) & (134)+1 \end{pmatrix},$$

$$d^1_{1,2} = \begin{pmatrix} -1-(123)+(134)-(124) & (234)-(123) \\ (142)-(132) & (142)-1+(124)+(143) \end{pmatrix},$$

$$d^1_{2,1} = \begin{pmatrix} -1 & 0 & -(123) \\ 0 & (142)-1 & (243) \\ (134) & 0 & -(123)-1 \end{pmatrix}.$$

We sum up, and obtain the low degree terms of a free resolution for $\mathcal{A}_4$:

$$\cdots \longrightarrow (\mathbb{Z}[\mathcal{A}_4])^{10} \xrightarrow{d_3} (\mathbb{Z}[\mathcal{A}_4])^6 \xrightarrow{d_2} (\mathbb{Z}[\mathcal{A}_4])^3 \xrightarrow{d_1} \mathbb{Z}[\mathcal{A}_4] \to 0,$$

where $d_1 = (d^1_{0,1}, \quad d^0_{1,0}) = ((123) - 1, \quad (12)(34) - 1, \quad (14)(23) - 1)\,,$

$$d_2 = \begin{pmatrix} d^1_{0,2} & d^0_{1,1} & 0 \\ 0 & d^1_{1,1} & d^0_{2,0} \end{pmatrix}$$
$$= \begin{pmatrix} (132)+(123)+1 & (12)(34)-1 & (14)(23)-1 & 0 & 0 & 0 \\ 0 & 1 & (142) & (12)(34)-1 & (14)(23)-1 & 0 \\ 0 & -(123) & (134)+1 & 0 & (12)(34)-1 & (14)(23)-1 \end{pmatrix},$$

and we assemble analogously $d_3 = \begin{pmatrix} d^1_{0,3} & d^0_{1,2} & 0 & 0 \\ 0 & d^1_{1,2} & d^0_{2,1} & 0 \\ 0 & d^2_{1,2} & d^1_{2,1} & d^0_{3,0} \end{pmatrix}$.

Lemma 45 (Schwermer/Vogtmann [37]). *Let M be $\mathbb{Z}$ or $\mathbb{Z}/2$. Consider group homology with trivial M-coefficients. Then the following holds.*

- *Any inclusion $\mathbb{Z}/2 \to \mathcal{S}_3$ induces an injection on homology.*
- *An inclusion $\mathbb{Z}/3 \to \mathcal{S}_3$ induces an injection on homology in degrees congruent to 3 or $0 \mod 4$, and is otherwise zero.*
- *Any inclusion $\mathbb{Z}/2 \to \mathcal{D}_2$ induces an injection on homology in all degrees.*
- *An inclusion $\mathbb{Z}/3 \to \mathcal{A}_4$ induces injections on homology in all degrees.*
- *An inclusion $\mathbb{Z}/2 \to \mathcal{A}_4$ induces injections on homology in degrees greater than 1, and is zero on H_1.*

Schwermer and Vogtmann prove this for $M = \mathbb{Z}$. We will make use of the following statements to prove it for $M = \mathbb{Z}/2$. As the only automorphism of $\mathbb{Z}/2$ is the identity, $\mathbb{Z}/2$-coefficients are always trivial coefficients.

Lemma 46. *Lemma 45 holds in the case $M = \mathbb{Z}/2$ for the inclusions into $\mathcal{D}_2$.*

Proof. Consider the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/2$-coefficients of the trivial extension $1 \to \mathbb{Z}/2 \to \mathcal{D}_2 \to \mathbb{Z}/2 \to 1$. It takes the form

$$E^2_{p,q} = \mathrm{H}_p\left(\mathbb{Z}/2; \mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2)\right) \Rightarrow \mathrm{H}_{p+q}(\mathcal{D}_2; \mathbb{Z}/2).$$

As $\mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for all $q \in \mathbb{N} \cup \{0\}$, we obtain $E^2_{p,q} = \mathbb{Z}/2$ for all $p, q \in \mathbb{N} \cup \{0\}$. As we know from corollary 29 that $\mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{q+1}$, all the differentials must be zero and $E^2 = E^\infty$. Hence we obtain the claimed injections on homology. □

Lemma 47. *Lemma 45 holds in the case $M = \mathbb{Z}/2$ for the inclusions into $\mathcal{S}_3$.*

Proof. Consider the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/2$-coefficients of the non-trivial extension $1 \to \mathbb{Z}/3 \to \mathcal{S}_3 \to \mathbb{Z}/2 \to 1$. It takes the form

$$E^2_{p,q} = \mathrm{H}_p\left(\mathbb{Z}/2; \mathrm{H}_q(\mathbb{Z}/3; \mathbb{Z}/2)\right) \Rightarrow \mathrm{H}_{p+q}(\mathcal{S}_3; \mathbb{Z}/2).$$

As $\mathrm{H}_q(\mathbb{Z}/3; \mathbb{Z}/2) = 0$ for $q > 0$, the E^2-page is concentrated in the row $q = 0$ and equals the E^∞-page. Thus we have isomorphisms $\mathrm{H}_p\left(\mathbb{Z}/2; \mathrm{H}_0(\mathbb{Z}/3; \mathbb{Z}/2)\right) \cong \mathrm{H}_p(\mathcal{S}_3; \mathbb{Z}/2)$, from which we obtain the claimed morphisms on homology. □

Let t be a generator of $\mathbb{Z}/3$. Let F be the periodic resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/3]$ given by

$$\cdots \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/3] \xrightarrow{t^2+t+1} \mathbb{Z}[\mathbb{Z}/3] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}/3] \xrightarrow{\text{augmentation}} \mathbb{Z}.$$

Lemma 48. *Let A be an abelian group consisting only of elements of order 2 and the neutral element. Then $F \otimes_{\mathbb{Z}[\mathbb{Z}/3]} A$ is exact in degrees greater than zero, regardless of the $\mathbb{Z}[\mathbb{Z}/3]$-module structure attributed to A.*

Proof. We will show the two equations

$$\text{image}(t^2+t+1) = \ker(t-1) \text{ and } \ker(t^2+t+1) = \text{image}(t-1).$$

As $t^3 = 1$, the equation $(t^2+t+1)(t-1) = 0$ holds, and yields the inclusions

$$\text{image}(t^2+t+1) \subset \ker(t-1) \text{ and } \ker(t^2+t+1) \supset \text{image}(t-1).$$

Now we want to show that $\text{image}(t^2+t+1) \supset \ker(t-1)$. Let $v \in \ker(t-1)$. Then $(t-1)\cdot v = 0$, or equivalently, $t \cdot v = v$. We apply this three times to obtain $(t^2+t+1)\cdot v = 3v$. As $2v = 0$ in A, we have $(t^2+t+1)\cdot v = v$ and hence the claimed inclusion.
It remains to show that $\ker(t^2+t+1) \subset \text{image}(t-1)$. Let $v \in \ker(t^2+t+1)$. We will see that $t \cdot v$ is a preimage of v for the multiplication by $t-1$. Namely,
$(t-1)\cdot(t\cdot v) = t^2 \cdot v - t \cdot v = -2t \cdot v - v$ because $t^2 \cdot v + t \cdot v + v = 0$. Using that multiplication by 2 is zero on A, we obtain the image v and hence the last inclusion. □

Lemma 49. *The E^2-page of the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/2$–coefficients for the extension $1 \to \mathcal{D}_2 \to \mathcal{A}_4 \to \mathbb{Z}/3 \to 1$ is concentrated in the column $p = 0$.*

Proof. The E^2-page of the Lyndon/Hochschild/Serre spectral sequence with $\mathbb{Z}/2$– coefficients is given by $E^2_{p,q} = \mathrm{H}_p\left(\mathbb{Z}/3; \mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2)\right)$.
The action of $\mathbb{Z}/3$ on $\mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{q+1}$ is determined by the non-trivial conjugation action of $\mathbb{Z}/3$ on $\mathcal{D}_2$. But applying lemma 48, we obtain $\mathrm{H}_p\left(\mathbb{Z}/3; (\mathbb{Z}/2)^{q+1}\right) = 0$ for $p > 0$ and any action of $\mathbb{Z}/3$ on $(\mathbb{Z}/2)^{q+1}$. □

We can now do the last remaining step to prove lemma 45 in the case $M = \mathbb{Z}/2$.

Corollary 50. *Lemma 45 holds in the case $M = \mathbb{Z}/2$ for the inclusions into $\mathcal{A}_4$.*

Proof.

- For an inclusion $\mathbb{Z}/3 \to \mathcal{A}_4$, this follows from the fact that $\mathrm{H}_p(\mathbb{Z}/3; \mathbb{Z}/2) = 0$ for all $p > 0$.
- For an inclusion $\mathbb{Z}/2 \to \mathcal{A}_4$, factorise by an inclusion $\mathbb{Z}/2 \to \mathcal{D}_2$. Lemma 49 gives an isomorphism $\mathrm{H}_q(\mathcal{A}_4; \mathbb{Z}/2) \cong \mathrm{H}_0\left(\mathbb{Z}/3; \mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2)\right)$ for $q \in \mathbb{N} \cup \{0\}$. From lemma 46 and the low terms of our resolution for $\mathcal{A}_4$, we deduce that the induced map on homology is injective whenever $\mathrm{H}_q(\mathcal{A}_4; \mathbb{Z}/2)$ is non-zero. From corollary 29, we see that this is the case for all $q \in \mathbb{N} \cup \{0\}$ except for $q = 1$, where we obtain the zero map.

□

As we see from lemma 30, the inclusions determined by lemma 45 are the only non-trivial inclusions which occur in our $\mathrm{PSL}_2(\mathcal{O})$-cell complex. Hence we can decompose the $d^1_{1,q}$ differential in the associated equivariant spectral sequence, for $q > 0$, into a 2-primary and a 3-primary part.

Definition 51. *For an abelian group A, the ℓ-*primary part *is the subgroup consisting of all elements of A of ℓ-power order.*

2.2.1 The 3-primary part

Denote by $(d^1_{1,q})_{(\ell)}$ the ℓ-primary part of our $d^1_{1,q}$ differential. It suffices to compute $(d^1_{1,1})_{(3)}$ and $(d^1_{1,3})_{(3)}$ to get the 3-primary part of our $d^1_{1,q}$ differential, because of the following.

Corollary 52. *The 3-primary part $(d^1_{1,q})_{(3)}$ is of period 4 in q; and is zero for $q > 0$ even.*

Proof. Lemmata 28 and 45. □

Proposition 53. *The $\mathbb{Z}/3$–rank of the matrix A computed by algorithm 5 equals the $\mathbb{Z}/3$–rank of the differential $(d^1_{1,1})_{(3)}$. This algorithm terminates in finite time.*

Proof.

- By fixing the elements $x_1, \ldots, x_\ell$ in $\Gamma^1_1, \ldots, \Gamma^\ell_1$, and $y_1, \ldots, y_k$ in $\Gamma^1_0, \ldots, \Gamma^k_0$ we have implicitly fixed group monomorphisms $\iota^{j,i}\colon \Gamma^i_1 \hookrightarrow \Gamma^j_0$, by setting $\iota^{j,i}(x_i) = y_j$, for all i running from 1 through ℓ and all j running from 1 through k. We consider generators $\xi_1, \ldots, \xi_\ell$ of the homology groups $\mathrm{H}_1(\Gamma^1_1), \ldots, \mathrm{H}_1(\Gamma^\ell_1)$. Then the entries in our matrix A refer to the induced images $\iota^{j,i}_*(\xi_i)$. More precisely, we have implicitly fixed a basis $\xi_1, \ldots, \xi_\ell$ of the $\mathbb{F}_3$-vector space $\mathrm{H}_1(\Gamma^1_1) \oplus \ldots \oplus \mathrm{H}_1(\Gamma^\ell_1)$ on which $(d^1_{1,1})_{(3)}$ is defined; and for each i, a basis $\iota^{1,i}_*(\xi_i), \ldots, \iota^{k,i}_*(\xi_i)$ of its target $\mathbb{F}_3$-vector space. Now if $\gamma x_i \gamma^{-1}$ is conjugate (within the vertex stabiliser Γ^j_0) to y_j, then obviously the morphism induced on homology by conjugation with γ, equals $\iota^{j,i}_*$. And if $\gamma x_i \gamma^{-1}$ is conjugate (within the vertex stabiliser Γ^j_0) to y_j^2, then the image of x_i induced by γ must equal $2\iota^{j,i}_*(\xi_i)$. So, if the implicit choice of bases $\iota^{1,i}_*(\xi_i), \ldots, \iota^{k,i}_*(\xi_i)$ is the same for all i, the matrix A computed by our algorithm expresses the differential $(d^1_{1,1})_{(3)}$ in the above bases.
  Else a base $\iota^{1,i}_*(\xi_i), \ldots, \iota^{k,i}_*(\xi_i)$ differs from the first base $\iota^{1,1}_*(\xi_1), \ldots, \iota^{k,1}_*(\xi_1)$. But then replacing x_i by x_i^2 will produce the first base. This replacement means a multiplication by 2 of the i-th column of A. Multiplying by 2 is an automorphism of $\mathbb{F}_3$, so this does not change the $\mathbb{Z}/3$–rank of A.

- Now we consider the run-time of the algorithm. This algorithm runs twice through the list of edge representatives, which is finite. In these loops, it performs only finite-time operations like checking the conjugations with a finite group (the vertex stabiliser).

□

Algorithm 5 Computation of the rank of $(d^1_{1,1})_{(3)}$.

Input: The stabilisers of the vertex representatives, of the type $\mathbb{Z}/3$ and $\mathcal{A}_4$. The stabilisers of the edge representatives, of the type $\mathbb{Z}/3$. Matrices in Γ sending the origin and end vertices of the concerned edges, to their vertex representative.
Output: The rank of $(d^1_{1,1})_{(3)}$.

Denote by $\Gamma^1_0,\ldots,\Gamma^k_0$ the stabilisers of 0-cells in the quotient $\Gamma\backslash X^0$, of type $\mathbb{Z}/3$ or $\mathcal{A}_4$.
Denote by $v_1,\ldots,v_k$ the corresponding vertex representatives.
Denote by $\Gamma^1_1,\ldots,\Gamma^\ell_1$ the stabilisers of 1-cells in the quotient $\Gamma\backslash X^1$, of type $\mathbb{Z}/3$.
Denote by $e_1,\ldots,e_k$ the corresponding edge representatives.
Let A be the zero matrix of dimensions $k\times\ell$.
Fix order-3-elements $x_1,\ldots,x_\ell$ in $\Gamma^1_1,\ldots,\Gamma^\ell_1$, and $y_1,\ldots,y_k$ in $\Gamma^1_0,\ldots,\Gamma^k_0$.
for all i running from 1 through ℓ **do**
  Find the representative v_j of the origin vertex of e_i.
  Denote by γ the input matrix sending the origin vertex of e_i to v_j.
  if $\gamma x_i\gamma^{-1}$ is conjugate (within the vertex stabiliser Γ^j_0) to y_j **then**
    Decrease the matrix entry $A_{j,i}$ by 1.
  else $\gamma x_i\gamma^{-1}$ is conjugate (within the vertex stabiliser Γ^j_0) to y_j^2
    Decrease the matrix entry $A_{j,i}$ by 2.
  end if
end for
for all i running from 1 through ℓ **do**
  Find the representative v_j of the end vertex of e_i.
  Denote by γ the input matrix sending the end vertex of e_i to v_j.
  if $\gamma x_i\gamma^{-1}$ is conjugate (within the vertex stabiliser Γ^j_0) to y_j **then**
    Increase the matrix entry $A_{j,i}$ by 1.
  else $\gamma x_i\gamma^{-1}$ is conjugate (within the vertex stabiliser Γ^j_0) to y_j^2
    Increase the matrix entry $A_{j,i}$ by 2.
  end if
end for
Compute the $\mathbb{Z}/3$–rank of A and output this rank.

The computation of the rank of the differential matrix $(d^1_{1,3})_{(3)}$ follows the same algorithm except that also the vertex stabilisers of type $\mathcal{S}_3$ are taken into account (their first homology group contains no 3-torsion). By lemmata 28 and 45, their third homology group behaves like the odd homology groups of $\mathbb{Z}/3$ and $\mathcal{A}_4$.

2.2.2 The 2-primary part

As by lemma 30, there are only edges with finite cyclic stabilisers, we see that the $d^1_{1,q}$ differential is zero for $q > 0$ even. Now for q odd, we want to compute its 2-primary part.
This is a $\mathbb{Z}/2$-module homomorphism $(d^1_{1,q})_{(2)} : (\mathbb{Z}/2)^k \leftarrow (\mathbb{Z}/2)^l$, where l is the number of orbits of edges with stabiliser $\mathbb{Z}/2$, and k is a sum to which every orbit of vertices makes a contribution according to its stabiliser type. This contribution can be read off from the $\mathbb{Z}/2$-rank of the homology groups listed in lemma 28. We want to establish a matrix A for $(d^1_{1,q})_{(2)}$. We initialise it as the zero matrix of dimensions $k \times l$. Obviously, there is only one injection $\mathbb{Z}/2 \to \mathbb{Z}/2$, so for any group monomorphism from edge stabilisers of type $\mathbb{Z}/2$ to vertex stabilisers of type $\mathbb{Z}/2$ or $\mathcal{S}_3$, we can increase the associated entry of A by 1.
For $q = 1$, the monomorphisms $\mathbb{Z}/2 \to \mathcal{A}_4$ induce zero maps (see lemma 45). So, consider the case $q > 1$ odd. By theorem 39, the block in the matrix A, associated to a vertex orbit of stabiliser type $\mathcal{A}_4$, can have only one non-zero column. For a suitable basis for $(\mathrm{H}_q(\mathcal{A}_4;\mathbb{Z}))_{(2)}$, this column takes the form $(1, 0, \ldots, 0)^{\mathrm{t}}$, where t stands for the transpose.

Lemma 54. *Let q be odd. Let v be a vertex representative of stabiliser type $\mathcal{D}_2$. Then the block associated to it in the matrix for the $(d^1_{1,q})_{(2)}$ differential, has exactly three non-zero columns. These block columns can be expressed as $(1, 0, \ldots, 0)^{\mathrm{t}}$, $(0, \ldots, 0, 1)^{\mathrm{t}}$ and $(1, \ldots, 1)^{\mathrm{t}}$. The latter are linearly independent if and only if $q \geqslant 3$.*

Proof. By theorem 39, there are exactly three Γ-representatives of non-trivially stabilised edges adjacent to v. Furthermore, the stabilisers of these three edges are precisely the three order-2-subgroups of the stabiliser of v. We apply the chain map computation of [37], to each of these three subgroup inclusions, and obtain the claimed three block columns. The length of these block columns is the $\mathbb{Z}/2$-rank of $\mathrm{H}_q(\mathcal{D}_2;\mathbb{Z})$, namely $\frac{q+3}{2}$, so we easily see that these block columns are linearly independent if and only if $q \geqslant 3$. □

***Remark* 55.** The author has however implemented to fix generators for the stabilisers of type $\mathcal{D}_2$, and to compute every block column with respect to these generators, which we denote by α and β. Then we obtain block columns
$$\begin{cases}(1,0,\ldots,0)^{\mathrm{t}}\\(0,\ldots,0,1)^{\mathrm{t}}\\(1,\ldots,1)^{\mathrm{t}}\end{cases}\text{ for a monomorphism with image }\begin{cases}\{1,\,\alpha\}\\\{1,\,\beta\}\\\{1,\,\alpha\beta\}\end{cases}\text{ according to the chain}$$
map computation in [37].
Comparison with lemma 54 allows another check on the geometry of the computed quotient space.

Proposition 56. *Let $q \geqslant 3$ odd. Then* $\operatorname{rank}(d^1_{1,q})_{(2)} = \operatorname{rank}(d^1_{1,3})_{(2)}$.

Proof. We see from lemma 45 that all inclusions of $\mathbb{Z}/2$ into any vertex stabiliser induce injections on homology in all degrees $q \geqslant 3$. As the groups $\mathbb{Z}/2$ and $\mathcal{S}_3$ have their q-th integral homology group $\mathbb{Z}/2$ for all odd q, there is just one possibility for induced injections into it; and hence the matrix block of $(d^1_{1,q})_{(2)}$ associated to vertex stabilisers of these types is the same for all odd q. Now for vertex stabilisers of type $\mathcal{A}_4$, we know from theorem 39 that there is just one 2-torsion edge representative stabiliser inclusion into them. Thus the associated matrix blocks $(1, 0, \ldots, 0)^{\mathrm{t}}$ only grow in the number of their zeroes when q grows, but this does not change the rank of $(d^1_{1,q})_{(2)}$. Finally we see from lemma 54 that associated to vertex representative stabilisers of type $\mathcal{D}_2$, there are exactly three matrix sub-blocks, which are linearly independent for all $q \geqslant 3$. □

The above methods allow us to compute the E^2-page in finite time.
Denote by $(\tilde{E}, \tilde{d})$ the same equivariant spectral sequence, but now with $\mathbb{Z}/2$-coefficients.

Lemma 57. *Let $q \geqslant 3$ odd. Then the rank of $d^1_{1,q} \otimes \mathbb{Z}/2$ equals the ranks of the differentials $\tilde{d}^1_{1,q}$ and $\tilde{d}^1_{1,q+1}$.*

Proof. Lemma 30 tells us that the edges in our cell complex have cyclic stabilisers, and that only those of type $\mathbb{Z}/2$ can contribute nontrivially to the $\mathbb{Z}/2$–modules $\tilde{E}^1_{1,q}$, $\tilde{E}^1_{1,q+1}$ and $\left(E^1_{1,q}\right)_{(2)}$. Then we see from lemmata 28 and 29, that $\tilde{E}^1_{1,q} \cong \tilde{E}^1_{1,q+1} \cong \left(E^1_{1,q}\right)_{(2)}$. Consider matrices for the homomorphisms $d^1_{1,q} \otimes \mathbb{Z}/2$, $\tilde{d}^1_{1,q}$ and $\tilde{d}^1_{1,q+1}$. Then applying lemma 45 with both the coefficients $M = \mathbb{Z}$ and $M = \mathbb{Z}/2$, we check entry by entry that these matrices are identical. □

Theorem 58. *Let $q \geqslant 3$, $\ell = 2$ or $\ell = 3$. Then the ℓ–primary part of the homology $\mathrm{H}_q(\Gamma; \mathbb{Z})$ is the direct sum over the $(E^\infty_{p,q-p})_{(\ell)}$–terms, p running from 0 to q, for all Bianchi groups Γ.*

Proof.
- The rows of the E^1-page with q even and $q \geqslant 2$ do not contain any 3-torsion, so nor does the E^∞-page. So the assertion follows for $\ell = 3$ knowing that the E^1-page is concentrated in the first two columns for $q > 0$.
- Lemma 57 implies that $\tilde{E}^\infty_{1,q} \cong \tilde{E}^\infty_{1,q+1} \cong \left(E^\infty_{1,q}\right)_{(2)}$ and $\tilde{E}^\infty_{0,q} \cong \tilde{E}^\infty_{0,q+1} \cong \left(E^\infty_{0,q}\right)_{(2)}$. Hence the only possible solution to the dévissage problem is the trivial solution, so we obtain the assertion for $\ell = 2$.

□

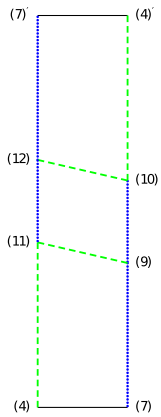
Corollary 59. *Let $q \geqslant 3$. Then the torsion in $\mathrm{H}_q(\Gamma; \mathbb{Z})$ consists only of direct summands of type $\mathbb{Z}/2$ and $\mathbb{Z}/3$.*

2.3 Torsion subgraph reduction

2.3.1 Extracting the torsion subgraphs

Definition 60. *Let ℓ be 2 or 3. We will call* ℓ-torsion subgraph *the subcomplex of the quotient space of our refined cell complex, consisting of all the cells the stabiliser of which contains elements of order ℓ.*

We immediately see that this is a graph because the cells of dimension greater than 1 are trivially stabilised in our refined cell complex. The cells which the 2-torsion and the 3-torsion subgraphs have in common, are precisely the vertices of stabiliser types $\mathcal{S}_3$ and $\mathcal{A}_4$. For example, the 2-torsion subgraph (- -) and the 3-torsion subgraph (·····) look as follows in the case $m = 19$. The vertices with matching labels are to be identified.



2.3.2 Computing the d^1-differentials with the torsion subgraphs

Let q be greater than the virtual cohomological dimension (abbreviated vcd, see [11] for the definition of this) of Γ. Then all the cell representatives which make a contribution to the ℓ-primary part of the differential $d^1_{1,q}$ lie in the ℓ-torsion subgraph, because tensoring the E^1-page with $\mathbb{Z}/\ell$ kills the homology groups in degrees above $\mathrm{vcd}(\Gamma)$ of the stabilisers of all other cells. We can even compute the rank of $(d^1_{1,q})_{(\ell)}$ for all $q \geqslant 1$ considering only the cells in the ℓ-torsion subgraph, because the only relevant cells which will be ignored this way, are the singular points. Their stabilisers have free abelian homology groups (the first one isomorphic to $\mathbb{Z}^2$, the second one isomorphic to $\mathbb{Z}$, and the higher degree ones are zero), so any map from ℓ-torsion to the latter must be zero. Hence for $q \geqslant 1$ we can determine the ℓ-primary part of the differential $d^1_{1,q}$ completely on the ℓ-torsion subgraph, the free abelian part contributed by the singular points passing unchanged to the E^2-page.

2.3.3 Reducing the torsion subgraphs

Now we use the geometric rigidity statements of section 2.1 to fuse cells in the ℓ-torsion subgraphs.

Let (a, v) and (v', b) be adjacent edges in the ℓ-torsion subgraph. This means, they are adjacent to a common vertex orbit $\Gamma \cdot v = \Gamma \cdot v'$.

Definition 61. *If there are exactly two edges adjacent to the vertex v, then we define an* edge fusion *by replacing the edges (a, v) and (v', b) by the edge (a, b) and forgetting the vertex v.*

Definition 62. *We will call a* reduced ℓ-torsion subgraph *a graph obtained from the ℓ-torsion subgraph by iterating edge fusions as often as this is permitted by definition* 61.

Invariance of the E^2-page under reduction of the torsion subgraphs

Theorem 63. *The E^2-page is invariant under replacing the ℓ-torsion subgraph by a reduced ℓ-torsion subgraph for the computation of the ℓ-primary part of the differential $d^1_{1,q}$, in all cases $q \geqslant 1$.*

Proof. A vertex representative v which is removed by an edge fusion must have exactly two orbits of edges of stabiliser type $\mathbb{Z}/\ell$ adjacent to it. Theorem 39 tells us that then, Γ_v is isomorphic to $\mathbb{Z}/2$ or $\mathcal{S}_3$ in the case $\ell = 2$, and to $\mathbb{Z}/3$ or $\mathcal{A}_4$ in the case $\ell = 3$. Now we see from definition 61 and lemma 28 that every edge fusion decreases each by 1 the $\mathbb{Z}/\ell$–ranks of the modules $E^1_{1,q}$ and $E^1_{0,q}$ for odd q, because there is a unique isomorphy type of the ℓ-primary part of the homology of the stabilisers for vertices with two adjacent edges in the ℓ-torsion subgraph:

- For the 2-primary part, $\mathrm{H}_q(\mathcal{S}_3; \mathbb{Z})_{(2)} \cong \mathbb{Z}/2 \cong \mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z})_{(2)}$, for odd q,
- for the 3-primary part, $\mathrm{H}_q(\mathcal{A}_4; \mathbb{Z})_{(3)} \cong \mathbb{Z}/3 \cong \mathrm{H}_q(\mathbb{Z}/3; \mathbb{Z})_{(3)}$, for odd q,

and the above ℓ-primary parts are all zero for $q > 0$ even.

Let q be odd. We will show that any edge fusion also decreases by 1 the $\mathbb{Z}/\ell$–rank of $(d^1_{1,q})_{(\ell)}$. Then we can conclude that the E^2-page is preserved under each edge fusion.
Let $\mathcal{G}'$ be the graph obtained by an edge fusion from an ℓ-torsion subgraph $\mathcal{G}$. We will show that passing from $\mathcal{G}'$ to $\mathcal{G}$ increases the rank of $(d^1_{1,q})_{(\ell)}$ by 1. Denote by (a, b) the fusioned edge. There is a column associated to it in $(d^1_{1,q})_{(\ell)}$ for $\mathcal{G}'$ of the shape

	(a,b)
a	-1
b	1

in a suitable choice of bases for the ℓ-primary part of the homology groups of the stabilisers. The remaining entries in this column are zeroes.
In the graph $\mathcal{G}$, we have an additional vertex v; and we replace our edge by the two edges (a, v) and (v', b), where v' is on the same orbit as v. By what we have seen in the beginning of this proof, $\mathrm{H}_q(\Gamma_v; \mathbb{Z})_{(\ell)} \cong \mathbb{Z}/\ell$. Furthermore, lemma 45 tells us that the

inclusions of the stabilisers of (a, v) and (v', b) into Γ_v induce injections on homology. Hence passing to $\mathcal{G}$, we replace the above matrix column by two columns of the shape

	(a,v)	(v',b)
v	1	-1
a	-1	0
b	0	1

(again in a suitable choice of bases, and the rest of these columns are zeroes). As the vertex v has exactly two edges adjacent to it in the ℓ-torsion subgraph, the remaining entries in the inserted row associated to the vertex v are zeroes. We further observe that the sum of the two inserted columns equals the replaced column (prolongated by a zero for the row of v). So the differential $(d^1_{1,q})_{(\ell)}$ for $\mathcal{G}$ has the same rank as the matrix

	(a,v)	$(a,v)+(v',b)$ and remaining columns
v	1	0
a	-1	$(d^1_{1,q})_{(\ell)}$ for $\mathcal{G}'$
other rows	0	

Hence the rank of the ℓ-primary part of the differential $d^1_{1,q}$ has increased exactly by 1. □

***Remark* 64.** From the proof of theorem 39, we see that any pair of fusioned edges lies on the same rotation axis. On the other hand, for every rotation axis, we find a chain of adjacent edges lying on it in the quotient space. Hence a reduced ℓ-torsion subgraph contains one edge for every ℓ-axis in the quotient space.

2.3.4 Classifying the reduced torsion subgraphs

Given an ℓ-torsion subgraph, the only difference that can occur between two of its reductions, is the following. If there is a loop in the graph, then this loop will become a single edge with identical origin and end vertex. But this vertex can be chosen arbitrarily from the vertices which are originally on the loop. However, the topology of the reduced graph does not depend on this choice of vertex, so as a topological space, it is well defined to speak of *the* reduced ℓ-torsion subgraph.

Theorem 65. *The ℓ-primary part of the terms $E^2_{p,q}$, in all cases $q \geqslant 1$, only depends on the homeomorphy type of the ℓ-torsion subgraph.*

Proof. By theorem 63, we can pass from the ℓ-torsion subgraph to a reduced ℓ-torsion subgraph. Then on the connected components which contain no loops, only vertices with one or three edges adjacent to them remain: we have eliminated all vertices with two adjacent edges by edge fusions; and vertices with no adjacent edges cannot be in the ℓ-torsion subgraph due to theorem 39.
By the latter theorem, there is a unique stabiliser type of vertices with one adjacent

edge in the ℓ-torsion subgraph. Furthermore, the group $\mathcal{D}_2$ is the unique stabiliser type of vertices with three adjacent edges in the 2-torsion subgraph; and in the 3-torsion subgraph, there is no vertex with three adjacent edges at all. We can recognise vertices with one, respectively three adjacent edges as end points respectively bifurcation points in the reduced ℓ-torsion subgraph considered as a topological space Y. The end points and bifurcation points are preserved by homeomorphisms. So, a reduced ℓ-torsion subgraph can be reconstructed from the homeomorphy type of Y, as well as the associated stabiliser types.
By lemma 54, it is sufficient to know that the bifurcation points come from vertices with stabiliser type $\mathcal{D}_2$ in order to establish the matrix block in the differential matrix. And for the vertices with just one adjacent edge, it suffices to use lemma 45 to get the associated matrix blocks. Computing the rank of this differential matrix, gives us the ℓ-primary part of the terms $E^2_{p,q}$, in all cases $q \geqslant 1$.
So it only remains to see that the above arguments still work when there are loops in the ℓ-torsion subgraph. We classify the types of loops by the number n of bifurcation points they contain. This number can be read off from the homeomorphy type of Y. A loop with $n = 0$ consists of one edge with identified origin and end point in a reduced ℓ-torsion subgraph. The stabiliser of this point is of type $\mathbb{Z}/2$ or $\mathcal{S}_3$ in the case $\ell = 2$ and of type $\mathbb{Z}/3$ or $\mathcal{A}_4$ in the case $\ell = 3$. As stated in the proof of theorem 63, the ℓ-primary part of the homology groups is the same for both possible stabiliser types, and by lemma 45 the inclusions of $\mathbb{Z}/\ell$ into them induce always injections, so we do not need to know which is precisely this stabiliser type to reobtain the original contribution to the E^2-page.
Finally, if $n > 0$, then such a loop contains no more edges with two adjacent vertices in the reduced ℓ-torsion subgraph, and the arguments of the situtation without loops work. □

***Observation* 66.** Consider the case $\ell = 3$. By theorem 39, there can be no bifurcation point in the 3-torsion subgraph. Hence, every connected component of a reduced 3-torsion subgraph consists of a single edge,

- either with two vertices of stabiliser type $\mathcal{S}_3$,
- or with identified origin and end point of stabiliser type $\mathbb{Z}/3$ or $\mathcal{A}_4$, so this connected component is a loop.

In the second case, the contribution to the E^2-page does not depend on the occurring stabiliser type, as we see from theorem 65.
Let us consider the 3-torsion Poincaré series

$$P^3_m(t) := \sum_{q=3}^{\infty} \dim_{\mathbb{F}_3} \mathrm{H}_q(\mathrm{PSL}_2\big(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}\big); \mathbb{Z}/3) t^q.$$

It depends only on the 3-primary part of the E^2-page because we have cut off the degrees smaller than or equal to the virtual cohomological dimension of Γ, namely 2.

We observe that we can decompose this as a sum over the series obtained from the connected components of the 3-torsion subgraph, because there can be no interference in the following sense.

Proposition 67. *The matrix for the ℓ-primary part of $d^1_{1,q}$ can be decomposed as a direct sum of the blocks associated to the connected components of the ℓ-torsion subgraph.*

Proof. As there is no adjacency between different connected components, all entries off these blocks are zero. □

***Observation* 68.** Hence it suffices to compute the 3-torsion Poincaré series $P^{3,1}(t)$ and $P^{3,2}(t)$ associated to the first and the second homeomorphy type appearing in observation 66, and for any Bianchi group $\mathrm{PSL}_2\big(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}\big)$ count the numbers n_1 of connected components of first type, n_2 of connected components of second type. Then the 3-torsion Poincaré series associated to this Bianchi group equals

$$P^3_m(t) = n_1 P^{3,1}(t) + n_2 P^{3,2}(t).$$

As $P^{3,1}(t)$ and $P^{3,2}(t)$ are linearly independent, the reduced 3-torsion subgraph can be easily computed from the 3-torsion Poincaré series.

For the connected components in the 2-torsion subgraph, a priori infinitely many homeomorphy types may occur. But we still have, by proposition 67, a direct sum decomposition of the 2-primary part of the E^2-page, for q greater than the virtual cohomological dimension of the Bianchi group.

***Observation* 69.** In the cases of class numbers 1 and 2, and in the cases where the absolute value of the discriminant is inferior to 200, there are just four homeomorphy types of connected components in the 2-torsion subgraph.

Chapter 3

Results for the homology of the Bianchi groups

3.1 The non-Euclidean principal ideal domain cases

We now give the results in the cases $m = 19$, 43, 67 and 163, which are the non-Euclidean principal ideal domain cases. The Euclidean principal ideal domain cases are already known from [37]. We observe that in these four cases, the torsion in the integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ is of the same isomorphy type. This comes from the fact that their 2-torsion and 3-torsion subgraphs are homeomorphic (see figure 3.1 below). Theorem 65 then explains this isomorphy.

Proposition 70. *For $m \in \{19, 43, 67, 163\}$, the E^2-page of the equivariant spectral sequence is concentrated in the columns $p = 0$, $p = 1$ and $p = 2$, given as follows.*

$q = 12n + 14$	$(\mathbb{Z}/2)^{4n+6}$		
$q = 12n + 13$	$(\mathbb{Z}/2)^{4n+3}$	$\mathbb{Z}/3$	
$q = 12n + 12$	$(\mathbb{Z}/2)^{4n+4}$		
$q = 12n + 11$	$(\mathbb{Z}/2)^{4n+5} \oplus \mathbb{Z}/3$		
$q = 12n + 10$	$(\mathbb{Z}/2)^{4n+2}$		
$q = 12n + 9$	$(\mathbb{Z}/2)^{4n+3}$	$\mathbb{Z}/3$	
$q = 12n + 8$	$(\mathbb{Z}/2)^{4n+4}$		
$q = 12n + 7$	$(\mathbb{Z}/2)^{4n+1} \oplus \mathbb{Z}/3$		
$q = 12n + 6$	$(\mathbb{Z}/2)^{4n+2}$		
$q = 12n + 5$	$(\mathbb{Z}/2)^{4n+3}$	$\mathbb{Z}/3$	
$q = 12n + 4$	$(\mathbb{Z}/2)^{4n}$		
$q = 12n + 3$	$(\mathbb{Z}/2)^{4n+1} \oplus \mathbb{Z}/3$		
$q = 2$	$(\mathbb{Z}/2)^2$		
$q = 1$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	
$q = 0$	$\mathbb{Z}$	$\mathbb{Z}^{\beta_1}$	$\mathbb{Z}^{\beta_2}$,
	$p = 0$	$p = 1$	$p = 2$

where the numbers β_i are

m	β_1	β_2
19	1	0
43	2	1
67	3	2
163	7	6.

We observe that the $E^2_{0,1}$-term vanishes completely. This term is the target of the only d^2-arrow which can for arbitrary m be non-zero, namely $d^2_{2,0}$. Hence our spectral sequence degenerates at the E^2-level.

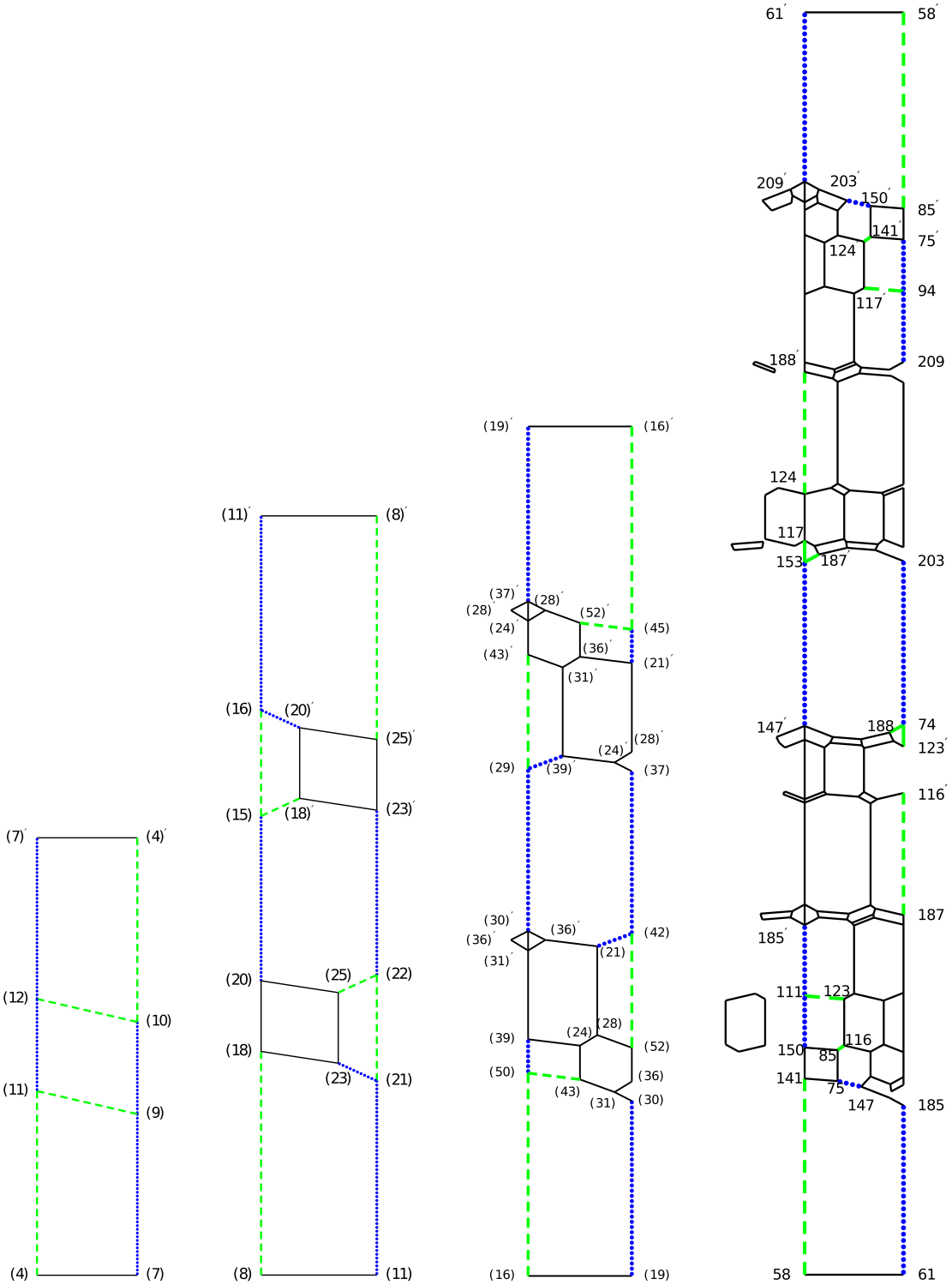
The only ambiguity in the dévissage concerns $\mathrm{H}_2(\mathrm{PSL}_2(\mathcal{O}_{-m});\mathbb{Z})$; the above E^∞-page says that its 2-primary part is either $(\mathbb{Z}/2)^3$ or $\mathbb{Z}/4 \oplus \mathbb{Z}/2$. Computing

$$\dim_{\mathbb{F}_2} \mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-m});\mathbb{Z}/2) = \begin{cases} 4k+5, & q = 6k+8, \\ 4k+3, & q = 6k+7, \\ 4k+5, & q = 6k+6, \\ 4k+3, & q = 6k+5, \\ 4k+1, & q = 6k+4, \\ 4k+3, & q = 6k+3, \\ \beta_2+2, & q = 2, \\ \beta_1, & q = 1, \end{cases}$$

and comparing with the help of the Universal Coefficient Theorem, we can exclude the first possibility.

Corollary 71. *With the above numbers β_1, β_2 for $m \in \{19, 43, 67, 163\}$, we obtain*

$$\mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-m});\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^{4n+6} \oplus \mathbb{Z}/3, & q = 12n+14, \\ (\mathbb{Z}/2)^{4n+3}, & q = 12n+13, \\ (\mathbb{Z}/2)^{4n+4}, & q = 12n+12, \\ (\mathbb{Z}/2)^{4n+5} \oplus \mathbb{Z}/3, & q = 12n+11, \\ (\mathbb{Z}/2)^{4n+2} \oplus \mathbb{Z}/3, & q = 12n+10, \\ (\mathbb{Z}/2)^{4n+3}, & q = 12n+9, \\ (\mathbb{Z}/2)^{4n+4}, & q = 12n+8, \\ (\mathbb{Z}/2)^{4n+1} \oplus \mathbb{Z}/3, & q = 12n+7, \\ (\mathbb{Z}/2)^{4n+2} \oplus \mathbb{Z}/3, & q = 12n+6, \\ (\mathbb{Z}/2)^{4n+3}, & q = 12n+5, \\ (\mathbb{Z}/2)^{4n}, & q = 12n+4, \\ (\mathbb{Z}/2)^{4n+1} \oplus \mathbb{Z}/3, & q = 12n+3, \\ \mathbb{Z}^{\beta_2} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3, & q = 2, \\ \mathbb{Z}^{\beta_1}, & q = 1. \end{cases}$$



(a) $m = 19$ (b) $m = 43$ (c) $m = 67$ (d) $m = 163$

Figure 3.1: The fundamental domain for the Flöge cellular complex with its 2-torsion subgraph (green, dashed edges) and its 3-torsion subgraph (blue, dotted edges), in the non-Euclidean principal ideal domain cases.

3.1.1 Intermediary results

We now give the intermediary results we have used to compute the above E^2-pages, in the case $m = 67$. Our fundamental domain for the Flöge cellular complex (which coincides with Mendoza's spine in the principal ideal domain cases) is drawn in figure 3.1c. We denote by (k) the vertex number k in the output files of the program [31], and by $\Gamma_{(k)}$ its stabiliser. We do the same for the edges, which we denote by their origin and end point. We will write $(k)'$ for the other vertices on the same Γ-orbit as (k).

We will use the following notations:

$$
\begin{aligned}
A &:= \pm\begin{pmatrix} -\omega & 16 \\ 1 & 1+\omega \end{pmatrix}, & B &:= \pm\begin{pmatrix} -\omega & 8 \\ 2 & 1+\omega \end{pmatrix}, \\
C &:= \pm\begin{pmatrix} 5-\omega & 10+2\omega \\ 2+\omega & -6+\omega \end{pmatrix}, & D &:= \pm\begin{pmatrix} 13 & 10+10\omega \\ \omega & -13 \end{pmatrix}, \\
F &:= \pm\begin{pmatrix} -\omega & 4 \\ 4 & 1+\omega \end{pmatrix}, & G &:= \pm\begin{pmatrix} -2-\omega & 4-\omega \\ 3 & 2+\omega \end{pmatrix}, \\
H &:= \pm\begin{pmatrix} -1-\omega & 15-\omega \\ 1 & 1+\omega \end{pmatrix}, & J &:= \pm\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \\
L &:= \pm\begin{pmatrix} -1-\omega & 7-\omega \\ 2 & 2+\omega \end{pmatrix}, & N &:= \pm\begin{pmatrix} 9-\omega & 14+6\omega \\ 2+\omega & -10+\omega \end{pmatrix}, \\
S &:= \pm\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}.
\end{aligned}
$$

We observe the following stabilisers of the vertex representatives.

$$
\begin{array}{lll}
\Gamma_{(36)} = \Gamma_{(24)} = & \Gamma_{(31)} = \Gamma_{(28)} & = 1 \\
\Gamma_{(43)} = & \langle JF \,|\, (JF)^2 = 1 \rangle & \cong \mathbb{Z}/2 \\
\Gamma_{(16)} = & \langle J \,|\, J^2 = 1 \rangle & \cong \mathbb{Z}/2 \\
\Gamma_{(52)} = & \langle G \,|\, G^2 = 1 \rangle & \cong \mathbb{Z}/2 \\
\Gamma_{(42)} = & \langle L,\, G \,|\, L^3 = G^2 = (LG)^3 = 1 \rangle & \cong \mathcal{A}_4 \\
\Gamma_{(29)} = & \langle B,\, N \,|\, B^3 = N^3 = (BN)^2 = 1 \rangle & \cong \mathcal{A}_4 \\
\Gamma_{(45)} = & \langle D,\, H \,|\, D^2 = H^2 = (DH)^3 = 1 \rangle & \cong \mathcal{S}_3 \\
\Gamma_{(50)} = & \langle J,\, F \,|\, J^2 = F^3 = (JF)^2 = 1 \rangle & \cong \mathcal{S}_3 \\
\Gamma_{(37)} = & \langle L \,|\, L^3 = 1 \rangle & \cong \mathbb{Z}/3 \\
\Gamma_{(21)} = & \langle C \,|\, C^3 = 1 \rangle & \cong \mathbb{Z}/3 \\
\Gamma_{(39)} = & \langle F \,|\, F^3 = 1 \rangle & \cong \mathbb{Z}/3 \\
\Gamma_{(19)} = \Gamma_{(30)} = & \langle S \,|\, S^3 = 1 \rangle & \cong \mathbb{Z}/3.
\end{array}
$$

The following edge representatives have stabiliser type $\mathbb{Z}/2$:

$$
\begin{aligned}
\Gamma_{(42),(52)} &= \langle G \,|\, G^2 = 1 \rangle \\
\Gamma_{(18),(45)} &= \langle H \,|\, H^2 = 1 \rangle \\
\Gamma_{(45),(51)} &= \langle D \,|\, D^2 = 1 \rangle \\
\Gamma_{(43),(50)} &= \langle JF \,|\, (JF)^2 = 1 \rangle \\
\Gamma_{(16),(50)} &= \langle J \,|\, J^2 = 1 \rangle \\
\Gamma_{(29),(44)} &= \langle BN \,|\, (BN)^2 = 1 \rangle.
\end{aligned}
$$

In order to compare these with the vertex representative stabilisers, we note that there are respectively two elements of Γ sending the vertex (18) to (16), (44) to (43) and (51) to (52).

The following edge representatives have stabiliser type $\mathbb{Z}/3$:

$$\begin{aligned}
\Gamma_{(19),(30)} &= \langle S \mid S^3 = 1\rangle \\
\Gamma_{(21),(42)} &= \langle C \mid C^3 = 1\rangle \\
\Gamma_{(37),(42)} &= \langle L \mid L^3 = 1\rangle \\
\Gamma_{(26),(45)} &= \langle DH \mid (DH)^3 = 1\rangle \\
\Gamma_{(20),(23)} &= \langle A \mid A^3 = 1\rangle \\
\Gamma_{(29),(54)} &= \langle N \mid N^3 = 1\rangle \\
\Gamma_{(29),(47)} &= \langle B \mid B^3 = 1\rangle \\
\Gamma_{(39),(50)} &= \langle F \mid F^3 = 1\rangle
\end{aligned}$$

In order to determine the homomorphisms into the vertex stabilisers, we give a matrix for each of the following vertex identifications. The whole coset of matrices performing this vertex identification is obtained by multiplying the edge stabiliser from the right onto this matrix.

The matrix	sends vertex number	to vertex number
$\pm\begin{pmatrix} 7 & 5+5\omega \\ \omega & -12 \end{pmatrix}$	(26)	(21)
$\pm\begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix}$	(20)	(19)
$\pm\begin{pmatrix} 8-2\omega & 30+7\omega \\ 3+\omega & -12+2\omega \end{pmatrix}$	(23)	(37)
$\pm\begin{pmatrix} 5 & 3+3\omega \\ \omega & -10 \end{pmatrix}$	(54)	(39)
$\pm\begin{pmatrix} -3-\omega & 6-\omega \\ 5 & 1+2\omega \end{pmatrix}$	(47)	(30).

The remaining seventeen edge orbits have trivial stabiliser. There are fifteen orbits of 2-cells. The above cardinalities sum up to the equivariant Euler characteristic

$$\chi_\Gamma(X) = 4 + \frac{3}{2} + \frac{2}{12} + \frac{2}{6} + \frac{5}{3} - 17 - \frac{6}{2} - \frac{8}{3} + 15 = 0,$$

whence there is a check of our calculations in view of proposition 75.

The d^1 differentials in the equivariant spectral sequence

The 2-primary part $(d^1_{1,1})_{(2)}\colon (\mathbb{Z}/2)^5 \leftarrow (\mathbb{Z}/2)^6$ can be expressed by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

of rank 5. For $q > 1$ odd, $(d^1_{1,q})_{(2)}$ has preimage $(\mathbb{Z}/2)^6$ and full rank 6, and can be expressed by the following matrices with $2k$ zero rows to be inserted.

differential	target	matrix
$(d^1_{1,6k+3})_{(2)}$ and $(d^1_{1,6k+7})_{(2)}$	$(\mathbb{Z}/2)^{7+2k}$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$
$(d^1_{1,6k+5})_{(2)}$	$(\mathbb{Z}/2)^{7+2k+2}$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

On the 3-primary part, we observe that for $q \equiv 1 \mod 4$, $(d^1_{1,q})_{(3)}\colon (\mathbb{Z}/3)^7 \leftarrow (\mathbb{Z}/3)^8$ can be expressed by the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

of rank 7; and for $q \equiv 3 \mod 4$, $(d^1_{1,q})_{(3)}\colon (\mathbb{Z}/3)^9 \leftarrow (\mathbb{Z}/3)^8$ can be expressed by the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

of rank 8. So in both cases the highest possible rank occurs.

These informations yield the E^2-page of proposition 70.

Figure 3.2: Stabiliser types of the representative cells in some principal ideal domain cases.

m	**vertices**, total	$\mathbb{Z}^2$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$	**edges**, total	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	**2 − cells**
2	4	0	0	1	1	1	0	1	6	1	3	2	2
7	4	0	0	1	1	0	2	0	6	1	3	2	2
11	4	0	0	1	1	0	0	2	6	1	2	3	2
19	6	0	0	1	1	0	2	2	9	1	4	4	3
43	10	0	0	3	3	0	2	2	17	5	6	6	7
67	16	0	4	3	5	0	2	2	31	17	6	8	15
163	52	0	32	9	7	0	2	2	109	87	12	10	57

3.2 Results in the general case

Cells in the quotient space

In figures 3.2 through 3.4, we give the multiplicities with which the stabiliser types occur in the quotient space of our $\mathrm{PSL}_2(\mathcal{O}_{-m})$-cell complex, separately for every cell dimension. The 2-cells all have the trivial stabiliser, as we already know from lemma 30. The vertices of stabiliser type $\mathbb{Z}^2$ are the singular points; there is one less orbit of them than the class number of $\mathcal{O}_{-m}$ (see subsection 1.1.3).

Equivariant Euler characteristic

We will use the Euler characteristic to check the geometry of the quotient $\Gamma\backslash X$. Recall the following definitions and proposition.

Definition 72 (Euler characteristic)**.** *Suppose Γ' is a torsion-free group. Then we define its Euler characteristic as*

$$\chi(\Gamma') = \sum_i (-1)^i \dim \mathrm{H}_i(\Gamma'; \mathbb{Q}).$$

Suppose further that Γ' is a torsion-free subgroup of finite index in a group Γ. Then we define the Euler characteristic *of Γ as*

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{[\Gamma : \Gamma']}.$$

The latter formula is well-defined because of [11, IX.6.3].

Figure 3.3: Stabiliser types of the representative cells in the class number 2 cases.

m	**vertices**, total	$\mathbb{Z}^2$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$	**edges**, total	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbf{2-cells}$
5	7	1	0	2	2	2	0	0	11	4	5	2	5
6	7	1	0	2	2	0	0	2	11	4	3	4	5
10	11	1	0	4	4	2	0	0	19	8	7	4	9
13	17	1	0	6	4	2	4	0	29	10	13	6	13
15	5	1	0	2	2	0	0	0	8	4	2	2	4
22	27	1	12	8	4	0	0	2	55	40	9	6	29
35	7	1	0	2	4	0	0	0	13	7	2	4	7
37	53	1	24	14	8	2	4	0	109	78	21	10	57
51	11	1	0	2	4	0	0	4	19	7	4	8	9
58	121	1	88	18	12	2	0	0	255	222	21	12	135
91	19	1	4	6	4	0	4	0	39	23	10	6	21
115	25	1	8	8	8	0	0	0	55	39	8	8	31
123	31	1	8	4	14	0	0	4	63	39	6	18	33
187	57	1	40	4	8	0	0	4	125	107	6	12	69
235	77	1	48	16	12	0	0	0	171	143	16	12	95
267	93	1	52	10	26	0	0	4	205	163	12	30	113
403	181	1	152	10	14	0	4	0	403	373	14	16	223
427	219	1	184	12	18	0	4	0	477	441	16	20	259

Figure 3.4: Stabiliser types of the representative cells in some cases of higher class number.

m	**vertices**, total	$\mathbb{Z}^2$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$	**edges**, total	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbf{2-cells}$
34	53	3	28	12	6	2	0	2	113	89	16	8	63
139	44	2	26	7	5	0	2	2	97	79	10	8	55
151	91	6	70	12	1	0	2	0	195	179	14	2	110
191	170	12	146	8	4	0	0	0	362	350	8	4	204

Definition 73 (Equivariant Euler characteristic)**.** *Suppose X is a Γ-complex such that*

- *every isotropy group Γ_σ is of finite homological type;*
- *X has only finitely many cells mod Γ.*

Then we define the Γ-equivariant Euler characteristic *of X as*

$$\chi_\Gamma(X) := \sum_\sigma (-1)^{\dim\sigma} \chi(\Gamma_\sigma),$$

where σ runs over the orbit representatives of cells of X.

Proposition 74 ([11, IX.7.3 e'])**.** *Suppose X is a Γ-complex such that $\chi_\Gamma(X)$ is defined. If Γ is virtually torsion-free, then Γ is of finite homological type and $\chi(\Gamma) = \chi_\Gamma(X)$.*

Let now Γ be $\mathrm{PSL}_2\big(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}\big)$. Then the above proposition applies to X taken to be Flöge's (or still, Mendoza's) Γ-equivariant deformation retract of $\mathcal{H}$.
Using $\chi(\Gamma_\sigma) = \frac{1}{\mathrm{card}(\Gamma_\sigma)}$ for Γ_σ finite, the fact that the singular points have stabiliser $\mathbb{Z}^2$, and the torsion-free Euler characteristic

$$\chi(\mathbb{Z}^2) = \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}}(\mathrm{H}_i\, \mathbb{Z}^2) = 1 - 2 + 1 = 0,$$

we get the formula

$$\chi(\Gamma) = \sum_\sigma (-1)^{\dim\sigma} \frac{1}{\mathrm{card}(\Gamma_\sigma)},$$

where σ runs over the orbit representatives of cells of X with finite stabilisers.

Proposition 75. *The Euler characteristic $\chi(\Gamma)$ vanishes for the Bianchi groups.*

This is a well-known fact, see for instance [30] for a proof. We obtain a "mass formula"

$$0 = \sum_\sigma (-1)^{\dim\sigma} \frac{1}{\mathrm{card}(\Gamma_\sigma)},$$

which allows us to check the topology of the computed quotient space. We display the expressions it takes in our computations, in figures 3.5 through 3.7.

Figure 3.5: The mass formula in some cases of class number 1.

m	equivariant Euler characteristic
2	$(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{12})-(1+\frac{3}{2}+\frac{2}{3})+2=0,$
7	$(\frac{1}{2}+\frac{1}{3}+\frac{2}{6})-(1+\frac{3}{2}+\frac{2}{3})+2=0,$
11	$(\frac{1}{2}+\frac{1}{3}+\frac{2}{12})-(1+\frac{2}{2}+\frac{3}{3})+2=0,$
19	$(\frac{1}{2}+\frac{1}{3}+\frac{2}{6}+\frac{2}{12})-(1+\frac{4}{2}+\frac{4}{3})+3=0,$
43	$(\frac{3}{2}+\frac{3}{3}+\frac{2}{6}+\frac{2}{12})-(5+\frac{6}{2}+\frac{6}{3})+7=0,$
67	$(4+\frac{3}{2}+\frac{5}{3}+\frac{2}{6}+\frac{2}{12})-(17+\frac{6}{2}+\frac{8}{3})+15=0,$
163	$(32+\frac{9}{2}+\frac{7}{3}+\frac{2}{6}+\frac{2}{12})-(87+\frac{12}{2}+\frac{10}{3})+57=0.$

3.2.1 Results in degrees above the virtual cohomological dimension

Let $P_m^2(t) := \sum_{q=3}^{\infty} \dim_{\mathbb{F}_2} \mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}); \mathbb{Z}/2)t^q$ denote the Poincaré series of the $\mathbb{F}_2$-dimensions in the degrees above the virtual cohomological dimension. The following lemma will be useful to transform the Poincaré series into fractions of finite polynomials in t.

Lemma 76. *The equation* $\sum_{k=0}^{\infty}(ak+b)t^{ik+j} = \frac{t^j(b+t^i(a-b))}{(1-t^i)^2}$ *holds for all* $a, b, i, j \in \mathbb{N}$.

Proof.

$$\begin{aligned}
&(1-t^i)^2 \sum_{k=0}^{\infty}(ak+b)t^{ik+j} \\
&= \sum_{k=0}^{\infty}(ak+b)t^{ik+j} - 2\sum_{k=0}^{\infty}(ak+b)t^{i(k+1)+j} + \sum_{k=0}^{\infty}(ak+b)t^{i(k+2)+j} \\
&= \sum_{k=0}^{\infty}(ak+b)t^{ik+j} - 2\sum_{k=1}^{\infty}(a(k-1)+b)t^{ik+j} + \sum_{k=2}^{\infty}(a(k-2)+b)t^{ik+j}
\end{aligned}$$

Now the multiplicity in this term of $\sum_{k=2}^{\infty} t^{ik+j}$ is $ak+b-2ak+2a-2b+ak-2a+b=0$, so the above term equals $bt^j+(a+b)t^{i+j}-2bt^{j+i} = t^j(b-bt^i+at^i)$. □

The results displayed in figure 3.8 are explained by theorem 65.

Figure 3.6: The mass formula in the cases of class number 2.

m	equivariant Euler characteristic
5	$(\frac{2}{2}+\frac{2}{3}+\frac{2}{4})-(4+\frac{5}{2}+\frac{2}{3})+5=0,$
6	$(\frac{2}{2}+\frac{2}{3}+\frac{2}{12})-(4+\frac{3}{2}+\frac{4}{3})+5=0,$
10	$(\frac{4}{2}+\frac{4}{3}+\frac{2}{4})-(8+\frac{7}{2}+\frac{4}{3})+9=0,$
13	$(\frac{6}{2}+\frac{4}{3}+\frac{2}{4}+\frac{4}{6})-(10+\frac{13}{2}+\frac{6}{3})+13=0,$
15	$(\frac{2}{2}+\frac{2}{3})-(4+\frac{2}{2}+\frac{2}{3})+4=0,$
22	$(12+\frac{8}{2}+\frac{4}{3}+\frac{2}{12})-(40+\frac{9}{2}+\frac{6}{3})+29=0,$
35	$(+\frac{2}{2}+\frac{4}{3})-(7+\frac{2}{2}+\frac{4}{3})+7=0,$
37	$(24+\frac{14}{2}+\frac{8}{3}+\frac{2}{4}+\frac{4}{6})-(78+\frac{21}{2}+\frac{10}{3})+57=0,$
51	$(+\frac{2}{2}+\frac{4}{3}+\frac{4}{12})-(7+\frac{4}{2}+\frac{8}{3})+9=0,$
58	$(88+\frac{18}{2}+\frac{12}{3}+\frac{2}{4})-(222+\frac{21}{2}+\frac{12}{3})+135=0,$
91	$(4+\frac{6}{2}+\frac{4}{3}+\frac{4}{6})-(23+\frac{10}{2}+\frac{6}{3})+21=0,$
115	$(8+\frac{8}{2}+\frac{8}{3})-(39+\frac{8}{2}+\frac{8}{3})+31=0,$
123	$(8+\frac{4}{2}+\frac{14}{3}+\frac{4}{12})-(39+\frac{6}{2}+\frac{18}{3})+33=0,$
187	$(40+\frac{4}{2}+\frac{8}{3}+\frac{4}{12})-(107+\frac{6}{2}+\frac{12}{3})+69=0,$
235	$(48+\frac{16}{2}+\frac{12}{3})-(143+\frac{16}{2}+\frac{12}{3})+95=0,$
267	$(52+\frac{10}{2}+\frac{26}{3}+\frac{4}{12})-(163+\frac{12}{2}+\frac{30}{3})+113=0,$
403	$(152+\frac{10}{2}+\frac{14}{3}+\frac{4}{6})-(373+\frac{14}{2}+\frac{16}{3})+223=0,$
427	$(184+\frac{12}{2}+\frac{18}{3}+\frac{4}{6})-(441+\frac{16}{2}+\frac{20}{3})+259=0.$

Figure 3.7: The mass formula in some cases of higher class number.

m	equivariant Euler characteristic
34	$(28+\frac{12}{2}+\frac{6}{3}+\frac{2}{4}+\frac{2}{12})-(89+\frac{16}{2}+\frac{8}{3})+63=0,$
139	$(26+\frac{7}{2}+\frac{5}{3}+\frac{2}{6}+\frac{2}{12})-(79+\frac{10}{2}+\frac{8}{3})+55=0,$
151	$(70+\frac{12}{2}+\frac{1}{3}+\frac{2}{6})-(179+\frac{14}{2}+\frac{2}{3})+110=0,$
191	$(146+\frac{8}{2}+\frac{4}{3})-(350+\frac{8}{2}+\frac{4}{3})+204=0.$

In 3–torsion, we define the Poincaré series

$$P_m^3(t) := \sum_{q=3}^{\infty} \dim_{\mathbb{F}_3} \mathrm{H}_q(\mathrm{PSL}_2\big(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}\big); \mathbb{Z}/3)t^q.$$

The results displayed in figure 3.9 are explained by observation 68.

Figure 3.8: The results for the Poincaré series $P_m^2(t)$ in 2–torsion

m	reduced 2-torsion subgraph	$P_m^2(t)$
7, 15, 35, 39, 91, 115, 191		$\frac{-2t^3}{t-1}$
87, 95, 151, 155, 159, 403		$\frac{-2t^3}{t-1}$
46		$2\left(\frac{-2t^3}{t-1}\right)$
427		$3\left(\frac{-2t^3}{t-1}\right)$
235		$3\left(\frac{-2t^3}{t-1}\right)$
11, 19, 43, 67, 139, 163		$\frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)}$
51, 123, 187, 267		$2\left(\frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)}\right)$
6, 22		$\frac{-2t^3}{t-1}+\frac{-t^3(t^3-2t^2+2t-3)}{(t-1)^2(t^2+t+1)}$
5 , 10, 13		$\frac{-t^3(3t-5)}{(t-1)^2}$
29, 58		$\frac{-t^3(3t-5)}{(t-1)^2}$
37		$\frac{-t^3(3t-5)}{(t-1)^2}+2\left(\frac{-2t^3}{t-1}\right)$
2		$\frac{-2t^3(t^3-t^2-2)}{(t-1)^2(t^2+t+1)}$
34		$2\left(\frac{-2t^3(t^3-t^2-2)}{(t-1)^2(t^2+t+1)}\right)+2\left(\frac{-2t^3}{t-1}\right)$

Figure 3.9: The results for the Poincaré series $P^3_m(t)$ in 3–torsion

m	reduced 3-torsion subgraph	$P^3_m(t)$
2, 5, 6, 10, 11, 15, 22, 29, 34, 35, 46, 51, 58 87, 95, 115, 123, 155, 159, 187, 191, 235, 267		$\frac{-2t^3}{t-1}$
7, 19, 43, 67, 139, 151, 163		$\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$
13, 37, 91, 403, 427		$2\left(\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}\right)$
39		$\frac{-2t^3}{t-1}+\frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$

The cases $m = 1$ and $m = 3$.

In the cases of additional units in the ring of integers, namely $m = 1$ and $m = 3$, we have to take care of the restrictions stated at the beginning of section 2.2. So, we treat them separately in a paper and pencil computation following the article of Schwermer and Vogtmann. We use this occasion to clean up some typographical impacts of the publication process (presumably the recomposition) to the editor's version of [37]. Some of the implied corrections have already been suggested by Berkove [7].

For the Gaussian integers $\mathcal{O}_{-1} = \mathbb{Z}[\sqrt{-1}]$, the E^1-page is concentrated in the columns $p = 0$ and $p = 1$, given as follows.

$$
\begin{array}{c|ccc}
q = 12n+12 & (\mathbb{Z}/2)^{8n+8} & & 0 \\
q = 12n+11 & (\mathbb{Z}/2)^{8n+12} \oplus (\mathbb{Z}/3)^3 & \hookleftarrow & (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \\
q = 12n+10 & (\mathbb{Z}/2)^{8n+6} & & 0 \\
q = 12n+9 & (\mathbb{Z}/2)^{8n+10} \oplus \mathbb{Z}/3 & \leftarrow & (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \\
q = 12n+8 & (\mathbb{Z}/2)^{8n+6} & & 0 \\
q = 12n+7 & (\mathbb{Z}/2)^{8n+8} \oplus (\mathbb{Z}/3)^3 & \hookleftarrow & (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \\
q = 12n+6 & (\mathbb{Z}/2)^{8n+4} & & 0 \\
q = 12n+5 & (\mathbb{Z}/2)^{8n+8} \oplus \mathbb{Z}/3 & \leftarrow & (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \\
q = 12n+4 & (\mathbb{Z}/2)^{8n+2} & & 0 \\
q = 12n+3 & (\mathbb{Z}/2)^{8n+6} \oplus (\mathbb{Z}/3)^3 & \hookleftarrow & (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \\
q = 12n+2 & (\mathbb{Z}/2)^{8n+2} & & 0 \\
q = 12n+1 & (\mathbb{Z}/2)^{8n+4} \oplus \mathbb{Z}/3 & \leftarrow & (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \\
q = 0 & \mathbb{Z} & & 0, \\
\hline
 & p = 0 & & p = 1
\end{array}
$$

Here, the symbol "$\hookleftarrow$" stands for a monomorphism of abelian groups; and all the three

symbols "$\leftarrow$" stand for homomorphisms of full rank over $\mathbb{Z}/2$, and of rank 1 over $\mathbb{Z}/3$. Hence,

$$\mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-1});\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^{8n+8}, & q = 12n+12, \\ (\mathbb{Z}/2)^{8n+10} \oplus \mathbb{Z}/3, & q = 12n+11, \\ (\mathbb{Z}/2)^{8n+6} \oplus \mathbb{Z}/3, & q = 12n+10, \\ (\mathbb{Z}/2)^{8n+8}, & q = 12n+9, \\ (\mathbb{Z}/2)^{8n+6}, & q = 12n+8, \\ (\mathbb{Z}/2)^{8n+6} \oplus \mathbb{Z}/3, & q = 12n+7, \\ (\mathbb{Z}/2)^{8n+4} \oplus \mathbb{Z}/3, & q = 12n+6, \\ (\mathbb{Z}/2)^{8n+6}, & q = 12n+5, \\ (\mathbb{Z}/2)^{8n+2}, & q = 12n+4, \\ (\mathbb{Z}/2)^{8n+4} \oplus \mathbb{Z}/3, & q = 12n+3, \\ (\mathbb{Z}/2)^{8n+2} \oplus \mathbb{Z}/3, & q = 12n+2, \\ (\mathbb{Z}/2)^{8n+2}, & q = 12n+1. \end{cases}$$

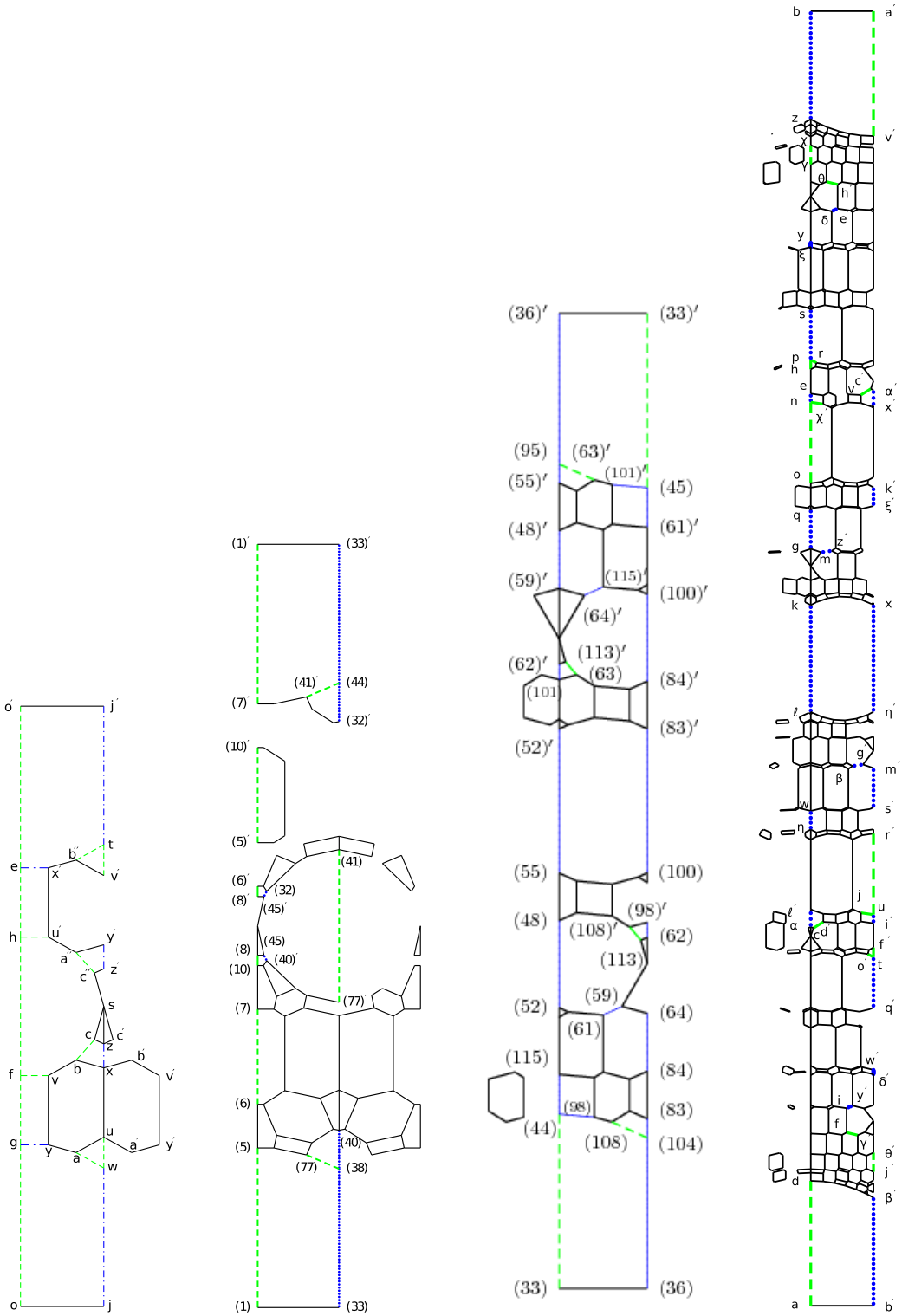
In the case of the Eisentein integers $\mathcal{O}_{-3} = \mathbb{Z}[\omega]$, with $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3} = \mathrm{e}^{\frac{2\pi i}{3}}$, the E^1-page is concentrated in the columns $p = 0$ and $p = 1$, given as follows.

$q = 12n+14$	$(\mathbb{Z}/2)^{4n+6}$		0
$q = 12n+13$	$(\mathbb{Z}/2)^{4n+5} \oplus (\mathbb{Z}/3)^2$	$\hookleftarrow$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
$q = 12n+12$	$(\mathbb{Z}/2)^{4n+4}$		0
$q = 12n+11$	$(\mathbb{Z}/2)^{4n+7} \oplus (\mathbb{Z}/3)^3$	$\hookleftarrow$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
$q = 12n+10$	$(\mathbb{Z}/2)^{4n+2}$		0
$q = 12n+9$	$(\mathbb{Z}/2)^{4n+5} \oplus (\mathbb{Z}/3)^2$	$\hookleftarrow$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
$q = 12n+8$	$(\mathbb{Z}/2)^{4n+4}$		0
$q = 12n+7$	$(\mathbb{Z}/2)^{4n+3} \oplus (\mathbb{Z}/3)^3$	$\hookleftarrow$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
$q = 12n+6$	$(\mathbb{Z}/2)^{4n+2}$		0
$q = 12n+5$	$(\mathbb{Z}/2)^{4n+5} \oplus (\mathbb{Z}/3)^2$	$\hookleftarrow$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
$q = 12n+4$	$(\mathbb{Z}/2)^{4n}$		0
$q = 12n+3$	$(\mathbb{Z}/2)^{4n+3} \oplus (\mathbb{Z}/3)^3$	$\hookleftarrow$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
$q = 2$	$(\mathbb{Z}/2)^2$		0
$q = 1$	$\mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$	$\leftarrow$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$
$q = 0$	$\mathbb{Z}$		$0,$
	$p = 0$		$p = 1$

The only of the above arrows that does not stand for a monomorphism, represents a

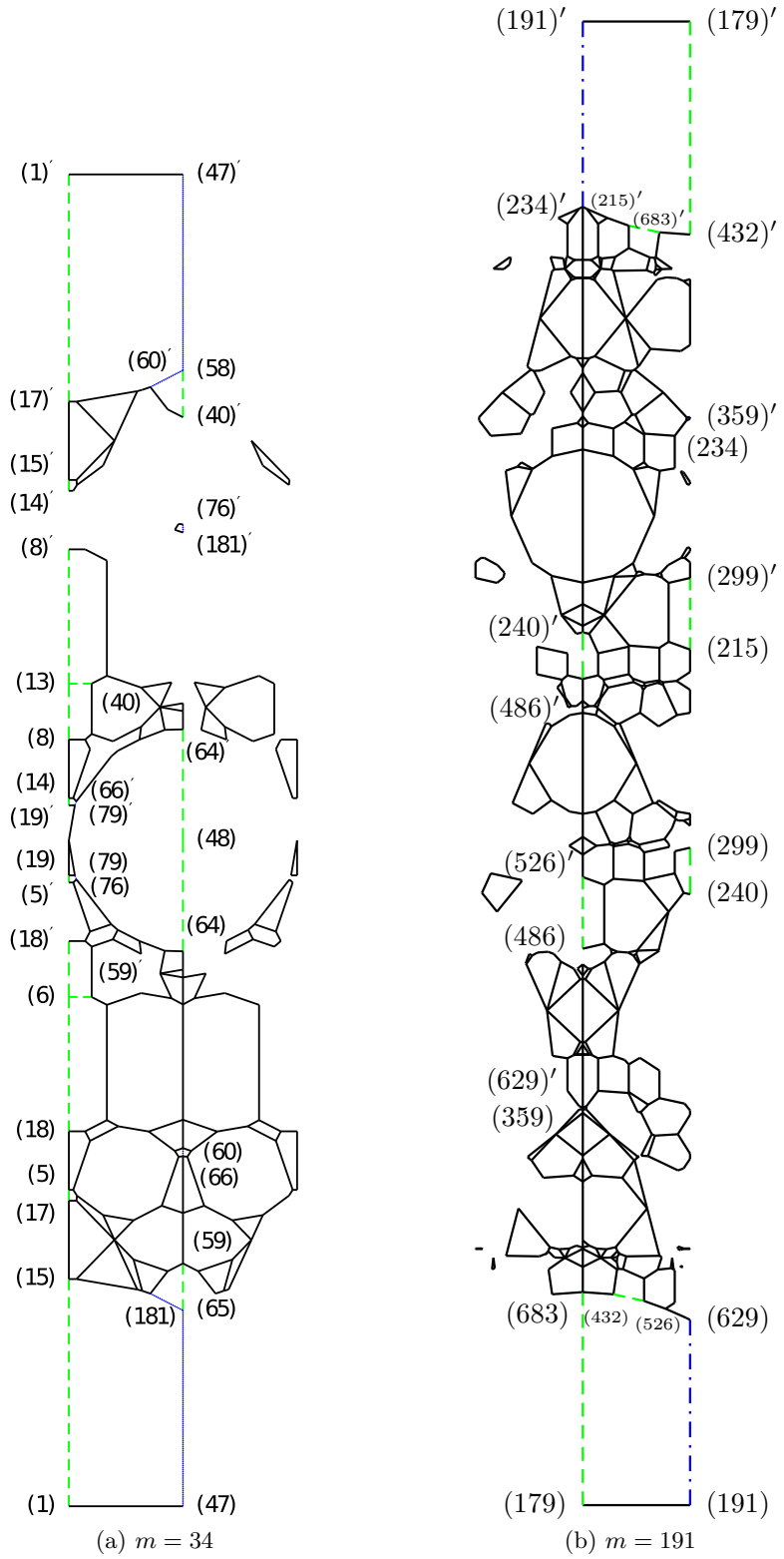
homomorphism of rank 1 over $\mathbb{Z}/3$, and also of rank 1 over $\mathbb{Z}/2$. Therefore,

$$\mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-3});\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^{4n+6}, & q = 12n+14, \\ (\mathbb{Z}/2)^{4n+3} \oplus \mathbb{Z}/3, & q = 12n+13, \\ (\mathbb{Z}/2)^{4n+4}, & q = 12n+12, \\ (\mathbb{Z}/2)^{4n+5} \oplus (\mathbb{Z}/3)^2, & q = 12n+11, \\ (\mathbb{Z}/2)^{4n+2}, & q = 12n+10, \\ (\mathbb{Z}/2)^{4n+3} \oplus \mathbb{Z}/3, & q = 12n+9, \\ (\mathbb{Z}/2)^{4n+4}, & q = 12n+8, \\ (\mathbb{Z}/2)^{4n+1} \oplus (\mathbb{Z}/3)^2, & q = 12n+7, \\ (\mathbb{Z}/2)^{4n+2}, & q = 12n+6, \\ (\mathbb{Z}/2)^{4n+3} \oplus \mathbb{Z}/3, & q = 12n+5, \\ (\mathbb{Z}/2)^{4n}, & q = 12n+4, \\ (\mathbb{Z}/2)^{4n+1} \oplus (\mathbb{Z}/3)^2, & q = 12n+3, \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3, & q = 1. \end{cases}$$



(a) $m = 13$ (b) $m = 22$ (c) $m = 123$ (d) $m = 427$

Figure 3.10: The fundamental domain for the Flöge cellular complex, with its 2-torsion subgraph (green, dashed edges) and its 3-torsion subgraph (blue, dotted edges), in some cases of class number 2.



(a) $m = 34$ (b) $m = 191$

Figure 3.11: The fundamental domain for the Flöge cellular complex, with its 2-torsion subgraph (green, dashed edges) and its 3-torsion subgraph (blue, dotted edges), in some cases of class numbers 4 ($m = 34$) and 13 ($m = 191$).

Chapter 4

Equivariant K-homology of the classifying space for proper actions

In this chapter, we compute the Bredon homology of the Bianchi groups, from which we deduce their equivariant K-homology. Then we use the fact that the Baum/Connes assembly map from the equivariant K-homology to the K-theory of the reduced C^*-algebras of the Bianchi groups is an isomorphism, in order to obtain the latter K-theory.

We will only outline the meaning of this assembly map. For any discrete group G, we define its reduced C^*-algebra, $C_r^*(G)$, and the K-theory of the latter as in [39]. According to Georges Skandalis, the algebras $C_r^*(G)$ are amongst the most important and natural examples of C^*-algebras. It is difficult to compute their K-theory directly, so we use a homomorphism constructed by Paul Baum and Alain Connes [4],

$$\mu_i : K_i^G(\underline{\mathrm{E}}G) \longrightarrow K_i(C_r^*(G)), \qquad i \in \mathbb{N} \cup \{0\},$$

called the (analytical) assembly map. A model for the classifying space for proper actions, written $\underline{\mathrm{E}}G$, is in the case of the Bianchi groups given by hyperbolic three-space (being isomorphic to their associated symmetric space), the stabilisers of their action on it being finite. For the definition of its equivariant K-homology $K_i^G(\,)$, we refer to [28]. The Baum/Connes conjecture now states that the assembly map is an isomorphism. The conjecture can be stated more generally, see [4].

The Baum/Connes conjecture implies several important conjectures in topology, geometry, algebra and functional analysis. Groups for which the assembly map is surjective verify the Kaplansky/Kadison conjecture on the idempotents. Groups for which it is injective satisfy the strong Novikov conjecture and one sense of the Gromov/Lawson/Rosenberg conjecture, namely the sense predicting the vanishing of the higher $\hat{A}$-genera.

In this chapter we prove the following result (corollary 81). Let $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$.

Then, for $\mathcal{O}_{-m}$ principal, the equivariant K-homology of Γ has isomorphy types

	$m=1$	$m=2$	$m=3$	$m=7$	$m=11$	$m\in\{19,43,67,163\}$
$K_0^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}^6$	$\mathbb{Z}^5\oplus\mathbb{Z}/2$	$\mathbb{Z}^5\oplus\mathbb{Z}/2$	$\mathbb{Z}^3$	$\mathbb{Z}^4\oplus\mathbb{Z}/2$	$\mathbb{Z}^{\beta_2}\oplus\mathbb{Z}^3\oplus\mathbb{Z}/2$
$K_1^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}$	$\mathbb{Z}^3$	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}\oplus\mathbb{Z}^{\beta_1}$,

where the Betti numbers are

m	19	43	67	163
β_1	1	2	3	7
β_2	0	1	2	6,

and the remainder of the equivariant K-homology of Γ is given by Bott 2-periodicity.

The assembly map for the Bianchi groups. Julg and Kasparov [25] have shown that the Baum/Connes conjecture is verified for all the discrete subgroups of $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$. The Lobachevski model of hyperbolic three-space gives a natural identification of its orientation preserving isometries, with matrices in $\mathrm{PSO}(3,1)$. So especially, the assembly map is an isomorphism for the Bianchi groups; and we have obtained the isomorphism type of $K_i(C_r^*(\Gamma))$.

A more complicated way to check that the Baum/Connes conjecture is verified for the Bianchi groups, is using "a-T-menability" in the sense of Gromov, also called the Haagerup property [13]. Cherix, Martin and Valette prove in [14] that among other groups, the Bianchi groups admit a proper action on a space "with measured walls". In [13], Haglund, Paulin and Valette show that groups with such an action have the Haagerup property. Finally Higson and Kasparov [23] prove that the latter property implies the bijectivity of several assembly maps, and in particular the Baum/Connes conjecture.

4.1 The Bredon chain complex

As described in [28], the equivariant K-homology of the classifying space for proper actions $K_*^G(\underline{E}G)$ can be computed by means of the Bredon homology with coefficients in the complex representation ring, $\mathrm{H}_*^{\mathfrak{Fin}}(G;R_{\mathbb{C}})$. More precisely, when we have a classifying space for proper actions of dimension at most 2, then the Atiyah/Hirzebruch spectral sequence from its Bredon homology to its equivariant K-homology degenerates on the E^2-page and directly yields the following.

Theorem 77 ([28])**.** *Let G be an arbitrary group such that* $\dim \underline{E}G \leqslant 2$. *Then there is a natural short exact sequence*

$$0\to \mathrm{H}_0^{\mathfrak{Fin}}(G;R_{\mathbb{C}})\to K_0^G(\underline{E}G)\to \mathrm{H}_2^{\mathfrak{Fin}}(G;R_{\mathbb{C}})\to 0$$

and a natural isomorphism $\mathrm{H}_1^{\mathfrak{Fin}}(G;R_{\mathbb{C}})\cong K_1^G(\underline{E}G)$.

We will follow Sánchez-García's treatment [35, 36].
Consider the Bianchi group $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$. Denote by Γ_σ the stabiliser of a cell σ, and by $R_{\mathbb{C}}(G)$ the complex representation ring of a group G. We will compute case by case the Bredon chain complex

$$0 \longrightarrow \bigoplus_{\sigma\in\Gamma\backslash X^{(2)}} R_{\mathbb{C}}(\Gamma_\sigma) \xrightarrow{\Psi_2} \bigoplus_{\sigma\in\Gamma\backslash X^{(1)}} R_{\mathbb{C}}(\Gamma_\sigma) \xrightarrow{\Psi_1} \bigoplus_{\sigma\in\Gamma\backslash X^{(0)}} R_{\mathbb{C}}(\Gamma_\sigma) \longrightarrow 0,$$

of our refined Γ-cell complex $X^{(p)}$. As X is a classifying space for proper Γ-actions, the homology of this Bredon chain complex is the Bredon homology $\mathrm{H}_p^{\mathfrak{Fin}}(\Gamma; R_{\mathbb{C}})$ of Γ [35]. Finally, as we can calculate a set of representatives of the elements of finite order for the conjugation in the Bianchi groups, the following lemma provides a check on our computations.

Lemma 78 ([28])**.** *Let G be an arbitrary group and write* $\mathrm{FC}(G)$ *for the set of conjugacy classes of elements of finite order in G. Then there is an isomorphism*

$$\mathrm{H}_0^{\mathfrak{Fin}}(G; R_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[\mathrm{FC}(G)].$$

4.2 The morphisms induced on the representation rings

We consider the complex representation ring of a finite group as the free $\mathbb{Z}$-module the basis of which are the irreducible characters of the group. These irreducible characters are given by the following character tables for the finite subgroups of the Bianchi groups.

The trivial group admits only the representations by the identity matrix, so its only irreducible character is given by the trace 1. For the other finite subgroups of the Bianchi groups, we make the following choices of conjugacy representatives of their elements, and list below the irreducible characters by specifying the value they take on these elements.

- For $\mathbb{Z}/2 = \langle\, g \,|\, g^2 = 1 \rangle$, we have the two irreducible characters

$\mathbb{Z}/2$	1	g
ρ_1	1	1
ρ_2	1	-1

- For $\mathbb{Z}/3 = \langle\, h \,|\, h^3 = 1 \,\rangle$, let $j = e^{\frac{2\pi i}{3}}$. Then we have

$\mathbb{Z}/3$	1	h	h^2
σ_1	1	1	1
σ_2	1	j	j^2
σ_3	1	j^2	j

- For the Klein four-group $\mathcal{D}_2 = \langle\, a,\, b \,|\, a^2 = b^2 = (ab)^2 = 1 \,\rangle$, the character table is

$\mathcal{D}_2$	1	a	b	ab
ξ_1	1	1	1	1
ξ_2	1	-1	-1	1
ξ_3	1	-1	1	-1
ξ_4	1	1	-1	-1

- We write the symmetric group $\mathcal{S}_3 = \langle\, (12), (123) \mid \text{ cycle relations } \rangle$ in cycle type notation. Then its character table is

$\mathcal{S}_3$	1	(12)	(123)
π_1	1	1	1
π_2	1	-1	1
π_3	2	0	-1

- We also write the alternating group $\mathcal{A}_4$ in cycle type notation, and let again $j = e^{\frac{2\pi i}{3}}$. Then we have

$\mathcal{A}_4$	1	$(12)(34)$	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	j	j^2
χ_3	1	1	j^2	j
χ_4	3	-1	0	0

For an injective morphism $H \hookrightarrow G$ of finite groups, we compute as follows the map induced on the complex representation rings. We restrict the characters ϕ_i of G to the image of H, and write $\phi_i \downarrow$ for the restricted character. Then we consider the scalar products $(\phi_i \downarrow | \tau_j)$ with the characters τ_j of H. By Frobenius reciprocity, the induced map of representation rings is given by the matrix $(\phi_i \downarrow | \tau_j)_{i,j}$.
When H is the trivial group, this matrix is the one-column-matrix of values of the characters of G on the neutral element.
For the non-trivial inclusions amongst finite subgroups of the Bianchi groups, let us compute this matrix case by case.

- Any inclusion $\mathbb{Z}/\ell \hookrightarrow \mathcal{S}_3$ maps the generator of $\mathbb{Z}/\ell$ to the only conjugacy class of elements of order ℓ in $\mathcal{S}_3$. So it induces the map obtained by restricting to the two cycles (1) and, for $\ell = 2$, (12),

$\mathbb{Z}/2 \hookrightarrow \mathcal{S}_3$	(1)	(12)	$(\pi_i \downarrow \| \rho_1)$	$(\pi_i \downarrow \| \rho_2)$
$\pi_1 \downarrow$	1	1	1	0
$\pi_2 \downarrow$	1	-1	0	1
$\pi_3 \downarrow$	2	0	1	1

respectively (123) for $\ell = 3$.

$\mathbb{Z}/3 \hookrightarrow \mathcal{S}_3$	(1)	(123)	$(\pi_i \downarrow \| \sigma_1)$	$(\pi_i \downarrow \| \sigma_2)$	$(\pi_i \downarrow \| \sigma_3)$
$\pi_1 \downarrow$	1	1	1	0	0
$\pi_2 \downarrow$	1	1	1	0	0
$\pi_3 \downarrow$	2	-1	0	1	1

- There are three possible inclusions $\mathbb{Z}/2 \hookrightarrow \mathcal{D}_2$, namely $g \mapsto a$, $g \mapsto b$ and $g \mapsto ab$. For each of these inclusions, we restrict to the image of $\mathbb{Z}/2$:

$g \mapsto a$	1	a	$(\xi_i \downarrow \lvert\rho_1)$	$(\xi_i \downarrow \lvert\rho_2)$
$\xi_1 \downarrow$	1	1	1	0
$\xi_2 \downarrow$	1	-1	0	1
$\xi_3 \downarrow$	1	-1	0	1
$\xi_4 \downarrow$	1	1	1	0

further

$g \mapsto b$	1	b	$(\xi_i \downarrow \lvert\rho_1)$	$(\xi_i \downarrow \lvert\rho_2)$
$\xi_1 \downarrow$	1	1	1	0
$\xi_2 \downarrow$	1	-1	0	1
$\xi_3 \downarrow$	1	1	1	0
$\xi_4 \downarrow$	1	-1	0	1

and

$g \mapsto ab$	1	ab	$(\xi_i \downarrow \lvert\rho_1)$	$(\xi_i \downarrow \lvert\rho_2)$
$\xi_1 \downarrow$	1	1	1	0
$\xi_2 \downarrow$	1	1	1	0
$\xi_3 \downarrow$	1	-1	0	1
$\xi_4 \downarrow$	1	-1	0	1

- Any inclusion $\mathbb{Z}/2 \hookrightarrow \mathcal{A}_4$ maps the generator of $\mathbb{Z}/2$ to the only conjugacy class of elements of order 2 in $\mathcal{A}_4$. So it induces the map obtained by restricting to the two cycles (1) and (12)(34).

$\mathbb{Z}/2 \hookrightarrow \mathcal{A}_4$	1	$(12)(34)$	$(\chi_i \downarrow \lvert\rho_1)$	$(\chi_i \downarrow \lvert\rho_2)$
$\chi_1 \downarrow$	1	1	1	0
$\chi_2 \downarrow$	1	1	1	0
$\chi_3 \downarrow$	1	1	1	0
$\chi_4 \downarrow$	3	-1	1	2

An inclusion $\mathbb{Z}/3 \hookrightarrow \mathcal{A}_4$ can either map the generator h of $\mathbb{Z}/3$ to the conjugacy class of (123), or to the conjugacy class of its square (132). So we have the two possibilities

$h \mapsto (123)$	1	(123)	(132)	$(\chi_i \downarrow \lvert\sigma_1)$	$(\chi_i \downarrow \lvert\sigma_2)$	$(\chi_i \downarrow \lvert\sigma_3)$
$\chi_1 \downarrow$	1	1	1	1	0	0
$\chi_2 \downarrow$	1	j	j^2	0	0	1
$\chi_3 \downarrow$	1	j^2	j	0	1	0
$\chi_4 \downarrow$	3	0	0	1	1	1

and

$h \mapsto (132)$	1	(132)	(123)	$(\chi_i \downarrow \lvert\sigma_1)$	$(\chi_i \downarrow \lvert\sigma_2)$	$(\chi_i \downarrow \lvert\sigma_3)$
$\chi_1 \downarrow$	1	1	1	1	0	0
$\chi_2 \downarrow$	1	j^2	j	0	1	0
$\chi_3 \downarrow$	1	j	j^2	0	0	1
$\chi_4 \downarrow$	3	0	0	1	1	1

***Remark* 79.** We have seen in observation 66 that there are only two homeomorphy types of connected components which can occur in the 3-torsion subgraph. An observation which facilitates our Bredon homology computations can be made on the connected components which are homeomorphic to a closed interval. Let C be such a component. Concerning our refined cell complex, by theorem 39 we have two inclusions of $\mathbb{Z}/3$ into each copy of $\mathcal{A}_4$, coming from adjacent edges. Given a copy of $\mathcal{A}_4$ in our Bianchi group stabilising a vertex v on C, we attribute the cycle $(123) \in \mathcal{A}_4$ to the image of the generator of one of the two stabilisers of type $\mathbb{Z}/3$ of edges adjacent to v. Then we choose the preimage of (123) under the inclusion of the other copy of $\mathbb{Z}/3$ to be the generator h. Since C is homeomorphic to a closed interval, v is the only point at which h can be related to our first copy of $\mathbb{Z}/3$.
So, we have obtained bases in which all the inclusions $\mathbb{Z}/3 \hookrightarrow \mathcal{A}_4$ induce the first of the two above possibilities.

4.3 Computations of the Bredon chain complex

The case $m = 19$.

On its fundamental domain (figure 3.1a), we find the only identification $g \cdot (4,7) = (4',7')$ with $g \in \Gamma$. Here we write $(4,7)$ for the edge with vertices (4) and (7), etc. We have the isomorphy types of vertex stabilisers
$\Gamma_{(4)} \cong \mathbb{Z}/2, \qquad \Gamma_{(7)} \cong \mathbb{Z}/3, \qquad \Gamma_{(9)} \cong \mathcal{A}_4 \cong \Gamma_{(12)}, \qquad \Gamma_{(11)} \cong \mathcal{S}_3 \cong \Gamma_{(10)};$
the edge stabilisers of the respectively four edges in the ℓ-torsion subgraph are of isomorphy type $\mathbb{Z}/\ell$. The trivially stabilised cells in the quotient complex are the edge $(4,7)$ and the three 2-cells. Thus our Bredon chain complex is

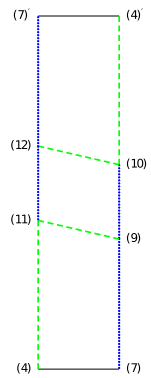


Figure 4.1: Fundamental domain in the case $m = 19$.

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\Psi_2} \mathbb{Z}^{4\cdot 3+4\cdot 2+1} \xrightarrow{\Psi_1} \mathbb{Z}^{2\cdot 4+2\cdot 3+3+2} \longrightarrow 0,$$

where (with t for the transpose), a matrix for Ψ_2^{t} is given as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

of rank 3, with elementary divisor 1 of multiplicity three.

The matrix for Ψ_1 is given as follows, where $\begin{smallmatrix}4\\7\end{smallmatrix}$ indicates the columns coming from the edge $(4,7)$, etc., and the symbol $\bullet$ is used for the value -1 in order to save horizontal space. We omit the zero blocks.

	4 7	12 10		10 4′		4 11		11 9		11 12			12 7′			7 9			9 10		
(4)	$\bullet$			1	0	$\bullet$	0														
	$\bullet$			0	1	0	$\bullet$														
(7)	1												1	0	0	$\bullet$	0	0			
	1												0	1	0	0	$\bullet$	0			
	1												0	0	1	0	0	$\bullet$			
(11)						1	0	$\bullet$	0	$\bullet$	0	0									
						0	1	0	$\bullet$	$\bullet$	0	0									
						1	1	$\bullet$	$\bullet$	0	$\bullet$	$\bullet$									
(10)		1	0	$\bullet$	0														1	0	0
		0	1	0	$\bullet$														1	0	0
		1	1	$\bullet$	$\bullet$														0	1	1
(9)								1	0							1	0	0	$\bullet$	0	0
								1	0							0	0	1	0	0	$\bullet$
								1	0							0	1	0	0	$\bullet$	0
								1	2							1	1	1	$\bullet$	$\bullet$	$\bullet$
(12)		$\bullet$	0							1	0	0	$\bullet$	0	0						
		$\bullet$	0							0	0	1	0	0	$\bullet$						
		$\bullet$	0							0	1	0	0	$\bullet$	0						
		$\bullet$	-2							1	1	1	$\bullet$	$\bullet$	$\bullet$						

This matrix has rank 16, the elementary divisor 2 of multiplicity one, and elementary divisor 1 of multiplicity fifteen. Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-19}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 2, \\ \mathbb{Z}^2, & p = 1, \\ \mathbb{Z}^3 \oplus \mathbb{Z}/2, & p = 0. \end{cases}$$

The case $m = 1$.

The computation of the quotient space can be found in [21] and [27]. Our Bredon chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Psi_2} \mathbb{Z}^{2+3+3+2} \xrightarrow{\Psi_1} \mathbb{Z}^{3+4+3+4} \longrightarrow 0,$$

where $\Psi_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ has rank 1, and elementary divisor 1. The matrix

$$\Psi_1 = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
-1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1
\end{pmatrix}$$

has rank 8, and all of its elementary divisors 1.

Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-1}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 2, \\ \mathbb{Z}, & p = 1, \\ \mathbb{Z}^6, & p = 0. \end{cases}$$

The case $m = 2$.

Our Bredon chain complex is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\Psi_2} \mathbb{Z}^{13} \xrightarrow{\Psi_1} \mathbb{Z}^{13} \longrightarrow 0,$$

where

$$\Psi_2^{\mathrm{t}} = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

has rank 2, and twice the elementary divisor 1.

The matrix for Ψ_1 is given as follows,

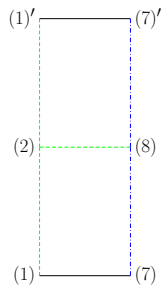


Figure 4.2: Fundamental domain in the case $m = 2$.

	$(1,7)$	$(7,8)$			$(8,7')$			$(8,2)$		$(1',2)$		$(2,1)$	
(1)	-1									-1	0	1	0
	-1									0	-1	0	1
(2)								1	0	1	0	-1	0
								0	1	0	1	-1	0
								0	1	1	0	0	-1
								1	0	0	1	0	-1
(7)	1	-1	0	0	1	0	0						
	1	0	-1	0	0	1	0						
	1	0	0	-1	0	0	1						
(8)		1	0	0	-1	0	0	-1	0				
		0	0	1	0	0	-1	-1	0				
		0	1	0	0	-1	0	-1	0				
		1	1	1	-1	-1	-1	-1	-2				

it has rank 8, elementary divisor 2 of multiplicity one, and elementary divisor 1 of multiplicity seven. Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-2}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 2, \\ \mathbb{Z}^3, & p = 1, \\ \mathbb{Z}^5 \oplus \mathbb{Z}/2, & q = 0. \end{cases}$$

The case $m = 3$.

The computation of the quotient space can be found in [21] and [27]. Our Bredon chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Psi_2} \mathbb{Z}^{2+2+3} \xrightarrow{\Psi_1} \mathbb{Z}^{4+4+3} \longrightarrow 0,$$

where $\Psi_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ has rank 1, and elementary divisor 1. The matrix

$$\Psi_1 = \begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

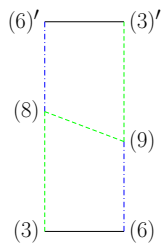
has rank 6, elementary divisor 2 of multiplicity one, and elementary divisor 1 of multiplicity five.

Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-3}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 1, \\ \mathbb{Z}^5 \oplus \mathbb{Z}/2, & p = 0. \end{cases}$$

The case $m = 7$.

A fundamental domain is given as



Our Bredon chain complex is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\Psi_2} \mathbb{Z}^{13} \xrightarrow{\Psi_1} \mathbb{Z}^{11} \longrightarrow 0,$$

where

$$\Psi_2^{\mathrm{t}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

has rank 2, and twice the elementary divisor 1. The matrix

$$\Psi_1 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

has rank 8, with the only occurring elementary divisor 1. Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-7}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 2, \\ \mathbb{Z}^3, & p = 1, \\ \mathbb{Z}^3, & q = 0. \end{cases}$$

The case $m = 11$.

Our Bredon chain complex is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\Psi_2} \mathbb{Z}^{14} \xrightarrow{\Psi_1} \mathbb{Z}^{13} \longrightarrow 0,$$

where

$$\Psi_2^{\mathrm{t}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

has rank 2, and twice the elementary divisor 1. The matrix Ψ_1 is

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}$$

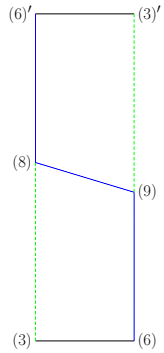


Figure 4.3: Fundamental domain in the case $m = 11$.

and has rank 9, elementary divisor 2 of multiplicity one, and elementary divisor 1 of multiplicity eight. Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-11}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 2, \\ \mathbb{Z}^3, & p = 1, \\ \mathbb{Z}^4 \oplus \mathbb{Z}/2, & q = 0. \end{cases}$$

The case $m = 43$.

The fundamental domain can be found in figure 3.1b. Our Bredon chain complex takes the form

$$0 \longrightarrow \mathbb{Z}^7 \xrightarrow{\Psi_2} \mathbb{Z}^{35} \xrightarrow{\Psi_1} \mathbb{Z}^{29} \longrightarrow 0,$$

where Ψ_2 is given by the transpose of the matrix

$$\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&1&1&1&0&0&0&0&\bullet&\bullet&\bullet&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&\bullet&\bullet&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1&0\\
0&0&0&0&0&0&\bullet&\bullet&\bullet&\bullet&\bullet&0&0&0&0&1&0&0&\bullet&0&0&0&0&0&0&0&\bullet&\bullet&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&\bullet&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bullet\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bullet&\bullet&1&1&0&0&\bullet&\bullet&\bullet&1\\
0&0&0&0&1&1&1&1&0&0&0&\bullet&\bullet&\bullet&\bullet&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&\bullet&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\bullet
\end{pmatrix}$$

Here we write the symbol $\bullet$ for the value -1 in order to save space. This matrix is of rank 6; it has elementary divisor 1 of multiplicity six.

The transpose of the matrix for Ψ_1 is given as follows,

$$\begin{pmatrix}
\bullet&&&&&&1&&&1&&&&&&&&&&&&&&&&&&&\\
&\bullet&&&&&&&1&1&&&&&&&&&&&&&&&&&&&\\
&&\bullet&&&&&1&&1&&&&&&&&&&&&&&&&&&&\\
1&1&1&&&&&&&&&&&&&\bullet&\bullet&&&&&&&&&&&&\\
&&&1&&1&\bullet&\bullet&\bullet&\bullet&&&&&&&&&&&&&&&&&&&\\
&&&&1&1&&&&-2&&&&&&&&&&&&&&&&&&&\\
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\end{pmatrix}$$

where we omit the zero entries. It has rank 26, elementary divisor 2 of multiplicity one, and elementary divisor 1 of multiplicity twenty-five. Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-43}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 3, \\ \mathbb{Z}, & p = 2, \\ \mathbb{Z}^3, & p = 1, \\ \mathbb{Z}^3 \oplus \mathbb{Z}/2, & p = 0. \end{cases}$$

The case $m = 67$.

The fundamental domain can be found in figure 3.1c. Our Bredon chain complex is

$$0 \longrightarrow \mathbb{Z}^{15} \xrightarrow{\Psi_2} \mathbb{Z}^{53} \xrightarrow{\Psi_1} \mathbb{Z}^{39} \longrightarrow 0,$$

where Ψ_2 has rank 13, and its only occurring elementary divisor is 1. The matrix for Ψ_1 has rank 36, elementary divisor 2 of multiplicity one, and elementary divisor 1 of multiplicity 35. These matrices are computed with, and can be displayed by [31].

Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-67}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 3, \\ \mathbb{Z}^2, & p = 2, \\ \mathbb{Z}^4, & p = 1, \\ \mathbb{Z}^3 \oplus \mathbb{Z}/2, & p = 0. \end{cases}$$

The case $m = 163$.

The fundamental domain can be found in figure 3.1d. Our Bredon chain complex is

$$0 \longrightarrow \mathbb{Z}^{57} \xrightarrow{\Psi_2} \mathbb{Z}^{141} \xrightarrow{\Psi_1} \mathbb{Z}^{85} \longrightarrow 0,$$

where Ψ_2 has rank 51, and its only occurring elementary divisor is 1. The matrix for Ψ_1 has rank 82, elementary divisor 2 of multiplicity one, and elementary divisor 1 of multiplicity eighty-one. These matrices are computed with, and can be displayed by [31].

Hence the above Bredon chain complex has the homology

$$\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-163}); R_{\mathbb{C}}) \cong \begin{cases} 0, & p \geqslant 3, \\ \mathbb{Z}^6, & p = 2, \\ \mathbb{Z}^8, & p = 1, \\ \mathbb{Z}^3 \oplus \mathbb{Z}/2, & p = 0. \end{cases}$$

4.4 Results in the principal ideal domain cases

The above computations give the following result, which is unified for the non-Euclidean principal ideal domain cases $m \in \{19, 43, 67, 163\}$. From proposition 70, we recall the values of their Betti numbers

$$\beta_i := \dim_{\mathbb{Q}} \mathrm{H}_i(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Q}), \qquad i \in \{1, 2\}.$$

In the Euclidean cases, these values are known from [37], and together they are the following in the principal ideal domain cases:

m	1	2	3	7	11	19	43	67	163
β_1	0	1	0	1	1	1	2	3	7
β_2	0	0	0	0	0	0	1	2	6.

Theorem 80. *The isomorphy types of the Bredon homology groups* $\mathrm{H}_p^{\mathfrak{Fin}}(\mathrm{PSL}_2(\mathcal{O}_{-m}); R_{\mathbb{C}})$ *are as follows in the cases where* $\mathcal{O}_{-m}$ *is a principal ideal domain.*

$m=1$	$m=2$	$m=3$	$m=7$	$m=11$	$m\in\{19,43,67,163\}$	
0,	0,	0,	0,	0,	0,	$p\geqslant 3$,
$\mathbb{Z}^{\beta_2}$,	$\mathbb{Z}^{\beta_2}$,	$\mathbb{Z}^{\beta_2}$,	$\mathbb{Z}^{\beta_2}$,	$\mathbb{Z}^{\beta_2}$,	$\mathbb{Z}^{\beta_2}$,	$p=2$,
$\mathbb{Z}\oplus\mathbb{Z}^{\beta_1}$,	$\mathbb{Z}^2\oplus\mathbb{Z}^{\beta_1}$,	$\mathbb{Z}^{\beta_1}$,	$\mathbb{Z}^2\oplus\mathbb{Z}^{\beta_1}$,	$\mathbb{Z}^2\oplus\mathbb{Z}^{\beta_1}$,	$\mathbb{Z}\oplus\mathbb{Z}^{\beta_1}$,	$p=1$,
$\mathbb{Z}^6$,	$\mathbb{Z}^5\oplus\mathbb{Z}/2$,	$\mathbb{Z}^5\oplus\mathbb{Z}/2$,	$\mathbb{Z}^3$,	$\mathbb{Z}^4\oplus\mathbb{Z}/2$,	$\mathbb{Z}^3\oplus\mathbb{Z}/2$,	$p=0$.

Theorem 77 now yields

Corollary 81. *Let* $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-m})$. *Then, for* $\mathcal{O}_{-m}$ *principal, the equivariant* K*-homology of* Γ *has isomorphy types*

	$m=1$	$m=2$	$m=3$	$m=7$	$m=11$	$m\in\{19,43,67,163\}$
$K_0^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}^6$	$\mathbb{Z}^5\oplus\mathbb{Z}/2$	$\mathbb{Z}^5\oplus\mathbb{Z}/2$	$\mathbb{Z}^3$	$\mathbb{Z}^4\oplus\mathbb{Z}/2$	$\mathbb{Z}^{\beta_2}\oplus\mathbb{Z}^3\oplus\mathbb{Z}/2$
$K_1^\Gamma(\underline{E}\Gamma)$	$\mathbb{Z}$	$\mathbb{Z}^3$	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}\oplus\mathbb{Z}^{\beta_1}$

The remainder of the equivariant K-homology of Γ is given by 2-periodicity.
As the Baum-Connes conjecture is verified by the Bianchi groups, these equivariant K-homology groups are isomorphic to the K-theory of the reduced group C^*-algebras.

Chapter 5

Chen/Ruan Orbifold Cohomology of the Bianchi groups

In this chapter, we will complexify our orbifolds by complexifying the real hyperbolic three-space. We obtain orbifolds given by the induced action of the Bianchi groups on complex hyperbolic three-space. Then we compute the Chen/Ruan Orbifold Cohomology for these complex orbifolds. We can determine its product structure with theorem 86.

5.1 The vector space structure of Chen/Ruan Orbifold Cohomology

Let Γ be a discrete group acting *properly*, i.e. with finite stabilisers, by diffeomorphisms on a manifold Y. For any element $g \in \Gamma$, denote by $C_\Gamma(g)$ the centraliser of g in Γ. Denote by Y^g the subset of Y consisting of the fixed points of g.

Definition 82. *Let $T \subset \Gamma$ be a set of representatives of the conjugacy classes of elements of finite order in Γ. Then we set*

$$\mathrm{H}^*_{orb}(Y//\Gamma) := \bigoplus_{g \in T} \mathrm{H}^* \left(Y^g / C_\Gamma(g); \mathbb{Q}\right).$$

It can be checked that this definition gives the vector space structure of the orbifold cohomology defined by Chen and Ruan [12], if we forget the grading of the latter. We can verify this analogously to the case where Γ is a finite group, treated by Fantechi and Göttsche [18]. The additional argument needed when considering some element g in Γ of infinite order, is the following. As the action of Γ on Y is proper, g does not admit any fixed point in Y. Thus,

$$\mathrm{H}^* \left(Y^g / C_\Gamma(g); \mathbb{Q}\right) = \mathrm{H}^* \left(\emptyset; \mathbb{Q}\right) = 0.$$

5.2 The orbifold product

In order to equip the orbifold cohomology vector space with the Chen/Ruan product structure, we need an almost complex orbifold structure on $Y//\Gamma$.
Let Y be a complex manifold of dimension D with a proper action of a discrete group Γ by diffeomorphisms, the differentials of which are holomorphic. For any $g \in \Gamma$ and $y \in Y^g$, we consider the eigenvalues $\lambda_1, \ldots, \lambda_D$ of the action of g on the tangent space $\mathrm{T}_y Y$. As the action of g on $\mathrm{T}_y Y$ is complex linear, its eigenvalues are roots of unity.

Definition 83. *Write $\lambda_j = e^{2\pi i r_j}$, where r_j is a rational number in the interval $[0, 1[$. The* age *of g in y is the rational number* $\mathrm{age}(g, y) := \sum_{j=1}^{D} r_j$.

We see in [18] that the age equals the degree shifting number defined by Chen and Ruan. It is also called the fermionic shift number in [44]. The age of an element g is constant on a connected component of its fixed point set Y^g. For the groups under our consideration, Y^g is connected, so we can omit the argument y. Details for this and the explicit value of the age are given in lemma 85. Then we can define the graded vector space structure of the orbifold cohomology as

$$\mathrm{H}^d_{orb}(Y//\Gamma) := \bigoplus_{g \in T} \mathrm{H}^{d-2\mathrm{age}(g)}\left(Y^g / C_\Gamma(g); \mathbb{Q}\right). \tag{5.1}$$

Denote by g, h two elements of finite order in Γ, and by $Y^{g,h}$ their common fixed point set. Chen and Ruan construct a certain vector bundle on $Y^{g,h}$ we call the *obstruction bundle*. We denote by $c(g, h)$ its top Chern class. In our cases, $Y^{g,h}$ is a connected manifold. In the general case, the fibre dimension of the obstruction bundle can vary between the connected components of $Y^{g,h}$, and $c(g, h)$ is the cohomology class restricting to the top Chern class of the obstruction bundle on each connected component. The obstruction bundle is at the heart of the construction [12] of the Chen/Ruan orbifold cohomology product. In [18], this product, when applied to a cohomology class associated to Y^g and one associated to Y^h, is described as a push-forward of the cup product of these classes restricted to $Y^{g,h}$ and multiplied by $c(g, h)$.
The following statement is made for global quotient orbifolds, but it is a local property, so we can apply it in our proper actions case.

Lemma 84 (Fantechi/Göttsche)**.** *Let $Y^{g,h}$ be connected. Then the obstruction bundle on it is a vector bundle of fibre dimension*

$$\mathrm{age}(g) + \mathrm{age}(h) - \mathrm{age}(gh) - \mathrm{codim}_{\mathbb{C}}\left(Y^{g,h} \subset Y^{gh}\right).$$

In [18], a proof is given in the more general setting that $Y^{g,h}$ needs not be connected. Examples where the product structure is worked out in the non-global quotient case, are for instance given in [12, 5.3] and [9].

5.2.1 Groups of hyperbolic motions

A class of examples with complex structures admitting the grading (5.1) is given by the discrete subgroups Γ of the orientation preserving isometry group $\mathrm{PSL}_2(\mathbb{C})$ of real hyperbolic 3-space $\mathcal{H}^3_{\mathbb{R}}$. The Lobachevski model of $\mathcal{H}^3_{\mathbb{R}}$ gives a natural identification of the orientation preserving isometries of $\mathcal{H}^3_{\mathbb{R}}$ with matrices in $\mathrm{PSO}(3,1)$. By the subgroup inclusion $\mathrm{PSO}(3,1) \hookrightarrow \mathrm{PSU}(3,1)$, these matrices specify isometries of the complex hyperbolic space $\mathcal{H}^3_{\mathbb{C}}$.

Lemma 85. *The age of any rotation of $\mathcal{H}^3_{\mathbb{C}}$ on its fixed points set is 1.*

Proof. For any rotation $\hat{\theta}$ of angle θ around a geodesic line in $\mathcal{H}^3_{\mathbb{R}}$, there is a basis for the construction of the Lobachevski model such that the matrix of $\hat{\theta}$ takes the shape

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{PSO}(3,1).$$

This matrix, considered as an element of $\mathrm{PSU}(3,1)$, performs a rotation of angle θ around the "complexified geodesic line" with respect to the inclusion $\mathcal{H}^3_{\mathbb{R}} \hookrightarrow \mathcal{H}^3_{\mathbb{C}}$. The fixed points of this rotation are exactly the points p lying on this complexified geodesic line, and the action on their tangent space $\mathrm{T}_p\mathcal{H}^3_{\mathbb{C}} \cong \mathbb{C}^3$ is again a rotation of angle θ. Hence we can choose a basis of this tangent space such that this rotation is expressed by the matrix

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{C}).$$

Therefore the age of the rotation $\hat{\theta}$ at p is 1. □

Theorem 86. *Let Γ be a group generated by translations and rotations of $\mathcal{H}^3_{\mathbb{C}}$. Then all obstruction bundles of the orbifold $\mathcal{H}^3_{\mathbb{C}}//\Gamma$ are of fibre dimension zero.*

Proof. Non-trivial obstruction bundles can only appear for two elements of Γ with common fixed points, and such that one of these is not a power of the other one. The translations of $\mathcal{H}^3_{\mathbb{C}}$ have their fixed point on the boundary and not in $\mathcal{H}^3_{\mathbb{C}}$. So let b and c be non-trivial hyperbolic rotations around distinct axes intersecting in the point $p \in \mathcal{H}^3_{\mathbb{C}}$. Then bc is again a hyperbolic rotation around a third distinct axis passing through p. Obviously, these rotation axes constitute the fixed point sets Y^b, Y^c and Y^{bc}. Hence the only fixed point of the group generated by b and c is p. Now lemma 84 yields the following fibre dimension for the obstruction bundle on $Y^{b,c}$:

$$\mathrm{age}(b) + \mathrm{age}(c) - \mathrm{age}(bc) - \mathrm{codim}_{\mathbb{C}}\left(Y^{b,c} \subset Y^{bc}\right).$$

After computing ages using lemma 85, we see that this fibre dimension is zero. □

Hence the obstruction bundle is trivial, and its top Chern class is the neutral element of the cohomological cup product. By Fantechi/Göttsche's description, the Chen/Ruan orbifold cohomology product is then a push-forward of the cup product of the cohomology classes restricted to the intersection of the fixed points sets.

5.3 Computations for the Bianchi groups

Let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$ be a Bianchi group. Then any element of Γ fixing a point in $\mathcal{H}^3_{\mathbb{R}}$ acts as a rotation of finite order. Hence the action of Γ on $\mathcal{H}^3_{\mathbb{C}}$ is proper.

Determination of the quotients of the twisted sectors. Given a non-trivial element $\gamma \in \Gamma$ of finite order, we give a method for determining the quotient of the fixed point set $\mathcal{H}^\gamma$ (a "twisted sector") by the centraliser $C_\Gamma(\gamma)$ of γ in Γ. We start by computing a subgroup in the centraliser of $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, a matrix which is contained in all the Bianchi groups via the inclusion $\mathrm{PSL}_2(\mathbb{Z}) < \Gamma$. This centraliser consists of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_{-m})$ satisfying the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We immediately deduce that such matrices are of the form $\begin{pmatrix} a & b \\ -b & a-b \end{pmatrix}$, so the determinant 1 equation gives us

$$a^2 - ab + b^2 = 1.$$

Assume that m is congruent to 1 or 2 $\mod 4$, so the ring of integers of $\mathbb{Q}(\sqrt{-m})$ is given by $\mathbb{Z}[\sqrt{-m}\,]$. Let $\omega := \sqrt{-m}$, and write $a = j + k\omega$, $b = \ell + n\omega$ with j, k, ℓ, $n \in \mathbb{Z}$. Then the determinant 1 equation takes the shape

$$j^2 + 2jk\omega - mk^2 + \ell^2 + 2\ell n\omega - mn^2 - j\ell + mkn - \omega(k\ell + jn) = 1,$$

and we decompose this into its real and imaginary part:

$$\text{(Re)}: \qquad j^2 - mk^2 + \ell^2 - mn^2 - j\ell + mkn = 1,$$

$$\text{(Im)}: \qquad (2k - n)j + 2\ell n - k\ell = 0.$$

First case: $n = 2k$**.** The equation (Im) yields $k\ell = 0$. If $k = 0$, we obtain the matrix group $\langle\gamma\rangle \cong \mathbb{Z}/3$ generated by our centralised matrix γ.
Otherwise $\ell = 0$, and equation (Re) gives

$$j^2 = 3mk^2 + 1. \tag{5.2}$$

Second case: $n \neq 2k$. This means that we can transform the equation (Im) into

$$j = \ell \frac{k-2n}{2k-n},$$

which we insert into the equation (Re):

$$\ell^2 \left(\frac{k-2n}{2k-n}\right)^2 - mk^2 + \ell^2 - mn^2 - \ell^2 \frac{k-2n}{2k-n} + mkn = 1.$$

We solve for ℓ^2 and find

$$\ell^2 = \frac{mk^2 + mn^2 - mkn + 1}{1 + \frac{k-2n}{2k-n}\left(\frac{k-2n}{2k-n} - 1\right)}. \tag{5.3}$$

As m is fixed, we can numerically compute the integer solutions of equations (5.2) and (5.3), up to a chosen bound for the absolute value of k, which shall also bound the absolute value of n in the second equation. We obtain a subset S of $C_\Gamma(\gamma)$ that contains all the matrices in $C_\Gamma(\gamma)$ the entries of which have absolute value at most the chosen bound. Then we calculate the group $\langle S \rangle$ generated by S. Once the quotient space $\mathcal{H}^\gamma / \langle S \rangle$ contains no more than one representative for any Γ-orbit of cells, we know that we have obtained the quotient space $\mathcal{H}^\gamma / C_\Gamma(\gamma)$, because of the subgroup inclusions $\langle S \rangle < C_\Gamma(\gamma) < \Gamma$.

We proceed analogously for γ running through the other representatives of conjugacy classes of elements of finite order in Γ.

5.3.1 $\Gamma = \mathbf{PSL}_2(\mathbb{Z}[\sqrt{-2}\,])$

A fundamental domain has been found by Luigi Bianchi [8]. We can imagine it by taking a polyhedron with vertices P_1, P_2, P_3, P_4 and ∞ and removing the vertex ∞, making it noncompact. The set of representatives of conjugacy classes can be chosen

$$T = \{1,\, a,\, c,\, b,\, b^2\},$$

with a and c of order 2, and b of order 3. Using lemma 78 and with the help of our Bredon homology computations, we check the cardinality of T. The fixed point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}^3_{\mathbb{C}}$:
$\mathcal{H}^1 = \mathcal{H}$,
$\mathcal{H}^a =$ the complex geodesic line through P_1 and P_2,
$\mathcal{H}^c =$ the complex geodesic line through P_1 and P_4,
$\mathcal{H}^b = \mathcal{H}^{b^2} =$ the complex geodesic line through P_2 and P_3.
The orbit space $\mathcal{H}/\Gamma$ being homotopy equivalent to a circle, we obtain

$$\mathrm{H}^{d-0}\left(\mathcal{H}^1/C_\Gamma(1); \mathbb{Q}\right) = \mathrm{H}^d\left(\mathcal{H}/\Gamma; \mathbb{Q}\right) = \mathbb{Q} \text{ for } d = 0, 1; \text{ and zero otherwise.}$$

The centraliser of $c = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ contains c itself and the element $u = \pm\begin{pmatrix} 3 & 2\sqrt{-2} \\ -2\sqrt{-2} & 3 \end{pmatrix}$ of infinite order. The matrices u and c commute, so this centraliser contains a subgroup of isomorphy type $\mathbb{Z} \oplus \mathbb{Z}/2$. In the upper-half space model

$$\{x + iy + rj \in \mathbb{C} \oplus \mathbb{R}j \mid r > 0\}$$

for $\mathcal{H}^3_{\mathbb{R}}$, we observe the following. The matrix u sends $P_1 = \frac{-\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j$ to $\frac{7\sqrt{2}}{10}i + \frac{\sqrt{2}}{10}j$ and $P_4 = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j$ to $\frac{41\sqrt{2}}{58}i + \frac{\sqrt{2}}{58}j$, so the image $u \cdot P_1P_4$ of the unit circle segment P_1P_4 lies again on the unit circle of real part zero, with the same orientation.
Thus u performs a hyperbolic translation on the geodesic line through P_1 and P_4, the quotient of this consisting of one edge with its origin and end vertices identified. This edge is pointwise fixed by its stabiliser in Γ, so we have obtained the orbit space $(\mathcal{H}^3_{\mathbb{R}})^c/C_\Gamma(c)$. We conclude that $(\mathcal{H}^3_{\mathbb{C}})^c/C_\Gamma(c)$ is homeomorphic to a cylinder.
Hence $\mathrm{H}^{d-2}\left(\mathcal{H}^c/C_\Gamma(c);\mathbb{Q}\right) = \begin{cases} \mathbb{Q}, & d = 2,3 \\ 0 & \text{else} \end{cases}$ is contributed to the orbifold cohomology.

5.3.2 $\Gamma = \mathbf{PSL}_2(\mathbb{Z}[\sqrt{-5}\,])$

The fundamental domain contains more cells than in the above case, but again a set of representatives of conjugacy classes can be chosen

$$T = \{1,\, a,\, c,\, ac,\, b,\, b^2\},$$

with a and c of order 2, and b of order 3.
The centraliser of b is generated by the two elements $b = \pm \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ itself, which is of order 3, and $C := \pm \begin{pmatrix} 2\omega & 4+\omega \\ -4-\omega & \omega-4 \end{pmatrix}$, where $\omega = \sqrt{-5}$. The matrix C is of infinite order and commutes with b. Thus the centraliser of b (which equals the centraliser of b^2) contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/3$.

Appendix A

Implementation

This appendix gives some extracts of the program Bianchi.gp [31], in which the algorithms of chapter 1 are implemented. The program is structured by some libraries of functions and procedures, see figure A.1.

Treatment of the mathematical data. As Riley pointed out, some numerical operations and comparisons in the computation of the Bianchi fundamental polyhedron must be carried out in arbitrary precision on the digits, because in hyperbolic space, arbitrarily small numerical values may appear. Therefore, we have to avoid recording floating point numbers into our data. For instance, instead of recording the height of a vertex, which is a positive real number, we record its square, which is a rational number in the case of the vertices of the Bianchi fundamental polyhedron. The complex coordinate of our vertices is an element of the number field $\mathbb{Q}(\sqrt{-m})$; and Pari/GP [1], the specialized computer algebra system on which Bianchi.gp is based, allows to handle such elements as fractions of integers in the number field.

The procedures of our program have the permission to handle global variables, for the following reason. All the data describing the Flöge cellular complex is stored in global variables to avoid blowing up the run-time by copying it through the numerous procedures, some of which are called within iterated loops. The procedure **BackUp()**, which we do not print as it is trivial, allows us to store the data on a hard disk and resume later for further computations with the cellular complex.

We now describe only a selection of procedures and functions, which have a certain

Figure A.1: Libraries of the program Bianchi.gp

Bianchi-Fundamental_Polyhedron_functions.gp
Bianchi-Fundamental_Polyhedron_procedures.gp
Bianchi-orbit_space_functions.gp
Bianchi-orbit_space_procedures.gp
Bianchi-common_procedures_and_diagnosis.gp
Bianchi-group_homology.gp
Bianchi-Bredon_homology.gp
Bianchi-output.gp

Figure A.2: Procedures and functions the source code of which is printed in this appendix
getFundamentalDomain(), figure A.3
computeNeighbours(), figure A.4
compareHeights(), figure A.6
deleteSpheresBelow(), figure A.5
computeVertexStabilizer, figure A.9
computeVertexStabilizer3mod4, figure A.7
getIdentificationMatrices, figure A.13
getIdentificationMatrices3mod4, figure A.10

mathematical relevance (and constitute about 10% of the source code). With the procedure **getFundamentalDomain()**, we compute a list of hemispheres, until Swan's termination criterion (theorem 12) is fulfilled. This is described in algorithm 1. The decisive step of algorithm 2, which is a subroutine of algorithm 1, is done by the procedure **deleteSpheresBelow()**. This procedure takes as input a local list of hemisphere centers with radii, and removes those that are everywhere below some other hemisphere, according to the condition in definition 13, by one of the hemispheres of greater radius that are in the global hemisphere list.

In order to compute the intersection arcs of the hemispheres, which contain the edges of the Bianchi fundamental polyhedron, we want to restrain our attention to the pairs of hemispheres with non-empty intersection. We can express the latter condition in terms of their radii and the distance between the sphere centers. The procedure **computeNeighbours()** checks this condition and records the indices of adjacent hemispheres for each element of our list of hemispheres. This facilitates carrying out algorithm 3.

From the list of all the hemispheres which touch the Bianchi fundamental polyhedron, computed with the procedure **getFundamentalDomain()**, we want to keep only the ones the intersection of which with the Bianchi fundamental polyhedron contains a 2-cell. Such a 2-cell must have at least three vertices, and these vertices can only lie on the intersection points of the hemispheres.

The procedure **compareHeights(numberOfRequiredCorners)** allows us to find how many intersection points lie on a hemisphere, and to remove the intersection points of hemispheres that are not in the Bianchi fundamental polyhedron. If this procedure is called with the parameter numberOfRequiredCorners = 0, then it only performs the latter operation. If it is called with the parameter 3, then it deletes all the hemispheres that have less than three intersection points in common with the Bianchi fundamental polyhedron.

Algorithm 4 for the computation of matrices identifying two points in the interior of hyperbolic space is carried out by the function **getIdentificationMatrices3mod4()**, and a simplified version of this algorithm for the computation of the isotropy group of such a point is carried out by the function **computeVertexStabilizer3mod4()**. We also give the versions computeVertexStabilizer(), getIdentificationMatrices() for the cases where m is not congruent to 3 mod 4. Below we print the source code of these functions and procedures.

```
getFundamentalDomain() =
{
  /* This procedure is the core of the program. */
  local(mufixsquare, oldlength, limit, cancelCriterion, minzetasquare);

  limit = estimateLimit(); /* Estimate the inferior limit for the hemisphere radius. */
  /* If this estimation is not precise, it will be corrected in the below while loop. */
  /* Then the program will take more computation time. */
  Mu = listcreate(4*m*limit +16);
  Lambda = listcreate(4*m*limit +16);
  deleteFlag = listcreate (4*m*limit +16);
  oldlength = 0; cancelCriterion = 0;
  mufixsquare = 1;
  /* The value mufixsquare will control the radius of the Euclidean hemispheres. */

  print("Running until the inverse of the radius square is greater than ", limit);

  while( cancelCriterion == 0,

    while( mufixsquare <= limit,

      print("Generating hemispheres of radius square = 1/",mufixsquare, " ...");
      if ( mIs3mod4,

                createSpheres3mod4( mufixsquare);
        ,
                createSpheres( mufixsquare);
      );
      mufixsquare = increaseWithinNormsOfIntegers( mufixsquare);
    );
    if (numberOfSpheres != oldlength,

          removeEverywhereBelows();
          cleanseListOfSpheres();
          getLinesAndVertices();
          minzetasquare = Get_minimal_proper_vertex_heightsquare();
          oldlength = numberOfSpheres;

    , /* else numberOfSpheres == oldlength */

           print("...kept old ",numberOfSpheres," hemispheres, ",
                  length(totalPointSet)," vertices,");
    );

    if( minzetasquare*mufixsquare >= 1,

                print("All remaining vertices have height square at least ",
                        minzetasquare," and thus lie above the radius square of the ",
                        "ignored hemispheres, which is at most 1/",mufixsquare);
                cancelCriterion = 1;

     , /* else minzetasquare becomes the new expected lowest value for 1/mufixsquare. */

                limit = 1/minzetasquare;
                print("Height denominator square difference between ignored ",
                     "hemispheres and these vertices: ",mufixsquare," - ",1.0*limit);
    );
  );
};
```

Figure A.3: The procedure getFundamentalDomain().

```
computeNeighbours() =
{
  /* Generate the lists of neighbouring hemispheres */
  local(neighbourslist:list, areNeighbours);
  areNeighbours = matrix( numberOfSpheres, numberOfSpheres);
  neighbours = vector(numberOfSpheres); /* (the global object which is computed) */

  maxNeighboursNumber = 0;

  for( j = 1, numberOfSpheres,

          for( k = j+1, numberOfSpheres,

                if ( norm(nfbasistoalg(K, sphereCenter[j] -sphereCenter[k]))
                        <= (radius[j] + radius[k])^2,

                        areNeighbours[j,k] = 1;
                        areNeighbours[k,j] = 1;
                );
          );
  );
  neighbourslist = listcreate(numberOfSpheres);

  for( j = 1, numberOfSpheres,

          for( k = 1, numberOfSpheres,

             if ( k != j,

                if ( areNeighbours[j,k],

                        /*Write the kth hemisphere into the list of
                                neighbours of the jth hemisphere. */
                        listput(neighbourslist,k);
                );
             );
          );
          neighbours[j] = neighbourslist;
          if ( length(neighbours[j]) > maxNeighboursNumber,
                maxNeighboursNumber = length(neighbours[j]);
          );
          listkill(neighbourslist);
  );
}; /* end of the procedure computeNeighbours(). */
```

Figure A.4: The procedure computeNeighbours().

```
deleteSpheresBelow(
  preliminaryMu:list, preliminaryLambda:list, MuNumber:list, everywhere_below:list) =
{
  /* Remove preliminary hemispheres which are everywhere below
        some hemisphere already in the list. */
  local(MU:list,LAMBDA:list, sphere_center, invertedNormSquareOfMU, distanceSquare,
         a,b,c);
  MU = listcreate(numberOfSpheres);
  LAMBDA = listcreate(numberOfSpheres);
  sphere_center    = vector( length( preliminaryLambda));
  invertedNormSquareOfMU = vector( length( preliminaryLambda));

  print("Removing everywhere covereds amongst ",
        length(preliminaryLambda)," hemispheres...");

  /* Precompute sphere center and radius. */
  for (k:small = 1, length(preliminaryLambda),

    if (everywhere_below[k] == 0,

        sphere_center[k] = nfeltdiv(K,
                                  preliminaryLambda[k], preliminaryMu[MuNumber[k]]);

        invertedNormSquareOfMU[k] = 1/norm(nfbasistoalg(K,
                                                  preliminaryMu[MuNumber[k]]));
    );
  );

  /* Perform the inner loop of Algorithm 2. */
  for (k:small = 1, length(preliminaryLambda),

    j = 1;
    while ( everywhere_below[k] == 0 && j <= length(Lambda),

      if ( deleteFlag[j] == 0,

        distanceSquare = norm(nfbasistoalg(K,sphere_center[k] -sphereCenter[j]));

        /* Check if invertedNormOfMU[k] +distance <= radius[j].  Let */
        a = invertedNormSquareOfMU[k]; b = distanceSquare; c = radiusSquare[j];
        /* The condition sqrt(a) +sqrt(b) < sqrt(c) is equivalent to */
        /* 4*a*b < (c -a -b)^2 && a +b < c, because a, b, c > 0. */
        if( 4*a*b < (c -a -b)^2 && a +b < c,

                        everywhere_below[k] = 1;
        );
        j++;
      );
    );
  );
  everywhere_below
};
```

Figure A.5: The function deleteSpheresBelow().

```
compareHeights( numberOfRequiredCorners) =
{
  local(z, comparisonHeightSquare, heightSquare, setOfSingularPoints);
  local(deletePoint, numberOfErased, point, auxiliaryList:list);
  numberOfErased = 0;

  deletePoint = vector( length(totalPointSet));

  listsort( singularpoint); setOfSingularPoints = Set(singularpoint);

  /* z runs through the projections of intersection points */
  for ( j = 1, length(totalPointSet),

        point = eval(totalPointSet[j]);
        z = component( point, 1);

        if( /* z not a singular point */ setsearch(setOfSingularPoints,z,1),

           /* then find the hemispheres touching the surface above z */

           comparisonHeightSquare = component( point, 2);

           if (comparisonHeightSquare <= 0,
                deletePoint[j] = 1;
              ,
                /* else consider z */

                for (k = 1, numberOfSpheres,
                  if(deleteFlag[k] == 0,

                     /* Height square of hemisphere k over the point z: */
                     heightSquare = radiusSquare[k] -norm(
                         nfbasistoalg( K, z -sphereCenter[k] ));

                     if( heightSquare > comparisonHeightSquare,
                         /* Hemisphere number k lies above the point z. */
                         /* delete z */  deletePoint[j] = 1;
                     );
                  );
                );
           );
        );
  );

  /* Throw out the vertices which are marked for erasure by the vector deletePoint. */
  cleanseVertexLists( deletePoint);

  if ( numberOfRequiredCorners > 0,

    for( k = 1, numberOfSpheres,

        if( length( pointsOfSphere[k]) < numberOfRequiredCorners,

                         deleteFlag[k] = 1;
                         /* Delete the hemisphere with center Lambda[k]/Mu[k]. */
                         numberOfErased++;
        );
    );

    print( numberOfErased, " hemispheres were deleted because they touched the surface less than ",
                numberOfRequiredCorners," times.");
  );
  cleanseListOfSpheres(1);
};
/* end of procedure compareHeights. */
```

Figure A.6: The procedure compareHeights().

```
computeVertexStabilizer3mod4( z, rsquare)=
{
  local(a,b,c,d, s_limitSummand, cz_R, cz_W, discriminant, squareRoot);
  local(  k_limitSummand, j_limit, currentStabilizer:list );
  currentStabilizer = listcreate(41);

  /* With c = 0, we get just the trivial stabilizing matrices (except for m= 1 or 3): */
  listput(currentStabilizer, matid(2) );
  listput(currentStabilizer, -matid(2) );

  /* Now we search stabilizer elements with nonzero entry c. */
  /* c = j +k*w runs through the lattice of integers with 0 < |c| <= 1/r  */

  j_limit = ceil(sqrt((1+1/m)/rsquare));
  /* j runs from -j_limit through j_limit: */
  for (j = -j_limit, j_limit,

    /* |c| = sqrt( (j -k/2)^2 +m*(k/2)^2)  = sqrt( j^2 +(m+1)/4*k^2 -j*k) */
    /* therefore, k_limit(+,-) = 2*j/(m+1) (+,-) 2*sqrt(-j^2*m +(m+1)/(rsquare) )/(m+1) */
    discriminant = -j^2*m +(m+1)/(rsquare);
    if( discriminant >= 0,

      k_limitSummand  = 2*sqrt(discriminant)/(m+1);

      for (k = floor( 2*j/(m+1) - k_limitSummand), ceil( 2*j/(m+1) + k_limitSummand),

        c = nfalgtobasis(K,j + k*w);

        /* Check that |c| <= 1/r and c nonzero. */
        if ( norm(nfbasistoalg(K,c)) <= 1/rsquare && c != [0,0]~ ,

          /* d runs through the lattice of integers with |cz-d|^2 +r^2|c|^2 = 1. */
          /* Decompose d as q +s*w. */
          /* print(" cz = ", nfeltmul( K, c, z));  */
          cz_R = component( nfeltmul( K, c, z), 1);
          cz_W = component( nfeltmul( K, c, z), 2);

          /* s_limit(+,-) = cz_W (+,-) 2*sqrt( (1 -r^2|c|^2)/m). */
          s_limitSummand  =  2*sqrt( (1 -rsquare*norm(nfbasistoalg(K,c)) )/m);

          for ( s = floor(cz_W -s_limitSummand), ceil(cz_W + s_limitSummand),

            discriminant = 1 -rsquare*norm(nfbasistoalg(K,c))
                                -(m/4)*cz_W^2 +(m/2)*cz_W*s -(m/4)*s^2;

            if( issquare( discriminant, &squareRoot),

              /* q(+,-) = cz_R -cz_W/2 +s/2  (+,-) sqrt( discriminant) */
              qPlus  = cz_R -cz_W/2 +s/2 +squareRoot;

              if( frac( qPlus) == 0,
                currentStabilizer = concat(currentStabilizer,
                                    FinishStabilizer( round(qPlus),s,c,z,rsquare));
              );

              if( discriminant != 0,
                qMinus = cz_R -cz_W/2 +s/2 -squareRoot;
                if( frac( qMinus) == 0,
                  currentStabilizer = concat(currentStabilizer,
                                    FinishStabilizer( round(qMinus),s,c,z,rsquare));
                );
              );
            );
          );
        );
      );
    );
  );
 /* Return */ currentStabilizer
};
```

Figure A.7: The function computeVertexStabilizer3mod4.

```
FinishStabilizer(q,s,c,z,rsquare) =
{
  /* Finish the computation of the stabilizer of (z,rsquare) in Hyperbolic Space. */
  /* m congruent 3 modulo 4. */
  local(a,b,d,StabilizingMatrix:list, cz, cz_R, cz_W, ad, TwoRe_cz);
  StabilizingMatrix = listcreate(1);

  cz = nfeltmul( K, c, z);
  cz_R = component( cz, 1);
  cz_W = component( cz, 2);

  /* Floege deduces a = conj(d) -2*Re(cz). */
  TwoRe_cz = 2*cz_R -cz_W;

  /* Check that a is in the ring of integers: */
  if( frac(TwoRe_cz) == 0,

    d = nfalgtobasis(K, q + s*w);

    /* Check that |cz-d|^2 +r^2*|c|^2 == 1, */
    if( norm(nfbasistoalg(K, cz -d)) +rsquare*norm(nfbasistoalg(K,c)) == 1
    /* and check that c,d generate the ring of integers. */
    && idealadd(K,c,d) == 1,

        a = conjugate(d) -TwoRe_cz*[1,0]~;
        ad = nfeltmul(K,a,d);

        /* b is determined by the determinant 1 equation: */
        b = nfeltdiv(K, ad -[1,0]~, c);

        /* Check that b is in the ring of integers: */
        if ( frac(b) == [0,0]~,

          /* Check that z == conj(d-cz)(az-b) -r^2*conj(c)a. */
          if( z == nfeltmul(K, conjugate(d -cz), nfeltmul(K,a,z) -b)
                  -rsquare*nfeltmul(K, conjugate(c), a),

                /* Check stabilization. */
                if( z != PoincareAction(a,b,c,d,z,rsquare),
                        print("***Error in function FinishStabilizer ",
                         "for the stabilization of z = ",z,
                         ", rsquare = ",rsquare,"\n [a,b; c,d](z,rsquare) = ",
                         PoincareAction(a,b,c,d,z,rsquare)
                        );
                );

                listput(StabilizingMatrix, [a,b; c,d] );

          , /* else */
                 error(Str("***Error in function FinishStabilizer ",
                           "caused by [a,b; c,d] = ",[a,b; c,d])
                 );
          );
        );
    );
  );
  /* Return */ StabilizingMatrix
};
```

Figure A.8: The subfunction FinishStabilizer of the function computeVertexStabilizer3mod4.

```
computeVertexStabilizer(z, rsquare)=
{
  local(a,b,c,d, q, sBound, cz_real, cz_omegacoeff, discriminant, currentStabilizer:list, kBound, jBound,
rsquareInverse, cz, cNormsquare, rcNormsquare); currentStabilizer = listcreate(24);
  /* The maximal order of finite subgroups in the Bianchi groups is 24. */
  rsquareInverse = 1/rsquare;

  /* With c = 0, we get just the trivial stabilizing matrices (except for m= 1 or 3): */
  c = [0,0]~;
  for ( k = 0, 1,
        d = [(-1)^k,0]~;    a = d;   b = [0,0]~;
        listput(currentStabilizer, [a,b; c,d] );

        if (m == 1,
                d = [0, (-1)^k]~;
                a = nfalgtobasis(K, conj( nfbasistoalg(K,d)));
                b = nfeltmul(K, a-d, z);
                listput(currentStabilizer, [a,b; c,d] );
        );
  );
  /* Now we search stabilizer elements with nonzero entry c. */
  /* c = j +k*w runs through the lattice of integers with 0 < |c| <= 1/r. */
  /* The equation |c| = sqrt(j^2 +k^2*m) gives us the following bound for k. */
  kBound = 1/(sqrt(rsquare*m));

  for (k = ceil(-kBound), floor(kBound),

        jBound = sqrt(rsquareInverse -m*k^2);

        for (j = ceil(-jBound), floor(jBound),

                c = nfalgtobasis(K,j + k*w);
                cNormsquare = norm(nfbasistoalg(K,c));

                if ( cNormsquare > rsquareInverse,
                        print("***Error in function computeVertexStabilizer ",
                                "on stabilizer entry c.");
                );
                if ( c != [0,0]~ ,

                  /* d runs through the lattice of integers with
                    |cz-d|^2 +r^2*|c|^2 = 1. Use d =: q +s*w. */

                  cz = nfeltmul( K, c, z);   cz_real = component( cz, 1);
                  cz_omegacoeff = component( cz, 2);  rcNormsquare = rsquare*cNormsquare;
                  sBound = abs(cz_omegacoeff) +sqrt((1 -rcNormsquare)/m);

                  for ( s = -ceil(sBound), ceil(sBound),

                   discriminant = 1 -rcNormsquare -m*(cz_omegacoeff -s)^2;

                   if( discriminant >= 0,
                        /* q+ */
                        q = round( cz_real + sqrt(discriminant));
                        d = nfalgtobasis(K, q+ s*w);

                        if (norm(nfbasistoalg(K, cz -d)) +rcNormsquare == 1
                                && idealadd(K,c,d) == [1,0;0,1],

                          a = nfalgtobasis(K,conj( nfbasistoalg(K,d)))-2*cz_real*[1,0]~;
                          b = nfeltdiv(K, nfeltmul(K,a,d) -[1,0]~, c);
                          /* Check that a, b are in the ring of integers: */
                          if ( frac(a) == [0,0]~ && frac(b) == [0,0]~,

                                  listput(currentStabilizer, [a,b; c,d] );
                          );
                        );
                    );
                    if( discriminant > 0,
                        /* q- */
                        q = round( cz_real - sqrt(discriminant));
                        d = nfalgtobasis(K, q+ s*w);

                        if ( norm(nfbasistoalg(K, cz -d)) +rcNormsquare == 1
                                && idealadd(K,c,d) == [1,0;0,1],

                          a = nfalgtobasis(K,conj(nfbasistoalg(K,d))) -2*cz_real*[1,0]~;
                          b = nfeltdiv(K, nfeltmul(K,a,d) -[1,0]~, c);
                          /* Check that a, b are in the ring of integers: */
                          if ( frac(a) == [0,0]~ && frac(b) == [0,0]~,

                                  listput(currentStabilizer, [a,b; c,d] );
                          );
                        );
                   );
                  );
                );
        );
  ); /* Return */ currentStabilizer
};
```

Figure A.9: The function computeVertexStabilizer in the cases not congruent 3 mod 4.

```
getIdentificationMatrices3mod4(p, p_2, stabilizerCardinal) =
{
 /* For m is not 3, but congruent to 3 modulo 4: */
 /* Computation of the identification matrices [a,b; c,d] */
 /* which transport the point p=(z,r) into the point p_2 =(Zeta, rho)*/
 /* in hyperbolic space, i.e., r and rho are both positive heights.*/
 /* If p and p_2 are cusps, use the function getSingularIdentifications later. */
 /* Check that the cardinal of the stabilizer is the same for p and p_2 */
 /* before calling this function. */

 local(z,rsquare, Zeta, rhosquare,a,b,c,d, hBound, cz, cz_real, cz_omegacoeff, rZetaByRhoSquare, squareRoot);
 local(vertexTransport:list,  g, discriminant, jbound, k_limitPlus, k_limitMinus, rsquareByRhosquare);
 local(lefthandside, ThereIsNoRamification, s_limitMinus, s_limitPlus,q, rByRho, rRhoInverse, rcSquare);
 vertexTransport = listcreate(24);

 z = component(p,1);
 rsquare = component(p,2);
 Zeta = component(p_2,1);
 rhosquare = component(p_2,2);

 if ( rsquare > 0 && rhosquare > 0,

  rsquareByRhosquare = rsquare/rhosquare;
  issquare(rsquareByRhosquare, &rByRho);
  rZetaByRhoSquare = rsquareByRhosquare*nfeltmul(K,Zeta,Zeta);
  /* First case: c = 0 */
  c = [0,0]~;
  /* d = g+hw = g +h(-1/2 +1/2sqrt(-m)) runs through the ring of integers with |d|^2 = r/rho. */
  /* |d|^2 = (g -h/2)^2 + m(h/2)^2 */
  /* |d|^2 = g^2 -gh +h^2/4 +mh^2/4 */
  hBound = sqrt( 4*rByRho/(m+1));

  for( h = -ceil( hBound), ceil( hBound),

    /* g_+- = h/2 +- sqrt( r/rho -mh^2/4). */
    /* If the above discriminant is zero, run just one case, else two cases for g_+-. */
    if( rsquareByRhosquare == (m*h^2/4)^2,
          ThereIsNoRamification = 1;
        , /* else */ ThereIsNoRamification = 0;
    );

    for ( Case = ThereIsNoRamification, 1,

      discriminant = rByRho -m*h^2/4;
      if( issquare( discriminant, &squareRoot),

       g = h/2 +(-1)^Case*squareRoot;
       /* Check that g is a rational integer */
       if( frac(g) == 0,

        d = nfalgtobasis(K, round(g)+h*w);
        /* Check that |d|^2 = r/rho */
        if( (norm(nfbasistoalg(K,d)))^2 == rsquareByRhosquare,

          a = nfeltdiv(K,1,d);

          /* check that 'a' is in the ring of integers: */
          if ( frac(a) == [0,0]~,

             b = nfeltmul(K, a, z) - nfeltmul(K, d, Zeta);

             /* check that b is in the ring of integers: */
             if ( frac(b) == [0,0]~,

                /* check that [a,b; 0,d]p = p_2:*/
                lefthandside = nfeltmul(K, conj(nfbasistoalg(K,d)), (nfeltmul(K,a,z) -b));
                if(  nfeltmul(K,lefthandside,lefthandside) == rZetaByRhoSquare,

                        /* [a,b; 0,d] sends p to p_2 */
                        listput(vertexTransport, [a,b; c,d]);
                );
             );
          );
        );
       );
      );
    );
  );
```

Figure A.10: The function getIdentificationMatrices3mod4, continued in figure A.11.

```
  /* Second case: c is nonzero.*/
  /* The equation (2') gives us $ r^2|c|^2 \leq \frac{r}{\rho} $. */
  /* c = j +kw runs through the ring of integers of the number field, verifying */
        /*  1/(r*rho) >= normsquare(c) > 0. */
  rRhoInverse = rByRho/rsquare;
  jbound = ceil( sqrt( (1+1/m)*rRhoInverse ));
  /* We find the extremal values of j by an argument analogous to the one in the documentation of
     computeVertexStabilizer3mod4. */
  for( j = -jbound, jbound,
   /* We have |c|^2 = (j -k/2)^2 +m(k/2)^2 = j^2 +((m+1)k^2)/4 -jk, */
   /* therefore k_limit^2 + 4/(m+1)*(-j*k_limit +j^2 -1/(r*rho)) = 0. */

   if( (m+1)*rRhoInverse >= j^2*m, /* (check discriminant >= 0 ) */
     /* Thus let */
     k_limitPlus  = 2/(m+1)*j +2*sqrt(( (m+1)*rRhoInverse -j^2*m)/(m+1));
     k_limitMinus = 4/(m+1)*j -k_limitPlus;
               /* = 2/(m+1)*j -2*sqrt(( (m+1)*rRhoInverse -j^2*m)/(m+1)). */

     for ( k = floor(k_limitMinus), ceil(k_limitPlus),

      c = j +k*w;

      if( c != 0 && (norm(c))^2*(rsquare*rhosquare) <= 1,

        /* Decompose cz as cz_real +cz_omegacoeff*w with rational integer coefficients. */
        cz = nfeltmul( K, c, z);
        cz_real = component( cz, 1);
        cz_omegacoeff = component( cz, 2);
        cz = nfbasistoalg(K, cz);
        rcSquare = rsquare*norm(c);

        /* d =: q+s*w runs through the ring of integers of the number field, verifying */
               /* normsquare(cz -d) +r^2*normsquare(c) = r/rho. */
        /* We decompose d as q +sw, where q and s are rational integers. */

        s_limitPlus  = cz_omegacoeff + 2*sqrt( (rByRho -rcSquare )/m);
        s_limitMinus = 2*cz_omegacoeff -s_limitPlus;
                  /* = cz_omegacoeff - 2*sqrt( (rByRho -rsquare*norm(c))/m) */

        for( s = floor(s_limitMinus), ceil(s_limitPlus),

          discriminant = rByRho -rcSquare -m/4*(cz_omegacoeff -s)^2 ;

          /* Check that 'discriminant' is a rational square: */
          if( type(discriminant) == "t_FRAC" || type(discriminant) == "t_INT",
            if( issquare( discriminant, &squareRoot),

              /* obtain q+ with the positive squareroot */
              q = cz_real -cz_omegacoeff/2 +s/2 +squareRoot;
              if( frac(q) == 0, /* Check that q is a rational integer */
                vertexTransport = concat( vertexTransport,
                  getRemainingEntries(round(q),s,c,z,cz,Zeta,rsquare,rhosquare,
                                  rsquareByRhosquare, rZetaByRhoSquare, rByRho)
                );
              );

              /* obtain q- with the negative squareroot */
              q = cz_real -cz_omegacoeff/2 +s/2 -squareRoot;
              if( frac(q) == 0, /* Check that q is a rational integer */
                vertexTransport = concat( vertexTransport,
                  getRemainingEntries( round(q),s,c,z,cz,Zeta,rsquare,rhosquare,
                                  rsquareByRhosquare, rZetaByRhoSquare, rByRho)
                );
              );
            );
          );
        );
      );
    );
   );
  );
  if( length(vertexTransport) > 0,
        /* Delete double occurencies in the vertexTransport list: */
        listsort(vertexTransport, 1);

        if( abs(z -Zeta) != [1,0]~ && abs(z -Zeta) != [0,1]~,
               /* There are length(vertexTransport) matrices sending p to p_2. */
        );

        if( length( vertexTransport) != stabilizerCardinal,
               error(Str("***Error in function getIdentificationMatrices3mod4, on vertexTransport from ",
                      p," to ",p_2,": ","It must be a coset of the two vertex stabilizers!"));
               for (r = 1, length(vertexTransport),
                      print(vertexTransport[r]); print();
               );
        );
  );
 );
 /* return the list */ vertexTransport
};
```

Figure A.11: The function getIdentificationMatrices3mod4, continued from figure A.10.

```
getRemainingEntries(g,h,c,z,cz,Zeta,rsquare,rhosquare, rsquareByRhosquare,
                          rZetaByRhoSquare, rByRho) =
{
  local(a,b,d, d_minus_czConj, lefthandside,   TransportMatrix:list);
  TransportMatrix = listcreate(1);

    d = g +h*w;
    d_minus_czConj = conj(d -cz);

    /* Check the square of the equation |cz -d|^2 +r^2|c|^2 = r/rho */
    /* to be verified by d: */

    if( (norm(cz -d) +rsquare*norm(c))^2 == rsquareByRhosquare,

      a = nfalgtobasis( K, 1/rByRho*d_minus_czConj) -nfeltmul(K, c, Zeta);

      /* Check that the matrix entry "a" has integer entries: */

      if( frac(a) == [0,0]~,

        a = round(a);
        /* Get b by the determinant 1 equation: */
        b = nfeltdiv(K, nfeltmul(K, a, d) -[1,0]~, c);

        /* check that b has integer entries: */
        if( (frac(component(b,1)))^3 == 0 && (frac(component(b,2)))^3 == 0,

          b = round(b);
          /* Check that [a,b; c,d]p == p_2 by the following equation: */
          /* nfeltmul(K, conj(d -cz), (nfeltmul(K,a,z)-b))
                 -rsquare*nfeltmul(K,conj(c),a) == r*Zeta/rho */

                lefthandside = nfeltmul(K, d_minus_czConj,
                                           (nfeltmul(K,a,z) -b))
                                  -rsquare*nfeltmul(K, conj(c), a);

                if( lefthandside == Zeta*rByRho,

                        c = nfalgtobasis(K, c);
                        d = nfalgtobasis(K, d);
                        /* [a,b; c,d] sends p to p_2 */
                        listput(TransportMatrix, [a,b; c,d]);
                );
        );
      );
    );
  TransportMatrix
};
```

Figure A.12: The common subfunction getRemainingEntries of the functions getIdentificationMatrices and getIdentificationMatrices3mod4.

```
getIdentificationMatrices(p, p_2, stabilizerCardinal) =
{
  /* For m is not 1, and not congruent to 3 modulo 4: */
 /* Computation of the identification matrices [a,b; c,d] */
 /* which transport the point p=(z,r) into the */
 /* point p_2 =(Zeta, rho) in hyperbolic space, */
 /* where r and rho are both positive heights.*/
 /* If p and p_2 are cusps, */
 /* use the function getSingularIdentifications later. */

 local(z,rsquare, Zeta, rhosquare,a,b,c,d, hBound, cz, cz_real,
        cz_omegacoeff, kbound, dConj, rByRho, oneByRrho,
        squareRoot, vertexTransport:list, g, discriminant, jbound,
        lefthandside, rsquareByRhosquare, rZetaByRhoSquare
 );
 vertexTransport = listcreate(42);

 z = component(p,1);
 rsquare = component(p,2);
 Zeta = component(p_2,1);
 rhosquare = component(p_2,2);

 if ( rsquare > 0 && rhosquare > 0,

  rsquareByRhosquare = rsquare/rhosquare;
  issquare(rsquareByRhosquare, &rByRho);
  rZetaByRhoSquare = rsquareByRhosquare*nfeltmul(K,Zeta,Zeta);
  /* First case: c = 0 */ c = [0,0]~;

  for( n = 0, 1,

     d = [(-1)^n, 0]~;
     a = d; /* = nfeltdiv(K,1,d) */
     b = nfeltmul(K, a, z) - nfeltmul(K, d, Zeta);
     dConj = conj(nfbasistoalg(K,d));

     /* check that b is in the ring of integers: */
     if ( frac(b) == [0,0]~,

        /* check that [1,-z-Zeta; 0,1]p = p_2:*/
        lefthandside = nfeltmul(K, dConj, (nfeltmul(K,a,z) -b));

        if( /* (norm(nfbasistoalg(K,d)))^2 = */ 1 == rsquareByRhosquare
          && nfeltmul(K,lefthandside,lefthandside) == rZetaByRhoSquare,

                /* [a,b; 0,d] sends p to p_2 */
                listput(vertexTransport, [a,b; c,d]);
        );
     );
  );
```

Figure A.13: The function getIdentificationMatrices, continued in figure A.14.

```
  /* Second case: c is nonzero.*/
  /* c = j +kw runs through the ring of integers of the number field, verifying */
        /*  1/(r*rho) >= normsquare(c) = j^2 +mk^2 */
  oneByRrho = rByRho/rsquare;
  kbound = ceil( sqrt( oneByRrho/m));
  for ( k = -kbound, kbound,

    jbound = ceil( sqrt( oneByRrho -m*k^2));
    for( j = -jbound, jbound,

      c = j +k*w;

      if( c != 0 && (norm(c))^2 <= 1/(rsquare*rhosquare),

        cz = nfeltmul( K, c, z);
        cz_real = component( cz, 1);
        cz_omegacoeff = component( cz, 2);
        cz = nfbasistoalg(K, cz);

        /* d =: g+hw runs through the ring of integers of the number field, verifying */
                /* normsquare(cz -d) +r^2*normsquare(c) = r/rho. */

        hBound = abs(cz_omegacoeff) +sqrt( (rByRho -rsquare*norm(c))/m);
        /* hBound >= abs(h) */

        for( h = -round(hBound), round( hBound),

          discriminant = rByRho -m*(cz_omegacoeff-h)^2 -rsquare*norm(c);

          /* Check that 'discriminant' is a rational square: */
          if( type(discriminant) == "t_FRAC" || type(discriminant) == "t_INT",
            if( issquare( discriminant, &squareRoot),

              /* obtain g with the positive squareroot */
              g = cz_real +squareRoot;
              /* Check that g is a rational integer */
              if( frac(g) == 0,
                vertexTransport = concat( vertexTransport, getRemainingEntries(
                        round(g),h,c,z,cz,Zeta,rsquare,rhosquare,
                        rsquareByRhosquare, rZetaByRhoSquare, rByRho));
              );
              /* obtain g with the negative squareroot */
              g = cz_real -squareRoot;
              /* Check that g is a rational integer */
              if( frac(g) == 0,
                vertexTransport = concat( vertexTransport, getRemainingEntries(
                        round(g),h,c,z,cz,Zeta,rsquare,rhosquare,
                        rsquareByRhosquare, rZetaByRhoSquare, rByRho));
              );
            );
          );
        );
      );
    );
  );
  if( length(vertexTransport) > 0,
        /* Delete double occurencies in the vertexTransport list: */
        listsort(vertexTransport, 1);

        if( abs(z -Zeta) != [1,0]~ && abs(z -Zeta) != [0,1]~,
                /* There are length(vertexTransport) matrices sending p to p_2. */
        );

        if( length( vertexTransport) != stabilizerCardinal,
                print("***Error in function getIdentificationMatrices, on vertexTransport from ",
                        p," to ",p_2,": ","It must be a coset of the two vertex stabilizers!");
                for (r = 1, length(vertexTransport),
                        print(vertexTransport[r]);
                        print();
                );
        );
  );
 );
 /* return the list */ vertexTransport
};
```

Figure A.14: The function getIdentificationMatrices, continued from figure A.13.

Bibliography

[1] Bill Allombert, Christian Batut, Karim Belabas, Dominique Bernardi, Henri Cohen, Francisco Diaz y Diaz, Yves Eichenlaub, Xavier Gourdon, Louis Granboulan, Bruno Haible, Guillaume Hanrot, Pascal Letard, Gerhard Niklasch, Michel Olivier, Thomas Papanikolaou, Xavier Roblot, Denis Simon, Emmanuel Tollis, Ilya Zakharevitch, and the PARI group, *PARI/GP, version* `2.4.3`, specialized computer algebra system, Bordeaux, 2010, `http://pari.math.u-bordeaux.fr/`.

[2] Avner Ash, *Deformation retracts with lowest possible dimension of arithmetic quotients of self-adjoint homogeneous cones*, Math. Ann. **225** (1977), no. 1, 69–76. MR 0427490 (55 #522), Zbl 0343.20026

[3] Ralf Aurich, Frank Steiner, and Holger Then, *Numerical computation of Maass waveforms and an application to cosmology*, Contribution to the Proceedings of the "International School on Mathematical Aspects of Quantum Chaos II", to appear in Lecture Notes in Physics (Springer) (2004).

[4] Paul Baum, Alain Connes, and Nigel Higson, *Classifying space for proper actions and K-theory of group C^*-algebras*, C^*-algebras: 1943–1993 (San Antonio, TX, 1993), Contemp. Math., vol. 167, Amer. Math. Soc., Providence, RI, 1994, pp. 240–291. MR **1292018 (96c:**46070)

[5] Karim Belabas and Herbert Gangl, *Generators and relations for $K_2\mathcal{O}_F$*, K-Theory **31** (2004), no. 3, 195–231, DOI 10.1023/B:KTHE.0000028979.91416.00. MR **2067570 (2005e:**11154)

[6] Ethan Berkove, *The integral cohomology of the Bianchi groups*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1033–1049 (electronic). MR **2187644 (2006h:**20073), Zbl pre02237880

[7] ———, *The mod-2 cohomology of the Bianchi groups*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4585–4602, DOI 10.1090/S0002-9947-00-02505-8. MR **1675241 (2001b:**11043)

[8] Luigi Bianchi, *Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginarî*, Math. Ann. **40** (1892), no. 3, 332–412 (Italian). MR 1510727, JFM 24.0188.02

[9] Samuel Boissière, Etienne Mann, and Fabio Perroni, *The cohomological crepant resolution conjecture for* $\mathbb{P}(1,3,4,4)$, Internat. J. Math. **20** (2009), no. 6, 791–801, DOI 10.1142/S0129167X09005479. MR 2541935

[10] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR **1744486 (2000k:**53038)

[11] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. **87**, Springer-Verlag, New York, 1982. MR **672956 (83k:**20002), Zbl 0584.20036

[12] Weimin Chen and Yongbin Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248** (2004), no. 1, 1–31. MR **2104605 (2005j:**57036), Zbl 1063.53091

[13] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette, *Groups with the Haagerup property*, Progress in Mathematics, vol. 197, Birkhäuser Verlag, Basel, 2001. Gromov's a-T-menability. MR **1852148 (2002h:**22007)

[14] Pierre-Alain Cherix, Florian Martin, and Alain Valette, *Spaces with measured walls, the Haagerup property and property (T)*, Ergodic Theory Dynam. Systems **24** (2004), no. 6, 1895–1908, DOI 10.1017/S0143385704000185. MR **2106770 (2005i:**22006)

[15] John E. Cremona, *Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields*, Compositio Math. **51** (1984), no. 3, 275–324. MR **743014 (85j:**11063)

[16] Jürgen Elstrodt, Fritz Grunewald, and Jens Mennicke, PSL(2) *over imaginary quadratic integers*, Arithmetic Conference (Metz, 1981), Astérisque, vol. 94, Soc. Math. France, Paris, 1982, pp. 43–60. MR **702365 (84g:**10051)

[17] ———, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR **1483315 (98g:**11058), Zbl 0888.11001

[18] Barbara Fantechi and Lothar Göttsche, *Orbifold cohomology for global quotients*, Duke Math. J. **117** (2003), no. 2, 197–227. MR **1971293 (2004h:**14062), Zbl 1086.14046

[19] Benjamin Fine, *Algebraic theory of the Bianchi groups*, Monographs and Textbooks in Pure and Applied Mathematics, vol. **129**, Marcel Dekker Inc., New York, 1989. MR **1010229 (90h:**20002), Zbl 0760.20014

[20] Dieter Flöge, *Zur Struktur der* PSL_2 *über einigen imaginär-quadratischen Zahlringen*, Math. Z. **183** (1983), no. 2, 255–279 (German). MR **704107 (85c:**11043), Zbl 0498.20036

[21] ———, *Dissertation: Zur Struktur der* PSL_2 *über einigen imaginär-quadratischen Zahlringen*, Johann-Wolfgang-Goethe-Universität, Fachbereich Mathematik, Frankfurt am Main, 1980 (German). Zbl 0482.20032

[22] Paul E. Gunnells, *Modular symbols for* $\mathbf{Q}$*-rank one groups and Voronoĭ reduction*, J. Number Theory **75** (1999), no. 2, 198–219. MR **1681629 (2000c:**11084), Zbl 0977.11023

[23] Nigel Higson and Gennadi Kasparov, *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. **144** (2001), no. 1, 23–74, DOI 10.1007/s002220000118. MR **1821144 (2002k:**19005)

[24] Georges Humbert, *Sur la réduction des formes d'Hermite dans un corps quadratique imaginaire*, C. R. Acad. Sci. Paris **16** (1915), 189–196 (French).

[25] Pierre Julg and Gennadi Kasparov, *Operator K-theory for the group* $\mathrm{SU}(n,1)$, J. Reine Angew. Math. **463** (1995), 99–152. MR **1332908 (96g:**19006)

[26] Colin Maclachlan and Alan W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, vol. **219**, Springer-Verlag, New York, 2003. MR **1937957 (2004i:**57021), Zbl 1025.57001

[27] Eduardo R. Mendoza, *Cohomology of* PGL_2 *over imaginary quadratic integers*, Bonner Mathematische Schriften [Bonn Mathematical Publications], **128**, Dissertation, Rheinische Friedrich-Wilhelms-Universität, Mathematisches Institut, Bonn, 1979. MR **611515 (82g:**22012), Zbl 0464.12005

[28] Guido Mislin and Alain Valette, *Proper group actions and the Baum-Connes conjecture*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2003. MR **2027168 (2005d:**19007), Zbl 1028.46001

[29] Henri Poincaré, *Mémoire*, Acta Math. **3** (1966), no. 1, 49–92 (French). Les groupes kleinéens. MR 1554613

[30] Alexander D. Rahm and Mathias Fuchs, *The integral homology of* PSL_2 *of imaginary quadratic integers with non-trivial class group*, to appear in the Journal of Pure and Applied Algebra. arXiv: 0903.4517 [math.KT] `http://hal.archives-ouvertes.fr/hal-00370722/fr/`, (2009).

[31] Alexander D. Rahm, *Bianchi.gp*, Open source program to be released on `http://www-fourier.ujf-grenoble.fr/~rahm/`, October 31st, 2010.

[32] John G. Ratcliffe, *Foundations of hyperbolic manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 149, Springer, New York, 2006. MR **2249478 (2007d:**57029), Zbl 1106.51009

[33] Robert Riley, *Applications of a computer implementation of Poincaré's theorem on fundamental polyhedra*, Math. Comp. **40** (1983), no. 162, 607–632. MR **689477 (85b:**20064), Zbl 0528.51010

[34] ———, *POINCARE*, A library of FORTRAN routines, available at `ftp://ftp.math.binghamton.edu/pub/poincare/`.

[35] Rubén J. Sánchez-García, *Bredon homology and equivariant K-homology of* $\mathrm{SL}(3,\mathbb{Z})$, J. Pure Appl. Algebra **212** (2008), no. 5, 1046–1059. MR **2387584 (2009b:**19007), Zbl 1145.19005

[36] ———, *Equivariant K-homology for some Coxeter groups*, J. Lond. Math. Soc. (2) **75** (2007), no. 3, 773–790, DOI 10.1112/jlms/jdm035. MR **2352735 (2009b:**19006)

[37] Joachim Schwermer and Karen Vogtmann, *The integral homology of* SL_2 *and* PSL_2 *of Euclidean imaginary quadratic integers*, Comment. Math. Helv. **58** (1983), no. 4, 573–598. MR **728453 (86d:**11046), Zbl 0545.20031

[38] Richard G. Swan, *Generators and relations for certain special linear groups*, Advances in Math. **6** (1971), 1–77. MR 0284516 (**44** #1741), Zbl 0221.20060

[39] Alain Valette, *Introduction to the Baum-Connes conjecture*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji; With an appendix by Guido Mislin. MR **1907596 (2003f:**58047)

[40] Karen Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), no. 3, 399–419. MR **799670 (87a:**22025), Zbl 0545.20031

[41] C. Terence C. Wall, *Resolutions for extensions of groups*, Proc. Cambridge Philos. Soc. **57** (1961), 251–255. MR 0178046 (31 #2304)

[42] Jeffrey Weeks, *SnapPea*, program for the study of hyperbolic 3-manifolds, available at `http://geometrygames.org/SnapPea/`.

[43] Dan Yasaki, *Hyperbolic tessellations associated to Bianchi groups*, Algorithmic Number Theory. 9th International Symposium, Nancy, France, ANTS-IX, July 19-23, 2010, Proceedings.

[44] Eric Zaslow, *Topological orbifold models and quantum cohomology rings*, Comm. Math. Phys. **156** (1993), no. 2, 301–331. MR **1233848 (94i:**32045)

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Résumé. Cette thèse consiste d'une étude de la géométrie d'une certaine classe de groupes arithmétiques, à travers d'une action propre sur un espace contractile. Nous calculons explicitement leur homologie de groupe, et leur K-homologie équivariante. Plus précisément, considérons un corps de nombres quadratique imaginaire et son anneau d'entiers $\mathcal{O}$. Les groupes de Bianchi sont les groupes $\mathrm{SL}_2(\mathcal{O})$ et $\mathrm{PSL}_2(\mathcal{O})$. Ces groupes agissent d'une manière naturelle sur l'espace hyperbolique à 3 dimensions. Ils constituent une clef pour l'étude d'une classe plus large de groupes, les groupes Kleiniens, étudiés depuis Poincaré. En fait, chaque groupe Kleinien arithmétique non-cocompact est commensurable avec un des groupes de Bianchi. L'auteur a implémenté à l'ordinateur, le calcul d'un domaine fondamental pour ces groupes. En calculant les stabilisateurs et identifications sur ce domaine fondamental, nous obtenons une structure explicite d'orbi-espace. Nous nous en servons pour étudier des aspects différents de la géométrie des groupes de Bianchi. D'abord, nous calculons l'homologie de groupe à coefficients entiers, à l'aide de la suite spectrale équivariante de Leray/Serre. Ensuite, nous calculons l'homologie de Bredon de groupes de Bianchi, de laquelle nous déduisons leur K-homologie équivariante. Par la conjecture de Baum/Connes, qui est vérifiée par nos groupes, nous obtenons la K-théorie des C*-algèbres réduites de nos groupes. Finalement, nous complexifions nos orbi-espaces, en complexifiant l'espace hyperbolique. Ceci nous permet de calculer la cohomologie d'orbi-espace de Chen/Ruan, qui est l'un des deux côtés de la conjecture de la résolution cohomologique crépante de Ruan.

Mots-clés. Homologie de groupes arithmétiques, théorie des nombres, espace classifiant pour actions propres, topologie algébrique, cohomologie d'orbifold de Chen/Ruan, conjecture de Baum/Connes.

Abstract. This thesis consists of the study of the geometry of a certain class of arithmetic groups, by means of a proper action on a contractible space. We will explicitly compute their group homology, and their equivariant K-homology. More precisely, consider an imaginary quadratic number field, and its ring of integers $\mathcal{O}$. The Bianchi groups are the groups $\mathrm{SL}_2(\mathcal{O})$ and $\mathrm{PSL}_2(\mathcal{O})$. These groups act in a natural way on hyperbolic three-space. The Bianchi groups are a key to the study of a larger class of groups, the Kleinian groups, which dates back to works of Poincaré. In fact, each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group. The author has implemented the computation of a fundamental domain for the Bianchi groups. By computing the stabilisers and identifications on this fundamental domain, we obtain an explicit orbifold structure. We use it to study different aspects of the geometry of our groups. Firstly, we compute group homology with integer coefficients, using the equivariant Leray/Serre spectral sequence. Secondly, we compute the Bredon homology of the Bianchi groups, from which we deduce their equivariant K-homology. By the Baum/Connes conjecture, which is verified by the Bianchi groups, we obtain the K-theory of the reduced C*-algebras of the Bianchi groups, as isomorphic images. Finally, we complexify our orbifolds, by complexifying the real hyperbolic three-space. We obtain orbifolds given by the induced action of the Bianchi groups on complex hyperbolic three-space. Then we compute the Chen/Ruan orbifold cohomology for these complex orbifolds. This is one side of Ruan's cohomological crepant resolution conjecture.

Keywords. Homology of arithmetic groups, number theory, classifying space for proper actions, algebraic topology, Chen/Ruan orbifold cohomology, Baum/Connes conjecture.